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Local asymptotic stabilization for a class of uncertain upper-triangular systems[☆]

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ABSTRACT

This paper considers local asymptotic stabilization of a class of uncertain upper-triangular systems. It shows that, by appropriately increasing the powers of the states in a linear controller, an uncertain upper-triangular system can be locally asymptotically stabilized. A nested nonlinear controller is designed by introducing the notion of homogeneity with strictly decreasing degrees. For the stability analysis, a common Lyapunov/Chetaev function is constructed and a necessary and sufficient condition for the local asymptotic stability is established.

1. Introduction

In this paper, we consider the following uncertain upper-triangular system

$$\begin{aligned} \dot{x}_i &= a_i x_{i+1} + f_i(x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, n-1, \\ \dot{x}_n &= a_n u, \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \Re^n$ is the system state vector, $u \in \Re$ is the control input, a_i, $i = 1, \ldots, n-1$, are unknown positive constants and f_i, $i = 1, \ldots, n-1$, are functions satisfying the basic assumption as follows:

Assumption 1. There exist positive constants $\delta_i > 0$ and $c_i \geq 0$ such that the functions f_i satisfy

$$|f_i(x_{i+1}, \ldots, x_n)| \leq c_i \Big(|x_{i+1}|^{1+\delta_i} + \sum_{j=i+2}^{n} |x_j| \Big), \quad 1 \leq i \leq n-1, \tag{2}$$

in a neighborhood of the origin, where c_i and δ_i do not need to be known.

Many practical control systems such as the cart-pendulum system (Mazenc & Bowong, 2003) and the ball and beam system (Barbu, Sepulchre, Lin, & Kokotovic, 1997) can be modeled as upper-triangular systems. The stabilization problem of upper-triangular systems has been extensively investigated in the literature, including Arcak, Teel, and Kokotovic (2001), Ding, Qian, and Li (2010), Krishnamurthy and Khorrami (2004), Qian and Lin (2012), Teel (1992) and Ye and Jiang (1998). However, most of those stabilization results were achieved under the assumptions that (i) the parameters a_i or their bounds are known, and (ii) the functions f_i are bounded by known higher-order nonlinearities. Without these two conditions, the stabilization problem of (1) becomes very challenging. In particular, when a_i are unknown, it is impossible to design a linear feedback controller $u = -(k_1 x_1 + k_1 x_2 + \cdots + k_n x_n)$ to achieve the local asymptotic stability of the equilibrium $x = 0$ for all unknown parameters $a_i > 0$ even when $f_i = 0$, $i = 1, \ldots, n-1$. For example, when $n = 3$ and $f_i = 0$ for $i = 1, 2$, the characteristic polynomial of the closed-loop system of

$$\dot{x}_1 = a_1 x_2, \ \dot{x}_2 = a_2 x_3, \ \dot{x}_3 = -a_3(k_1 x_1 + k_2 x_2 + k_3 x_3) \tag{3}$$

is $\lambda^3 + a_3 k_3 \lambda^2 + a_2 a_3 k_2 \lambda + a_1 a_2 a_3 k_1$, which is Hurwitz stable if and only if $\frac{a_3}{a_1} > \frac{k_1}{k_2 k_3}$, due to the classical Routh stability criterion. Hence, for any given linear feedback controller, system (3) cannot be locally asymptotically stable for all the uncertain parameters a_i.

In addition, the linear feedback controller is sensitive to the upper-triangular , e.g., f_i in (1). For example, the system

$$\dot{x}_1 = x_2 + f_1, \ \dot{x}_2 = x_3, \ \dot{x}_3 = -x_1 - x_2 - 2x_3, \tag{4}$$

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is asymptotically stable when $f_i = 0$, but its stability is lost when $f_1 = -x_3$. To overcome the limitation of linear controllers, we change the linear structure of the controller by increasing the powers of the states. For example, when $n = 3$, the linear controller $u = -(k_1x_1 + k_2x_2 + k_3x_3)$ is changed into

$$u = -(((k_1x_1)^{5/3} + k_2x_2)^{33/25} + k_3x_3) \tag{5}$$

with any given positive gains k_i. We are going to show that the nonlinear controller (5) can guarantee the local asymptotic stability of the equilibrium $x = 0$ of system (1) with $n = 3$ for all possible unknown positive parameters a_i and functions f_i satisfying Assumption 1 with some δ_i and c_i.

It should be noted that Assumption 1 is less restrictive than the traditional higher-order growth condition (Teel, 1992). From the right side of (2), we see that only the power of $|x_{i+1}|$ is greater than 1, while in Teel (1992) this is also required for $|x_{i+2}|$ through $|x_n|$. Moreover, the constants $\delta_i > 0$ and $c_i \geq 0$ in Assumption 1 do not need to be known.

One underlying philosophy in appropriately increasing the powers of the states of a linear controller is the theory of homogeneous systems (Hermes, 1995; Rosier, 1992). However, the traditional homogeneity is still not sufficient to handle the unknown parameters a_i due to the strong similarity between linear systems and homogeneous systems.

To overcome the limitation of traditional homogeneity, in Polendo and Qian (2007) and Zhang, Qian, and Li (2013), the concept of homogeneity with monotone degrees (HMD) is proposed for the stability analysis and the controller design of inherently nonlinear systems. In our early paper (Zhu & Qian, 2018), the following power integrator system driven by a linear feedback controller is investigated:

$$\begin{aligned} \dot{x}_i &= x_{i+1}^{p_i}, \quad i = 1, 2, \dots, n-1, \\ \dot{x}_n &= -(k_1x_1 + k_2x_2 + \cdots + k_nx_n)^{p_n}, \end{aligned} \tag{6}$$

where $x = (x_1, x_2, \dots, x_n)$ is the state vector, $p_1 > p_2 > \cdots > p_n \geq 1$ are ratios of positive odd integers, and k_i are constants. System (6) has a homogeneity with strictly decreasing degrees (HSDD) with respect to the homogeneous weight vector $(1, 1, \dots, 1) \in \Re^n$; see Definition 2. HSDD plays an instrumental role in distinguishing the convergence rate of each state and consequently proving the stability result. A natural question is whether the result of Zhu and Qian (2018) can be extended by using the HSDD with respect to the general homogeneous weight vector $(r_1,\ r_2,\ \dots, r_n)$ instead of $(1, 1, \dots, 1) \in \Re^n$.

In this paper, we reveal that a linear controller can be changed into a nested nonlinear form to achieve the local asymptotic stability of $x = 0$ for system (1) with respect to all the unknown positive parameters and a class of uncertain upper-triangular perturbations satisfying Assumption 1. By using the Lyapunov second method and Chetaev instability theorem, a necessary and sufficient condition for the local asymptotic stability of the equilibrium $x = 0$ of the closed-loop system for all uncertainties is established.

2. Stabilization with a nested controller

In this section, we first introduce the concept of HSDD. Then, based on the notion of HSDD, we can increase the powers of the states in a linear controller to locally asymptotically stabilize the uncertain upper-triangular system (1).

Definition 2. A continuous vector field $v : \Re^n \to \Re^n$ with $v = [v_1, \dots, v_n]^{\mathrm{T}}$ is said to satisfy homogeneity with strictly decreasing degrees (HSDD), if we can find positive real numbers $(r_1, \dots, r_n)$ and real numbers $\mu_1 > \mu_2 > \cdots > \mu_n$ such that $v_i(\epsilon^{r_1}x_1, \dots, \epsilon^{r_n}x_n) = \epsilon^{r_i+\mu_i}v_i(x)$ for all $x \in \Re^n$, $\epsilon > 0$ and $i = 1, 2, \dots, n$. The constants r_i and μ_i are called homogeneous weights and degrees, respectively.

HSDD is a special case of the homogeneity with monotone degrees (HMD); the latter merely requires $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ (Polendo & Qian, 2008). If all the degrees μ_i are the same, HMD becomes the traditional homogeneity (Hermes, 1995).

As discussed in the introduction, a linear controller cannot guarantee the local asymptotic stability of the equilibrium $x = 0$ for (1) for all the unknown constants a_i. Choosing constants $r_1 \geq 1$ as a ratio of two positive odd integers, and μ_i, $i = 1, \dots, n-1$, as ratios of an even positive and an odd positive integers satisfying $\mu_1 > \mu_2 > \cdots > \mu_{n-1} > 0 =: \mu_n$, we define the powers r_i as $r_{i+1} = r_i + \mu_i$, $\quad i = 1, \dots, n$. It is easy to check that every $r_i \geq 1$ is a ratio of two positive odd integers. With the help of the above powers, we replace the linear controller $u = -(k_1x_1 + k_2x_2 + \cdots + k_nx_n)$ by

$$\begin{aligned} u = &-((\cdots((k_1x_1^{\frac{r_2}{r_1}} + k_2x_2)^{\frac{r_3}{r_2}} + k_3x_3)^{\frac{r_4}{r_3}} + \cdots \\ &+ k_{n-1}x_{n-1})^{\frac{r_n}{r_{n-1}}} + k_nx_n) =: -\phi_n(x_1, \dots, x_n). \end{aligned} \tag{7}$$

Remark 3. The function ϕ_n in controller (7) can be obtained by the following recurrence:

$$\begin{aligned} \phi_1(x_1) &= k_1x_1, \\ \phi_{i+1}(x_1, \dots, x_{i+1}) &= \big(\phi_i(x_1, \dots, x_i)\big)^{r_{i+1}/r_i} + k_{i+1}x_{i+1} \end{aligned} \tag{8}$$

for each $i = 1, 2, \dots, n-1$.

The closed-loop system (1) under controller (7) is

$$\begin{aligned} \dot{x}_i &= a_ix_{i+1} + f_i(x_{i+1}, \dots, x_n), \quad i = 1, 2, \dots, n-1, \\ \dot{x}_n &= -a_n\phi_n(x_1, \dots, x_n). \end{aligned} \tag{9}$$

In order to analyze the local asymptotic stability of $x = 0$, we transform system (9) to a new system via a diffeomorphism around the origin. Denote each ϕ_i by e_i and let $e = [e_1, e_2, \dots, e_n]^{\mathrm{T}} = \Phi(x)$. It is easy to check that the inverse mapping of $e = \Phi(x)$ is

$$x_1 = k_1^{-1}e_1,\ \ x_i = k_i^{-1}(e_i - e_{i-1}^{r_i/r_{i-1}}), \quad i = 2, \dots, n. \tag{10}$$

Since $r_i > r_{i-1}$, both the transformation $e = \Phi(x)$ and its inverse mapping (10) are continuously differentiable, which implies that the transformation $e = \Phi(x)$ is a diffeomorphism. Therefore, the closed-loop system (9), or (1) and (7), can be equivalently transformed into the following system:

$$\dot{e}_1 = \frac{k_1a_1}{k_2}(e_2 - e_1^{\frac{r_2}{r_1}}) + k_1\tilde{f}_1(e) = -\frac{k_1a_1}{k_2}e_1^{\frac{r_2}{r_1}} + g_1(e), \tag{11}$$

$$\begin{aligned} \dot{e}_i = &\frac{k_ia_i}{k_{i+1}}(e_{i+1} - e_i^{\frac{r_{i+1}}{r_i}}) + k_i\tilde{f}_i(e) \\ &+ \frac{r_i}{r_{i-1}}e_{i-1}^{\frac{r_i}{r_{i-1}}-1}\Big(-\frac{k_{i-1}a_{i-1}}{k_i}e_{i-1}^{\frac{r_i}{r_{i-1}}} + g_{i-1}(e)\Big) \\ = &-\frac{k_ia_i}{k_{i+1}}e_i^{\frac{r_{i+1}}{r_i}} + g_i(e), \qquad i = 2, 3, \dots, n-1, \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{e}_n = &-k_na_ne_n + \frac{r_n}{r_{n-1}}e_{n-1}^{\frac{r_n}{r_{n-1}}-1}\Big(-\frac{k_{n-1}a_{n-1}}{k_n}e_{n-1}^{\frac{r_n}{r_{n-1}}} + g_{n-1}(e)\Big) \\ = &-k_na_ne_n + g_n(e), \end{aligned} \tag{13}$$

where $\tilde{f}_i = f_i \circ \Phi^{-1}$, $\quad i = 1, 2, \dots, n-1$, and g_i are recursively defined by

$$g_1(e) = \frac{k_1a_1}{k_2}e_2 + k_1\tilde{f}_1(e), \tag{14}$$

$$g_i(x) = \frac{k_ia_i}{k_{i+1}}e_{i+1} + k_i\tilde{f}_i(e)$$

$$+\frac{r_i}{r_{i-1}}e_{i-1}^{\frac{r_i}{r_{i-1}}-1}\Big(-\frac{k_{i-1}a_{i-1}}{k_i}e_{i-1}^{\frac{r_i}{r_{i-1}}}+g_{i-1}(e)\Big),$$
$$i=2,\ldots,n, \tag{15}$$
$$g_n(e)=\frac{r_n}{r_{n-1}}e_{n-1}^{\frac{r_n}{r_{n-1}}-1}\Big(-\frac{k_{n-1}a_{n-1}}{k_n}e_{n-1}^{\frac{r_n}{r_{n-1}}}+g_{n-1}(e)\Big). \tag{16}$$

Remark 4. The above system transformation is valid when the uncertain parameters a_i are constant. When the latter are time-varying, the transformation is no longer applicable.

Corresponding to Assumption 1 imposed on uncertain functions of the original system, we can have an estimation on g_i as shown in the following proposition, whose proof is listed in Appendix.

Proposition 5. *For $i=1,\ldots,n$, there are positive constants C_i and $\hat{\delta}_i$ such that*

$$|g_i(e)|\le C_i\Big(\sum_{j=1}^{i-1}|e_j|^{\frac{\mu_j+r_i}{r_j}}+|e_i|^{\frac{\mu_i+r_i}{r_i}(1+\hat{\delta}_i)}+\sum_{j=i+1}^{n}|e_j|\Big) \tag{17}$$

in a neighborhood of the origin.

Now, we can present our main result in this paper.

Theorem 6. *Assume that all the f_i satisfy Assumption 1, the gains k_i are nonzero, and r_i are ratios of positive odd integers determined by $r_{i+1}=r_i+\mu_i$, $i=1,\ldots,n$, where $r_1>0$ and $\mu_1>\mu_2>\cdots>\mu_{n-1}>\mu_n=0$. Then the uncertain system (9) is locally asymptotically stable for all $a_i>0$ and f_i satisfying Assumption 1 if and only if $k_i>0$ for all $i=1,2,\ldots,n$.*

Proof. Construct the following Lyapunov/Chetaev function $V(e)=\sum_{i=1}^{n}\frac{l_i}{\alpha_i}e_i^{\alpha_i}$, where

$$l_i=-k_{i+1}k_i^{-1}a_i^{-1},\ i=1,2,\ldots,n-1,\ l_n=-k_n^{-1}a_n^{-1} \tag{18}$$

and $\alpha_i=2r_nr_i^{-1}\ge 2$, $i=1,2,\ldots,n$. The derivative of $V(e)$ along (12) can be easily calculated as

$$\dot V(e)=\sum_{i=1}^{n}e_i^{\alpha_i-1}e_i^{\frac{r_{i+1}}{r_i}}+\sum_{i=1}^{n}l_ie_i^{\alpha_i-1}g_i(e), \tag{19}$$

where $r_{n+1}=r_n$ and $\alpha_n=2$. From (19), it follows that

$$\dot V(e)\ge\sum_{i=1}^{n}e_i^{m_i}-\sum_{i=1}^{n}|l_i||e_i|^{\alpha_i-1}|g_i(e)|, \tag{20}$$

where $m_i=\alpha_i-1+\frac{r_{i+1}}{r_i}=\frac{\mu_i+2r_n}{r_i}$, $i=1,2,\ldots,n$ are ratios of an even integer and an odd one. Applying (17) to (20), we can find constants ρ_i, $\hat{\rho}_i$ and $\tilde{\rho}_i$ such that

$$\dot V(e)\ge\sum_{i=1}^{n}e_i^{m_i}-\sum_{i=1}^{n}\rho_i|e_i|^{\alpha_i-1+\frac{\mu_i+r_i}{r_i}(1+\hat{\delta}_i)}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{i-1}\tilde{\rho}_i|e_i|^{\alpha_i-1}|e_j|^{\frac{\mu_j+r_i}{r_j}}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{\rho}_i|e_i|^{\alpha_i-1}|e_j|.$$

Denoting $\zeta_i=|e_i|^{m_i}$ for $i=1,\ldots,n$, we can rewrite the above inequality as

$$\dot V(e)\ \ge\ \sum_{i=1}^{n}\zeta_i-\sum_{i=1}^{n}\rho_i\zeta_i^{\frac{\alpha_i-1}{m_i}+\frac{r_{i+1}}{m_ir_i}(1+\hat{\delta}_i)}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{i-1}\tilde{\rho}_i|\zeta_i|^{\frac{\alpha_i-1}{m_i}}|\zeta_j|^{\frac{\mu_j+r_i}{m_jr_j}}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{\rho}_i\zeta_i^{\frac{\alpha_i-1}{m_i}}\zeta_j^{\frac{1}{m_j}}$$
$$=:\sum_{i=1}^{n}\zeta_i-H(\zeta), \tag{21}$$

where $\zeta=[\zeta_1,\zeta_2,\ldots,\zeta_n]$.

In what follows, we verify that $H(\zeta)$ is composed of higher-order terms with respect to ζ.

First, by the expression of m_i below (20), it is clear that

$$\frac{\alpha_i-1}{m_i}+\frac{r_{i+1}}{m_ir_i}(1+\hat{\delta}_i)=1-\frac{r_{i+1}}{m_ir_i}+\frac{r_{i+1}}{m_ir_i}(1+\hat{\delta}_i)$$
$$=1+\frac{r_{i+1}}{m_ir_i}\hat{\delta}_i>1. \tag{22}$$

Similarly, we have

$$\frac{\alpha_i-1}{m_i}+\frac{\mu_j+r_i}{m_jr_j}=1-\frac{r_{i+1}}{\mu_i+2r_n}+\frac{\mu_j+r_i}{\mu_j+2r_n}.$$

In addition, since $\mu_j>\mu_i$ for $j<i$ and $2r_n>r_i$, the following holds

$$\frac{\alpha_i-1}{m_i}+\frac{\mu_j+r_i}{m_jr_j}>1-\frac{r_{i+1}}{\mu_i+2r_n}+\frac{\mu_i+r_i}{\mu_i+2r_n}=1. \tag{23}$$

Finally, by the fact that $r_j\ge r_{i+1}$ and $\mu_i>\mu_j$ for $j\ge i+1$, we have

$$\frac{\alpha_i-1}{m_i}+\frac{1}{m_j}=1-\frac{r_{i+1}}{m_ir_i}+\frac{1}{m_j}$$
$$=1-\frac{r_{i+1}}{\mu_i+2r_n}+\frac{r_j}{\mu_j+2r_n}>1. \tag{24}$$

With the help of inequalities (22), (23) and (24), we conclude that $H(\zeta)$ has an order higher than 1 in terms of ζ. Therefore, there exists a sufficiently small neighborhood $D\subset\Re^n$ of the origin such that

$$\dot V(e)>\varepsilon\sum_{i=1}^{n}\zeta_i=\varepsilon\sum_{i=1}^{n}e_i^{m_i},\quad\forall\ e\in D \tag{25}$$

for a constant $\varepsilon>0$. From (18), it is easily seen that

$$l_i<0\ (i=1,2,\ldots,n)\ \Leftrightarrow\ k_i>0\ (i=1,2,\ldots,n). \tag{26}$$

If $k_i>0$ $(i=1,2,\ldots,n)$, it is clear that $V(e)$ is negative definite due to (26). This, together with (25), implies that the zero solution of (9) is asymptotically stable by Lyapunov Stability Theorem. Therefore the positivity of k_i is sufficient for the local asymptotic stability of (9) .

On the other hand, if there exists a $k_i<0$, by (26) there exists an $l_j>0$. In this case, we know that the set $G:=\{e\in\Re^n\mid V(e)>0\}$ is not empty and $e=0$ is a boundary point of G. Therefore, from (25) and Chetaev Instability Theorem, it follows that $x=0$ for (9) is unstable. This implies that the positivity of k_i is also necessary for the local asymptotic stability of (9). □

3. Examples

Example 1. The dynamics of the orientation of a car can be described as follows (Mellodge & Kachroo, 2004; Zhu & Yuan, 2014):

$$\dot\theta=\frac{v}{l}\tan\phi,\quad\dot\phi=\omega,\quad\dot\omega=\frac{1}{J}u, \tag{27}$$

where θ is the car's angle with respect to the x-axis, ϕ is the steering wheel's angle with respect to the car longitudinal axis, ω is the angular velocity of the steering wheels, v is the linear velocity of the center of the rear axle during the cruise stage, l the length between the steering wheels and the rear wheels, J is the moment of inertia, and u is the external torque. If v, l and J are unknown constants, system (27) can be written as the form of (1) satisfying Assumption 1 with $f_1 = \frac{v}{l}(\tan\phi - \phi))$ and $f_2 = 0$. We choose $r_1 = 3$, $r_2 = 5$ and $r_3 = \frac{33}{5}$. Then by (7) we get the controller as follows:

$$u = -(((k_1\theta)^{5/3} + k_2\phi)^{33/25} + k_3\omega). \tag{28}$$

Let $k_1 = 1$, $k_2 = 2$ and $k_3 = 1$. The local asymptotic stability can be easily verified. With some simulations when $v = 2$, $l = 3$ and $J = 4$, we find that the closed ball $\mathcal{B}_{0.3} = \{x \in \Re^3 \mid \|x\| \le 0.3\}$ is a subset of the attracting region.

Example 2. Consider the upper-triangular system:

$$\begin{aligned} \dot{x}_1 &= a_1x_2 + b_{11}x_3 + b_{12}x_4 + c_{11}x_2^{1/3}x_4 + c_{12}x_3^{1/3}x_4^{2/3}, \\ \dot{x}_2 &= a_2x_3 + b_{21}x_4 + c_{21}x_3x_4, \\ \dot{x}_3 &= a_3x_4 + c_{31}x_4^2, \\ \dot{x}_4 &= a_4u, \end{aligned} \tag{29}$$

where $a_i > 0$, $b_{ij} \ne 0$ and $c_{ij} \ne 0$ are unknown parameters. Choose $r_1 = 5$, $r_2 = 17/3$, $r_3 = 31/5$, $r_4 = 33/5$ with $\mu_1 = 2/3 > \mu_2 = 8/15 > \mu_3 = 2/5 > \mu_4 = 0$, under which the nested nonlinear controller is designed as

$$u = -(((k_1x_1^{\frac{17}{15}} + k_2x_2)^{\frac{93}{85}} + k_3x_3)^{\frac{33}{31}} + k_4x_4). \tag{30}$$

By (29), we have that

$$\begin{aligned} f_1 &= b_{11}x_3 + b_{12}x_4 + c_{11}x_2^{1/3}x_4 + c_{12}x_3^{1/3}x_4^{2/3}, \\ f_2 &= b_{21}x_4 + c_{21}x_3x_4, \quad f_3 = c_{31}x_4^2. \end{aligned}$$

By the Young's Inequality, we get that

$$\begin{aligned} |f_1| \le\ & |b_{11}||x_3| + |b_{12}||x_4| + |c_{11}||x_2|^{4/3} + |c_{11}||x_4|^{4/3} \\ & + |c_{12}||x_3| + |c_{12}||x_4|. \end{aligned}$$

If $|x_4| < 1$, then $|f_1| \le c_1(|x_2|^{1+\delta_1} + |x_3| + |x_4|)$, $|f_2| \le c_2(|x_3|^{1+\delta_2} + |x_4|)$, $|f_3| \le c_3|x_4|^{1+\delta_3}$ for some $c_i > 0$, $\delta_1 = 1/3$ and $\delta_2 = \delta_3 = 1$. Therefore, Assumption 1 is satisfied and from Theorem 6 the local asymptotic stability of $x = 0$ for (29) with those unknown parameters follows. The attracting basin can be estimated by a numerical method. In particular, when $a_1 = 1$, $b_{11} = 2$, $b_{12} = 1.2$, $c_{11} = 3$, $c_{12} = -3$, $a_2 = 1.2$, $b_{21} = 1.1$, $c_{21} = -3$, $a_3 = 1.2$, $c_{31} = -3$, $a_4 = 1.1$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 1$, we find that at least the closed ball $\mathcal{B}_5 = \{x \in \Re^4 \mid \|x\| \le 5\}$ lies in the attracting basin.

4. Conclusion

This paper introduces a new concept of homogeneity with strictly decreasing degrees (HSDD) and an approach of designing a nested state feedback stabilizer for a class of uncertain upper-triangular nonlinear systems. It is shown that a linear controller can be converted into a nested nonlinear form to realize the local asymptotic stability of the equilibria of the uncertain upper-triangular systems.

Appendix. Proof of Proposition 5

Under (10) and the expression of $\tilde{f}_i$ below (13), Assumption 1 can be represented as follows

$$\begin{aligned} |\tilde{f}_i(e)| &\le c_i\Big(|k_{i+1}^{-1}(e_{i+1} - e_i^{\frac{r_{i+1}}{r_i}})|^{1+\delta_i} + \sum_{j=i+2}^{n} |k_j^{-1}||e_j - e_{j-1}^{\frac{r_j}{r_{j-1}}}|\Big) \\ &\le \tilde{c}_i(|e_i^{\frac{r_{i+1}}{r_i}}|^{1+\delta_i} + |e_{i+1}|^{1+\delta_i} + \sum_{j=i+2}^{n}(|e_j| + |e_{j-1}^{\frac{r_j}{r_{j-1}}}|)) \end{aligned}$$

in a neighborhood of the origin, where $\tilde{c}_i$ is a positive constant. Since $\frac{r_j}{r_{j-1}} > 1$, in a neighborhood of $e = 0$ there exists positive constant $\tilde{C}_i$ such that

$$|\tilde{f}_i(e)| \le \tilde{C}_i(|e_i|^{\frac{r_{i+1}}{r_i}(1+\delta_i)} + \sum_{j=i+1}^{n} |e_j|). \tag{A.1}$$

By the relationship of $g_i(e)$ and $g_{i-1}(e)$ defined in (12), there exist positive constants σ_i such that

$$|g_i(e)| \le \sigma_i(|e_{i+1}| + |\tilde{f}_i(e)| + |e_{i-1}|^{\frac{2r_i}{r_{i-1}}-1} + |e_{i-1}|^{\frac{r_i}{r_{i-1}}-1}|g_{i-1}(e)|) \tag{A.2}$$

for any $i = 1, 2, 3, \ldots, n$ with $g_0 = 0$ and $\tilde{f}_n = 0$. In the following, we prove (17) based on (A.1) and (A.2).

For the case of $i = 1$, from (14) and (A.1), it follows that there is a constant $C_1 > 0$ such that

$$|g_1(e)| \le C_1(|e_1|^{\frac{\mu_1+r_1}{r_1}(1+\delta_1)} + \sum_{j=2}^{n} |e_j|), \tag{A.3}$$

which implies that (17) holds for the case of $i = 1$.

Suppose that (17) holds for the case of $i - 1$, that is,

$$|g_{i-1}| \le C_{i-1}\Big(\sum_{j=1}^{i-2} |e_j|^{\frac{\mu_j+r_{i-1}}{r_j}} + |e_{i-1}|^{\frac{\mu_{i-1}+r_{i-1}}{r_{i-1}}(1+\hat{\delta}_{i-1})} + \sum_{j=i}^{n} |e_j|\Big). \tag{A.4}$$

By the Young's inequality, we have

$$|x|^c|y|^d \le |x|^{c+d} + |y|^{c+d}, \forall\, x,\ y \in \Re, \tag{A.5}$$

where c and d are positive constants. Letting $|x| = |e_{i-1}|^{\frac{1}{r_{i-1}}}$, $|y| = |g_{i-1}(e)|^{\frac{1}{r_i}}$, $c = \mu_{i-1}$, and $d = r_i$ and applying (A.5) to (A.2) yield

$$\begin{aligned} |g_i(e)| &\le \sigma_i(|e_{i+1}| + |\tilde{f}_i(e)| + |e_{i-1}|^{\frac{\mu_{i-1}+r_i}{r_{i-1}}} + |e_{i-1}|^{\frac{\mu_{i-1}}{r_{i-1}}}|g_{i-1}^{\frac{r_i}{r_i}}(e)|) \\ &\le \sigma_i(|e_{i+1}| + |\tilde{f}_i(e)| + 2|e_{i-1}|^{\frac{\mu_{i-1}+r_i}{r_{i-1}}} + |g_{i-1}(e)|^{\frac{\mu_{i-1}+r_i}{r_i}}). \end{aligned} \tag{A.6}$$

By (A.4) and Jensen's inequality, we have

$$\begin{aligned} |g_{i-1}(\cdot)|^{\frac{\mu_{i-1}+r_i}{r_i}} \le\ & n^{\frac{\mu_{i-1}}{r_i}} C_{i-1}^{\frac{\mu_{i-1}+r_i}{r_i}}\Big(\sum_{j=1}^{i-2} |e_j|^{\frac{\mu_j+r_{i-1}}{r_j}\cdot\frac{\mu_{i-1}+r_i}{r_i}} \\ & + |e_{i-1}|^{\frac{\mu_{i-1}+r_{i-1}}{r_{i-1}}\cdot\frac{\mu_{i-1}+r_i}{r_i}(1+\hat{\delta}_{i-1})} + |e_i|^{\frac{\mu_{i-1}+r_i}{r_i}} \\ & + \sum_{j=i+1}^{n} |e_j|^{\frac{\mu_{i-1}+r_i}{r_i}}\Big). \end{aligned} \tag{A.7}$$

For $j \le i - 2$, by the fact that $\mu_j > \mu_{i-1}$, we have

$$\begin{aligned} & \frac{\mu_j + r_{i-1}}{r_j} \cdot \frac{\mu_{i-1} + r_i}{r_i} - \frac{\mu_j + r_i}{r_j} \\ = & \frac{\mu_j\mu_{i-1} + r_{i-1}\mu_{i-1} + r_ir_{i-1} - r_i^2}{r_jr_i} \\ > & \frac{\mu_{i-1}\mu_{i-1} + r_{i-1}\mu_{i-1} - r_i\mu_{i-1}}{r_jr_i} = 0. \end{aligned} \tag{A.8}$$

In addition, considering the power of $|e_{i-1}|$,

$$\frac{\mu_{i-1} + r_{i-1}}{r_{i-1}} \cdot \frac{\mu_{i-1} + r_i}{r_i}(1+\hat{\delta}_{i-1}) = \frac{\mu_{i-1} + r_i}{r_{i-1}}(1+\hat{\delta}_{i-1}) \tag{A.9}$$

For the powers of $|e_i|$, we have

$$\frac{\mu_{i-1} + r_i}{r_i} = \frac{r_{i+1} + \mu_{i-1} - \mu_i}{r_i} = \frac{r_{i+1}}{r_i}(1 + \frac{\mu_{i-1} - \mu_i}{r_{i+1}}). \tag{A.10}$$

By (A.7)–(A.10), we conclude that in a neighborhood of the origin there exist positive constants $\tilde{\sigma}_i$ and $\tilde{\delta}_i = \frac{\mu_{i-1}-\mu_i}{r_{i+1}}$ such that

$$|g_{i-1}(e)|^{\frac{\mu_{i-1}+r_i}{r_i}} \le \tilde{\sigma}_i\Big(\sum_{j=1}^{i-2}|e_j|^{\frac{\mu_j+r_i}{r_j}} + |e_{i-1}|^{\frac{\mu_{i-1}+r_i}{r_{i-1}}} + |e_i|^{\frac{r_{i+1}}{r_i}(1+\tilde{\delta}_i)} + \sum_{j=i+1}^{n}|e_j|\Big). \tag{A.11}$$

Substituting (A.1) and (A.11) into (A.6) yields (17) for the case of i.

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