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Stability Analysis of a Chain of Integrators with Pulse-Width-Modulation Controller

Shuaipeng He[1], Chunjiang Qian[2], Yunlei Zou[3]

***Abstract*— In application, many controllers are implemented through Pulse-Width-Modulation (PWM). Owing to the nonlinearity and discontinuity of PWM signal, the closed-loop stability analysis is challenging even for linear systems. In this paper, we investigate the stability problem for a linear chain of integrators with PWM controller. Different from the existing works, the proposed procedure is developed based on Zero-Order-Hold (ZOH) and we prove that the system is stable when the switching period of the PWM signal is small enough. Simulation results of 1-order and 2-order systems demonstrate the effectiveness of the proposed method.**

I. INTRODUCTION

Digital control is a branch of control theory in which a continuous-time plant is controlled with a digital device. It has widespread applications as the availability of cheap microprocessors. A digital controller is usually cascaded with a plant in a feedback system and requires both an A/D converter and a D/A converter. Normally the A/D converter is a sampler that converts analog signals to a digital format, and the D/A converter is often a Zero-Order-Holder that converts digital signals to a form that can be used as inputs to a plant [5].

In addition to a digital controller implemented ZOH, there is another kind of form which is designed using PWM. PWM is a way of describing a digital signal that is created through a modulation technique, which involves encoding a message into a pulsing signal. One early application of PWM was in an audio amplifier in the 1960s [8], [10], and around the same time PWM started to be used in AC motor control [19]. Since then PWM has been widely used in electric, electronic and electromechanical systems, such as signal processing [13], voltage regulation [20], attitude control [7], servo control [3] and the like. The reason PWM has been used in a wide variety of applications is due to the simplicity of realization, low power loss, low sensitivity to noise, elimination of dead zone and so on. Owing to the inherent nonlinearities introduced by the PWM signal, a control system designed by continuous or digital approaches needs to be re-designed before it can be implemented in a PWM based control system [14]. Normally the re-design procedure is realized by approximating the control input to the width of the pulse signal in each sampling period based on the principle of equivalent areas [2].

Even though a controller can stabilize a system when it is implemented in analog format, it could fail if it's implemented in digital format due to a large sampling interval. The sampling rate characterizes the transient response and stability of the hybrid system and must be selected carefully to avoid instability. For PWM based control systems, the stability analysis is even more challenging due to the inherent nonlinear and discontinuous characteristics of a pulse-width-modulator. The stability analysis for PWM based feedback control systems has received much attention for the decade since 1960s. For example, [18] proposed a frequency domain stability criterion, yielding a geometric interpretation in the Popov plane. [4] used an exact analytical method for the determination of the response of such systems to arbitrary inputs, which has the limitation associated with the use of describing function. [15] presented a graphical analysis approach which only applied to the first order sampled-data systems. [9] presented a stability result based on the Lyapunov's second method but its linear model was given in a transfer function form and all the poles were assumed real and different. The stability criterion presented by [12] is not applicable to systems with integrator. [22] used Lyapunov's second method and gave a stability criterion for PWM based feedback systems that only contain one integrating element.

In recent years, a growing number of researchers are devoting time to the stability analysis of PWM based control systems. For example, [6] studied the stability problem of PWM based feedback control for both linear and nonlinear plant. The authors assume the linear plant is Hurwitz stable or only has one pole at the origin, and the nonlinear system has a stable Jacobian matrix. [1] presented a stability analysis for PWM systems incorporate DC-DC converters by employing linear matrix inequalities (LMIs). [16] proposed stability analysis for the fuzzy PWM system which based on intelligent digital redesign method. [21] studied the stability problem of PWM feedback systems with time-varying delays and stochastic perturbations by establishing a Lyapunov-Krasovskii function. [17] proposed an exact linearization for two-dimensional systems with the PWM inputs via the method of input transformation.

In this paper, we use the Lyapunov's second method to present a new stability analysis procedure for a linear chain of integrators whose poles are all at the origin. The main difference between our work and the existing works is that we establish a new proof procedure based on the foundation

[1]Shuaipeng He is with Department of Electrical and Computer Engineering,The University of Texas at San Antonio, San Antonio, TX 78249, USA. shuaipeng.he@utsa.edu

[2]Chunjiang Qian is with Department of Electrical and Computer Engineering,The University of Texas at San Antonio, San Antonio, TX 78249, USA. chunjiang.qian@utsa.edu

[3]Yunlei Zou is with School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu, 225002, PR China, and also with Department of Electrical and Computer Engineering, The University of Texas at San Antonio, San Antonio, TX 78249, USA. ylzou@yzu.edu.cn

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of ZOH and only one parameter needs to be decided. A sufficient condition of the switching period of PWM signal is obtained and two numerical simulations are given to demonstrate the effectiveness of the proposed method.

II. PRELIMINARIES

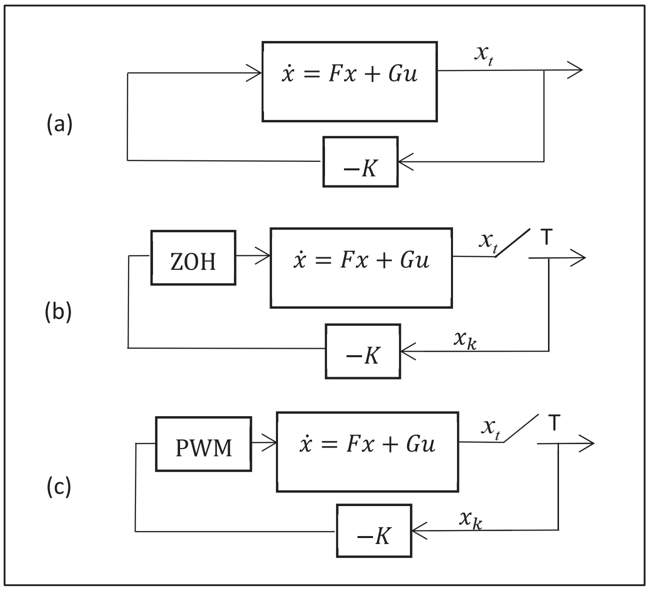

Fig. 1. System (1) with (a): A continuous-time controller. (b): A digital controller via ZOH. (c): A digital controller via PWM.

As Fig.1 shows, we consider a class of linear system

$$\dot{x} = Fx + Gu, \tag{1}$$

where

$$F = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}, \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1},$$

$x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}$ is the control input. For a controllable system (1), the continuous state feedback controller

$$u(t) = -Kx \tag{2}$$

can always stabilize it with an appropriate selection of control gains $K = [k_1, k_2, \cdots, k_n]$.

Theorem 1: For the same control gain vector K selected in (2) and under a proper sampling period T, there exists a discrete-time controller

$$u(t_k) = -Kx(kT), k = 0, 1, 2, \cdots \tag{3}$$

that globally asymptotically stabilizes system (1).

Proof. System (1) under a discrete-time controller $u(t_k)$ via ZOH can be discreterized as

$$x_{k+1} = \Phi x_k + \Gamma u(t_k), \tag{4}$$

where $\Phi = e^{FT}$ and $\Gamma = \int_0^T e^{Fs} G\, ds$. With (3), the discrete-time closed-loop system is

$$x_{k+1} = (\Phi - \Gamma K)x_k \triangleq \Psi x_k. \tag{5}$$

Given the control gain vector K, it is easy to calculate all the eigenvalues of Ψ that are inside the unit circle with a small enough sampling period T, which means the discrete-time closed-loop system (5) is globally asymptotically stable. Thus, there exists a symmetric positive definite matrix P such that

$$\Psi^T P \Psi - P = -I. \tag{6}$$

Choose a Lyapunov function

$$V(x_k) = x_k^T P x_k, \tag{7}$$

which is positive definite and proper. Let

$$\Delta V = V(x_{k+1}) - V(x_k). \tag{8}$$

The equation (8) along the closed-loop system (5) is

$$\begin{aligned} \Delta V &= V(x_{k+1}) - V(x_k) \\ &= (\Psi x_k)^T P (\Psi x_k) - x_k^T P x_k \\ &= x_k^T (\Psi^T P \Psi - P) x_k \\ &= -\|x_k\|^2, \end{aligned} \tag{9}$$

which is negative definite and this completes the proof.

III. MAIN RESULTS

Inspired by [12], the output of the PWM controller is described by

$$u_{(PWM)} = \begin{cases} U_{max} \cdot sgn(u(t_k)), & 0 \le t - \lambda T_1 < \alpha T_1, \\ 0, & \alpha T_1 \le t - \lambda T_1 < T_1, \end{cases} \tag{10}$$

where $\lambda = 0, 1, 2, \cdots, N-1$, $\alpha = \left|\frac{u(t_k)}{U_{max}}\right| \in [0, 1]$ is the duty cycle. Here, U_{max} is a positive constant defined as

$$U_{max} = max \ \{u = |Kx|, x \in \Omega_x\},$$

where $\Omega_x \in \mathbb{R}^n$ is a compact set and T_1 is the switching period which is related to the sampling period T designed in (5) by

$$T_1 = \frac{T}{N}, \tag{11}$$

where N is a positive constant to be designed later. And the signum function is defined by

$$sgn(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{12}$$

Theorem 2: For the same control gain vector K selected in (2), the PWM controller (10) with a proper selection of N semi-globally asymptotically stabilizes system (1).

Proof. The system (1) under the PWM controller (10) can be discreterized as

$$x_{k+1} = \Phi x_k + \Gamma u_{(PWM)}, \tag{13}$$

where $\Phi = e^{FT}$ and $\Gamma = \int_0^T e^{Fs} G\, ds$. Based on (5), equation (13) can be rewritten as

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u(t_k) + \Gamma u_{(PWM)} - \Gamma u(t_k) \\ &\triangleq \Psi x_k + w, \end{aligned} \tag{14}$$

where $w = \varepsilon_2 - \varepsilon_1$, $\varepsilon_2 = \Gamma u_{(PWM)}$ and $\varepsilon_1 = \Gamma u(t_k)$.

Choose the Lyapunov function (7), then its difference (8) along equation (14) becomes

$$\begin{aligned} \Delta V &= (\Psi x_k + w)^T P (\Psi x_k + w) - x_k^T P x_k \\ &= x_k^T \Psi^T P \Psi x_k + x_k^T \Psi^T P w + w^T P \Psi x_k \\ &\quad + w^T P w - x_k^T P x_k \\ &\leq -\|x_k\|^2 + 2\|\Psi\|\|P\|\|x_k\|\|w\| + \|P\|\|w\|^2. \end{aligned} \tag{15}$$

In order to determine if ΔV is negative definite, we need to estimate the last two terms in equation (15). According to equation (14), $w = \varepsilon_2 - \varepsilon_1$, so firstly we calculate ε_2. Substituting the control law $u_{(PWM)}$ (10) into $\varepsilon_2 = \Gamma u_{(PWM)}$, we have

$$\begin{aligned} \varepsilon_2 &= \Big(\int_0^{\alpha T_1} e^{Fs} G\, ds + \int_{T_1}^{T_1 + \alpha T_1} e^{Fs} G\, ds + \cdots \\ &\quad + \int_{(N-1)T_1}^{(N-1)T_1 + \alpha T_1} e^{Fs} G\, ds \Big) \cdot U_{max} \cdot sgn(u(t_k)) \\ &= \Big(\int_0^{\alpha T_1} e^{Fs} G\, ds + \int_0^{\alpha T_1} e^{F(s+T_1)} G\, ds + \cdots \\ &\quad + \int_0^{\alpha T_1} e^{F[s+(N-1)T_1]} G\, ds \Big) \cdot U_{max} \cdot sgn(u(t_k)) \\ &= \Big(I + e^{FT_1} + \cdots + e^{F(N-1)T_1} \Big) \cdot \int_0^{\alpha T_1} e^{Fs} G\, ds \\ &\quad \cdot U_{max} \cdot sgn(u(t_k)). \end{aligned} \tag{16}$$

For any positive integer m, it is easy to obtain that

$$e^{FmT_1} = \begin{bmatrix} 1 & mT_1 & \cdots & \frac{1}{(n-1)!}(mT_1)^{n-1} \\ 0 & 1 & \cdots & \frac{1}{(n-2)!}(mT_1)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{17}$$

Based on equation (17), we have

$$\begin{aligned} &\Big(I + e^{FT_1} + \cdots + e^{F(N-1)T_1} \Big) \\ &= \begin{bmatrix} N & \left(\sum\limits_{i=1}^{N-1} i\right) T_1 & \cdots & \left(\sum\limits_{i=1}^{N-1} i^{n-1}\right) \frac{1}{(n-1)!} T_1^{n-1} \\ 0 & N & \cdots & \left(\sum\limits_{i=1}^{N-1} i^{n-2}\right) \frac{1}{(n-2)!} T_1^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N \end{bmatrix} \end{aligned} \tag{18}$$

and

$$\int_0^{\alpha T_1} e^{Fs} G\, ds = \begin{bmatrix} \frac{1}{n!}(\alpha T_1)^n \\ \frac{1}{(n-1)!}(\alpha T_1)^{n-1} \\ \vdots \\ \frac{1}{1!}(\alpha T_1)^1 \end{bmatrix}. \tag{19}$$

By (18) and (19), we have

$$\begin{aligned} \varepsilon_2 = &\begin{bmatrix} \frac{N}{n!}(\alpha T_1)^n + \sum\limits_{j=1}^{n-1} \left(\sum\limits_{i=1}^{N-1} i^j \right) \frac{T_1^j}{j!} \frac{(\alpha T_1)^{n-j}}{(n-j)!} \\ \frac{N}{(n-1)!}(\alpha T_1)^{n-1} + \sum\limits_{j=1}^{n-2} \left(\sum\limits_{i=1}^{N-1} i^j \right) \frac{T_1^j}{j!} \frac{(\alpha T_1)^{n-1-j}}{(n-1-j)!} \\ \vdots \\ \frac{N}{1!}(\alpha T_1)^1 \end{bmatrix} \\ &\cdot U_{max} \cdot sgn(u(t_k)). \end{aligned} \tag{20}$$

Similarly, ε_1 can be calculated as

$$\varepsilon_1 = \int_0^T e^{Fs} G\, ds \cdot u(t_k) = \begin{bmatrix} \frac{1}{n!} T^n \\ \frac{1}{(n-1)!} T^{n-1} \\ \vdots \\ \frac{1}{1!} T^1 \end{bmatrix} \cdot u(t_k). \tag{21}$$

Since $\alpha = \left| \frac{u(t_k)}{U_{max}} \right|$ and $u(t_k) = -K x_k$, ε_1 can be rewritten as

$$\varepsilon_1 = \begin{bmatrix} \frac{1}{n!} T^n \alpha \\ \frac{1}{(n-1)!} T^{n-1} \alpha \\ \vdots \\ \frac{1}{1!} T^1 \alpha \end{bmatrix} \cdot U_{max} \cdot sgn(u(t_k)). \tag{22}$$

Based on (20) and (22), we have the expression of w shown in (23).

Next, we estimate the vector w. For the sake of simplicity, we denote

$$d_n \triangleq \left(\sum_{i=1}^{N-1} i^{n-1} \right) \frac{1}{(n-1)!} T_1^{(n-1)} \frac{1}{1!} (\alpha T_1)^1 - \frac{1}{n!} T^n \alpha \tag{24}$$

and

$$\begin{aligned} \xi_n &\triangleq N \frac{1}{n!} (\alpha T_1)^n + d_n \\ &\quad + \sum_{j=1}^{n-2} \left(\sum_{i=1}^{N-1} i^j \right) \frac{1}{j!} T_1^j \frac{1}{(n-j)!} (\alpha T_1)^{n-j}. \end{aligned} \tag{25}$$

Note that $\alpha \in [0\ 1]$, so for any positive integer l, the following relation holds:

$$\alpha^l \leq \alpha. \tag{26}$$

Keeping $T_1 = \frac{T}{N}$ in mind and by (26) we have

$$\begin{aligned} \xi_n &\leq \frac{N + (N-1)^2 + \cdots + (N-1)^{n-1}}{N^n} T^n \alpha + d_n \\ &\leq \frac{N + N^2 + \cdots + N^{n-1}}{N^n} T^n \alpha + d_n \\ &\leq \frac{C_n}{N} T^n \alpha + d_n, \end{aligned} \tag{27}$$

where $C_n = \frac{1}{N^{n-2}} + \frac{1}{N^{n-3}} + \cdots + \frac{1}{N^2} + \frac{1}{N^1} + 1$, and $n \geq 2$ is a positive constant.

$$w=\begin{bmatrix} N\frac{1}{n!}(\alpha T_1)^n+\sum\limits_{j=1}^{n-1}\left(\sum\limits_{i=1}^{N-1}i^j\right)\frac{1}{j!}T_1^j\frac{1}{(n-j)!}(\alpha T_1)^{n-j}-\frac{1}{n!}T^n\alpha \\ N\frac{1}{(n-1)!}(\alpha T_1)^{n-1}+\sum\limits_{j=1}^{n-2}\left(\sum\limits_{i=1}^{N-1}i^j\right)\frac{1}{j!}T_1^j\frac{1}{(n-1-j)!}(\alpha T_1)^{n-1-j}-\frac{1}{(n-1)!}T^{n-1}\alpha \\ \vdots \\ N\frac{1}{1!}(\alpha T_1)^1-\frac{1}{1!}T^1\alpha \end{bmatrix}\cdot U_{max}\cdot sgn(u(t_k)). \tag{23}$$

Next we estimate d_n. According to the Faulhaber's Formula [11], the sum of the first $N-1$ positive integers of power $n-1$ can be expressed as

$$\begin{aligned}\sum_{i=1}^{N-1}i^{n-1}&=\frac{(N-1)^n}{n}+\frac{1}{2}(N-1)^{n-1}\\&+\sum_{i=2}^{n-1}\frac{B_i}{i!}\frac{(n-1)!}{(n-i)!}(N-1)^{n-i},\end{aligned} \tag{28}$$

where B_i are the Bernoulli numbers. Substituting (28) into (24) results in

$$\begin{aligned}d_n&=\frac{(N-1)^n}{n!N^n}T^n\alpha+\frac{1}{2}\frac{(N-1)^{n-1}}{(n-1)!N^n}T^n\alpha\\&+\sum_{i=2}^{n-1}\frac{B_i}{i!(n-i)!}\frac{(N-1)^{n-i}}{N^n}T^n\alpha-\frac{1}{n!}T^n\alpha\\&\le\left[\frac{(N-1)^n}{N^n}-1\right]\frac{1}{n!}T^n\alpha+\frac{(N-1)^{n-1}}{N^n}T^n\alpha\\&+\sum_{i=2}^{n-1}\frac{(N-1)^{n-i}}{N^n}T^n\alpha\\&\le\frac{1}{N}T^n\alpha+\frac{1}{N}T^n\alpha+\frac{1}{N}T^n\alpha\\&\le\frac{3}{N}T^n\alpha.\end{aligned} \tag{29}$$

Based on the format of C_n in (27), a straightforward calculation leads to $C_n<(n-1),\ n\ge 2$. With (29) the general term ξ_n can be estimated as

$$\xi_n\le\frac{n+2}{N}T^n\alpha, \tag{30}$$

where n is the system order. Then (23) can be estimated as

$$w\le\begin{bmatrix}(n+2)T^n\\(n+2)T^{n-1}\\\vdots\\(n+2)T^2\\0\cdot T\end{bmatrix}\cdot\frac{\alpha U_{max}}{N}. \tag{31}$$

With $\alpha=\left|\frac{u(t_k)}{U_{max}}\right|$ and $u(t_k)=-Kx_k$ in mind, finally $\|w\|$ can be estimated as

$$\|w\|\le\|\theta\|\cdot\frac{\|K\|\|x_k\|}{N}, \tag{32}$$

where

$$\theta=\begin{bmatrix}(n+2)T^n\\(n+2)T^{n-1}\\\vdots\\(n+2)T^3\\(n+2)T^2\\0\cdot T\end{bmatrix}$$

is a constant vector.

Based on the estimation of $\|w\|$ in (32), it follows from (15) that

$$\begin{aligned}\Delta V&\le-\|x_k\|^2+2\|\Psi\|\|P\|\|x_k\|\|w\|+\|P\|\|w\|^2\\&\le-M\|x_k\|^2,\end{aligned} \tag{33}$$

where $M=\left(1-\frac{2\|\Psi\|\|P\|\|\theta\|\|K\|}{N}-\frac{\|P\|\|\theta\|^2\|K\|^2}{N}\right)$ is a constant. Choose N such that $N>2\|\Psi\|\|P\|\|\theta\|\|K\|+\|P\|\|\theta\|^2\|K\|^2$, then ΔV becomes negative definite, which means the closed-loop system (14) is stable. As a conclusion, the PWM controller (10) with a proper selection of N asymptotically stabilizes the system (1).

IV. SIMULATION

In this section, numerical simulations are studied to illustrate the stability of two examples with PWM controller under different selections of N.

A. example 1

Consider the 1-order system

$$\dot{x}=u, \tag{34}$$

where the system matrix $F=0$ and $G=1$. The control gain matrix K here is a scalar and we assume $K=2$. It is easy to calculate that $\Phi=e^{0T}=1$ and $\Gamma=\int_0^T e^{0s}1\,ds=T$. Thus the closed loop system with $u(t_k)$ is

$$x_{k+1}=(1-2T)x_k. \tag{35}$$

It is easy to see that the closed-loop system (35) is stable when the sampling period T satisfies $0<T<1$. If we choose $T=\frac{1}{4}$, according to (14), the closed-loop system with $u_{(PWM)}$ can be written as

$$x_{k+1}=\frac{1}{2}x_k+w. \tag{36}$$

Based on (16) and (21), ε_2 and ε_1 can be obtained as

$$\begin{aligned}\varepsilon_2&=N\alpha T_1\cdot U_{max}\cdot sgn(u(t_k)),\\\varepsilon_1&=T\cdot u(t_k).\end{aligned} \tag{37}$$

Obviously $w = \varepsilon_2 - \varepsilon_1 = 0$, so the closed-loop system (36) is stable independently of N. Based on (32), $\theta = 0$ when system order is one, thus N can be any positive integer according to (33), which draws the same conclusion as above.

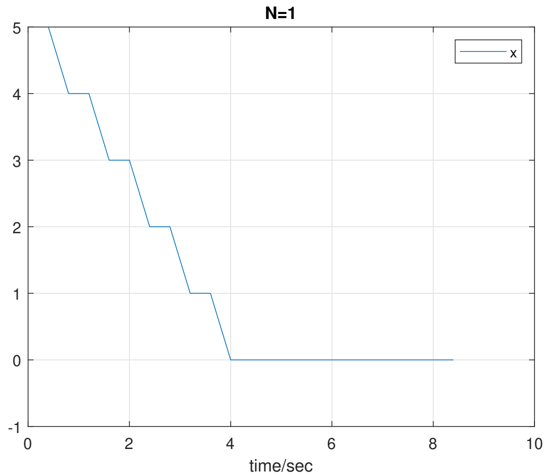

Fig. 2. State trajectory of system (34) with the PWM controller ($N = 1$).

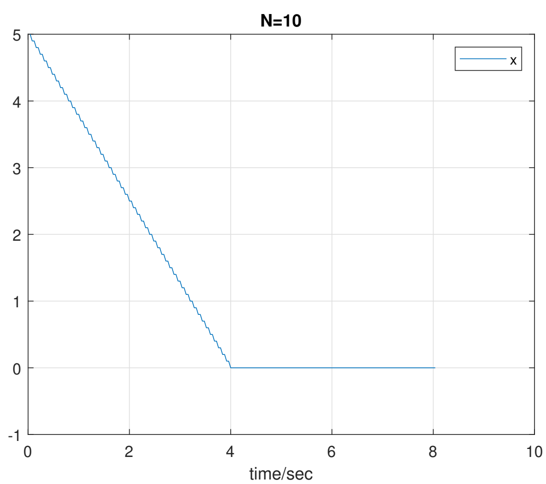

Fig. 3. State trajectory of system (34) with the PWM controller ($N = 10$).

Fig. 2 and Fig. 3 give the simulation results, from which we can see that the closed-loop system (36) is always convergent no matter the values of N.

B. example 2

Consider the 2-order system

$$\dot{x} = Fx + Gu, \tag{38}$$

where $F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Assume the control gain matrix is $K = [1 \ \ 1]$.

Similarly, it is easy to obtain $\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$. Thus the closed-loop system with $u(t_k)$ is

$$x_{k+1} = \Psi x_k = \begin{bmatrix} 1 - \frac{T^2}{2} & T - \frac{T^2}{2} \\ -T & 1 - T \end{bmatrix} x_k, \tag{39}$$

and the sampling period T to make (39) stable satisfies $0 < T < 2$. Choose $T = 1$, then we have $\Psi = \begin{bmatrix} 0.5 & 0.5 \\ -1 & 0 \end{bmatrix}$. According to (6), the symmetric positive definite matrix P is $\begin{bmatrix} 3 & 0.5 \\ 0.5 & 1.75 \end{bmatrix}$. By (32), we have $\theta = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. According to (14), the closed-loop system with the PWM controller is

$$x_{k+1} = \begin{bmatrix} 0.5 & 0.5 \\ -1 & 0 \end{bmatrix} x_k + w. \tag{40}$$

For the Lyapunov function $V(x_k) = x_k^T P x_k$, if we choose $N = 4\|\Psi\|\|P\|\|\theta\|\|K\| + 2\|P\|\|\theta\|^2\|K\|^2 = 286$, the difference ΔV along (40) becomes

$$\Delta V \leq -\frac{1}{2}\|x_k\|^2, \tag{41}$$

which means the closed-loop system (40) is stable.

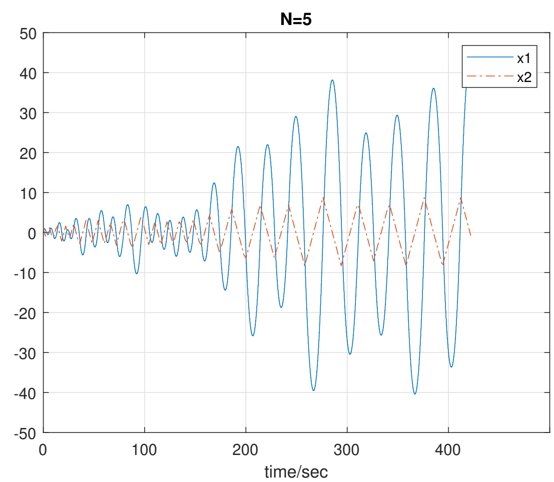

Fig. 4. State trajectories of system (38) with the PWM controller ($N = 5$).

From Fig. 4 we can see that, if N is not big enough such as $N = 5$, the closed-loop system (40) will not converge. As shown in Fig. 5 and Fig. 6, system states converge with different overshooting and settling time if N is big enough. There are two points worth mentioning here. The first one is that the range of N calculated above is just a sufficient condition but not a necessary condition. Thus even when N is beyond the range that we give, the closed-loop system could still be stable. The second one is, in our simulation only the stability is considered to test different values of N, thus the other control performances, such as overshooting and settling time, might not be the best.

V. CONCLUSIONS

This paper presents a new Lyapunov stability analysis for a chain of integrators with the PWM controller. Based on the

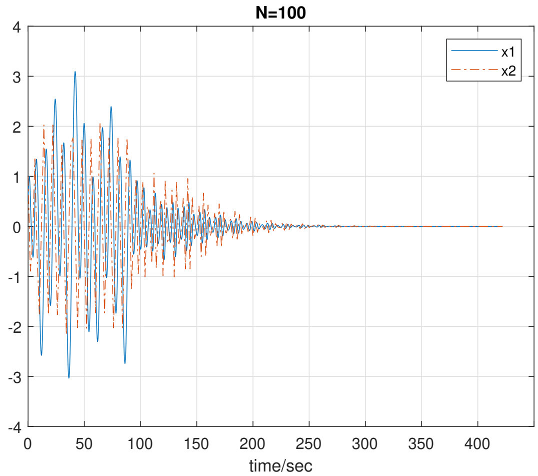

Fig. 5. State trajectories of system (38) with the PWM controller ($N = 100$).

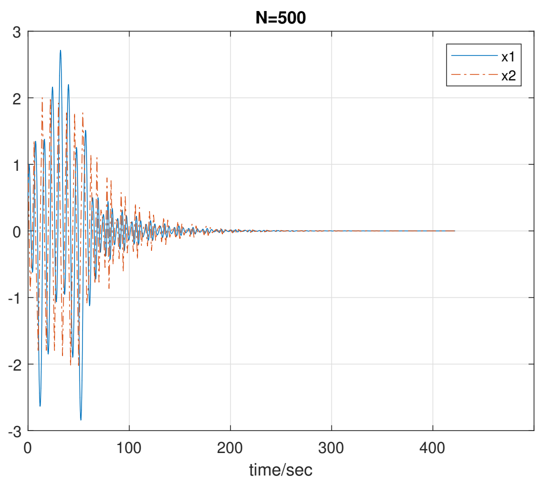

Fig. 6. State trajectories of system (38) with the PWM controller ($N = 500$).

foundation of ZOH, a sufficient condition of the switching period of PWM signal is given to ensure the closed-loop stability. Numerical simulation results show the effectiveness of the proposed method.

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