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A necessary and sufficient condition for stability of a class of planar nonlinear systems ☆

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ABSTRACT

In this paper, the stability problem of a class of planar nonlinear systems is investigated. Motivated by the Routh–Hurwitz stability criterion for planar linear systems, a necessary and sufficient condition for stability of a class of planar nonlinear systems is established. To prove the necessity, a new method called particular solution method is provided to justify instability. In addition, a sufficient condition for existence of oscillations is introduced. The necessary and sufficient condition can provide a simple way to design controllers by adjusting control parameters for some planar nonlinear systems.

1. Introduction

In this paper, we analyze stability problem of a class of planar nonlinear systems described by

$$\begin{cases} \dot{x}_1 = c_1x_1 + c_2x_2^p, \\ \dot{x}_2 = c_3x_1^{1/p} + c_4x_2, \end{cases} \tag{1}$$

where $x = (x_1\ x_2)^\top \in \mathbb{R}^2$ is state vector, p is a positive odd number, and $c_1, c_2, c_3, c_4 \in \mathbb{R}$ are constants. As $p = 1$, system (1) becomes a linear system $\dot{x}_1 = c_1x_1 + c_2x_2$, $\dot{x}_2 = c_3x_1 + c_4x_2$. It is well known that planar linear system $\dot{x} = Ax$ is asymptotically stable if and only if its characteristic polynomial $\det|\lambda I - A|$ is Hurwitz, that is c_1, c_2, c_3, c_4 satisfy $c_1 + c_4 < 0$ and $c_1c_4 > c_2c_3$ (Routh–Hurwitz stability criterion, Hurwitz, 1964; Routh, 1877). For the planar nonlinear system (1) with $p > 1$, a natural question is that if it has a result similar to the Routh–Hurwitz stability criterion.

Planar nonlinear systems are a kind of dynamical systems where the information propagation occurs in two independent directions (Dayawansa, Martin, & Knowles, 1990). Such systems are widely used to describe various practical and physical systems, like circuit analysis, image processing, seismographic data transmission, multidimensional digital filtering and thermal processes, etc. (Shtessel, Shkolnikov, & Levant, 2007). Stability problem is one of the most active topics in nonlinear control theory (Hahn, 1967). It is crucial as a step in achieving many other control problems such as output regulation, optimal control, disturbance decoupling and attenuation, etc. (Khalil & Grizzle, 2002).

Over the past few decades, a great number of interesting results have been obtained for stability of nonlinear systems (Li & Wu, 2016; Li, Yang, & Song, 2019; Ooba, 2012; Sontag & Sussmann, 1980; Wu, Yang, Shi, & Su, 2015), and many techniques, such as center manifold theory (Aeyels, 1985), the idea of zero dynamics (Byrnes & Isidori, 1991), homogeneous domination approach (Qian & Lin, 2006; Zhu & Qian, 2018), etc., have been proposed. Specially, by using sum of squares (SOS) techniques, Chesi (2013) provided an exact linear matrix inequality (LMI) condition for robust asymptotic stability of uncertain systems. In Aylward, Parrilo, and Slotine (2008), the robust stability properties of uncertain nonlinear systems with polynomial or rational dynamics were analyzed with convexity-based methods. In Lacerda and Seiler (2017), a systematic procedure was presented to investigate robust stability of uncertain systems in polytopic domains. However, due to lack of constructive applied approaches (Qian & Lin, 2001), there are few references to investigate necessary and sufficient conditions for stability of nonlinear systems.

For the following planar nonlinear system with input

$$\dot{x}_1 = x_1 - x_2^3, \quad \dot{x}_2 = u, \tag{2}$$

Bacciotti (1992) showed that there is no continuously differentiable state feedback controller stabilizing system (2). Kawski

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(1989) designed a controller $u = -ax_2 + bx_1^{1/3}$ and proved that system (2) is asymptotically stable if $b > a > 1$. It is easy to see that the system (2) with $u = -ax_2 + bx_1^{1/3}$ is a special case of (1). If there exists a necessary and sufficient condition for stability of system (1), then we can easily solve stabilization problem for system (2). Thus, it is meaningful to investigate necessary and sufficient conditions for stability of system (1).

Motivated by the Routh–Hurwitz stability criterion for planar linear control, we propose a necessary and sufficient condition for stability of system (1), that is system (1) is asymptotically stable if and only if c_1, c_2, c_3 and c_4 satisfy $c_1 + c_4 < 0$ and $c_1c_4^p > c_2c_3^p$. We prove the necessary and sufficient condition with particular solution method and Lyapunov method. This paper is organized as follows: In Section 2, we give a number of basic concepts. The main results are presented in Section 3, where a necessary and sufficient condition for stability is given, the problem of oscillation detection is addressed, and a sufficient condition for oscillation existence is formulated. Additionally, the application of stability condition in state-feedback control design is analyzed, and three examples are given. Concluding remarks are given in Section 4.

2. Preliminaries

This section presents some fundamental theorems and some useful lemmas which will play important roles in obtaining the main results of this paper.

Theorem 1 (*Lyapunov Stability Theorem, Lyapunov (1992)*). *Given a nonlinear system*

$$\dot{x} = f(x), x \in \mathbb{R}^n, \tag{3}$$

where $f(x)$ is Lipschitz continuous and $f(0) = 0$. System (3) is asymptotically stable, if there exists a continuously differentiable function $V(x)$ such that $V(x)$ is positive definite and $\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) < 0$ for all $x \neq 0$.

Theorem 2 (*LaSalle's Invariance Theorem, LaSalle (1960)*). *The nonlinear system (3) is asymptotically stable, if there exists a continuously differentiable function $V(x)$ such that $V(x)$ is positive definite, $\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) \leq 0$, and the only solution of $\dot{x} = f(x)$ and $\dot{V}(x) = 0$ is $x = 0$.*

Theorem 3 (*Chetaev Instability Theorem, Chetaev (1961)*). *The $x = 0$ is an unstable equilibrium point of (3), if there exists a continuously differentiable function $V(x)$ such that: (i) the origin is a boundary point of the set $G = \{x \in \mathbb{R}^n | V(x) > 0\}$, (ii) there exists a neighborhood U of the $x = 0$ such that $\dot{V}(x) > 0$ for any $x \in U \cap G$.*

Lemma 4 (*Du, Qian, Li, & Chu, 2019, Qian, 2002*). *For any positive integers m, n and real-valued function $\gamma(x, y) > 0$, the inequality $|x^m y^n| \leq \frac{m}{m+n}\gamma(x, y)|x|^{m+n} + \frac{n}{m+n}\gamma^{-\frac{m}{n}}(x, y)|y|^{m+n}$ holds.*

Lemma 5. *Consider a continuously differentiable function*

$$V(x) = \frac{p}{p+1}x_1^{\frac{p+1}{p}} + ax_1x_2 + \frac{b}{p+1}x_2^{p+1}, \tag{4}$$

where $x = (x_1\ x_2)^\top \in \mathbb{R}^2$, $b > 0$ and p is a positive odd number. If $a^{p+1} < b$, then $V(x)$ is positive definite.

Proof. Since $a^{p+1} < b$, $\varepsilon := 1 - \frac{|a|}{b^{1/(p+1)}}$ is a positive number satisfying $\varepsilon \in (0, 1)$. Noting $|a| = b^{1/(p+1)}(1 - \varepsilon)$, it follows from Lemma 4 that

$$|ax_1x_2| = \left|\left(x_1^{1/p}(1-\varepsilon)^{1/(p+1)}\right)^p \left(\left(b(1-\varepsilon)\right)^{1/(p+1)}x_2\right)\right|$$
$$\leq \frac{p}{p+1}|x_1^{1/p}|^{p+1}(1-\varepsilon) + \frac{1}{p+1}|x_2|^{p+1}b(1-\varepsilon).$$

Since p is a positive odd number, the above inequality yields

$$ax_1x_2 \geq -\frac{p(1-\varepsilon)}{p+1}x_1^{\frac{p+1}{p}} - \frac{b(1-\varepsilon)}{p+1}x_2^{p+1}. \tag{5}$$

Substituting (5) into (4), we have

$$V(x) \geq \frac{p}{p+1}x_1^{\frac{p+1}{p}} - \frac{p(1-\varepsilon)}{p+1}x_1^{\frac{p+1}{p}} - \frac{b(1-\varepsilon)}{p+1}x_2^{p+1} + \frac{b}{p+1}x_2^{p+1}$$
$$= \frac{p\varepsilon}{p+1}x_1^{\frac{p+1}{p}} + \frac{b\varepsilon}{p+1}x_2^{p+1},$$

which implies that $V(x) > 0$ for all $x \neq 0$. Therefore, $V(x)$ is positive definite. □

Lemma 6. *The nonlinear system (1) has a particular solution if there is a solution to $f(\lambda) = 0$ where*

$$f(\lambda) = \left(\frac{\lambda}{p} - c_4\right)^p (\lambda - c_1) - c_2c_3^p. \tag{6}$$

If $f(\lambda) = 0$ has non-negative solutions, there must exist a particular solution to system (1) which is not asymptotically stable.

Proof. When $(c_2, c_3) = (0, 0)$, it is clear that (6) has solutions $\lambda_1 = c_1$ and $\lambda_2 = c_4p$, and system (1) has a particular solution described as $x_1(t) = e^{c_1t}x_1(0)$, $x_2(t) = e^{c_4t}x_2(0)$. In addition, if one of λ_1 or λ_2 is non-negative, the above solution is not asymptotically stable.

When $(c_2, c_3) \neq (0, 0)$, without loss of generality, we assume $c_2 \neq 0$. Assume λ^* is a solution to $f(\lambda) = 0$. Then (1) has the following explicit solution

$$\begin{cases} x_1(t) = e^{\lambda^*t}x_1(0) \\ x_2(t) = \left(\frac{\lambda^* - c_1}{c_2}\right)^{1/p} e^{\frac{\lambda^*t}{p}} x_1^{1/p}(0) \end{cases} \tag{7}$$

for any $x_1(0)$. First, it can be verified that (7) satisfies $\dot{x}_1 = \lambda^* e^{\lambda^*t}x_1(0) = c_1x_1 + c_2x_2^p$, which means that (7) satisfies the first equation of (1). By $f(\lambda^*) = 0$, i.e., $\left(\frac{\lambda^*}{p} - c_4\right)^p(\lambda^* - c_1) - c_2c_3^p = 0$, we have

$$\left(\frac{\lambda^*}{p} - c_4\right)\left(\frac{\lambda^* - c_1}{c_2}\right)^{1/p} - c_3 = 0. \tag{8}$$

It follows from (7) and (8) that

$$\dot{x}_2 = \frac{\lambda^*}{p}\left(\frac{\lambda^* - c_1}{c_2}\right)^{1/p} e^{\frac{\lambda^*t}{p}} x_1^{1/p}(0)$$
$$= \left(c_4\left(\frac{\lambda^* - c_1}{c_2}\right)^{1/p} + c_3\right) e^{\frac{\lambda^*t}{p}} x_1^{1/p}(0)$$
$$= c_3\left(e^{\lambda^*t}x_1(0)\right)^{1/p} + c_4\left(\frac{\lambda^* - c_1}{c_2}\right)^{1/p} e^{\frac{\lambda^*t}{p}} x_1^{1/p}(0)$$
$$= c_3x_1^{1/p} + c_4x_2,$$

which means that (7) also satisfies the second equation of (1). Therefore, (7) is a particular solution of (1). In addition, if λ^* is non-negative, by the construction the solution (7) is not asymptotically stable. □

Remark 7. If (6) is replaced by

$$f(\lambda) = (\lambda - c_4)^p(p\lambda - c_1) - c_2c_3^p, \tag{9}$$

the result presented in Lemma 6 also holds. Thus, we can also use the solutions of (9) to solve particular solutions for system (1).

It is clear that system (1) becomes a planar linear system when $p = 1$, and (6) is its characteristic polynomial. Here, we call (6) p-order characteristic polynomial of the planar nonlinear system

(1). Lemma 6 implies that we can construct particular solutions for system (1) based on the solutions of p-order characteristic polynomial, and then justify instability of system (1).

3. Main results

In this section, we shall give the main results of this paper. We first analyze a necessary and sufficient condition for stability of system (1).

Theorem 8. *The nonlinear system (1) is asymptotically stable if and only if c_1, c_2, c_3 and c_4 satisfy $c_1 + c_4 < 0$ and $c_1c_4^p > c_2c_3^p$.*

Proof. (**sufficiency**) The sufficient proof is broken into two cases. **Case 1**: $c_1c_4 \geq 0$. Since $c_1 + c_4 < 0$, then $c_1 \leq 0$, $c_4 \leq 0$ and $(c_1, c_4) \neq (0, 0)$. Without loss of generality, assume $c_4 < 0$. Choose the Lyapunov function $V(x) = \frac{p}{p+1}x_1^{\frac{p+1}{p}} + \frac{b}{p+1}x_2^{p+1}$, $b > 0$, which is positive definite. The time derivative of $V(x)$ along with the trajectory of (1) is

$$\begin{aligned}\dot{V}(x) &= c_1x_1^{\frac{p+1}{p}} + (c_2 + bc_3)x_1^{1/p}x_2^p + bc_4x_2^{p+1}\\ &= \frac{(p+1)bc_4}{p}\Big(\frac{px_2^{p+1}}{p+1} + \frac{p(c_2 + bc_3)}{(p+1)bc_4}x_2^px_1^{1/p} + \frac{pc_1x_1^{\frac{p+1}{p}}}{(p+1)bc_4}\Big).\end{aligned}$$

If $c_2c_3 = 0$, we can choose b such that $\left(\frac{p(c_2+bc_3)}{(p+1)bc_4}\right)^{p+1} < \frac{pc_1}{bc_4}$ due to $c_1c_4^p > c_2c_3^p = 0$. If $c_2c_3 > 0$, set $b = pc_2/c_3$, then $\left(\frac{p(c_2+bc_3)}{(p+1)bc_4}\right)^{p+1} = \left(\frac{c_3}{c_4}\right)^{p+1} < \frac{c_1c_3}{c_2c_4} = \frac{pc_1}{bc_4}$ since $c_1c_4^p > c_2c_3^p > 0$. By Lemma 5, we have $\dot{V}(x) < 0$. By Theorem 1, system (1) is asymptotically stable. If $c_2c_3 < 0$, set $b = -pc_2/c_3$, then $\dot{V}(x) = c_1x_1^{\frac{p+1}{p}} - \frac{c_2c_4}{c_3}x_2^{p+1}$. Since $c_1 \leq 0$ and $c_4 < 0$, then $\dot{V}(x) \leq 0$. By Theorem 2, system (1) is asymptotically stable.

Case 2: $c_1c_4 < 0$. Since $c_2c_3^p < c_1c_4^p < 0$, then $c_2c_3 < 0$. Denote $a = \frac{(p+1)c_1c_4}{c_3(pc_1-c_4)}$ and $b = -\frac{c_2}{c_3}$. Consider the Lyapunov function

$$V(x) = \frac{p}{p+1}x_1^{\frac{p+1}{p}} + ax_1x_2 + \frac{b}{p+1}x_2^{p+1}.$$

By Lemma 5, $V(x)$ is positive definite. The time derivative of $V(x)$ along with the trajectory of (1) is

$$\begin{aligned}\dot{V}(x) &= (c_1 + ac_3)x_1^{\frac{p+1}{p}} + (c_1 + c_4)ax_1x_2 + (c_2a + c_4b)x_2^{p+1}\\ &= \frac{c_1(c_1 + c_4)(p+1)}{pc_1 - c_4}\Big(\frac{px_1^{\frac{p+1}{p}}}{p+1} + \frac{c_4x_1x_2}{c_3} + \frac{c_4c_2x_2^{p+1}}{c_1c_3(p+1)}\Big).\end{aligned}$$

Since $\left(\frac{c_4}{c_3}\right)^{p+1} < \frac{c_2c_4}{c_1c_3}$, by Lemma 5, we have $\dot{V}(x) < 0$. By Lyapunov Stability Theorem 1, system (1) is asymptotically stable.

(**Necessity**) We just need to prove that when $c_1 + c_4 \geq 0$ or $c_1c_4^p \leq c_2c_3^p$, system (1) is not asymptotically stable. This part is mainly based on Lemma 6 and is broken into three cases.

Case 1. $c_1c_4^p \leq c_2c_3^p$. By (6), we have $f(0) = c_1c_4^p - c_2c_3^p \leq 0$. Since $f(\lambda)$ is continuous and tends to positive infinity as x goes to positive infinity, $f(\lambda) = 0$ must have a non-negative solution. By Lemma 6, system (1) has a particular solution which is not asymptotically stable, that is the planar system (1) is not asymptotically stable.

Case 2. $c_1 + c_4 \geq 0$ and $c_2c_3^p \geq 0$. By (6), we have

$$f'(\lambda) = \Big(\frac{\lambda}{p} - c_4\Big)^{p-1}\Big(\Big(1 + \frac{1}{p}\Big)\lambda - (c_1 + c_4)\Big),$$

which implies $f(\lambda)$ has one extreme point $\lambda = \frac{p(c_1+c_4)}{p+1}$. Substituting the extreme point into (6) yields

$$f\Big(\frac{p(c_1 + c_4)}{p+1}\Big) = -\Big(\frac{c_1 - pc_4}{p+1}\Big)^{p+1} - c_2c_3^p \leq 0.$$

Since $c_1 + c_4 \geq 0$, $f(\lambda) = 0$ must have a non-negative solution. By Lemma 6, the planar system (1) is not asymptotically stable.

Case 3. $c_1 + c_4 \geq 0$ and $c_2c_3^p < 0$. Since (1) is not asymptotically stable when $c_1c_4^p \leq c_2c_3^p$, we assume $c_1c_4^p > c_2c_3^p$. Choose the Lyapunov function

$$V(x) = \frac{p}{p+1}x_1^{\frac{p+1}{p}} - ax_1x_2 + \frac{b}{p+1}x_2^{p+1}. \tag{10}$$

The time derivative of $V(x)$ is

$$\begin{aligned}\dot{V}(x) &= (c_1 - ac_3)x_1^{\frac{p+1}{p}} - (c_1 + c_4)ax_1x_2 + (c_4b - c_2a)x_2^{p+1}\\ &\quad + (c_2 + bc_3)x_1^{1/p}x_2^p.\end{aligned}$$

Set $a = \frac{c_1-pc_4}{(p+1)c_3}$ and $b = -\frac{c_2}{c_3} > 0$, then $c_1 - ac_3 = \frac{(c_1+c_4)p}{p+1}$ and $c_4b - c_2a = \frac{(c_1+c_4)b}{p+1}$. Thus $\dot{V}(x) = (c_1 + c_4)V(x)$. If $c_1 + c_4 > 0$, then $\dot{V}(x) > 0$ when $V(x) > 0$. By Theorem 3, system (1) is unstable. If $c_1 + c_4 = 0$, then $\dot{V}(x) = 0$ and $V(x) > 0$ due to $a^{p+1} = -\frac{c_1c_4^p}{c_3^{p+1}} < -\frac{c_2}{c_3} = b$. By Theorem 2, system (1) is not asymptotically stable. □

The necessary proof of Theorem 8 implies that we can use particular solutions of system (1) to justify instability. We call this method particular solution method. This method can be used to simply and intuitively determine instability of some systems. For example, we can use the particular solution method to show that the following system

$$\dot{x}_1 = c_1x_1^3 + c_2x_2^3, \quad \dot{x}_2 = c_3x_1^3 + c_4x_2^3c_1 \tag{11}$$

is not asymptotically stable when $c_1c_4 \leq c_2c_3$. System (11) is a general form of $\dot{x}_1 = -0.1x_1^3 + 2x_2^3$, $\dot{x}_2 = -2x_1^3 - 0.1x_2^3$, which was studied in the recent works (Chen, Rubanova, Bettencourt, & Duvenaud, 2018, https://github.com/rtqichen/torchdiffeq/tree/master/exam-ples) on neural ODE.

The work (Efimov & Perruquetti, 2010) proposed that there existed an invariant set between unstable subset and stable subset which contained oscillating trajectory. Theorem 8 implies that, under the condition $c_1c_4^p > c_2c_3^p$, system (1) is unstable when $c_1 + c_4 > 0$ and it is asymptotically stable when $c_1 + c_4 < 0$. For the case of $c_1c_4^p > c_2c_3^p$ and $c_1 + c_4 = 0$, we have $\dot{V}(x) = 0$ and $V(x) > 0$ where $V(x)$ is defined in (10). Hence, we have the following result.

Proposition 9. *The solution of the planar system (1) is an oscillation if $c_1c_4^p > c_2c_3^p$ and $c_1 + c_4 = 0$.*

Oscillations are very common phenomenon in practice. Yakubovich and Tomberg (1989) introduced an important and useful concept to study oscillations. The proposed conditions for oscillations in the sense of Yakubovich are based on existence of two Lyapunov functions (Efimov & Fradkov, 2009). The first Lyapunov function ensures local instability of the origin, while the second Lyapunov function provides global boundedness for system trajectories. Here, based on the particular solution method and Lyapunov method, a sufficient condition for oscillation existence of system (1) is formulated.

In what follows, we demonstrate the application of stability condition in state-feedback control design. Consider a class of planar nonlinear systems described by

$$\begin{cases}\dot{x}_1 = c_1x_1 + c_2x_2^p,\\ \dot{x}_2 = u,\end{cases} \tag{12}$$

where $x = (x_1\ x_2)^\top \in \mathbb{R}^2$ is state vector, p is a positive odd number, and c_1, $c_2 \in \mathbb{R}$ are constants. The stabilization problem of system (12) was first studied by Kawski (1989). Here, we have the following result, which can be directly derived from Theorem 8 and Proposition 9.

Theorem 10. *The controller* $u = k_1 x_1^{1/p} + k_2 x_2$, *where* k_1 *and* k_2 *satisfy* $c_1 + k_2 < 0$ *and* $c_1 k_2^p > c_2 k_1^p$, *globally stabilizes system* (12). *Moreover, if* k_1 *and* k_2 *satisfy* $c_1 k_2^p > c_2 k_1^p$ *and* $c_1 + k_2 = 0$, *the solution of system* (12) *is an oscillation.*

Next, we give three examples to illustrate the usefulness of Theorem 10.

Example 11. Consider the system

$$\begin{cases} \dot{x}_1 = \theta_1 x_1 + \theta_2 x_2^3, \\ \dot{x}_2 = u, \end{cases} \tag{13}$$

where θ_1, θ_2 are unknown positive parameters with known bounds. Assume $a_1 \leq \theta_1 \leq b_1$, $a_2 \leq \theta_2 \leq b_2$, where a_1, b_1, a_2 and b_2 are known constants. By Theorem 10, we can design a controller as follows

$$u = -k_1 x_1^{1/3} - k_2 x_2, \tag{14}$$

where the control gains k_1 and k_2 satisfy $\theta_1 - k_2 < 0$ and $\theta_1(-k_2)^3 > \theta_2(-k_1)^3$, that is $k_2 > \theta_1$ and $k_1 > (\theta_1 k_2^3/\theta_2)^{1/3}$. The bound conditions of θ_1 and θ_2 yield $\max(\theta_1) = b_1$ and $\max\left(\left(\frac{\theta_1 k_2^3}{\theta_2}\right)^{1/3}\right) = \left(\frac{b_1 k_2^3}{a_2}\right)^{1/3}$. Let

$$k_2 > b_1, \quad k_1 > \left(\frac{b_1 k_2^3}{a_2}\right)^{1/3}. \tag{15}$$

The controller (14) with k_1 and k_2 satisfying (15) stabilizes system (13).

Example 12. Reconsider the stabilization problem of system (2). By Theorem 10, we can design a controller $u = -ax_2 + bx_1^{1/3}$ with a and b satisfy $1 - a < 0$, $(-a)^3 > -b^3$, that is $b > a > 1$, which is in accord with the result given in Kawski (1989). Specifically, system (2) with $u = -ax_2 + bx_1^{1/3}$ is an oscillator when $a = 1$ and $b > 1$. For instance, when $a = 3/2$ and $b = 2$, the trajectory of the system (2) with $u = -ax_2 + bx_1^{1/3}$ is graphed in Fig. 1. When $a = 1$ and $b = 2$, the solution of the system (2) with $u = -ax_2 + bx_1^{1/3}$ is oscillating, which is shown in Fig. 2.

Example 13. Consider a second-order dynamic model of a reduced-order boiler system in thermal power plants (Su, Qian, & Shen, 2016), described by

$$\begin{cases} \dot{x}_1 = \text{sign}(x_2)|x_2|^{1.031}, \\ \dot{x}_2 = x_2 + f(u). \end{cases} \tag{16}$$

By Theorem 10, we design controllers as follows

$$f(u) = -k_1 \text{sign}(x_1)|x_1|^{1/1.031} - k_2 x_2. \tag{17}$$

Substituting (17) into (16), we have

$$\begin{cases} \dot{x}_1 = \text{sign}(x_2)|x_2|^{1.031}, \\ \dot{x}_2 = -k_1 \text{sign}(x_1)|x_1|^{1/1.031} - (k_2 - 1)x_2. \end{cases} \tag{18}$$

By Theorem 8, system (18) is stable when $k_2 - 1 > 0$ and $k_1 > 0$, that is $k_1 > 0$ and $k_2 > 1$. For instance, as $k_1 = 1$ and $k_2 = 2$, the nonlinear system (18) is asymptotically stable and the solution is graphed in Fig. 3.

4. Conclusion

In this paper, we study stability problem for a class of planar nonlinear systems. A necessary and sufficient condition for stability is put forward. A new method called particular solution method is developed to justify instability. It is worth pointing out that the particular solution method proposed in this paper can also be exploited to justify instability of more general planar nonlinear systems. In addition, the analysis method for stability of the planar nonlinear system (1) can be used to study stability of high order nonlinear systems. The necessary and sufficient condition for stability of high order nonlinear systems is currently under investigation.

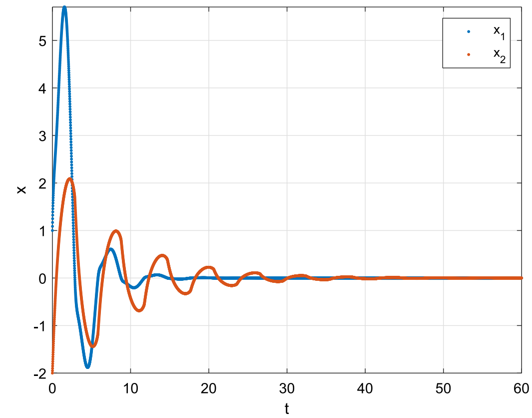

Fig. 1. Trajectory of the system (2) with $u = -\frac{3}{2}x_2 + 2x_1^{1/3}$ and $x(0) = (1\ \ -2)^\top$.

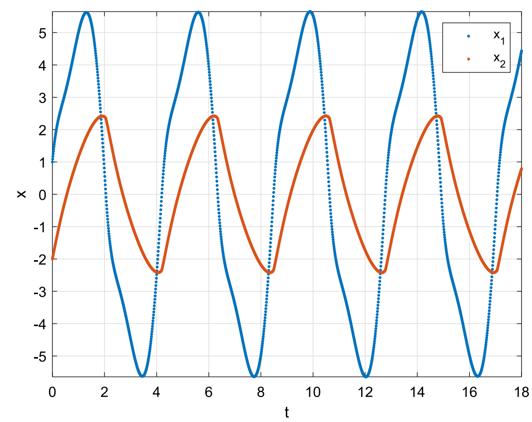

Fig. 2. Oscillating solution for the system (2) with $u = -x_2 + 2x_1^{1/3}$ and $x(0) = (1\ \ -2)^\top$.

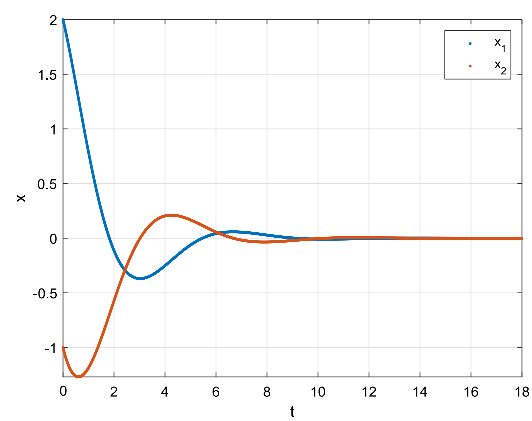

Fig. 3. Solution to the system (18) with $(k_1, k_2) = (1, 2)$ and $x(0) = (2\ \ -1)^\top$.

References

Aeyels, D. (1985). Stabilization of a class of nonlinear systems by a smooth feedback control. *Systems & Control Letters*, *5*(5), 289–294.

Aylward, E. M., Parrilo, P. A., & Slotine, J. J. E. (2008). Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming. *Automatica*, *44*(8), 2163–2170.

Bacciotti, A. (1992). *Local stabilizability of nonlinear control systems*. World Scientific.

Byrnes, C. I., & Isidori, A. (1991). On the attitude stabilization of rigid spacecraft. *Automatica*, *27*(1), 87–95.

Chen, T. Q., Rubanova, Y., Bettencourt, J., & Duvenaud, D. K. (2018). Neural ordinary differential equations. In *Advances in neural information processing systems* (pp. 6571–6583).

Chesi, G. (2013). Exact robust stability analysis of uncertain systems with a scalar parameter via LMIs. *Automatica*, *49*(4), 1083–1086.

Chetaev, N. G. (1961). *The stability of motion*. Pergamon Press.

Dayawansa, W. P., Martin, C. F., & Knowles, G. (1990). Asymptotic stabilization of a class of smooth two-dimensional systems. *SIAM Journal on Control and Optimization*, *28*(6), 1321–1349.

Du, H., Qian, C., Li, S., & Chu, Z. (2019). Global sampled-data output feedback stabilization for a class of uncertain nonlinear systems. *Automatica*, *99*, 403–411.

Efimov, D. V., & Fradkov, A. L. (2009). Oscillatority of nonlinear systems with static feedback. *SIAM Journal on Control and Optimization*, *48*(2), 618–640.

Efimov, D., & Perruquetti, W. (2010). Oscillations conditions in homogenous systems. *IFAC Proceedings Volumes*, *43*(14), 1379–1384.

Hahn, W. (1967). *Stability of motion*. Berlin: Springer.

Hurwitz, A. (1964). On the conditions under which an equation has only roots with negative real parts. In *Selected papers on mathematical trends in control theory, 65* (pp. 273–284).

Kawski, M. (1989). Stabilization of nonlinear systems in the plane. *Systems & Control Letters*, *12*(2), 169–175.

Khalil, H. K., & Grizzle, J. W. (2002). *Nonlinear systems*. Upper Saddle River, NJ: Prentice hall.

Lacerda, M. J., & Seiler, P. (2017). Stability of uncertain systems using Lyapunov functions with non-monotonic terms. *Automatica*, *82*, 187–193.

LaSalle, J. (1960). Some extensions of Liapunov's second method. *IRE Transactions on Circuit Theory*, *7*(4), 520–527.

Li, X., & Wu, J. (2016). Stability of nonlinear differential systems with state-dependent delayed impulses. *Automatica*, *64*, 63–69.

Li, X., Yang, X., & Song, S. (2019). Lyapunov conditions for finite-time stability of time-varying time-delay systems. *Automatica*, *103*, 135–140.

Lyapunov, A. M. (1992). The general problem of the stability of motion. *International Journal of Control*, *55*(3), 531–534.

Ooba, T. (2012). Asymptotic stability of two-dimensional discrete systems with saturation nonlinearities. *IEEE Transactions on Circuits and Systems. I. Regular Papers*, *60*(1), 178–188.

Qian, C. (2002). *Global synthesis of nonlinear systems with uncontrollable linearization* (Ph.D. dissertation), Case Western Reserve University.

Qian, C., & Lin, W. (2001). A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, *46*(7), 1061–1079.

Qian, C., & Lin, W. (2006). Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, *51*(9), 1457–1471.

Routh, E. J. (1877). *A treatise on the stability of a given state of motion: particularly steady motion*. Macmillan and Company.

Shtessel, Y. B., Shkolnikov, I. A., & Levant, A. (2007). Smooth second-order sliding modes: Missile guidance application. *Automatica*, *43*(8), 1470–1476.

Sontag, E., & Sussmann, H. (1980). Remarks on continuous feedback. In *19th IEEE conference on decision and control including the symposium on adaptive processes, Albuquerque (pp. 916–921)*.

Su, Z., Qian, C., & Shen, J. (2016). Interval homogeneity-based control for a class of nonlinear systems with unknown power drifts. *IEEE Transactions on Automatic Control*, *62*(3), 1445–1450.

Wu, L., Yang, R., Shi, P., & Su, X. (2015). Stability analysis and stabilization of 2-D switched systems under arbitrary and restricted switchings. *Automatica*, *59*, 206–215.

Yakubovich, V. A., & Tomberg, E. A. (1989). Conditions for self-induced oscillations in nonlinear systems. *Siberian Mathematical Journal*, *30*, 641–653.

Zhu, J., & Qian, C. (2018). A necessary and sufficient condition for local asymptotic stability of a class of nonlinear systems in the critical case. *Automatica*, *96*, 234–239.