Enhancing Practical Tractability of Lyapunov-Based Economic Model Predictive Control

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***Abstract*— Lyapunov-based economic model predictive control (LEMPC) is an optimization-based control design that computes economically-optimal control actions for a process while maintaining the closed-loop state within a bounded region of state-space; however, it may be difficult to design in practice without closed-loop simulations, as it requires an auxiliary stabilizing controller, Lyapunov function, and a number of sets to be developed to ensure closed-loop stability. Practical application of this method could benefit from methods which make it more likely that, without simulations to identify aspects of the control design that would provide stability, controller parameters can be selected that would maintain stability. In this work, we propose a method to seek to enhance tractability of LEMPC by providing initial suggestions for reducing the likelihood that *ad hoc* selection of a value for one of its parameters would be problematic for closed-loop stability.**

I. INTRODUCTION

Economic model predictive control (EMPC) [1] is a model-based control design that has attracted research attention due to its ability to optimize process economic performance on-line via the control actions while respecting process constraints. EMPC seeks to optimize a cost function based on the process economics subject to the process dynamic model. Various versions of this controller (e.g., with terminal constraints [2] or Lyapunov-based stability constraints [3]) have been developed and characterized in terms of closed-loop stability. EMPC has also been extensively examined for practical considerations such as usage in various applications (e.g., wastewater treatment [4] or fault accommodation for batch processing [5]).

The EMPC formulation known as LEMPC [3] has closed-loop stability properties even in the presence of disturbances, without the need to utilize a model that accounts for disturbances in the controller itself. However, designing this controller requires two level sets of a Lyapunov function (one a superset of the other) to be selected in which a stabilizing control law can asymptotically stabilize the origin of the system under consideration. The proper selection of the level sets is part of guaranteeing closed-loop stability; however, the most likely way that the smaller level set would be chosen in practice would be via closed-loop simulations which check whether, from many different initial conditions in the larger level set, the LEMPC designed with this smaller level set always maintains the closed-loop state in the larger level set. This guess-and-check method of selecting the smaller level set size is neither rigorous nor industrially practical. However, the ability to guarantee that the closed-loop state for this controller remains within the larger level set for all times has been demonstrated to be a useful property for considerations such as cyberattack-resilience [6] and safety [7]. Motivated by these considerations, we develop an implementation strategy that takes advantage of the explicit stabilizing controller and frequent measurement sampling to potentially make an *ad hoc* selection of the smaller level set more likely to not cause closed-loop stability issues.

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II. PRELIMINARIES

A. Notation

The vector Euclidean norm is represented by $|\cdot|$. A function is of class $\mathcal{K}$ if it is a strictly increasing function $\alpha : [0, a) \to [0, \infty)$ with $\alpha(0) = 0$. The transpose of a vector x is denoted by x^T. The notation " / " signifies set subtraction $x \in A/B := \{x \in R^n : x \in A, x \notin B\}$). A level set of a positive definite function V is represented by by $\Omega_\rho := \{x \in R^n : V(x) \leq \rho\}$.

B. Class of Systems

We consider the following class of systems:

$$\dot{x}(t) = f(x(t), u(t), w(t)) \tag{1}$$

where f is a nonlinear locally Lipschitz vector function ($f(0, 0, 0) = 0$), $x(t) \in R^n$ is the process state vector, $u(t) \in U \subset R^m$ is the manipulated input vector, and $w(t) \in W \subset R^l$, where $W := \{w \in R^l : |w| \leq \theta\}$, is the bounded disturbance vector. We assume there exists a sufficiently smooth Lyapunov function $V(x)$, class $\mathcal{K}$ functions $\alpha_j(\cdot)$, $j = 1, \ldots, 4$, and a Lyapunov-based controller $h_1(x)$ which renders the origin of the nominal system of Eq. 1 (i.e., $w(t) \equiv 0$) asymptotically stable such that:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2}$$

$$\frac{\partial V(x)}{\partial x} f(x, h_1(x), 0) \leq -\alpha_3(|x|) \tag{3}$$

$$\left|\frac{\partial V(x)}{\partial x}\right| \leq \alpha_4(|x|) \tag{4}$$

$$h_1(x) \in U \tag{5}$$

$\forall\ x \in D \subset R^n$ where D is an open neighborhood of the origin and $\Omega_\rho \subset D$ is defined as the stability region. Also:

$$|f(x, u, w)| \leq M \tag{6}$$

$$|f(x_1, u_1, w) - f(x_2, u_1, 0)| \leq L_x|x_1 - x_2| + L_w|w| \tag{7}$$

$$|\frac{\partial V(x_1)}{\partial x} f(x_1, u_1, w) - \frac{\partial V(x_2)}{\partial x} f(x_2, u_1, 0)| \le L'_x |x_1 - x_2| + L'_w |w| \quad (8)$$

for all $x, x_1, x_2 \in \Omega_\rho$, $u_1 \in U$, and $w \in W$.

C. Lyapunov-Based Economic Model Predictive Control

LEMPC [3] is the following control law:

$$\min_{u(t) \in S(\Delta)} \int_{t_k}^{t_{k+N}} L_e(\tilde{x}(\tau), u(\tau))\, d\tau \quad (9a)$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), u(t), 0) \quad (9b)$$

$$\tilde{x}(t_k) = x(t_k) \quad (9c)$$

$$\tilde{x}(t) \in X, \forall\, t \in [t_k, t_{k+N}) \quad (9d)$$

$$u(t) \in U, \ \forall\, t \in [t_k, t_{k+N}) \quad (9e)$$

$$V(\tilde{x}(t)) \le \bar{\rho}_e, \ \ \forall\, t \in [t_k, t_{k+N}), \text{ if } x(t_k) \in \Omega_{\bar{\rho}_e} \text{ or } t_k \le t' \quad (9f)$$

$$\frac{\partial V(x(t_k))}{\partial x} f(x(t_k), u(t_k), 0) \le \frac{\partial V(x(t_k))}{\partial x} f(x(t_k), h_1(x(t_k)), 0) \text{ if } x(t_k) \notin \Omega_{\bar{\rho}_e} \text{ or } t_k > t' \quad (9g)$$

where $u(t) \in S(\Delta)$ signifies that the input vector is piecewise constant over the prediction horizon comprised of N sampling periods of length Δ. The stage cost function L_e (Eq. 9a) is minimized for the prediction horizon using state predictions from the nominal system of Eq. 1 (Eq. 9b). Eq. 9d is an initial condition which sets the predicted state $\tilde{x}(t_k)$ equal to the state measurement at t_k. States and inputs are constrained by Eqs. 9d and 9e, respectively. The part of the optimal solution vector of Eq. 9 that holds for $t \in [t_i, t_{i+1})$ at t_k is denoted by $u^*(t_i|t_k)$, where $i = k, \ldots, k+N-1$. Eqs. 9f and 9g ensure closed-loop stability. $\Omega_{\bar{\rho}_e} \subset \Omega_\rho$ is a subset of the stability region in which the state is allowed to evolve under the first mode of operation (i.e., when $x(t_k) \in \Omega_{\bar{\rho}_e}$ and $t_k \le t'$, where t' is a time after which the constraint of Eq. 9g is always enforced).

To support the following discussion, we here present several propositions and a theorem from [3] that describe the theoretical properties of LEMPC.

Proposition 1: [8], [3] Consider the systems

$$\dot{x}_y(t) = f(x_y(t), u(t), w(t)) \quad (10)$$

$$\dot{x}_z(t) = f(x_z(t), u(t), 0) \quad (11)$$

with initial states $x_y(t_0) = x_z(t_0) \in \Omega_\rho$. There exists a class $\mathcal{K}$ function $f_W(\cdot)$ such that

$$|x_y(t) - x_z(t)| \le f_W(t - t_0) \quad (12)$$

for all $x_y(t), x_z(t) \in \Omega_\rho$ and all $w(t) \in W$ with:

$$f_W(\tau) = \frac{L_w \theta}{L_x}(e^{L_x \tau} - 1) \quad (13)$$

Proposition 2: [8], [3] Consider the Lyapunov function $V(\cdot)$ of the system of Eq. 1. There exists a quadratic function $f_V(\cdot)$ such that:

$$V(x) \le V(\hat{x}) + f_V(|x - \hat{x}|) \quad (14)$$

for all $x, \hat{x} \in \Omega_\rho$ with

$$f_V(s) = \alpha_4(\alpha_1^{-1}(\rho))s + M_v s^2 \quad (15)$$

where M_v is a positive constant.

Theorem 1: [3] Consider the system of Eq. 1 in closed-loop under the LEMPC design of Eq. 9 based on a controller $h(x)$ that satisfies the assumptions of Eqs. 2-5. Let $\bar{\rho}_e = \rho_e$, where $\epsilon_w > 0$, $\Delta > 0$, $\rho > \rho_e > \rho_{\min} > \rho_s > 0$ satisfy:

$$\rho_e \le \rho - f_V(f_W(\Delta)) \quad (16)$$

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x M \Delta + L'_w \theta \le -\epsilon_w/\Delta \quad (17)$$

If $x(t_0) \in \Omega_\rho$ and $N \ge 1$, where:

$$\rho_{\min} = \max\{V(x(t+\Delta)) : V(x(t)) \le \rho_s\} \quad (18)$$

then the state $x(t) \in \Omega_\rho \ \forall\, t \ge t_0$ and is ultimately bounded in $\Omega_{\rho_{min}}$ for $t > t'$.

III. Removing Barriers to the Use of LEMPC

A. Less Stringent Requirements on $\Omega_{\bar{\rho}_e}$

The most significant challenge for LEMPC is the practical difficulty of determining the various components of this controller in Eq. 9 (e.g., $\bar{\rho}_e$, ρ, $h(x)$, and $V(x)$). This work focuses on the difficulty of selecting $\bar{\rho}_e$; a rigorous determination of this parameter to obtain the closed-loop stability properties which LEMPC can have would require that a value for $\bar{\rho}_e = \rho_e$ that satisfies Eq. 16 be found. Given the difficulty of locating the constants such as M_v, L_w, L_x, and θ, and functions such as α_4 and α_1, that appear in f_V and f_W in Propositions 1-2, it is likely that closed-loop simulations would be used in selecting $\bar{\rho}_e$, rather than the theoretical requirements of Eq. 16, or that an overly conservative value of the parameter would be selected.

Part of the challenge for the selection of $\bar{\rho}_e$ is that, as shown in Eq. 16, its size depends on the magnitude of the sampling period, upper bound on the disturbances, and process dynamics. In a practical situation, it can be expected that the upper bound on the disturbances would be reduced as much as possible by the team that derives the process model, and the process dynamics characteristics would not be able to be altered. This suggests that the sampling period is the only remaining parameter which can be tuned when selecting $\bar{\rho}_e$; specifically, the value of $\bar{\rho}_e$ must be sufficiently less than the value of ρ in a manner that depends on Δ, where it does not need to be as much less than ρ if Δ is very small compared to if it is larger (Eqs. 16 and 13). In fact, these equations indicate that $\bar{\rho}_e$ can approach ρ in Eq. 19, and closed-loop stability under the resulting LEMPC would still be maintained, if Δ approaches 0.

In practice, however, it may be necessary to use a larger value of Δ, partially due to the potential computational burden of solving the resulting LEMPC of Eq. 9; in such a case, Eq. 16 is not constructive in allowing the size of $\bar{\rho}_e$ to be readily determined for a given ρ and Δ. One way to attempt to deal with this in a manner that may provide an industrially-relevant solution would be to develop an implementation strategy for LEMPC that allows measurements to be obtained frequently (i.e., many times within a given Δ) and then to select $\bar{\rho}_e$ in an *ad hoc* manner but have a back-up explicit stabilizing controller available to drive the closed-loop state to lower level sets of V during a sampling period if the closed-loop state leaves $\Omega_{\bar{\rho}_e}$ in that timeframe. Though the *ad hoc* selection of $\bar{\rho}_e$ does not guarantee that stability will be maintained under the resulting LEMPC of Eq. 9, if the time between measurements becomes very small, it will have an effect similar to that described with respect to Eq. 16 (i.e., $\bar{\rho}_e$ could almost be the same as ρ before any closed-loop stability issues would arise that are associated with $\bar{\rho}_e$). This manner for selecting $\bar{\rho}_e$ may provide sufficient flexibility in the selection of $\bar{\rho}_e$ so that its *ad hoc* selection may be less likely to cause closed-loop stability issues.

Specifically, we consider measuring x at time periods $\Delta_{meas} < \Delta$, where Δ_{meas} corresponds to the time that it takes to obtain a new measurement from the sensor, to monitor the state throughout the sampling period under a control input computed with a $\bar{\rho}_e < \rho$ that has been arbitrarily selected (this $\bar{\rho}_e$ will be henceforth referred to as $\bar{\rho}'_e$). Then, if at any point in a sampling period, $V(x(i\Delta_{meas})) > \bar{\rho}'_e$, the control action in use at the time is no longer used for the remainder of the sampling period, and instead the control actions become calculated by $h_1(x)$ (implemented in a sample-and-hold fashion with a hold time of Δ_{meas}). It is assumed that $\Delta = M'\Delta_{meas}$, for M' a positive integer (i.e., that Δ is an integer multiple of Δ_{meas}) for consistency with the assumption that measurements are also available at every Δ. This strategy may help to reduce the need for significant conservatism in the selection of $\bar{\rho}'_e$ or reduce the likelihood that the closed-loop state will leave Ω_ρ if $\bar{\rho}'_e$ is not rigorously selected according to Eq. 16, though it does assume that measurements of the process states can be obtained much more frequently than Δ (i.e., M' is large). The use of a backup controller for maintaining closed-loop stability is consistent with other works in LEMPC where backup controllers have been critical to maintaining closed-loop stability of a process under LEMPC when it could not otherwise have been guaranteed; for example, in [9], $h_1(x)$ is used in sample-and-hold throughout a sampling period when an LEMPC formulation with additional constraints beyond those in Eq. 9 is not feasible at t_k. In the concept proposed in the present manuscript, we consider that $h_1(x)$ can be activated during a sampling period.

Remark 1: Though a potential reduction in the conservatism of $\Omega_{\bar{\rho}_e}$ for the potential to increase profits is one of the motivations for the proposed methodology, it is not guaranteed that the proposed method will enhance profits. Specifically, if $\bar{\rho}'_e$ is selected to be too large, given the process disturbances, such that the closed-loop state regularly exits $\Omega_{\bar{\rho}'_e}$ throughout a sampling period even when the state predictions from Eq. 9b indicate that it will not (Eq. 9f), the Lyapunov-based controller, which drives the closed-loop state to level sets of V with smaller upper bounds throughout a sampling period, will be activated more often. This may have the effect of causing the closed-loop state to be operated under $h_1(x)$ in sample-and-hold with a period of Δ_{meas} frequently, which could have the effect of decreasing profits by not allowing an economically-optimal control action coming from the LEMPC to be utilized.

1) Less Stringent Requirements on $\Omega_{\bar{\rho}_e}$: Implementation Strategy: The proposed strategy trades off the use of LEMPC with the use of $h_1(x)$ in sample-and-hold with period Δ_{meas} for $h_1(x)$ and of Δ for the LEMPC, assuming that $\bar{\rho}'_e$ has been selected in a less conservative manner than implied by Eq. 16. Specifically, if at a time $t_s \in (t_k+p\Delta_{meas}, t_k+(p+1)\Delta_{meas}]$, the closed-loop state exits $\Omega_{\bar{\rho}'_e}$, then for $t \in [t_k+(p+1)\Delta_{meas}, t_k+M'\Delta_{meas})$, the backup controller $h_1(x)$ is applied with period Δ_{meas}. The use of $h_1(x)$ guarantees that the closed-loop state remains in the stability region and will eventually drive the state into $\Omega_{\bar{\rho}'_e}$, where Eq. 9 can again be used to compute an optimal input policy for $t_k \le t'$.

The implementation strategy is as follows:

1) At t_k, the controller receives the state measurement $x(t_k)$.
2) If $t_k < t'$, go to Step 3. Else, go to Step 3b.
3) If $x(t_k) \in \Omega_{\bar{\rho}'_e}$, go to Step 3a. Else, go to Step 3b.
   a) The LEMPC of Eq. 9 computes inputs for every sampling period from t_k to t_{k+N} to maximize the economic cost function such that $V(x) \le \bar{\rho}'_e$.
   b) The LEMPC computes inputs that decrease the value of the Lyapunov function at t_k.
4) The controller implements the optimal input computed for t_k. Measurements of x are obtained at every $t_k + i\Delta_{meas}$, $i = 0, \ldots, M'-1$, throughout the sampling period. If $V(x(t_k+i\Delta_{meas})) > \bar{\rho}'_e$, $i = 1, \ldots, M'-1$, go to Step 4a. Else, go to Step 5.
   a) $h_1(x)$ is implemented in sample-and-hold with period Δ_{meas} for the remainder of the sampling period. Go to Step 5.
5) $t_{k+1} \leftarrow t_k$. Go to Step 1.

2) Less Stringent Requirements on $\Omega_{\bar{\rho}_e}$: Stability Analysis: Theorem 1 below provides sufficient conditions for which the implementation strategy in the above section guarantees the process state is always bounded within Ω_ρ and ultimately bounded in $\Omega_{\rho_{\min}}$ when $t_k > t'$.

Theorem 2: Consider the system of Eq. 1 in closed loop under the LEMPC design of Eq. 9 based on a controller $h_1(x)$ that satisfies Eqs. 2-5, applied according to the implementation strategy in Section III-A.1. Let $\epsilon_w > 0$, $\bar{\epsilon}'_w > 0$, $\Delta_{meas} > 0$, $\rho > \bar{\rho}'_e > \rho_{\min} > \rho_s > 0$ satisfy Eq. 17 and Eq. 18, and

$$\bar{\rho}'_e \le \rho - f_V(f_W(\Delta_{meas})) \tag{19}$$

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x M\Delta_{meas} + L'_w\theta \le -\bar{\epsilon}'_w/\Delta_{meas} \tag{20}$$

If $x(t_0) \in \Omega_\rho$, then $x(t)$ is always bounded in Ω_ρ for $N \ge 1$, and the state $x(t)$ is ultimately bounded in $\Omega_{\rho_{\min}}$ for $t > t'$.

Proof 1: This proof follows the proof for the LEMPC of Eq. 9 in [3] and consists of several parts, proving: 1) feasibility of Eq. 9 for all $x(t) \in \Omega_\rho$; 2) when $x(t_k) \in \Omega_{\bar{\rho}'_e}$ and $t_k \le t'$, then $x(t) \in \Omega_\rho$, $\forall\ t \in [t_k, t_{k+1})$, under the proposed implementation strategy; 3) when $x(t_k) \in \Omega_\rho/\Omega_{\bar{\rho}'_e}$, the LEMPC of Eq. 9 drives the closed-loop state toward or into $\Omega_{\bar{\rho}'_e}$ throughout the subsequent sampling period; 4) if $t_k > t'$, the closed-loop state is ultimately bounded in $\Omega_{\rho_{\min}}$.

Part 1. When $x(t)$ is maintained in Ω_ρ, there exists a feasible solution $u(t) = h_1(\tilde{x}(t_j))$, $\forall\ t \in [t_j, t_{j+1})$, $j = k, \ldots, k + N - 1$, to the optimization problem of the LEMPC of Eq. 9 at every t_k due to the closed-loop stability property of the Lyapunov-based controller $h_1(x)$ [3], [10], as proven in [3].

Part 2. We first analyze the case that $x(t_k) \in \Omega_{\bar{\rho}'_e}$ and $t_k \le t'$ such that the constraint of Eq. 9f is applied. If $x(t_k) \in \Omega_{\bar{\rho}'_e}$, then from the constraint in Eq. 9f, $\tilde{x}(t) \in \Omega_{\bar{\rho}'_e}$, $\forall\ t \in [t_k, t_{k+1})$. When the optimal solution of Eq. 9 meeting this constraint is applied to the process for $t \in [t_k, t_{k+1})$, either 1) $x(t) \in \Omega_{\bar{\rho}'_e}$, $\forall\ t \in [t_k, t_{k+1})$, in which case $x(t) \in \Omega_\rho$, $\forall\ t \in [t_k, t_{k+1})$ since $\Omega_{\bar{\rho}'_e} \subset \Omega_\rho$, or 2) $x(t) \notin \Omega_{\bar{\rho}'_e}$ starting at some $t_s \in [t_k, t_{k+1})$. If this second case occurs, the implementation strategy of Section III-A.1 indicates that the control action will be changed to $h_1(x(t_k + i\Delta_{meas}))$, starting at $t_k + (p+1)\Delta_{meas}$. In this case, we demonstrate first that if Eq. 19 holds, then if $x(t_k + p\Delta_{meas}) \in \Omega_{\bar{\rho}'_e}$, $x(t_k + (p+1)\Delta_{meas}) \in \Omega_\rho$. Subsequently, we demonstrate that after $t_k + (p+1)\Delta_{meas}$, the closed-loop state remains bounded in Ω_ρ under the proposed implementation strategy.

If $x(t_k + p\Delta_{meas}) \in \Omega_{\bar{\rho}'_e}$, then from [3], $x(t_k + (p+1)\Delta_{meas}) \in \Omega_\rho$. Specifically, as in [3], Proposition 2 gives:

$$\begin{aligned} & V(x(t_k + (p+1)\Delta_{meas})) \le V(\tilde{x}(t_k + (p+1)\Delta_{meas})) \\ & + f_V(|x(t_k + (p+1)\Delta_{meas}) - \tilde{x}(t_k + (p+1)\Delta_{meas})|) \\ & \le V(\tilde{x}(t_k + (p+1)\Delta_{meas})) + f_V(f_W(\Delta_{meas})) \\ & \le \bar{\rho}'_e + f_V(f_W(\Delta_{meas})) \le \rho \end{aligned} \tag{21}$$

which follows from Proposition 1, Eq. 9f, and Eq. 19.

If $x(t_k + (p+1)\Delta_{meas}) \in \Omega_\rho / \Omega_{\bar{\rho}'_e}$, the implementation strategy of Section III-A.1 indicates that $h_1(x(t_k + i\Delta_{meas}))$, $i \in \{p+1, \ldots, M'-2\}$, is subsequently applied for all remaining i before t_{k+1}. When $h_1(x)$ is applied in this manner, the time-derivative of the Lyapunov function along the closed-loop state trajectories under $h_1(x)$ is determined following [3]. Specifically, denoting $t_k + (i+1)\Delta_{meas}$ as $\tilde{t}_i$, Eq. 3 gives:

$$\frac{\partial V(x(\tilde{t}_i))}{\partial x} f(x(\tilde{t}_i), h_1(x(\tilde{t}_i)), 0) \le -\alpha_3(|(x(\tilde{t}_i))|) \tag{22}$$

for $i \in \{p, \ldots, M'-2\}$. From Eq. 3, the time derivative of the Lyapunov function using the backup controller h_1 in sample-and-hold $\forall\ \tau \in [\tilde{t}_i, \tilde{t}_{i+1})$ is as follows:

$$\begin{aligned} \dot{V}(x(\tau)) = & \frac{\partial V(x(\tau))}{\partial x} f(x(\tau), h_1(x(\tilde{t}_i)), w(t)) \\ & + \frac{\partial V(x(\tilde{t}_i))}{\partial x} f(x(\tilde{t}_i), h_1(x(\tilde{t}_i)), 0) \\ & - \frac{\partial V(x(\tilde{t}_i))}{\partial x} f(x(\tilde{t}_i), h_1(x(\tilde{t}_i)), 0) \end{aligned} \tag{23}$$

For all $\tau \in [\tilde{t}_i, \tilde{t}_{i+1})$, Eq. 6 gives:

$$|x(\tau) - x(\tilde{t}_i)| < M\Delta_{meas} \tag{24}$$

From Eqs. 23, 24, 2, and 8, and considering $|w| \le \theta$ and $x(\tilde{t}_i) \in \Omega_\rho / \Omega_{\bar{\rho}'_e}$ and $\Omega_{\rho_s} \subset \Omega_{\bar{\rho}'_e}$:

$$\dot{V}(x(\tau)) \le -\alpha_3(\alpha_2^{-1}(\rho_s)) + L'_x M\Delta_{meas} + L'_w\theta \tag{25}$$

If Eq. 20 holds, then

$$\dot{V}(x(t)) \le -\bar{\epsilon}'_w / \Delta_{meas} \tag{26}$$

and

$$V(x(t)) \le V(x(\tilde{t}_i)),\ \forall\ t \in [\tilde{t}_i, \tilde{t}_{i+1}) \tag{27}$$

Eq. 27 holds for all i after that in which the Lyapunov-based controller first begins to be applied within a sampling period at intervals Δ_{meas}, as long as $x(\tilde{t}_i) \in \Omega_\rho / \Omega_{\rho_s}$. If $x(\tilde{t}_i) \in \Omega_{\rho_s}$, then from Eq. 18, it remains within $\Omega_{\rho_{\min}}$ thereafter. By selecting $\bar{\rho}'_e > \rho_{\min}$, the closed-loop state under the proposed implementation strategy is maintained within Ω_ρ for $t \in [t_k, t_{k+1})$ if $x(t_k) \in \Omega_{\bar{\rho}'_e}$.

The fact that the closed-loop state is maintained within Ω_ρ for $t \in [t_k, t_{k+1})$ when $x(t_k) \in \Omega_\rho / \Omega_{\rho'_e}$ and ultimately bounded in $\Omega_{\rho_{\min}}$ if $t_k > t'$ under the control actions computed by the LEMPC of Eq. 9 follows from the fact that Eqs. 17 and 18 guarantee this in [3]. Applying this recursively ensures that $x(t) \in \Omega_\rho$, $\forall\ t > 0$, if $x(t_0) \in \Omega_\rho$ under the LEMPC of Eq. 9 with the implementation strategy proposed in Section III-A.1.

Remark 2: As noted above, the use of the implementation strategy in Section III-A.1 allows the requirements on $\bar{\rho}'_e$ to reduce to those in Eq. 19, where if Δ_{meas} is quite small, then in practice, the value of $\bar{\rho}'_e$ might be arbitrarily selected to be relatively close to ρ, and in many cases, such a $\bar{\rho}'_e$ may already meet the condition in Eq. 19. It is in this sense in which the proposed implementation strategy may be a step in moving LEMPC toward practical implementation.

3) Less Stringent Requirements on $\Omega_{\bar{\rho}_e}$: Application to a Process Example: We consider an example in which a continuous stirred tank reactor (CSTR) is used to facilitate a second order exothermic reaction $A \to B$. The manipulated inputs are C_{A0}, the concentration of the reactant in the feed stream, and Q, the rate at which heat may be added or removed by a heating/cooling jacket. The dynamics of the process are:

$$\dot{C}_A = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{-\frac{E}{R_g T}} C_A^2 \tag{28}$$

$$\dot{T} = \frac{F}{V}(T_0 - T) - \frac{\Delta H k_0}{\rho_L C_p} e^{-\frac{E}{R_g T}} C_A^2 + \frac{Q}{\rho_L C_p V} \tag{29}$$

where C_A and T, representing concentration and temperature inside the reactor respectively, are process state variables. k_0 is the pre-exponential constant, E and ΔH are the activation energy and enthalpy of the reaction, respectively, R_g represents the ideal gas constant, F is the inlet/outlet volumetric flow rate F, and the liquid density ρ_L, heat capacity C_p, and liquid volume inside the reactor V are fixed. Vectors of the deviation variables of the states C_A and T and inputs C_{A0} and Q are $x = [x_1\ x_2]^T = [C_A - C_{As}\ T - T_s]^T$ and $u = [u_1\ u_2]^T = [C_{A0} - C_{A0s}\ Q - Q_s]^T$. The steady-state values are $C_{As} = 1.22$ kmol/m^3, $T_s = 438.2$ K,

$C_{A0s} = 4$ kmol/m^3, and $Q_s = 0$ kJ/h. Values of the process parameters are found in [11]. The LEMPC maximizes the production rate of B by manipulating inputs C_{A0} and Q with the following stage cost:

$$L_e = -k_0 e^{-\frac{E}{R_g T(\tau)}} C_A(\tau)^2 \tag{30}$$

Input constraints require that $0.5 \leq C_{A0} \leq 7.5$ kmol/m^3 and $-5 \times 10^5 \leq Q \leq 5 \times 10^5$ kJ/h. Lyapunov-based stability constraints are developed using $V = x^T P x$, where $P = [1200\ 5; 5\ 0.1]$. The Lyapunov-based controller $h_1(x)$ is applied in a sample-and-hold fashion for a period of Δ_{meas} when the closed-loop state $x(t_k + p\Delta_{meas}) \in \Omega_\rho / \Omega_{\bar{\rho}'_e}$. For simplicity, the first component $h_{1,1}(x) = 0$ kmol/m^3, and the second component $h_{1,2}(x)$ is computed via Sontag's control law [12] but saturated at the input bounds if they are hit (the form of the control law is given in [11]). The constraint of Eq. 9f was enforced at the end of every sampling period when the state measurement at t_k was in $\Omega_{\bar{\rho}'_e}$, and was also enforced at the end of every sampling period after the first when the state measurement at t_k was in $\Omega_\rho / \Omega_{\bar{\rho}'_e}$.

The simulations were performed using the Explicit Euler numerical integration method with an integration step of 10^{-4} h to simulate the process, and the optimization problem was solved in MATLAB using fmincon. The process was simulated with additive noise added to the right-hand side of Eqs. 28 and 29 with a normal distribution generated by the MATLAB function randn with mean zero, where the standard deviation for the noise added to Eq. 28 was 0.3, and that for the noise added to Eq. 29 was 20. The lower and upper bounds on the noise (below and above which the noise was clipped to the bound) were set to -0.6 and 0.6 for Eq. 28 and -40 to 40 for Eq. 29. ρ was set to 300, and Δ_{meas} was set to the integration step of 10^{-4} h.

Initially, the process was simulated with $\bar{\rho}_e = 0.98\rho$ under LEMPC without a backup control law activated between sampling periods. The negative of the time integral of Eq. 30 after an hour of operation (reflecting profit during that time) was 33.05. When the size of $\bar{\rho}_e$ is increased to 99% of ρ, the closed-loop state exits the stability region after 11 sampling periods when the random number generator rng in MATLAB, used for seeding randn, is given an argument of 1. Several other values of the argument to rng were attempted (e.g., 10, 20, 30, 50, and 100), but the closed-loop state also left the stability region with these when $\bar{\rho}_e = 0.99\rho$.

We therefore explore whether $\bar{\rho}'_e$ can be set to 0.99ρ and whether the closed-loop state can be maintained within the stability region with the proposed implementation strategy, and what the impact of this on profit would be. When this was done, the profit was again 33.05, and the state trajectories in state-space are shown in Fig. 1. Again, other arguments for rng besides 1 were tried (10, 20, 30, 50, and 100) and in each case, the closed-loop state did not exit the stability region. This indicates that the proposed method was able to, with approximately the same profit as the case where $\bar{\rho}_e$ was selected via closed-loop simulations in which it did not result in the closed-loop state exiting the stability region, keep the closed-loop state inside when selected to be a value that would have caused the closed-loop state to leave the stability region when the backup control law was not implemented.

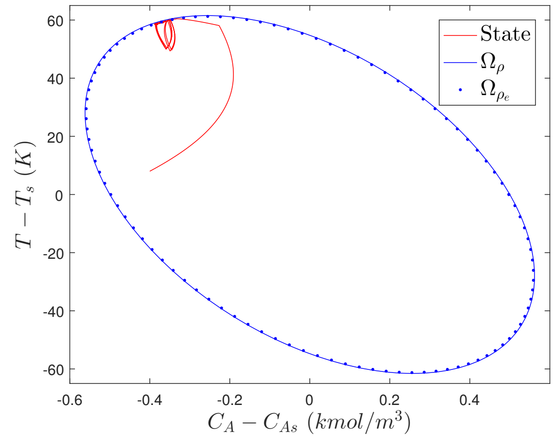

Fig. 1. Input trajectories with the modified value of $\bar{\rho}_e$ as $\bar{\rho}'_e = 0.99\rho$ and the proposed methodology.

B. Removing Barriers to LEMPC Use: Other Constraint Tuning Concepts

Our work in [13] explored another concept for aiding in the design of LEMPC's. In [13], the constraints below, first developed in [14], were added to the LEMPC of Eq. 9:

$$|u_i(t_k) - h_i(x(t_k))| \leq \epsilon_r,\ i = 1, \ldots, m \tag{31}$$

$$|u_i(t_j) - h_i(\tilde{x}(t_j))| \leq \epsilon_r, \quad i = 1, \ldots, m,\ j = k+1, \ldots, k+N-1 \tag{32}$$

where $\epsilon_r \geq 0$. In [14], these constraints were added to the LEMPC to prevent the difference in the inputs computed between two sampling periods from becoming larger than a desired threshold, with the goal of preventing actuator wear. However, actuator wear is not directly represented by the value of ϵ_r; therefore, this is a constraint for LEMPC with a parameter that may be somewhat difficult to tune. Unlike the parameter $\bar{\rho}'_e$, which was also described as being difficult to tune above, however, ϵ_r does not impact closed-loop stability or recursive feasibility of the LEMPC of Eq. 9 augmented with the constraints of Eqs. 31-32, which was proven in [14]. Therefore, [13] suggested that this difficult-to-tune parameter could perhaps be tuned by performing on-line "experiments" (in the sense that the process could be operated with different values of the tuning parameter over time), and then operators or engineers could provide feedback regarding how well they liked the process response for different values of the tuning parameter. This may lead to an optimal value of ϵ_r being able to be chosen after the controller is set in place, rather than through extensive closed-loop simulations or testing before the controller is put on-line.

This has a flavor similar to that noted in the above section in the sense of being a method for reducing the time required to determine parameters in LEMPC before it can be put on-line, potentially making the controller more tractable for industrial use. It also gives another idea for aiding in selecting $\bar{\rho}'_e$ that may help to optimize the parameter more than the *ad hoc* selection suggested previously. Specifically, $\bar{\rho}'_e$ from the prior section may also be able to be tuned on-line using past data to aid in selecting better values of $\bar{\rho}'_e$ for seeking to find a value that most optimizes profits (i.e., trades off between a larger region in which profit is optimized in Eq. 9 and the potential that larger regions may result in more frequent activation of h_1 between sampling periods). The on-line tuning method might, for example, monitor a profit metric over time with different values of $\bar{\rho}'_e$ that would meet the theoretical conditions which guarantee closed-loop stability, and then select a value that seems to be most economically attractive, based on the data. The on-line tuning and the fact that the data may not be truly representative of the future plant behavior does not pose closed-loop stability issues in this case, as $\bar{\rho}'_e$ can be varied within a range where closed-loop stability is still guaranteed.

For example, we return again to the CSTR from Section III-A.3 (using rng(100)), but this time we seek to use the activation of $h_1(x)$ not as a long-term solution for disturbance handling, but instead as a technique for allowing on-line operating data to be gathered that allows $\bar{\rho}'_e$ to be adjusted over time. The goal of adjusting $\bar{\rho}'_e$ is to attempt to select a value that does not cause $h_1(x)$ to be activated often, thus leaving it as a backup method for attempting to handle irregular scenarios to design some conservatism into the controller, but based on on-line operating data rather than closed-loop simulations carried out *a priori*. Specifically, we initialize $\bar{\rho}'_e$ at 0.75ρ, but then increase it by 1 at the end of each sampling period until the value of $V(x)$ exceeds 0.99ρ at some point in a sampling period. At that point, we again activate $h_1(x)$, but then decrease $\bar{\rho}'_e$ by 1 compared to the value that was used for the sampling period when $V(x) > 0.99\rho$ at some point in Δ. The goal of this is to utilize on-line process operating data to serve as a warning that the value of $\bar{\rho}'_e$ may not be tight enough for typical disturbances and the size of the sampling period to allow the closed-loop state to remain within the majority of Ω_ρ, suggesting that reducing $\bar{\rho}'_e$ may promote closed-loop stability. Fig. 2 shows the values of $V(x)$ and $\bar{\rho}'_e$ in comparison to ρ over time when this strategy is used. The maximum value of $V(x)$ experienced during the hour of operation under this strategy was 297.02, which is slightly above $0.99\rho = 297$. The value of $\bar{\rho}'_e$ increased from 225 to 295 but then was subsequently decreased to 294, and ended at 293. The figure shows that this on-line tuning method for $\bar{\rho}'_e$, which takes advantage of the implementation strategy developed in Section III-A.1 to prevent the closed-loop state from moving too close to the boundary of Ω_ρ, but then uses the on-line data regarding the values of ρ'_e for which the closed-loop state approaches the boundary of Ω_ρ over time to decrease that parameter, was successful in preventing the closed-loop state from leaving Ω_ρ in this simulation.

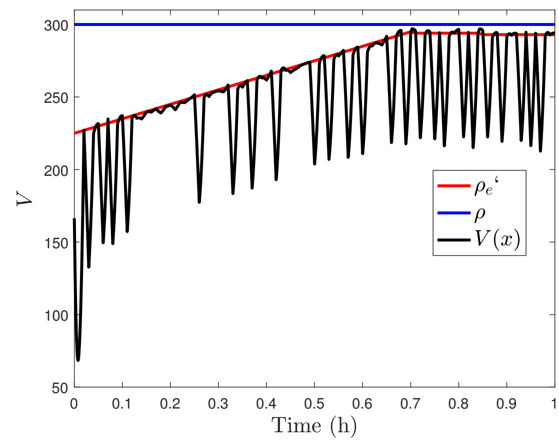

Fig. 2. $V(x)$ and ρ'_e compared with ρ over time.

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