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Robust Analysis of Uncertain ODE-PDE Systems Using PI Multipliers, PIEs and LPIs

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Abstract— **This paper presents a computational framework for analyzing stability and performance of uncertain Partial Differential Equations (PDEs) when they are coupled with uncertain Ordinary Differential Equations (ODEs). To analyze the behavior of the interconnected ODE-PDE systems under uncertainty, we introduce a class of multipliers of Partial Integral (PI) operator type and consider various classes of uncertainties by enforcing constraints on these multipliers. Since the ODE-PDE models are equivalent to Partial Integral Equations (PIEs), we show that the robust stability and performance can be formulated as Linear PI Inequalities (LPIs) and LPIs can be solved by LMIs using PIETOOLS. The methods are demonstrated on examples of ODE-PDE systems that are subjected to wide classes of uncertainty.**

I. INTRODUCTION

The modeling of spatio-temporal processes often involves mathematical representations in combinations of ordinary differential equations (ODEs) and partial differential equations (PDEs). In mission-critical operations, such processes require precise and optimal performance while being safe, reliable, and fault-tolerant. In the derivation of these mathematical models, one often faces the challenge of incorporating unforeseen perturbations or uncertainties in the model due to a) the way the model interacts with a dynamic environment, b) lack of knowledge about physical parameters and unforeseen changes in their values, or c) problems related to non-modeled or inaccurately modeled physical phenomena (model mismatch or uncertainty). Despite these problems, all engineering solutions related to questions on estimation, monitoring, prediction, control, diagnosis, process operation, and optimization must provide rigorous and quantified certifications on process (or system) performance in view of these uncertainties. Indeed, providing firm guarantees on performance is an enabling factor in mission and safety-critical operations.

It is the aim of this paper to provide stability and performance guarantees for very generic classes of uncertain spatio-temporal systems that are governed by coupled ODEs and PDEs. These guarantees should be given despite uncertainty and/or non-linearity in the model components. In particular, we aim to develop a scalable computational method that is able to answer the following questions

1) Is it possible to guarantee that an ODE-PDE system is robustly stable for given classes of parameter perturbations or different descriptions of uncertainties?
2) What is the smallest value of $\gamma > 0$, for which the input-output pair (w, z) of the ODE-PDE system satisfies the following inequality
$$||z||_{L_2} \leq \gamma ||w||_{L_2}$$
in a robust fashion concerning parameter variations or uncertainties?

The theory of dissipativity (c.f. [1]) is at the core of understanding robustness properties of systems with a graph separation on the (deterministic) model and its uncertainty. In the case of finite-dimensional systems, robustness properties are investigated by using multipliers and Integral Quadratic Constraints (IQCs) (see [2]–[6] and the references therein). However, for spatio-temporal systems, such a robustness framework is still in its nascent stage. In recent years, only a few works can be found that focus on robustness aspects (for example, see [7] and [8]).

Recently, we have shown that distributed parameter systems can be represented by Partial Integral Equations (PIEs). It has been shown that the analysis and control of PIEs can be formulated using Linear PI Inequalities (LPIs) that can be solved using Linear Matrix Inequalities (LMIs) without requiring any numerical discretization technique. Interested readers may refer to [9] (Chapter 6) for detailed description of PIEs, and the computational aspects of solving LPIs using LMIs. Moreover, an open-source toolbox, named PIETOOLS, is developed that offers user-friendly, scalable and a computationally efficient framework for the analysis and design of estimator-based controllers that are again represented through PIEs [10].

We present an extension of the PIE framework to consider various classes of uncertainty and analyze their stability and input-output properties. The purpose of this paper is to show that

- ODE-PDE systems under a) norm-bounded uncertainty, b) polytopic uncertainty and c) sector-bounded non-linearity admit a PIE representation in Fractional Representation form.
- Properties of robust stability and performance can be computationally verified for a uncertain ODE-PDE system where discretization and approximation techniques are not necessary.
- Robust stability and performance of uncertain PIE systems can be verified through the feasibility of linear PI inequalities (LPIs).

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- The tests are computationally feasible and attractive as they involve solving convex optimizations to verify feasibility of LMI's.

The remainder of this paper is organized as follows. After some preliminary discussions on notations, PI operators, PIEs and LPIs, in Section III, uncertain PDE-ODE systems are represented in the PIE framework. In Section IV, PI multipliers are introduced to capture various classes of uncertainty. In Section V, LPIs are derived for verifying the robust stability of uncertain models. Moreover, the stability result is extended in to determine the robust performance of these uncertain models in terms of input-output properties. In Section VI, example models are illustrated for testing the developed methodologies. Finally, Section VII provides some concluding remarks.

II. PRELIMINARIES

A. Notations

The set of square-integrable functions is $L_2^n[a,b]$ on the domain $[a,b]\subset\mathbb{R}$. The Sobolev space $W_{2,k}[a,b]$ is defined as

$$W_{2,k}[a,b] := \{f\in L_2[a,b] \mid \frac{\partial^n f}{\partial s^n}\in L_2[a,b] \text{ for all } n\le k\}$$

The space $\mathbb{R}^m\times L_2^n[a,b]$ equipped with the inner-product

$$\langle \begin{bmatrix}x_1\\ \mathbf{x}_2\end{bmatrix}, \begin{bmatrix}y_1\\ \mathbf{y}_2\end{bmatrix}\rangle = x_1^\top y_1 + \langle \mathbf{x}_2,\mathbf{y}_2\rangle_{L_2},\ \begin{bmatrix}x_1\\ \mathbf{x}_2\end{bmatrix}, \begin{bmatrix}y_1\\ \mathbf{y}_2\end{bmatrix}\in\mathbb{R}^m\times L_2^n[a,b].$$

We use $\mathbf{x}_s$ to denote the partial derivative of $\frac{\partial \mathbf{x}}{\partial s}$ where the number of repetitions of the subscript s corresponds to the order of the partial derivative. $\varnothing$ denotes an empty matrix of suitable dimension. The set of all operator $V: A\to B$ is denoted by $\mathcal{L}(A,B)$, in other words $V\in\mathcal{L}(A,B)$.

B. PI Operators

Partial Integral operators, also known as PI operators, are a class of bounded linear integral operators that are defined jointly on a vector space and Hilbert space. There are two classes of PI operators.

Definition II.1. *(PI Operator) A PI operator is a bounded linear operator that maps from $\mathbb{R}^m\times L_2^n[a,b]$ to $\mathbb{R}^p\times L_2^q[a,b]$ and is parametrized as*

$$\mathcal{P}\begin{bmatrix} P, & Q_1\\ Q_2, & \{\{R_0,R_1,R_2\}\}\end{bmatrix}\begin{bmatrix}x\\ \mathbf{y}\end{bmatrix}(s) := \begin{bmatrix} Px+\int_a^b Q_1(s)\mathbf{y}(s)ds\\ Q_2(s)x+\mathcal{P}_{\{\{R_0,R_1,R_2\}\}}\mathbf{y}(s)\end{bmatrix} \tag{1}$$

where $P\in\mathbb{R}^{p\times m}$ is a matrix, $Q_1:[a,b]\to\mathbb{R}^{p\times n}$, $Q_2:[a,b]\to\mathbb{R}^{q\times m}$, $R_0:[a,b]\to\mathbb{R}^{q\times n}$, and $R_1,R_2:[a,b]\times[a,b]\to\mathbb{R}^{q\times n}$ are bounded integrable functions and $\mathcal{P}_{\{R_0,R_1,R_2\}}: L_2^n[a,b]\to L_2^q[a,b]$ is another PI operator of the form

$$\left(\mathcal{P}_{\{R_0,R_1,R_2\}}\mathbf{x}\right)(s) := R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

C. Properties Of PI Operators

PI operators have the following properties (see [9], Chapter 6):

- The addition of two PI operators is also a PI operator.
- The adjoint of a PI operator is also a PI operator.
- The composition of two PI operators is also a PI operator.
- Positivity of PI operators can be verified using LMIs.

With these properties, PI operators form a *-subalgebra with binary operations of addition and composition.

D. PIEs

Partial Integral Equations (PIEs) are set of linear differential equations that are parametrized by PI operators. The general form of a PIE is

$$\begin{aligned}\mathcal{T}\dot{\mathbf{z}}(t)+\mathcal{B}_1\dot{w}(t) &= \mathcal{A}\mathbf{z}(t)+\mathcal{B}_w w(t),\\ z(t) &= \mathcal{C}\mathbf{z}(t)+\mathcal{D}w(t),\\ \mathbf{z}(0) &= \mathbf{z}_0\in\mathbb{R}^m\times L_2^n[a,b],\end{aligned} \tag{2}$$

where $\mathcal{T},\mathcal{A}:\mathbb{R}^m\times L_2^n[a,b]\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{B}_1:\mathbb{R}^p\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{B}_w:\mathbb{R}^p\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{C}:\mathbb{R}^m\times L_2^n[a,b]\to\mathbb{R}^k$, and $\mathcal{D}\in\mathbb{R}^{k\times p}$ are PI operators.

E. LPIs

Using the algebra of PI operators, one can set up linear operator inequalities, now, involving PI operators. These inequalities, when defined by PI operators, are called *Linear PI Inequalities* or LPIs.

Definition II.2. *(LPIs) For given PI operators $\{\mathcal{E}_{i,j},\mathcal{F}_{i,j},\mathcal{G}_i\}$ and a convex linear functional $\mathcal{L}(\cdot)$, a linear PI inequality is a convex optimization of the following form*

$$\begin{aligned}&\min_{P_i,Q_{1i},Q_{2i},R_{0i},R_{1i},R_{2i}} \mathcal{L}(\{P_i,Q_{1i},Q_{2i},R_{0i},R_{1i},R_{2i}\})\\ &\text{s.t. } \sum_{j=1}^K \mathcal{E}_{ij}^*\mathcal{P}\begin{bmatrix}P_i, & Q_{1i}\\ Q_{2i}, & \{\{R_{0i},R_{1i},R_{2i}\}\}\end{bmatrix}\mathcal{F}_{ij}+\mathcal{G}_i \succcurlyeq 0\end{aligned} \tag{3}$$

Only LMIs are required to solve an optimization problem involving LPIs.

F. PIETOOLS

PIETOOLS is a MATLAB toolbox used for declaration, manipulation and solving LPIs. For learning how to use PIETOOLS and its functionalities visit `http://control.asu.edu/pietools/`.

III. UNCERTAIN ODE-PDE MODEL, FR-PIE STRUCTURE

A. Uncertain ODE-PDE Models

One way to take the uncertainties into account is to consider a feedback perturbation of ODE-PDE models through an uncertain component whose accurate realization is not available. To this end, consider the following class of uncertain ODE-PDE systems in one spatial dimension $[a,b], a<b$, with sufficient number of boundary conditions.

$$
\begin{aligned}
\begin{bmatrix}\dot{x}(t)\\ \dot{\mathbf{x}}(s,t)\end{bmatrix} &= \begin{bmatrix}Ax(t)+(\mathcal{E}\mathbf{x})(t)\\ E(s)x(t)+(\mathcal{A}_p\mathbf{x})(s,t)\end{bmatrix}+\begin{bmatrix}B_1\\ B_2(s)\end{bmatrix}w(t)\\
&\quad+\mathcal{P}\begin{bmatrix}J, & G_1\\ G_2, & \{\{H_0,H_1,H_2\}\}\end{bmatrix}p(s,t),\\
z(t) &= \left(\mathcal{C}_d\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(t)+Dw(t),\\
q(s,t) &= \left(\mathcal{E}_d\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(s,t)+\begin{bmatrix}F_1\\ F_2(s)\end{bmatrix}w(t),\\
p(s,t) &= \Delta(q(s,t)),\\
B\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix} &= B_x x(t)+B_c w(t),\\
\begin{bmatrix}x(0)\\ \mathbf{x}(\cdot,0)\end{bmatrix} &= \mathbf{x}_0\in D(\mathcal{A}_d),
\end{aligned}
\tag{4}
$$

where, $\mathbf{x}(s,t) := \begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}$, $\mathbf{x}_c(s,t) := \begin{bmatrix}\mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}$.

Here,

$$
\begin{aligned}
&(\mathcal{A}_p\mathbf{x})(s,t) :=\\
&A_0(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}+A_1(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}+A_2(s)\mathbf{x}_{3ss}(s,t),\\
&(\mathcal{E}\mathbf{x})(t) := E_{10}\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix}+\int_a^b E_a(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}ds\\
&\qquad+\int_a^b E_b(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}ds,\\
&\left(\mathcal{C}_d\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(t) := Cx(t)+C_{10}\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix}\\
&\qquad+\int_a^b C_a(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}ds+\int_a^b C_b(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}ds,\\
&\left(\mathcal{E}_d\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}\right)(s,t) := Ex(t)+E_{10}\begin{bmatrix}\mathbf{x}_c(a,t)\\ \mathbf{x}_c(b,t)\end{bmatrix}\\
&\qquad+\int_a^b E_a(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix}ds+\int_a^b E_b(s)\begin{bmatrix}\mathbf{x}_{2s}(s,t)\\ \mathbf{x}_{3s}(s,t)\end{bmatrix}ds\\
&\qquad+E_{1m}(s)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\\ \mathbf{x}_{2s}\\ \mathbf{x}_{3s}\\ \mathbf{x}_{3ss}\end{bmatrix}(s,t)+\int_a^s E_{ln}(s,\theta)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\\ \mathbf{x}_{2s}\\ \mathbf{x}_{3s}\\ \mathbf{x}_{3ss}\end{bmatrix}(\theta,t)d\theta\\
&\qquad+\int_s^b E_{rn}(s,\theta)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\\ \mathbf{x}_{2s}\\ \mathbf{x}_{3s}\\ \mathbf{x}_{3ss}\end{bmatrix}(\theta,t)d\theta.
\end{aligned}
\tag{5}
$$

The signals are $x(t)\in\mathbb{R}^m$, $\mathbf{x}_i(s,t)\in\mathbb{R}^{n_i}$, $w(t)\in\mathbb{R}^p$, $z(t)\in\mathbb{R}^k$, $p(s,t)\in\mathbb{R}^{p_1+p_2}$ and $q(s,t)\in\mathbb{R}^{q_1+q_2}$. Moreover, the total number of distributed states is $n=n_1+n_2+n_3$. The matrix-valued functions A_0, A_1, A_2, E, E_a, E_b, E_{1m}, E_{ln}, E_{rn}, C_a, C_b, B_2, F_2, G_1, G_2, H_0, H_1 and H_2 are bounded and have appropriate dimensions. Moreover, A, E_{10}, C_{10}, B_1, D, B, B_x, B_c, F_1 and J are constant real-valued matrices of appropriate dimensions. Furthermore, it is assumed that $\text{rank}(B)=n_2+2n_3$.

The solution of the ODE-PDE system (4) lies in $D(\mathcal{A}_d)$ where

$$
D(\mathcal{A}_d):=\left\{\begin{matrix}\begin{bmatrix}x\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}\in\mathbb{R}^m\times L_2^{n_1}[a,b]\times W_{2,1}^{n_2}[a,b]\times W_{2,2}^{n_3}[a,b]\,\Big|\\ B\begin{bmatrix}x_c(a)\\ x_c(b)\end{bmatrix}=B_x x+B_c w,\ \text{where } x_c(s)=\begin{bmatrix}\mathbf{x}_2(s)\\ \mathbf{x}_3(s)\\ \mathbf{x}_{3s}(s)\end{bmatrix}\end{matrix}\right\}
\tag{6}
$$

Here, $\Delta:\mathbb{R}^{q_1}\times L_2^{q_2}[a,b]\to\mathbb{R}^{p_1}\times L_2^{p_2}[a,b]$ represents an uncertain system in the sense that either Δ is unknown, has unknown parameters or is assumed to belong to a class of functions.

B. Including Uncertainty in PIEs

Definition III.1. *(FR-PIE Structure)*

For given PI operators $\mathcal{T},\mathcal{A}:\mathbb{R}^m\times L_2^n[a,b]\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{B}_1:\mathbb{R}^p\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{B}_w:\mathbb{R}^p\to\mathbb{R}^m\times L_2^n[a,b]$, $\mathcal{C}:\mathbb{R}^m\times L_2^n[a,b]\to\mathbb{R}^k$, $\mathcal{C}_3:\mathbb{R}^m\times L_2^n[a,b]\to\mathbb{R}^{q_1}\times L_2^{q_2}[a,b]$, $\mathcal{D}_3:\mathbb{R}^p\to\mathbb{R}^{q_1}\times L_2^{q_2}[a,b]$, $\mathcal{B}_p:\mathbb{R}^{p_1}\times L_2^{p_2}[a,b]\to\mathbb{R}^m\times L_2^n[a,b]$ and $\mathcal{D}\in\mathbb{R}^{k\times p}$, let Σ denote the following PIEs

$$
\Sigma:=\begin{cases}\mathcal{T}\dot{\mathbf{z}}(t)+\mathcal{B}_1\dot{w}(t)= & \mathcal{A}\mathbf{z}(t)+\mathcal{B}_w w(t),\\ & z(t)=\mathcal{C}\mathbf{z}(t)+\mathcal{D}w(t).\end{cases}
$$

Suppose there exist a set $\mathbf{\Delta}\subset\mathcal{L}\Big(\mathbb{R}^{q_1}\times L_2^{q_2}[a,b],\mathbb{R}^{p_1}\times L_2^{p_2}[a,b]\Big)$ and functions $p(t)\in\mathbb{R}^{p_1}\times L_2^{p_2}[a,b]$, $q(t)\in\mathbb{R}^{q_1}\times L_2^{q_2}[a,b]$ such that $(\mathbf{z}(t),w(t),z(t),p(t),q(t))$ satisfy the following PIE for all $\Delta\in\mathbf{\Delta}$

$$\mathcal{T}\dot{\mathbf{z}}(t)+\mathcal{B}_1\dot{w}(t)=\mathcal{A}\mathbf{z}(t)+\mathcal{B}_w w(t)+\mathcal{B}_p p(t), \tag{7a}$$

$$z(t)=\mathcal{C}\mathbf{x}(t)+\mathcal{D}w(t), \tag{7b}$$

$$q(t)=\mathcal{C}_3\mathbf{x}(t)+\mathcal{D}_3 w(t), \tag{7c}$$

$$p(t)=\Delta(q(t)). \tag{7d}$$

Then, for all $\Delta\in\mathbf{\Delta}$, (7) is defined as an FR-PIE.

Therefore, the uncertain FR-PIE system consists of the interconnection of the nominal system Σ with the operator $\Delta\in\mathbf{\Delta}$ acting on the interconnection variables (p,q). Here, we emphasize that the operator Δ is only known to belong to a class of operators.

C. Equivalence Between Uncertain ODE-PDE Systems and FR-PIEs

Lemma III.1. *Suppose, in* (4), $B\in\mathbb{R}^{(n_2+2n_3)\times 2(n_2+2n_3)}$ *and*

$$
B\begin{bmatrix}I & 0 & 0\\ I & 0 & 0\\ 0 & I & 0\\ 0 & I & (b-a)I\\ 0 & 0 & I\\ 0 & 0 & I\end{bmatrix}\quad \textit{is invertible}.
$$

Then, for all $col(x(t), \mathbf{x}(t)) \in D(\mathcal{A}_d)$*, the following identity is satisfied*

$$\begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}(t) = \begin{bmatrix} \mathcal{T} & \mathcal{B}_1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ w \end{bmatrix}(t), \qquad \mathbf{z}(t) := \begin{bmatrix} x \\ \mathbf{x}_1 \\ \mathbf{x}_{2s} \\ \mathbf{x}_{3ss} \end{bmatrix}(t).$$

Moreover, for any $(x, \mathbf{x}, p, q, w, z)$ *satisfying PDE-ODE model* (4)*, there exists a unique* $(\mathbf{z}, p, q, w, z)$ *that satisfies the FR-PIE in* (7) *for all* $\Delta \in \mathbf{\Delta}$.

Proof. The proof as well as the expression of PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_w$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{C}_3$, $\mathcal{D}_3$ can be determined by minor modification of the Theorem 6.4 [9]. □

IV. PI MULTIPLIERS

A key result in the theory of dissipative systems promises that a neutral interconnection of dissipative dynamical systems is dissipative. We will employ this result by viewing both Σ and $\Delta \in \mathbf{\Delta}$ as dissipative elements that establish a neutral interconnection through the variables (p, q). Here, neutrality of the interconnection is typically enforced by specific scaling matrices, also known as *multipliers*. Due to the equivalence between uncertain ODE-PDE systems and FR-PIEs, we choose the multipliers to be a class of PI operators, also called PI multipliers.

Definition IV.1. *(PI Multipliers) Given a set* $\mathbf{\Delta} \subset \mathcal{L}\Big(\mathbb{R}^{q_1} \times L_2^{q_2}[a,b], \mathbb{R}^{p_1} \times L_2^{p_2}[a,b]\Big)$*, define the set of operators* $\mathcal{M} : \mathbb{R}^{q_1} \times L_2^{q_2}[a,b] \to \mathbb{R}^{p_1} \times L_2^{p_2}[a,b]$ *as follows:*

$$\mathbb{M}_{\mathbf{\Delta}} := \left\{ \begin{array}{c} \mathcal{M} \mid \forall q \in \mathbb{R}^{q_1} \times L_2^{q_2}[a,b], \Delta \in \mathbf{\Delta} \\ \mathcal{M} \text{ is a non-zero PI operator, and} \\ \left\langle \begin{bmatrix} \Delta(q) \\ q \end{bmatrix}, \mathcal{M} \begin{bmatrix} \Delta(q) \\ q \end{bmatrix} \right\rangle \geq 0 \end{array} \right\}. \quad (8)$$

Then, an element $\mathcal{M}$ *in the set* $\mathbb{M}_{\mathbf{\Delta}}$ *is called as a PI Multiplier for all* $\Delta \in \mathbf{\Delta}$.

However, in this definition, searching for a PI multiplier $\mathcal{M}$ is not a finite test as it requires verifying the last inequality in (8) for every realization of Δ in the set $\mathbf{\Delta}$. This is a fundamental reason of conservatism behind the robust analysis results of uncertain systems (see Chapter 6, [11]).

Instead of testing the multiplier for every inequality in (8), the set $\mathbb{M}_{\mathbf{\Delta}}$ is often approximated to obtain a computationally tractable finite test. To this end, by ranging over classes of multipliers, we present a collection of possible relaxations of the set $\mathbb{M}_{\mathbf{\Delta}}$.

A. Norm-Bounded Uncertainty

Without considering any specific structure of the uncertainty, in (7), suppose that the L_2-induced norm of Δ is bounded and known. Then, let $\mathbf{\Delta}$ be defined as

$$\mathbf{\Delta} := \Big\{ \Delta \mid \exists \alpha,\ ||\Delta|| \leq \alpha \Big\}. \quad (9)$$

where $\|\Delta\|$ represents the induced norm of Δ Now for any $\Delta \in \mathbf{\Delta}$ which is bounded by α. One can select the following class of PI multipliers

$$\left\{ \mathcal{P}\begin{bmatrix} \tau \begin{bmatrix} -\frac{1}{\alpha} I & 0 \\ 0 & \alpha \end{bmatrix}, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix} \mid \tau \geq 0 \right\} \subset \mathbb{M}_{\mathbf{\Delta}}. \quad (10)$$

Example IV.1. *Let* $p(t) \in \mathbb{R}^{p_1}, q(t) \in \mathbb{R}^{q_1}$ *be finite dimensional and the uncertainty has the form* $p(t) = \Delta q(t)$. *Here,* $\Delta \in \mathbb{R}^{p_1 \times q_1}$ *is a unknown matrix whose spectral norm is always lower that* 0.5. *In that case, the PI multiplier admits the following expression:*

$$\left\{ \tau \left(\mathcal{P}\begin{bmatrix} \begin{bmatrix} -2I & 0 \\ 0 & 0.5 \end{bmatrix}, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix} \right) \mid \tau \geq 0 \right\}.$$

B. Structured (Polytopic) Uncertainty

Another common type of relaxation appears by considering the set $\mathbf{\Delta}$ is a convex-hull generated by a finite number, say N, generators $\mathbf{\Delta}_g = \{\Delta^1, \cdots, \Delta^N, N < \infty\}$, or $\mathbf{\Delta} = \text{conv}(\mathbf{\Delta}_g)$. Then, the class of PI multipliers can be selected from the following set.

$$\left\{ \begin{array}{c} \mathcal{M} \mid \forall q \in \mathbb{R}^{q_1} \times L_2^{q_2}[a,b], \Delta \in \mathbf{\Delta}_g \\ \mathcal{M} := \begin{bmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^* & \mathcal{R} \end{bmatrix}; \mathcal{Q}, \mathcal{S}, \mathcal{R} \text{ are PI operators,} \\ \mathcal{Q} \preccurlyeq 0, \left\langle \begin{bmatrix} \Delta(q, \cdot) \\ q \end{bmatrix}, \mathcal{M} \begin{bmatrix} \Delta(q, \cdot) \\ q \end{bmatrix} \right\rangle \geq 0, \end{array} \right\} \subset \mathbb{M}_{\mathbf{\Delta}}. \quad (11)$$

The requirement $Q \preccurlyeq 0$ in (11) enforces that the positivity condition $\langle \begin{bmatrix} \Delta(q) \\ q \end{bmatrix}, \mathcal{M} \begin{bmatrix} \Delta(q) \\ q \end{bmatrix} \rangle \geq 0$ not only holds for all $\Delta \in \mathbf{\Delta}_g$ but also for all $\Delta \in \mathbf{\Delta}$, the convex hull of $\mathbf{\Delta}_g$.

Example IV.2. *In (7d), let* Δ *be a linear, parameter-dependent operator defined by* $p = \Delta(\delta) q$*, where* $\delta := col(\delta^1, \cdots, \delta^N) \in \mathbb{R}^N$ *contains unknown parameters, where* $\delta^j \in [a_j, b_j]$ *for* $j = \{1, \cdots, N\}$ *with* $a_j < b_j$. *Then, the PI multipliers can be defined to be in the following set*

$$\left\{ \begin{array}{c} \mathcal{M} \mid \mathcal{M} := \sum_{j=1}^{N} \begin{bmatrix} \mathcal{E}_j & 0 \\ 0 & \mathcal{E}_j \end{bmatrix} \mathcal{M}_j \begin{bmatrix} \mathcal{E}_j & 0 \\ 0 & \mathcal{E}_j \end{bmatrix}^*, \\ \mathcal{M}_j := \mathcal{P}\begin{bmatrix} \begin{bmatrix} q_j D_j & s_j D_j + G_j \\ s_j D_j - G_j & r_j D_j \end{bmatrix}, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix}, \\ \mathcal{P}\begin{bmatrix} D_j, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix} \succcurlyeq 0, \mathcal{P}\begin{bmatrix} G_j, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix} = \mathcal{P}\begin{bmatrix} G_j, & \varnothing \\ \varnothing, & \{\varnothing\} \end{bmatrix}^* \end{array} \right\}. \quad (12)$$

Here,

$$\begin{bmatrix} q_j & s_j \\ s_j & r_j \end{bmatrix} := \begin{bmatrix} -1 & m_j \\ m_j & d_j^2 - m_j^2 \end{bmatrix}$$

with $m_j = (a_j + b_j)/2$ *and* $d_j = (b_j - a_j)/2$.

The derivation follows from [11], page 181 with minor modifications.

C. Sector-bounded Nonlinearity

When the uncertainty is considered to be a non-linear functional, a common relaxation of $\mathbf{\Delta}$ is the set of all *sector bounded* functions. The, any $\Delta \in \mathbf{\Delta}$ admits

$$\Big(\Delta(q) - \mathcal{U}q\Big)^* \Big(\Delta(q) - \mathcal{L}q\Big) \leq 0, \quad (13)$$

for all q where $\mathcal{L}$ and $\mathcal{U}$ are given. If $\mathcal{L},\mathcal{U}$ are PI operators, then the set of PI multipliers that satisfy a sector condition (13) can be defined as

$$\left\{\tau\begin{bmatrix} I & -\mathcal{U}\\ I & -\mathcal{L}\end{bmatrix}^*\begin{bmatrix} 0 & -I\\ -I & 0\end{bmatrix}\begin{bmatrix} I & -\mathcal{U}\\ I & -\mathcal{L}\end{bmatrix} \mid \tau\geq 0\right\}\subset \mathbb{M}_{\mathbf{\Delta}}. \tag{14}$$

In literature, these multipliers directly resemble the so-called *Zames-Falb Multipliers* (c.f [2], [12]).

Example IV.3. *When $\mathcal{L}=I,\mathcal{U}=0$, then* (14) *includes, among others, the saturation function that satisfies* (13) *and admits the following definition:*

$$\Delta(q)=\begin{cases}1, & \text{for } q\geq 1\\ q, & \text{for } |q|\leq 1,\\ -1, & \text{for } q\leq -1.\end{cases}$$

V. ROBUST ANALYSIS USING LPIs

So far, we have shown that a class of uncertain ODE-PDE models are equivalent to FR-PIES. Moreover, by using the PI multipliers, one can enforce passivity on various classes of uncertainty. In the sequel, these two results are used to formulate LPIs that test the stability and input-output properties of FR-PIEs as well as uncertain ODE-PDE systems.

A. Robust Stability

Definition V.1. *(Robust stability)*

From an initial condition $\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}(t)\in D_{\mathcal{A}_d}$ *according to* (6)*, if any solution to* (4) *satisfies* $\lim_{t\to\infty}\left\|\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}(t)\right\|_{\mathbb{R}\times L_2}\to 0$ *for all* $\Delta\in\mathbf{\Delta}$*, then the uncertain ODE-PDE system* (4) *with* $w(t)\equiv 0, z(t)\equiv 0$ *are defined to be robustly stable.*

A sufficient condition for verifying robust stability is the existence of a Lyapunov functional $V(x(t),\mathbf{x}(t))$ that has a quadratic form and decreases along the nonzero solution to (4). Now, using PI multipliers and such quadratic Lyapunov functional, the robust stability of uncertain PDE-ODE coupled systems can be verified.

Theorem V.1. *(Robust stability) Let the set* $\mathbf{\Delta}$ *be given. Suppose there exists a PI multiplier* $\mathcal{M}\in\mathbb{M}_{\mathbf{\Delta}}$ *as well as* $\epsilon,\delta>0$*, a constant real-valued matrix* $P\in\mathbb{R}^{m\times m}$ *and matrix-valued polynomials* $Q:[a,b]\to\mathbb{R}^{m\times n}$*,* $R_0:[a,b]\to\mathbb{R}^{n\times n}$*, and* $R_1,R_2:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$*, such that*

- $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^\top, & \{\{R_0,R_1,R_2\}\}\end{smallmatrix}\right]$, $\mathcal{P}=\mathcal{P}^*\succcurlyeq\epsilon I$
- *and*

$$\begin{bmatrix}0 & \mathcal{B}_p^*\mathcal{P}\mathcal{T}\\ \mathcal{T}^*\mathcal{P}\mathcal{B}_p & \mathcal{A}^*\mathcal{P}\mathcal{T}+\mathcal{T}^*\mathcal{P}\mathcal{A}+\delta\mathcal{T}^*\mathcal{T}\end{bmatrix} + \begin{bmatrix}I & 0\\ 0 & \mathcal{C}_3\end{bmatrix}^*\mathcal{M}\begin{bmatrix}I & 0\\ 0 & \mathcal{C}_3\end{bmatrix}^*\preccurlyeq 0. \tag{15}$$

Then, (4) *is robustly stable for all* $\Delta\in\mathbf{\Delta}$.

Proof. Consider the following candidate Lyapunov functional $V(\mathbf{z}(t)):=\left\langle\mathcal{T}\mathbf{z}(t),\mathcal{P}\mathcal{T}\mathbf{z}(t)\right\rangle$ with

$$\beta\left\|\mathcal{T}\mathbf{z}(t)\right\|^2_{\mathbb{R}\times L_2}\geq V(\mathbf{z}(t))\geq\epsilon\left\|\mathcal{T}\mathbf{z}(t)\right\|^2_{\mathbb{R}\times L_2}.$$

The LPI (15) suggests that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{z}(t))+\left\langle\begin{bmatrix}p(t)\\ q(t)\end{bmatrix},\mathcal{M}\begin{bmatrix}p(t)\\ q(t)\end{bmatrix}\right\rangle\leq-\delta\left\|\mathcal{T}\mathbf{z}(t)\right\|^2_{\mathbb{R}\times L_2}.$$

If $\mathcal{M}\in\mathbb{M}_{\mathbf{\Delta}}$, then

$$\left\langle\begin{bmatrix}\Delta(q(t),t)\\ q(t)\end{bmatrix},\mathcal{M}\begin{bmatrix}\Delta(q(t),t)\\ q(t)\end{bmatrix}\right\rangle\geq 0.$$

Hence, we obtain, for some $\epsilon>0$

$$\dot V(\mathbf{z}(t))\leq-\delta\left\|\mathcal{T}\mathbf{z}(t)\right\|^2_{\mathbb{R}\times L_2}\implies\dot V(\mathbf{z}(t))\leq-\frac{\delta}{\beta}V(\mathbf{z}(t)).$$

Then, by using Gronwall-Bellman Inequality, there exists constants $M>0$ and $k>0$ such that

$$V(\mathbf{z}(t))\leq V(\mathbf{z}(0))Me^{(-kt)},$$

which implies $\left\|\mathcal{T}\mathbf{z}(t)\right\|_{\mathbb{R}\times L_2}\to 0$ as $t\to\infty$. According to Lemma (III.1), $\mathcal{T}\mathbf{z}(t)=\begin{bmatrix}x\\ \mathbf{x}\end{bmatrix}(t)$ which sufficiently proves the robust stability of (4) according to the Definition V.1. □

B. Extension: LPIs for Determining the Bound of L_2 Gain

An extension of the robust stability is to determine what is the smallest value of $\gamma>0$ for which the uncertain PDE-ODE coupled systems (4) admits

$$\left\|z\right\|_{L_2[0,\infty)}\leq\gamma\left\|w\right\|_{L_2[0,\infty)}.$$

Theorem V.2. *(Robust Performance: Bounded L_2 Gain) Let the set* $\mathbf{\Delta}$ *be given. Suppose there exists a PI multiplier* $\mathcal{M}\in\mathbb{M}_{\mathbf{\Delta}}$ *as well as* $\rho>0$*, a constant real-valued matrix* $P\in\mathbb{R}^{m\times m}$ *and matrix-valued polynomials* $Q:[a,b]\to\mathbb{R}^{m\times n}$*,* $R_0:[a,b]\to\mathbb{R}^{n\times n}$*,* $R_1,R_2:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$*, such that*

- $\mathcal{P}:=\mathcal{P}\left[\begin{smallmatrix}P, & Q\\ Q^\top, & \{\{R_0,R_1,R_2\}\}\end{smallmatrix}\right]$, $\mathcal{P}=\mathcal{P}^*,\mathcal{P}\succcurlyeq 0$,
- *and*

$$\begin{bmatrix}\mathcal{T}^*\mathcal{P}\mathcal{A}+\mathcal{A}^*\mathcal{P}\mathcal{T} & \mathcal{T}^*\mathcal{P}\mathcal{B}_p & \mathcal{T}^*\mathcal{P}\mathcal{B}_w+\mathcal{A}^*\mathcal{P}\mathcal{B}_1\\ \mathcal{B}_p^*\mathcal{P}\mathcal{T} & \mathcal{B}_p^*\mathcal{P} & \mathcal{B}_p^*\mathcal{P}\mathcal{B}_1\\ \mathcal{B}_w^*\mathcal{P}\mathcal{T}+\mathcal{B}_1^*\mathcal{P}\mathcal{A} & \mathcal{B}_1^*\mathcal{P}\mathcal{B}_p & \mathcal{B}_w^*\mathcal{P}\mathcal{B}_1+\mathcal{B}_1^*\mathcal{P}\mathcal{B}_w\end{bmatrix} + \begin{bmatrix}0 & I & 0\\ \mathcal{C}_3 & 0 & \mathcal{D}_3\\ 0 & 0 & 0\end{bmatrix}^*\begin{bmatrix}\mathcal{M} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & I & 0\\ \mathcal{C}_3 & 0 & \mathcal{D}_3\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}\mathcal{C}_1^*\mathcal{C}_1 & 0 & \mathcal{C}_1^*\mathcal{D}_1\\ 0 & 0 & 0\\ \mathcal{D}_1^*\mathcal{C}_1 & 0 & -\rho I\end{bmatrix}\preccurlyeq 0. \tag{16}$$

Then, with $x(0),\mathbf{x}(0)\equiv 0$*,* (4) *satisfies following inequality for any* $w\in L_2^p[0,\infty)$*,* $z\in L_2^k[0,\infty)$ *and all* $\Delta\in\mathbf{\Delta}$

$$\left\|z\right\|_{L_2[0,\infty)}\leq\sqrt{\rho}\left\|w\right\|_{L_2[0,\infty)}. \tag{17}$$

Proof. The proof follows from some modifications of the result presented in [6]. □

VI. ILLUSTRATIVE EXAMPLES

Based on the previous expositions, verifying the robust stability and robust input-output properties amounts to solving a set of LPIs. These LPIs can be implemented in PIETOOLS and efficiently solved using LMIs (for detailed implementation of LPIs, see [10]). In the sequel, we illustrate the results by considering two examples.

A. Stability of Fitzhug-Nagumo Model

1) Model: In neuroscience, Fitzhug-Nagumo model describes an equivalent oscillator model of a neuron. Here, in a domain $[0, 1]$, the membrane voltage $v(s, t)$ is related by

$$\frac{\partial v}{\partial t}(s,t) = \lambda v(s,t) + \frac{\partial^2 v}{\partial s^2}(s,t) + f(v(s,t)), \tag{18}$$

The boundary conditions are $v(0, t) = v(1, t) = 0$. Moreover, the nonlinear functional $f(\cdot)$ belongs to sector $[-I, I]$, i.e. $(f(v) - v)(f(v) + v) \leq 0$.

2) Stability Analysis: In [13], asymptotic behaviour of such systems have been explored using spatio-temporal separability of solutions. Our purpose is to use PIE framework and PIETOOLS to verify, for different values of $\lambda \in \mathbb{R}$, whether (18) is stable or not despite the class of nonlinear function $f(\cdot)$. To this end, we select the class of PI multiplier (14). Using PIETOOLS, the stability theorem can be implemented. For different value of λ, we conclude

- For $\lambda \leq 1.7$, the system is stable.
- For $\lambda > 1.7$, the system is not robustly stable.

B. L_2-Gain for Euler–Bernoulli beam Under Parametric Uncertainty

1) Model: Consider the following Euler–Bernoulli beam equation

$$\frac{\partial^2 x(s,t)}{\partial t^2} = \frac{EI}{\mu}\frac{\partial^4 x(s,t)}{\partial s^4} + B_w(s)w(t), \tag{19}$$

Let the output $z(t)$ be a measurement at the boundary $s = L$ according to

$$z(t) = x(L,t) + w(t).$$

We consider that the elastic modulus E is deviated from its nominal value due to long-term fatigue. In particular, we consider $E \in [-0.5E^*, 1.5E^*]$ where, $E^* \in \mathbb{R}$ is its nominal value.

2) Bound on L_2 gain: The chosen class of multiplier has the form (12), with $N = 1$ and $a_j = -0.5E^*, b_j = 1.5E^*$. We obtain that the minimum value of $\gamma = 0.45$ for which $||y||_{L_2} \leq \gamma ||w||_{L_2}$. Now, by increasing range of the interval in which E takes value, we re-evaluate the smallest value of γ. As expected, with increasing size of the uncertainty interval, the γ-value increases, implying deterioration of robust performance.

VII. CONCLUSIONS

In this paper, we present a computational framework for testing robust stability and performance of ODE-PDE systems under uncertainty. Various classes of uncertainties are considered, including a) the perturbation of the plant by norm-bounded unstructured uncertainty, b) polytopic uncertainty, and c) sector-bounded non-linearity. The method relies on three steps:

1) A class of uncertain ODE-PDE models are equivalent to PIEs with uncertainty (called as FR-PIEs).
2) The uncertainties are captured by enforcing positivity on a class PI operators, known as PI multipliers.
3) Verifying robust stability and performance of FR-PIEs are formulated in terms of LPIs that require only a set of LMIs in PIETOOLS.

The illustrative examples show the merit of the presented results for providing quantitative guarantees of uncertain spatio-temporal processes.

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