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Duality and H_∞-Optimal Control Of Coupled ODE-PDE Systems

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***Abstract*— In this paper, we present a convex formulation of H_∞-optimal control problem for coupled linear ODE-PDE systems with one spatial dimension. First, we reformulate the coupled ODE-PDE system as a Partial Integral Equation (PIE) system and show that stability and H_∞ performance of the PIE system implies that of the ODE-PDE system. We then construct a dual PIE system and show that asymptotic stability and H_∞ performance of the dual system is equivalent to that of the primal PIE system. Next, we pose a convex dual formulation of the stability and H_∞-performance problems using the Linear PI Inequality (LPI) framework. Next, we use our duality results to formulate the stabilization and H_∞-optimal state-feedback control problems as LPIs. LPIs are a generalization of LMIs to Partial Integral (PI) operators and can be solved using PIETOOLS, a MATLAB toolbox. Finally, we illustrate the accuracy and scalability of the algorithms by constructing controllers for several numerical examples.**

I. INTRODUCTION

In this paper, we consider the problem of H_∞-optimal state-feedback controller synthesis for Partial Integral Equation (PIE) systems of the form

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^m \times L_2^n \\ z(t) &= \mathcal{C}\mathbf{x}(t) + \mathcal{D}u(t) \end{aligned} \tag{1}$$

where $\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are Partial Integral (PI) operators and $u(t) \in \mathbb{R}^p$. The dual (or adjoint) PIE system is then defined to be

$$\begin{aligned}\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) &= \mathcal{A}^*\bar{\mathbf{x}}(t) + \mathcal{C}^*\bar{u}(t), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \in \mathbb{R}^m \times L_2^n \\ \bar{z}(t) &= \mathcal{B}^*\bar{\mathbf{x}}(t) + \mathcal{D}^*\bar{u}(t) \end{aligned} \tag{2}$$

where * denotes the adjoint with respect to L_2-inner product. Recently, it has been shown that almost any PDE system in a single spatial dimension coupled with an ODE at the boundary has an equivalent PIE system representation [11] (see Sec. IV). It should be noted, however, that the formulation in Eqn. (1) does not allow for inputs directly at the boundary - rather these must enter through the ODE or into the domain of the PDE. Use of the PIE system representation, defined by the algebra of Partial Integral (PI) operators, allows us to generalize LMIs developed for ODEs to infinite-dimensional systems. These generalizations are referred to as Linear PI Inequalities (LPIs) and can be solved efficiently using the Matlab toolbox PIETOOLS [13]. In previous work, LPIs have been proposed for stability [10], H_∞-gain [12] and H_∞-optimal estimation [3] of PIE systems. However, until now the stabilization and H_∞-optimal controller synthesis problems have remained unresolved. In this paper, we resolve the problems of stabilizing and H_∞ state-feedback controller synthesis by proving the following results.

(A) *Dual Stability Theorem:* We show that the PIE system (1) is stable for $u = 0$ and any initial condition $\mathbf{x}(0) \in L_2$ if and only if the dual PIE system (2) is stable for any initial conditions $\bar{\mathbf{x}}(0) \in L_2$ and $\bar{u} = 0$.

(B) *Dual L_2-gain Theorem:* For $u \in L_2([0,\infty))$ and $\mathbf{x}(0) = 0$, any solution of the PIE system (1) satisfies $\|z\|_{L_2} \le \gamma \|u\|_{L_2}$ if and only if any solution to the dual PIE system Eq. (2) satisfies $\|\bar{z}\|_{L_2} \le \gamma \|\bar{u}\|_{L_2}$ for $\bar{\mathbf{x}}(0) = 0$ and $\bar{u} \in L_2([0,\infty))$.

(C) *H_∞-optimal Control of PIEs:* The stabilization and H_∞-optimal state-feedback controller synthesis problem for PIE systems (1) may be formulated as an LPI.

Previous work on controller synthesis for coupled ODE-PDE systems includes the well-established method of backstepping (See e.g. [6]) and discretization-based methods (See e.g. [4], [5], [1]). In the former case, backstepping methods allow for inputs at the boundary and are guaranteed to find a stabilizing controller if one exists. However, the resulting controllers are not optimal in any sense. In the latter case, H_∞-optimal controllers are designed for an ODE approximation of the coupled ODE-PDE system. However, these controllers do not have provable performance properties when applied to the actual ODE-PDE, i.e. the H_∞-norm of the ODE-PDE system is not same as the H_∞-norm of the ODE approximation and indeed, the resulting closed-loop system is often unstable. Frequency-domain approaches, for example [14], [7], design optimal control using transfer function of the system. However, the concept of transfer function is not extendable to systems with multiple inputs/outputs.

The fundamental issue in controller synthesis for both finite-dimensional and infinite-dimensional systems is one of non-convexity. In simple terms, for either a finite or infinite-dimensional system of the form

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t),$$

finding a stabilizing control $u(t) = \mathcal{K}x(t)$ and a corresponding Lyapunov functional $V(t) = \langle x(t), \mathcal{P}x(t)\rangle_X$ with negative time-derivative leads to a bilinear problem in variables $\mathcal{K}$ and $\mathcal{P}$ of the form $(\mathcal{A}+\mathcal{B}\mathcal{K})^*\mathcal{P} + \mathcal{P}(\mathcal{A}+\mathcal{B}\mathcal{K}) \le 0$.

In case of finite-dimensional linear systems, the linear operators $\mathcal{P}, \mathcal{K}, \mathcal{A}$ and $\mathcal{B}$ are just matrices P, K, A and B. In absence of a controller, the Lyapunov stability test (referred to as primal stability test) can be written as an LMI in positive matrix variable $P > 0$ such that $A^TP + PA \le 0$. In finite-dimensions, the eigenvalues of A and A^* are the same and hence there is an equivalent dual Lyapunov inequality of the form $AP + PA^T \le 0$. Then

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the test for existence of a stabilizing controller K and a Lyapunov functional P which proves the stability of the closed-loop system can now be written as: find $P > 0$ such that $(A + BK)P + P(A + BK)^T \leq 0$. The key difference, however, is the bilinearity can now be eliminated by introducing new variable $Z = KP$ which leads to the LMI constraint $AP + BZ + (AP + BZ)^T \leq 0$.

For infinite-dimensional systems, Theorem 5.1.3 of [2] is analogous to primal stability test for ODEs. The result is similar in the sense that matrices in the constraints of primal stability test for ODE are replaced by linear operators for infinite-dimensional systems, i.e. a test for existence of a positive operator $\mathcal{P} > 0$ that satisfies the operator-valued constraint $\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} \leq 0$. However, there does not exist a dual form of the stability test for infinite-dimensional systems. In [9], a dual Lyapunov criterion for stability in infinite-dimensional systems was presented. However, the result was restricted to infinite-dimensional systems of the form

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t)$$

and included constraints on the image of the operator $\mathcal{P}$ of the form $\mathcal{P}(X) = X$ where $X = D(\mathcal{A})$ is the domain of the infinitesimal generator $\mathcal{A}$. Furthermore, because $\mathcal{A}$ for PDEs is a differential operator, this approach provides no way of enforcing negativity of the dual stability condition. These difficulties in analysis and controller synthesis for PDE systems led to the development of the PIE formulation of the problem - wherein both system parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{T}$ and the Lyapunov parameter $\mathcal{P}$ lie in the algebra of bounded linear PI operators.

In this work, we adopt the PIE formulation of the ODE-PDE system and propose dual stability and performance tests wherein all operators lie in the PI algebra and do not include additional constraints such as $\mathcal{P}(X) = X$. Specifically, the results (A) and (B) lead to LPIs which, by allowing for the variable change trick used in finite-dimensional systems, allows us to propose convex and testable formulations of the stabilization and optimal control problems - resulting in stabilizing or H_∞ optimal controllers for coupled PDE-ODE systems where the inputs enter through the ODE or in the domain. More specifically, these methods apply for linear ODE-PDE systems in 1 spatial variable with a very general set of boundary conditions including Dirichlet, Neumann, Robin, Sturm-Lioville etc. The resulting LPIs are solved numerically using PIETOOLS [13], an open-source MATLAB toolbox to handle PI variables and setup PI operator-valued optimization problems. Finally, we note that this is the first result to achieve H_∞-optimal control of coupled ODE-PDE systems without discretizing the PDE. Although we are currently restricted to inputs using an ODE filter or in-domain, we believe the duality results presented here can ultimately be extended to cover inputs applied directly at the boundary.

The paper is organized as follows. After introducing preliminary notations in Section II, in Section III and IV, we present the general form of PIE and ODE-PDE under consideration. In Section V, we define the conditions under which PIE and ODE-PDE as equivalent followed by equivalence in stability and H_∞-gain in Section VI. Section VII discusses the properties of adjoint PIE systems. In Section VIII and IX, we derive the dual stability theorem and dual H_∞-gain theorem for PIEs. Sections XI through XIV present the LPIs developed using dual stability theorem and dual H_∞-gain theorem. Examples are illustrated in Section XV and followed by conclusions in Section XVI.

II. Notation

The calligraphic font, for example $\mathcal{A}$, is used to represent linear operators on Hilbert spaces and the bold font, $\mathbf{x}$, is used to denote functions in $L_2^n[a, b]$ which is the set of all square-integrable functions on the domain $[a, b] \subset \mathbb{R}$. The Sobolev space $W_{2,k}[a, b]$ is defined as
$W_{2,k}[a, b] := \{f \in L_2[a, b] \mid \dfrac{\partial^n f}{\partial s^n} \in L_2[a, b] \text{ for all } n \leq k\}$.
$Z^{m,n}[a, b]$ denotes the space $\mathbb{R}^m \times L_2^n[a, b]$ that is equipped with the inner-product

$$\left\langle \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix} \right\rangle_Z = x_1^T y_1 + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle_{L_2}, \quad \begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ \mathbf{y}_2 \end{bmatrix} \in Z^{m,n}.$$

We use $\mathbf{x}_s$ to denote partial derivative of $\frac{\partial \mathbf{x}}{\partial s}$ where the number of repetitions of the subscript s corresponds to the order of the partial derivative and $\dot{\mathbf{x}}$ to denote the partial derivative $\frac{\partial \mathbf{x}}{\partial t}$.

III. Partial Integral Equations

In this section, we will define a PIE system with inputs and disturbances of the form

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad \mathbf{x}(0) \in Z^{m,n}[a, b]$$
$$z(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \tag{3}$$

where the $\mathcal{T}, \mathcal{A} : Z^{m,n}[a, b] \to Z^{m,n}[a, b]$, $\mathcal{B}_1 : \mathbb{R}^q \to Z^{m,n}[a, b]$, $\mathcal{B}_2 : \mathbb{R}^p \to Z^{m,n}[a, b]$, $\mathcal{C} : Z^{m,n}[a, b] \to \mathbb{R}^r$, $\mathcal{D}_{11} \in \mathbb{R}^{r\times q}$ and $\mathcal{D}_{12} \in \mathbb{R}^{r\times p}$ are Partial Integral (PI) operators, defined as follows.

Definition 1. *(PI Operators:) A 4-PI operator is a bounded linear operator between $Z^{m,n}[a, b]$ and $Z^{p,q}[a, b]$ of the form*

$$\mathcal{P}\left[\begin{smallmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{smallmatrix}\right] \begin{bmatrix} x \\ \mathbf{y} \end{bmatrix}(s) = \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{y}(s)ds \\ Q_2(s)x + \mathcal{P}_{\{R_i\}}\mathbf{y}(s) \end{bmatrix} \tag{4}$$

where $P \in \mathbb{R}^{p\times m}$ is a matrix, $Q_1 : [a, b] \to \mathbb{R}^{p\times n}$, $Q_2 : [a, b] \to \mathbb{R}^{q\times m}$ are bounded integrable functions and $\mathcal{P}_{\{R_i\}} : L_2^n[a, b] \to L_2^q[a, b]$ is a 3-PI operator of the form

$$\left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s) :=$$
$$R_0(s)\mathbf{x}(s) + \int_a^s R_1(s, \theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s, \theta)\mathbf{x}(\theta)d\theta.$$

Definition 2. *For given $u \in L_2([0, \infty); \mathbb{R}^p)$, $w \in L_2([0, \infty); \mathbb{R}^q)$ and initial conditions $\mathbf{x}_0 \in Z^{m,n}[a, b]$, we say that $\mathbf{x} : [0, \infty) \to Z^{m,n}[a, b]$ and $z : [0, \infty) \to \mathbb{R}^r$ satisfy the PIE (1) defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ if $\mathbf{x}$ is Fréchet differentiable almost everywhere on $[0, \infty)$, $\mathbf{x}(0) = \mathbf{x}_0$ and the equations (3) are satisfied for almost all $t \geq 0$.*

IV. A General Class of Linear ODE-PDE Systems

In this paper, we consider control of the following class of coupled linear ODE-PDE systems in a single spatial variable $s \in [a, b]$.

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\mathbf{x}}(s,t)\end{bmatrix} = \begin{bmatrix}Ax(t) + (\mathcal{E}\mathbf{x})(t)\\ E(s)x(t) + (\mathcal{A}_p\mathbf{x})(s,t)\end{bmatrix} + \begin{bmatrix}B_{11} & B_{12}\\ B_{21}(s) & B_{22}(s)\end{bmatrix}\begin{bmatrix}w(t)\\ u(t)\end{bmatrix},$$

$$z(t) = \left(\mathcal{C}\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(t) + D_{11}w(t) + D_{12}u(t),$$

$$B\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix} = B_x x(t), \qquad \begin{bmatrix}x(0)\\ \mathbf{x}(\cdot,0)\end{bmatrix} = \mathbf{x}_0 \in D(\mathcal{A}_d)$$

$$\mathbf{x}(s,t) = \begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}, \quad \mathbf{x}_c(s,t) = \begin{bmatrix}\mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix} \tag{5}$$

where the operators $\mathcal{A}_p$, $\mathcal{C}$ and $\mathcal{E}$ are defined as

$$(\mathcal{A}_p\mathbf{x})(s,t) := A_0(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix} + A_1(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix} + A_2(s)\mathbf{x}_{3ss}(s,t),$$

$$(\mathcal{E}\mathbf{x})(t) := E_{10}\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix} + \int_a^b E_a(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}ds + \int_a^b E_b(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}ds,$$

$$\left(\mathcal{C}\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(t) := Cx(t) + C_{10}\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix} + \int_a^b C_a(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}ds + \int_a^b C_b(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}ds,$$

and the domain is given by

$$D(\mathcal{A}_d) := \left\{\begin{matrix}\begin{bmatrix}x\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} \in \mathbb{R}^{n_o} \times L_2^{n_1}[a,b] \times W_{2,1}^{n_2}[a,b] \times W_{2,2}^{n_3}[a,b] : \\ B\begin{bmatrix}x_c(a)\\ x_c(b)\end{bmatrix} = B_x x, \text{ where } x_c(s) = \begin{bmatrix}\mathbf{x}_2(s)\\ \mathbf{x}_3(s)\\ \mathbf{x}_{3s}(s)\end{bmatrix}\end{matrix}\right\} \tag{6}$$

The ODE states are $x(t) \in \mathbb{R}^{n_o}$, while the PDE states are $\mathbf{x}_i(s,t) \in \mathbb{R}^{n_i}$. The total number of PDE is defined to be $n_p = n_1 + n_2 + n_3$. The ODE-PDE system is defined by the parameters $A_0 : [a,b] \to \mathbb{R}^{n_p \times n_p}$, $A_1 : [a,b] \to \mathbb{R}^{n_p \times (n2+n3)}$, $A_2 : [a,b] \to \mathbb{R}^{n_p \times n_3}$, $E : [a,b] \to \mathbb{R}^{n_p \times n_o}$, $E_a : [a,b] \to \mathbb{R}^{n_o \times n_p}$, $E_b : [a,b] \to \mathbb{R}^{n_o \times (n_2+n_3)}$, $C_a : [a,b] \to \mathbb{R}^{n_z \times n_p}$, $C_b : [a,b] \to \mathbb{R}^{n_z \times (n_2+n_3)}$ and B_{2j} are bounded integrable functions. $A \in \mathbb{R}^{n_o \times n_o}$, $E_{10} \in \mathbb{R}^{n_o \times 2n_r}$, $C_{10} \in \mathbb{R}^{n_z \times 2n_r}$, B_{1j}, D_{ij} and $B \in \mathbb{R}^{n_r \times 2n_r}$ are matrices. B has row rank $n_r := n_2 + 2n_3$ and $B_x \in \mathbb{R}^{n_r \times n_o}$. This class of systems includes almost all coupled linear ODE-PDE systems with the constraint that the input does not directly act at the boundary, but rather through the ODE or in the domain of the PDE.

Illustrative Example To illustrate how this representation is applied to a typical ODE-PDE model, we consider a wave equation coupled with an ODE as shown below.

$$\dot{x}(t) = ax(t) + dw(1,t), \tag{7}$$

$$\ddot{w}(s,t) = cw_{ss}(s,t), \qquad w(0,t) = kx(t), w_s(1,t) = 0,$$

where $w(s,t)$ is the PDE state and x is the ODE state. These equations may be rewritten in the form (5)

$$\dot{x}(t) = ax(t) + d\mathbf{x}_3(1,t),$$

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_3\end{bmatrix}(s,t) = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_3\end{bmatrix}(s,t) + \begin{bmatrix}c\\ 0\end{bmatrix}\mathbf{x}_{3ss}(s,t)$$

$$\mathbf{x}_3(0,t) = kx(t), \mathbf{x}_{3s}(1,t) = 0$$

where $\mathbf{x}_1 = \dot{w}$ and $\mathbf{x}_3 = w$. The parameters that define the ODE-PDE (5) are

$$A = a, A_0 = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, A_2 = \begin{bmatrix}c\\ 0\end{bmatrix}, E_{10} = \begin{bmatrix}0 & 0 & d & 0\end{bmatrix},$$

$$B = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, B_x = \begin{bmatrix}k\\ 0\end{bmatrix},$$

and the rest of the system parameters are zero.

Definition 3. *For given $u \in L_2([0,\infty);\mathbb{R}^{n_u})$ and initial conditions $\mathbf{x}_0 \in D(\mathcal{A}_d)$ as defined in* (6)*, we say that $x : [0,\infty) \to \mathbb{R}^{n_o}$, $\mathbf{x} : [0,\infty) \to \prod_{i=0}^{2} W_{2,i}^{n_{i+1}}[a,b]$ and $z : [0,\infty) \to \mathbb{R}^{n_z}$ satisfy the ODE-PDE* (5) *defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ if x is differentiable and $\mathbf{x}$ is Fréchet differentiable almost everywhere on $[0,\infty)$, $\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}(0) = \mathbf{x}_0$, $\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}(t) \in D(\mathcal{A}_d)$ and Equations* (5) *hold for almost all $t \geq 0$.*

V. PIE Representation of the ODE-PDE System

A coupled ODE-PDE of the form Eq. (5) can be written as a PIE Eq. (3). Furthermore, the solutions of the PIE define solutions of the ODE-PDE and vice-versa.

Theorem 4. *For given $u \in L_2([0,\infty);\mathbb{R}^{n_u})$ and initial conditions $\begin{bmatrix}x_0\\ \mathbf{x}_0\end{bmatrix} \in D(\mathcal{A}_d)$ as defined in* (6)*, suppose x, $\mathbf{x}$ and z satisfy the ODE-PDE defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$. Then z also satisfies the PIE*

$$\mathcal{T}\dot{\mathbf{v}}(t) = \mathcal{A}\mathbf{v}(t) + \mathcal{B}u(t), \quad \mathbf{v}(0) = \mathbf{v}_0$$

$$z(t) = \mathcal{C}\mathbf{v}(t) + \mathcal{D}u(t),$$

with

$$\mathbf{v}_0 = \begin{bmatrix}x_0\\ \mathbf{x}_{01}\\ \mathbf{x}_{02,s}\\ \mathbf{x}_{03,ss}\end{bmatrix}, \quad \mathbf{v}(t) := \begin{bmatrix}x(t)\\ \mathbf{x}_1(t)\\ \mathbf{x}_{2,s}(t)\\ \mathbf{x}_{3,ss}(t)\end{bmatrix},$$

where the 4-PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are as defined in Eqns. (19)*. Conversely, for given $u \in L_2([0,\infty);\mathbb{R}^{n_u})$ and initial conditions $\mathbf{v}_0 \in Z^{n_o,n_p}[a,b]$, suppose $\mathbf{v}$ and z satisfy the PIE defined by the 4-PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ as defined in Equations* (19)*. Then, z also satisfies the ODE-PDE defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ with*

$$\mathbf{x}_0 = \mathcal{T}\mathbf{v}_0, \quad \begin{bmatrix}x(t)\\ \mathbf{x}(t)\end{bmatrix} := \mathcal{T}\mathbf{v}(t).$$

Proof. Refer Lemma 3.3 and 3.4 in [11] for proof. □

PIE representations differ from typical ODE-PDE form in several ways. First, while PDEs rely on a differential operator in $\mathcal{A}_d$, the a PIE system is parameterized by PI operators which are bounded on L_2 and form an algebra. Second, the PIE eliminates boundary conditions by incorporating the effect of boundary conditions directly into the dynamics. Finally, solutions of the PIE system are defined on $Z^{n_0,n_p}[a,b]$, which is a Hilbert space with respect to Z-inner product, whereas $D(\mathcal{A}_d)$ is not a Hilbert space.

VI. Equivalence in Stability

In this section, we show that stability in Z and L_2-gain of the ODE-PDE system is same as that of the PIE system. First, we define asymptotic stability of PIEs and of ODE-PDEs.

Definition 5. *For $w = u = 0$, the PIE (3) defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, \mathcal{D}_{ij}\}$ is said to be asymptotically stable if for any initial condition $\mathbf{x}_0 \in Z^{m,n}[a,b]$, if $\mathbf{x}$ and z satisfy the PIE, we have $\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\|_Z = 0$.*

Definition 6. *For $w = u = 0$, the ODE-PDE (5) defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ is said to be asymptotically stable if for initial condition $\mathbf{x}_0 \in D(\mathcal{A}_d)$, if x, $\mathbf{x}$ and z satisfy the ODE-PDE , then*

$$\lim_{t\to\infty} \left\| \begin{bmatrix} x(t) \\ \mathbf{x}(t) \end{bmatrix} \right\|_Z = 0.$$

Lemma 7. *Suppose $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, \mathcal{D}_{ij}\}$ and $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ satisfy Eqns. (19). Then the ODE-PDE defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ is asymptotically stable if the PIE defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, \mathcal{D}_{ij}\}$ is asymptotically stable.*

Proof. From Theorem 4, x and $\mathbf{x}$ satisfy the ODE-PDE for the given x_0, $\mathbf{x}_0$ if and only if $\mathbf{v}$ satisfies the PIE for $\mathbf{v}_0$ where

$$\begin{bmatrix} x(t) \\ \mathbf{x}(t) \end{bmatrix} = \mathcal{T}\mathbf{v}(t), \qquad \begin{bmatrix} x_0 \\ \mathbf{x}_0 \end{bmatrix} = \mathcal{T}\mathbf{v}_0.$$

If the PIE is stable, then $\lim_{t\to 0} \|\mathbf{v}(t)\|_Z = 0$. Since $\mathcal{T}$ is a bounded linear operator, this implies

$$\lim_{t\to 0} \left\| \begin{bmatrix} x(t) \\ \mathbf{x}(t) \end{bmatrix} \right\|_Z = \lim_{t\to 0} \|\mathcal{T}\mathbf{v}(t)\|_Z = 0.$$

□

Lemma 8. *Suppose $\{\mathcal{T}, \mathcal{A}, \mathcal{B}_i, \mathcal{C}, \mathcal{D}_{ij}\}$ and $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ satisfy Eqns. (19). For $w \in L_2([0,\infty))$, $u = 0$ and $\mathbf{x}(0) = 0$, any solution $\mathbf{x}, z$ of the PIE system satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$ if and only if any solution to the ODE-PDE system, $x, \mathbf{x}, z$ satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$ for $\mathbf{x}(0) = 0$, $x(t) = 0$, $u = 0$ and $\bar{w} \in L_2([0,\infty))$.*

Proof. From Theorem 4, x, $\mathbf{x}$, and z satisfy the ODE-PDE for a given w if and only if $\mathbf{v}$ and z satisfy the PIE for the given w where

$$\begin{bmatrix} x(t) \\ \mathbf{x}(t) \end{bmatrix} = \mathcal{T}\mathbf{v}(t).$$

□

VII. The Dual PIE

For a PIE system of the form Eq. (1) we may associate the following dual (adjoint) PIE.

$$\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t) + \mathcal{C}^*\bar{w}(t)$$
$$\bar{z}(t) = \mathcal{B}^*\bar{\mathbf{x}}(t) + \mathcal{D}^*\bar{w}(t)$$

where $\mathcal{T}^*, \mathcal{A}^* : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$, $\mathcal{B}^* : Z^{m,n}[a,b] \to \mathbb{R}^{n_w}$, $\mathcal{C}^* : \mathbb{R}^{n_z} \to Z^{m,n}[a,b]$ and $\mathcal{D}^* \in \mathbb{R}^{n_w \times n_z}$ are 4-PI operators.

When the PIE system Eq. (1) is constructed from a PDE system, then the dual PIE system Eq. (2) may also be constructed from a PDE system.

Example 9. *Consider the transport equation*

$$\dot{\mathbf{v}}(s,t) + \mathbf{v}_s(s,t) = 0, \qquad s \in [0,1], t > 0,$$
$$\mathbf{v}(0,t) = 0, \ \mathbf{v}(s,0) \in L_2[0,1]. \tag{8}$$

The PIE form Eq.(8) *is*

$$(\mathcal{P}_{\{0,1,0\}}\dot{\mathbf{x}})(t) = (\mathcal{P}_{\{-1,0,0\}}\mathbf{x})(t), \qquad t > 0.$$

The corresponding dual PIE is

$$(\mathcal{P}_{\{0,0,1\}}\dot{\mathbf{y}})(t) = (\mathcal{P}_{\{-1,0,0\}}\mathbf{y})(t), \qquad t > 0.$$

The dual PIE may be constructed from the following PDE

$$\dot{\mathbf{z}}(s,t) - \mathbf{z}_s(s,t) = 0, \qquad s \in [0,1], t > 0,$$
$$\mathbf{z}(1,t) = 0, \ \mathbf{z}(s,0) \in L_2[0,1].$$

VIII. Dual Stability Theorem

In this section, show that dual PIE is stable if and only if the primal PIE is stable.

Theorem 10. *(Dual Stability of PIEs:) Suppose $\mathcal{T}$ and $\mathcal{A}$ are 4-PI operators. Then the following statements are equivalent.*

1) $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) \to 0$ for any $\mathbf{x}$ that satisfies $\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in Z^{m,n}[a,b]$.

2) $\lim_{t\to\infty} \mathcal{T}^*\mathbf{x}(t) \to 0$ for any $\mathbf{x}$ that satisfies $\mathcal{T}^*\dot{\mathbf{x}}(t) = \mathcal{A}^*\mathbf{x}(t)$ with initial condition $\mathbf{x}(0) \in Z^{m,n}[a,b]$.

Proof. Suppose $\mathbf{x}$ satisfies $\mathcal{T}\dot{x}(t) = \mathcal{A}x(t)$ with initial condition $\mathbf{x}(0) \in Z^{m,n}[a,b]$ and $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) \to 0$. Let $\bar{\mathbf{x}}$ satisfy $\mathcal{T}^*\dot{\mathbf{x}}(t) = \mathcal{A}^*\mathbf{x}(t)$ with initial condition $\bar{\mathbf{x}}(0) \in Z^{m,n}[a,b]$. In the following, we use $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_Z$. Then for any finite $t > 0$, by IBP and a variable change,

$$\begin{aligned}
&\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\mathbf{x}}(s)\rangle \, ds \\
&= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle - \int_0^t \langle \partial_s \bar{\mathbf{x}}(t-s), \mathcal{T}\mathbf{x}(s)\rangle \, ds \\
&= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle - \int_t^0 \left\langle \dot{\bar{\mathbf{x}}}(\theta), \mathcal{T}\mathbf{x}(t-\theta)\right\rangle d\theta \\
&= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle + \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta
\end{aligned}$$

where $\theta = t - s$. Furthermore, using a variable change,

$$\begin{aligned}
\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\mathbf{x}}(s)\rangle \, ds &= \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{A}\mathbf{x}(s)\rangle \, ds \\
= \int_0^t \langle \bar{\mathbf{x}}(\theta), \mathcal{A}\mathbf{x}(t-\theta)\rangle \, d\theta &= \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta)\rangle \, d\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta)\rangle \, d\theta \\
&= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle + \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta.
\end{aligned}$$

However, $\mathcal{A}^*\bar{\mathbf{x}}(\theta) = \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta)$ for all $\theta \in [0,t]$ and so we have

$$\langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle = 0 \qquad \forall t > 0.$$

If $\lim_{t\to\infty} \mathcal{T}\mathbf{x}(t) = 0$, then

$$\lim_{t\to\infty} \langle \mathcal{T}^*\bar{\mathbf{x}}(t), \mathbf{x}(0)\rangle = \lim_{t\to\infty} \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle = 0$$

for any $\mathbf{x}(0) \in Z^{m,n}[a,b]$. We conclude that $\lim_{t\to\infty} \mathcal{T}^*\bar{\mathbf{x}}(t) = 0$. Since the dual and primal systems are interchangeable, necessity follows from sufficiency. □

IX. DUAL L_2-GAIN THEOREM

We proved that stability of a PIE system and its dual are equivalent. Now, we show that for $\mathbf{x}_0 = 0$, input-output performance of primal and dual PIE in the L_2-gain metric is equivalent.

Theorem 11. *(Duality on L_2-gain bound of PIEs:) Suppose $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are 4-PI operators. Then the following statements are equivalent.*

1) *For any $w \in L_2([0,\infty);\mathbb{R}^q)$ and $\mathbf{x}(0) = 0$ any solution $\mathbf{x}(t) \in Z^{m,n}$ and $z(t) \in \mathbb{R}^p$ of the PIE system*
$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t), \qquad \mathbf{x}(0) = 0$$
$$z(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}w(t) \tag{9}$$
satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.
2) *For any $\bar{w} \in L_2([0,\infty);\mathbb{R}^p)$ and $\bar{\mathbf{x}}(0) = 0$, any $\bar{\mathbf{x}}(t) \in Z^{m,n}$ and $\bar{z}(t) \in \mathbb{R}^q$ of the dual PIE system*
$$\mathcal{T}^*\dot{\bar{\mathbf{x}}}(t) = \mathcal{A}^*\bar{\mathbf{x}}(t) + \mathcal{C}^*\bar{w}(t), \qquad \bar{\mathbf{x}}(0) = 0$$
$$\bar{z}(t) = \mathcal{B}^*\bar{\mathbf{x}}(t) + \mathcal{D}^*\bar{w}(t) \tag{10}$$
satisfies $\|\bar{z}\|_{L_2} \le \gamma \|\bar{w}\|_{L_2}$.

Proof. Suppose that for any $w \in L_2([0,\infty);\mathbb{R}^q)$ and $\mathbf{x}(0) = 0$ any solution $\mathbf{x}(t) \in Z^{m,n}$ and $z(t) \in \mathbb{R}^p$ of the PIE system satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$. For $\bar{w} \in L_2([0,\infty);\mathbb{R}^p)$ and $\bar{\mathbf{x}}(0) = 0$, let $\bar{\mathbf{x}}(t) \in Z^{m,n}$ and $\bar{z}(t) \in \mathbb{R}^q$ satisfy the dual PIE system. Then for any finite $t \ge 0$, since $\mathbf{x}(0) = \bar{\mathbf{x}}(0) = 0$, we have

$$\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\mathbf{x}}(s)\rangle\, ds$$
$$= \langle \bar{\mathbf{x}}(0), \mathcal{T}\mathbf{x}(t)\rangle - \langle \bar{\mathbf{x}}(t), \mathcal{T}\mathbf{x}(0)\rangle + \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta$$
$$= \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta$$

where $\theta = t-s$. Furthermore, by the change variable change,

$$\int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{T}\dot{\mathbf{x}}(s)\rangle\, ds$$
$$= \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{A}\mathbf{x}(s)\rangle\, ds + \int_0^t \langle \bar{\mathbf{x}}(t-s), \mathcal{B}w(s)\rangle\, ds$$
$$= \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta)\rangle\, d\theta + \int_0^t \langle \mathcal{B}^*\bar{\mathbf{x}}(\theta), w(t-\theta)\rangle\, d\theta.$$

Combining the two equalities, we obtain

$$\int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta$$
$$= \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta)\rangle\, d\theta + \int_0^t \langle \mathcal{B}^*\bar{\mathbf{x}}(\theta), w(t-\theta)\rangle\, d\theta.$$

By the definition of $\bar{z}$, we obtain

$$\int_0^t \langle \bar{z}(\theta), w(t-\theta)\rangle\, d\theta - \int_0^t \langle \mathcal{D}^*\bar{w}(\theta), w(t-\theta)\rangle\, d\theta$$
$$= \int_0^t \langle \mathcal{B}^*\bar{\mathbf{x}}(\theta), w(t-\theta)\rangle\, d\theta$$
$$= \int_0^t \left\langle \mathcal{T}^*\dot{\bar{\mathbf{x}}}(\theta), \mathbf{x}(t-\theta)\right\rangle d\theta - \int_0^t \langle \mathcal{A}^*\bar{\mathbf{x}}(\theta), \mathbf{x}(t-\theta)\rangle\, d\theta$$
$$= \int_0^t \langle \mathcal{C}^*\bar{w}(\theta), \mathbf{x}(t-\theta)\rangle\, d\theta = \int_0^t \langle \bar{w}(\theta), \mathcal{C}\mathbf{x}(t-\theta)\rangle\, d\theta$$
$$= \int_0^t \langle \bar{w}(\theta), z(t-\theta)\rangle\, d\theta - \int_0^t \langle \bar{w}(\theta), \mathcal{D}w(t-\theta)\rangle\, d\theta.$$

We conclude that for any $t > 0$, if z and w satisfy the primal PIE and $\bar{z}$ and $\bar{w}$ satisfy the dual PIE, then

$$\int_0^t \langle \bar{z}(\theta), w(t-\theta)\rangle\, d\theta = \int_0^t \langle \bar{w}(\theta), z(t-\theta)\rangle\, d\theta.$$

Now, for any $\bar{w} \in L_2^p$, suppose $\bar{z}$ solves the dual PIE for some $\bar{\mathbf{x}}$. For any fixed $T > 0$, define $w(t) = \bar{z}(T-t)$ for $t \le T$ and $w(t) = 0$ for $t > T$. Then $w \in L_2^q$ and for this input, let z solve the primal PIE for some $\mathbf{x}$. Then if we define the truncation operator P_T, we have

$$\|P_T\bar{z}\|_{L_2}^2 = \int_0^T \langle \bar{z}(s), \bar{z}(s)\rangle\, ds = \int_0^T \langle \bar{z}(s), w(T-s)\rangle\, ds$$
$$= \int_0^T \langle \bar{w}(s), z(T-s)\rangle\, ds \le \|P_T\bar{w}\|_{L_2} \|P_T z\|_{L_2}$$
$$\le \gamma \|\bar{w}\|_{L_2} \|w\|_{L_2} = \gamma \|\bar{w}\|_{L_2} \|\bar{z}\|_{L_2}.$$
$$\|P_T\bar{z}\|_{L_2}^2 = \int_0^T \langle \bar{z}(s), \bar{z}(s)\rangle\, ds = \int_0^T \langle \bar{z}(s), w(T-s)\rangle\, ds$$
$$= \int_0^T \langle \bar{w}(s), z(T-s)\rangle\, ds \le \|P_T\bar{w}\|_{L_2} \|P_T z\|_{L_2}$$
$$\le \|P_T\bar{w}\|_{L_2} \|z\|_{L_2} \le \gamma \|P_T\bar{w}\|_{L_2} \|w\|_{L_2}$$
$$= \gamma \|P_T\bar{w}\|_{L_2} \|P_T w\|_{L_2} = \gamma \|P_T\bar{w}\|_{L_2} \|P_T\bar{z}\|_{L_2}.$$

Therefore, we have that $\|P_T\bar{z}\|_{L_2} \le \gamma \|P_T\bar{w}\|_{L_2}$ for all $T \ge 0$. Hence, we conclude that $\|\bar{z}\|_{L_2} \le \gamma \|\bar{w}\|_{L_2}$. Since the dual and primal systems are interchangeable, necessity follows from sufficiency. □

X. LINEAR PARTIAL INTEGRAL INEQUALITIES

Optimization problems with PI operator decision variables and Linear PI Inequality constraints are called Linear PI Inequalities (LPIs) and take the form

$$\mathcal{P}\begin{bmatrix} P_0, & Q_0 \\ Q_0^T, & \{R_{0i}\} \end{bmatrix} + \sum_{k=1}^{N} x_j \mathcal{P}\begin{bmatrix} P_k, & Q_k \\ Q_k^T, & \{R_{ki}\} \end{bmatrix} \succcurlyeq 0, \tag{11}$$

where $x \in \mathbb{R}^N$ is the decision variable and $\mathcal{P}\begin{bmatrix} P_k, & Q_k \\ Q_k^T, & \{R_{ki}\} \end{bmatrix} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ are known self-adjoint 4-PI operators. LPI optimization problems can be solved using the MATLAB software package PIETOOLS [13]. In the following sections, we present applications of Theorems 10 and 11 in the form of LPI tests for dual stability, dual L_2-gain, stabilization, and H_∞-optimal control of PIE systems, each with associated code snippets using the PIETOOLS implementation.

XI. A DUAL LPI FOR STABILITY

Using Theorem 10, we give primal and dual LPIs for stability of a PIE system.

Theorem 12. *(Primal LPI for Stability:) Suppose there exists a self-adjoint bounded and coercive operator $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ such that*
$$\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \preccurlyeq -\epsilon\mathcal{T}^*\mathcal{T} \tag{12}$$
for some $\epsilon > 0$. Then any $\mathbf{x} \in Z^{m,n}[a,b]$ that satisfies the system
$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in Z^{m,n}[a,b]$$

satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\|_Z = 0$.

Proof. The proof can be found in the [8]. □

Theorem 13. *(Dual LPI for Stability:) Suppose there exists a self-adjoint bounded and coercive operator* $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ *such that*

$$\mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \preccurlyeq -\epsilon\mathcal{T}\mathcal{T}^* \tag{13}$$

for some $\epsilon > 0$. *Then any* $\mathbf{x} \in Z^{m,n}[a,b]$ *that satisfies the system*

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in Z^{m,n}[a,b]$$

satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\|_Z = 0$.

Proof. The proof can be found in the Appendix. □

Pseudo Code 1.

```
prog = sosprogram([s,t]);
[prog,P] = sos_posopvar(prog,dim,I,s,t);
D=T*P*A'+A*P*T'+eps*T*T';
prog = sos_opineq(prog, -D);
prog = sossolve(prog);
```

XII. Dual KYP Lemma

Using Theorem 11, we present the following dual LPI for L_2-gain of PIE in the form Eq. (1) where $\mathcal{T},\mathcal{A} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$, $\mathcal{B}_i : \mathbb{R}^q \to Z^{m,n}[a,b]$, $\mathcal{C} : Z^{m,n}[a,b] \to \mathbb{R}^r$ and $\mathcal{D}_{1i} : \mathbb{R}^q \to \mathbb{R}^r$.

Theorem 14. *(LPI for* L_2*-gain:) Suppose there exist* $\epsilon > 0, \gamma > 0$, *bounded linear operators* $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$, *such that* $\mathcal{P}$ *is self-adjoint, coercive and*

$$\begin{bmatrix} -\gamma I & \mathcal{D} & \mathcal{C}\mathcal{P}\mathcal{T}^* \\ (\cdot)^* & -\gamma I & \mathcal{B}^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P})^* \end{bmatrix} \preccurlyeq 0. \tag{14}$$

Then, for $w \in L_2$, *any* $\mathbf{x}$ *and* z *that satisfy the PIE* (1) *also satisfies* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. The proof is same as the proof for Theorem 16 with $\mathcal{B}_1 = \mathcal{B}$, $\mathcal{B}_2 = 0$, $\mathcal{D}_{11} = \mathcal{D}$ and $\mathcal{D}_{12} = 0$. □

Pseudo Code 2.

```
prog = sosprogram([s,t], gam);
[prog,P] = sos_posopvar(prog,dim,I,s,t);
D = [-gam*I+eps*I   D      C*P*T';
      D       -gam*I+eps*I   B';
     (P*C')'   B'   (·)'+T*(A*P)'+eps*T*T'];
prog = sos_opineq(prog, -D);
prog = sossetobj(prog,gam);
prog = sossolve(prog);
```

XIII. Stabilizing Controller Synthesis

For PIEs with inputs,

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t)$$

the following LPI can be used to find a stabilizing state-feedback controller of the form $u(t) = \mathcal{K}\mathbf{x}(t)$ where $\mathcal{K} : Z^{m,n}[a,b] \to \mathbb{R}^p$ is a 4-PI operator.

Corollary 15. *(LPI for Stabilizing Controller Synthesis:) Suppose there exist bounded linear operators* $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ *and* $\mathcal{Z} : Z^{m,n}[a,b] \to \mathbb{R}^p$, *such that* $\mathcal{P}$ *is self-adjoint, coercive and*

$$(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})\mathcal{T}^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})^* \le -\epsilon\mathcal{T}\mathcal{T}^*. \tag{15}$$

Then, for $u(t) = \mathcal{K}\mathbf{x}(t)$, *where* $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, *any* $\mathbf{x} \in Z^{m,n}[a,b]$ *that satisfies the system*

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in Z^{m,n}[a,b]$$

also satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\|_Z = 0$.

Proof. The proof is same as the proof for Theorem 13. Replace $\mathcal{A}$ by $\mathcal{A} + \mathcal{B}\mathcal{K}$ and substitute $\mathcal{Z} = \mathcal{K}\mathcal{P}$. □

Pseudo Code 3.

```
prog = sosprogram([s,t]);
[prog,Z] = sos_opvar(prog,dim,I,s,t,deg);
[prog,P] = sos_posopvar(prog,dim,I,s,t);
D=T*(A*P+B*Z)'+(A*P+B*Z)*T'+eps*T*T';
prog = sos_opineq(prog, -D);
prog = sossolve(prog);
```

XIV. H_∞-Optimal Controller Synthesis

Now we use Theorem 11 to pose the H_∞-optimal controller synthesis problem as an LPI. Specifically, we formulate the following LPI for finding the H_∞-optimal controller for a PIE in the form Eq. (3) where $\mathcal{T},\mathcal{A} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$, $\mathcal{B}_1 : \mathbb{R}^q \to Z^{m,n}[a,b]$, $\mathcal{B}_2 : \mathbb{R}^p \to Z^{m,n}[a,b]$, $\mathcal{C} : Z^{m,n}[a,b] \to \mathbb{R}^r$, $\mathcal{D}_{11} : \mathbb{R}^q \to \mathbb{R}^r$ and $\mathcal{D}_{12} : \mathbb{R}^p \to \mathbb{R}^r$.

Theorem 16. *(LPI for* H_∞ *Optimal Controller Synthesis:) Suppose there exist* $\gamma > 0$, *bounded linear operators* $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ *and* $\mathcal{Z} : Z^{m,n}[a,b] \to \mathbb{R}^p$, *such that* $\mathcal{P}$ *is self-adjoint, coercive and*

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}\mathcal{P} + \mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & -\gamma I & \mathcal{B}_1^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \preccurlyeq 0. \tag{16}$$

Then, for any $w \in L_2$, *for* $u(t) = \mathcal{K}\mathbf{x}(t)$ *where* $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, *any* $\mathbf{x}$ *and* z *that satisfy the PIE* (3) *also satisfy* $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.

Proof. The proof can be found in the appendix. □

Pseudo Code 4.

```
prog = sosprogram([s,t], gam);
[prog,P] = sos_posopvar(prog,dim,I,s,t);
[prog,Z] = sos_opvar(prog,dim,I,s,t,deg);
D = [-gam*I+eps*I   D11'      (C*P+D12*Z)*T';
     (·)'       -gam*I+eps*I   B1';
     (·)'   (·)'   (·)'+T*(A*P + B2*Z)'+eps*T*T'];
prog = sos_opineq(prog, -D);
prog = sossetobj(prog,gam);
prog = sossolve(prog);
```

XV. Numerical Examples

In this section, use numerical examples to demonstrate the application of the LPIs presented in this paper. First, we verify the stability of PDEs, where the stability holds for certain values of the system parameters (referred to as a stability parameter). We test for the stability of the system using the dual stability criterion and change the stability parameter continuously to identify the point at which the stability of the system changes. The second set of examples will focus on finding in-domain controllers to stabilize an unstable system. Finally, we also present a numerical example of systems with inputs and outputs to find H_∞-optimal controllers.

A. Stability Tests Using Dual Stability Criterion

Example 17. *Consider the scalar diffusion-reaction equation with fixed boundary conditions.*

$$u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t), \qquad s \in [0,1], t > 0,$$
$$u(0,t) = u(1,t) = 0, u(s,0) = u_0$$

This system is stable for $\lambda \le \pi^2$. Using LPIs, we find the maximum value for which the system is stable by varying λ continuously and testing the feasibility of dual LPI for stability. From our tests, we find that the system is stable for $\lambda \le (1 + 1e^{-5})\pi^2$.

Example 18. *Using the same PDE from the previous example, with different boundary conditions, the stability parameter changes to $\lambda \le 2.467$.*

$$u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t), \qquad s \in [0,1], t > 0,$$
$$u(0,t) = u_s(1,t) = 0, u(s,0) = u_0$$

Testing the stability using the dual LPI for stability, we find that the system is stable for $\lambda \le 2.467 + 5e^{-4}$.

B. Finding Stabilizing Controller For Unstable PDE Systems

Example 19. *In the Example 17, suppose $\lambda = 10$. Then the system is unstable. To stabilize the system, we introduce an in-domain control input as*

$$u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t) + d(t)$$

where $d(t) = \int_a^b K(s) u_{ss}(s,t) ds$ is the control input. Solving the LPI in Theorem 15 we find a stabilizing controller $K(s) = 0.29s^5 - 1.01s^4 + 0.95s^3 + 0.16s^2 - 0.51s + 0.98$.

C. H_∞-optimal Controller Synthesis

Example 20. *Consider the following cascade of diffusion-reaction equations with a dynamic controller acting at the boundary.*

$$\dot{\mathbf{x}}_i(s,t) = \lambda \mathbf{x}_i(s,t) + \sum_{k=i}^{N} \mathbf{x}_{k,ss}(s,t) + w(t), \quad i \in \{1...N\}$$
$$\dot{x}_0(t) = u(t), \qquad x_0(0) = 0, \mathbf{x}_i(s,0) = 0, s \in [0,1]$$
$$z(t) = x_0(t),$$
$$\mathbf{x}_i(0,t) = 0, \quad \mathbf{x}_i(1,t) = 0 \quad \forall\, i \in \{1...N-1\},$$
$$\mathbf{x}_N(0,t) = 0, \quad \mathbf{x}_N(1,t) = x_0(t),$$

where x_0 is the state of the dynamic boundary controller, $\mathbf{x}_i$ are distributed states, z is the output and w is the input disturbances. The control input, $u(t) = K_0 x_0(t) + \int_0^1 K(s)\mathbf{x}(s,t)ds$ where $K : [a,b] \to \mathbb{R}^{1\times N}$, enters the system through the ODE and acts at the boundary of the PDE state x_N. For $\lambda = 10$, $N = 3$ the H_∞-optimal controller has a norm bound of 6.5095. *In Figure 1, we plot the system response for a disturbance $w(t) = \frac{sin(5t)}{3t}$ with zero initial conditions.*

XVI. CONCLUSIONS

In this article, we have proven the equivalence, in stability and H_∞-norm, between a PIE and its dual. Coupled ODE-PDE have equivalent PIE representations and properties of the ODE-PDE are inherited from the PIE. Our duality results allow can be used with LPIs to find stabilizing and H_∞-optimal state-feedback controllers for PIEs and these controllers can then be used to regulate the associated ODE-PDEs. We have demonstrated the accuracy and scalability of the resulting algorithms by applying the results to several illustrative examples. While the scope of the paper is limited to inputs entering through the ODE or in-domain, we believe the results can be extended to inputs at the boundary.

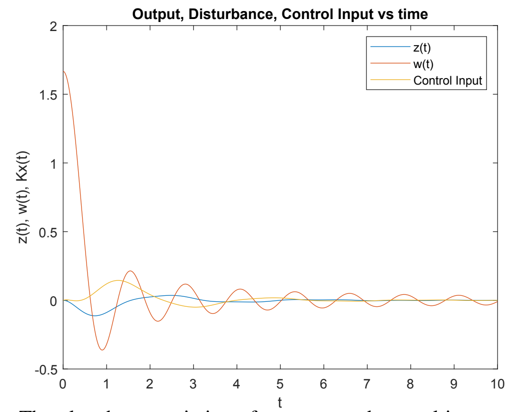

Fig. 1. The plot shows variation of output z and control input u with time when a bounded disturbance w is applied to system in Example 20 with zero initial conditions.

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APPENDIX

A. Proof of Theorem 13

Theorem 13. *Suppose there exists a self-adjoint bounded and coercive operator $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ such that*

$$\mathcal{T}\mathcal{P}\mathcal{A}^* + \mathcal{A}\mathcal{P}\mathcal{T}^* \preccurlyeq -\epsilon\mathcal{T}\mathcal{T}^*$$

for some $\epsilon > 0$. Then any $\mathbf{x} \in Z^{m,n}[a,b]$ that satisfies the system

$$\mathcal{T}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \in Z^{m,n}[a,b]$$

satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\|_Z = 0$.

Proof. Define a Lyapunov candidate as $V(y) = \langle \mathcal{T}^* y, \mathcal{P}\mathcal{T}^* y\rangle_Z$. Then there exists an $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|\mathcal{T}^* y\|_Z \le V(y) \le \beta \|\mathcal{T}^* y\|_Z .$$

The time derivative of $V(y)$ along the solutions of the PIE

$$\mathcal{T}^*\dot{y}(t) = \mathcal{A}^* y(t), \qquad y(0) \in Z^{m,n}[a,b]$$

is given by

$$\begin{aligned}\dot{V}(y(t)) &= \langle \mathcal{T}^* y(t), \mathcal{P}\mathcal{T}^*\dot{y}(t)\rangle_Z + \langle \mathcal{T}^*\dot{y}(t), \mathcal{P}\mathcal{T}^* y(t)\rangle_Z \\ &= \langle \mathcal{T}^* y(t), \mathcal{P}\mathcal{A}^* y(t)\rangle_Z + \langle \mathcal{A}^* y(t), \mathcal{P}\mathcal{T}^* y(t)\rangle_Z \\ &= \langle y(t), \mathcal{T}\mathcal{P}\mathcal{A}^* y(t)\rangle_Z + \langle y(t), \mathcal{A}\mathcal{P}\mathcal{T}^* y(t)\rangle_Z \\ &\le -\epsilon \|\mathcal{T}^* y(t)\|_Z \le -\frac{\epsilon}{\beta} V(y(t)).\end{aligned}$$

Then, by using Gronwall-Bellman Inequality, there exists constants M and k such that

$$V(y(t)) \le V(y(0)) M e^{(-kt)}.$$

As $t \to \infty$, $V(y(t)) \to 0$ which implies $\|\mathcal{T}^* y(t)\|_Z \to 0$. Then, from Theorem 10, $\|\mathcal{T}x(t)\|_Z \to 0$. □

B. Proof of Theorem 16

Theorem 16. *Suppose there exist $\gamma > 0$, bounded linear operators $\mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b]$ and $\mathcal{Z} : Z^{m,n}[a,b] \to \mathbb{R}^p$, such that $\mathcal{P}$ is self-adjoint, coercive and*

$$\begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}\mathcal{P} + \mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & -\gamma I & \mathcal{B}_1^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \preccurlyeq 0. \tag{17}$$

Then, for any $w \in L_2$, for $u(t) = \mathcal{K}\mathbf{x}(t)$ where $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, any $\mathbf{x}$ and z that satisfy the PIE (3) *also satisfy $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$.*

Proof. Define a Lyapunov candidate function $V(x) = \langle \mathcal{T}^* x, \mathcal{P}\mathcal{T}^* x\rangle_Z$. Since $\mathcal{P}$ is coercive and bounded, there exists $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|\mathcal{T}^* x\|_Z \le V(x) \le \beta \|\mathcal{T}^* x\|_Z .$$

The time derivative of $V(x)$ along the solutions of

$$\begin{aligned}\mathcal{T}^*\dot{\mathbf{x}}(t) &= (\mathcal{A} + \mathcal{B}_2\mathcal{K})^*\mathbf{x}(t) + (\mathcal{C} + \mathcal{D}_{12}\mathcal{K})^* w(t), \\ z(t) &= \mathcal{B}_1^*\mathbf{x}(t) + \mathcal{D}_{11}^* w(t), \quad \mathbf{x}(0) = 0\end{aligned} \tag{18}$$

is given by

$$\begin{aligned}\dot{V}(x(t)) &= \langle \mathcal{T}^* x(t), \mathcal{P}\mathcal{T}^*\dot{x}(t)\rangle_Z + \langle \mathcal{T}^*\dot{x}(t), \mathcal{P}\mathcal{T}^* x(t)\rangle_Z \\ &= \langle \mathcal{T}^* x(t), \mathcal{P}(\mathcal{A} + \mathcal{B}_2\mathcal{K})^* x(t)\rangle_Z \\ &\quad + \langle (\mathcal{A} + \mathcal{B}_2\mathcal{K})^* x(t), \mathcal{P}\mathcal{T}^* x(t)\rangle_Z \\ &\quad + \langle \mathcal{T}^* x(t), \mathcal{P}(\mathcal{C} + \mathcal{D}_{12}\mathcal{K})^* w(t)\rangle_Z \\ &\quad + \langle (\mathcal{C} + \mathcal{D}_{12}\mathcal{K})^* w(t), \mathcal{P}\mathcal{T}^* x(t)\rangle_Z .\end{aligned}$$

For any $w(t) \in \mathbb{R}^p$ and $x(t) \in Z$ that satisfies Eq. (18),

$$\begin{aligned}&\left\langle \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} -\gamma I & \mathcal{D}_{11} & (\mathcal{C}\mathcal{P} + \mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & -\gamma I & \mathcal{B}_1^* \\ (\cdot)^* & (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix}, \begin{bmatrix} 0 & (\mathcal{C}\mathcal{P} + \mathcal{D}_{12}\mathcal{Z})\mathcal{T}^* \\ (\cdot)^* & (\cdot)^* + \mathcal{T}(\mathcal{A}\mathcal{P} + \mathcal{B}_2\mathcal{Z})^* \end{bmatrix} \begin{bmatrix} w(t) \\ \mathbf{x}(t) \end{bmatrix} \right\rangle \\ &\quad - \gamma \|w(t)\|^2 - \gamma \|v(t)\|^2 + \langle v(t), \mathcal{B}_1^*\mathbf{x}(t) + \mathcal{D}_{11}^* w(t)\rangle \\ &\quad + \langle \mathcal{B}_1^*\mathbf{x}(t) + \mathcal{D}_{11}^* w(t), v(t)\rangle \\ &= \dot{V}(\mathbf{x}(t)) - \gamma \|w(t)\|^2 - \gamma \|v(t)\|^2 + \langle v(t), z(t)\rangle \\ &\quad + \langle z(t), v(t)\rangle \le 0\end{aligned}$$

for any $v(t) \in \mathbb{R}^p$ and $t \ge 0$. Let $v(t) = \frac{1}{\gamma} z(t)$. Then

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) &\le \gamma \|w(t)\|^2 + \frac{1}{\gamma}\|z(t)\|^2 - \frac{2}{\gamma}\|z(t)\|^2 - \epsilon \|\mathbf{x}(t)\|^2 \\ &\le \gamma \|w(t)\|^2 - \frac{1}{\gamma}\|z(t)\|^2 .\end{aligned}$$

Integrating forward in time with the initial condition $\mathbf{x}(0) = 0$, we get

$$\frac{1}{\gamma}\|z(t)\|^2 \le \gamma \|w(t)\|^2 .$$

Using Theorem 11, the adjoint PIE system of Eq.(18) has the same bound on L_2-gain from input to output. In other words, for $w \in L_2$, any $\mathbf{x}$ and z that satisfy equations

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \qquad \mathbf{x}(0) = 0 \\ z(t) &= \mathcal{C}\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t)\end{aligned}$$

with $u = \mathcal{K}\mathbf{x}$ and $\mathcal{K} = \mathcal{Z}\mathcal{P}^{-1}$, we have $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$. □

C. PI operator definitions in Theorem 4

Given $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$, we define the following functions and 4-PI operators.

$$\begin{aligned}
&H_0(s) = K(s)(BT)^{-1}B_x, H_1(s) = V(s)(BT)^{-1}B_x, \\
&T_1 = T(BT)^{-1}B_x, \quad T_2(s) = T(BT)^{-1}BQ(0,s) + Q(0,s) \\
&G_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; L_0 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&G_2(s,\theta) = -K(s)(BT)^{-1}BQ(s,\theta), \\
&G_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (s-\theta)I \end{bmatrix} + G_2(s,\theta), \\
&L_2(s,\theta) = -V(s)(BT)^{-1}BQ(s,\theta), \; L_1(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + L_2(s,\theta), \\
&K(s) = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & (s-a)I \end{bmatrix}, \quad V(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
&T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & (b-a)I \\ 0 & 0 & I \end{bmatrix}, Q(s,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (b-\theta)I \\ 0 & 0 & I \end{bmatrix} \\
&\mathcal{T} = \mathcal{P}\begin{bmatrix} I, & 0 \\ H_0, & \{G_i\} \end{bmatrix}, \mathcal{B} = \mathcal{P}\begin{bmatrix} B_1, & 0 \\ B_2, & \{0\} \end{bmatrix}, \mathcal{D} = \mathcal{P}\begin{bmatrix} D, & 0 \\ 0, & \{0\} \end{bmatrix} \\
&\mathcal{A} = \mathcal{P}\begin{bmatrix} A + E_{10}T_1, & E_{10}T_2 \\ E, & \{\begin{bmatrix} 0 & 0 & A_2 \end{bmatrix}, 0, 0\} \end{bmatrix} \\
&\quad + \mathcal{P}\begin{bmatrix} 0, & E_a \\ 0, & \{A_0, 0, 0\} \end{bmatrix} \mathcal{P}\begin{bmatrix} 0, & 0 \\ H_0, & \{G_i\} \end{bmatrix} + \mathcal{P}\begin{bmatrix} 0, & E_b \\ 0, & \{A_1, 0, 0\} \end{bmatrix} \mathcal{P}\begin{bmatrix} 0, & 0 \\ H_1, & \{L_i\} \end{bmatrix} \\
&\mathcal{C} = \mathcal{P}\begin{bmatrix} C + C_{10}T_1, & C_{10}T_2 \\ 0, & \{0\} \end{bmatrix} + \mathcal{P}\begin{bmatrix} 0, & C_a \\ 0, & \{0\} \end{bmatrix} \mathcal{P}\begin{bmatrix} 0, & 0 \\ H_0, & \{G_i\} \end{bmatrix} \\
&\qquad + \mathcal{P}\begin{bmatrix} 0, & C_b \\ 0, & \{0\} \end{bmatrix} \mathcal{P}\begin{bmatrix} 0, & 0 \\ H_1, & \{L_i\} \end{bmatrix}.
\end{aligned} \tag{19}$$