



Exact theory for superconductivity in a doped Mott insulator

Philip W. Phillips  , Luke Yeo and Edwin W. Huang 

Because the cuprate superconductors are doped Mott insulators, it would be advantageous to solve even a toy model that exhibits both Mottness and superconductivity. We consider the Hatsugai–Kohmoto model^{1,2}, an exactly solvable system that is a prototypical Mott insulator. Upon either doping or reducing the interaction strength, our exact calculations show that the system becomes a non-Fermi liquid metal with a superconducting instability. In the presence of a weak pairing interaction, the instability produces a thermal transition to a superconducting phase, which is distinct from the traditional state described by Bardeen–Cooper–Schrieffer (BCS) theory, as evidenced by a gap-to-transition temperature ratio exceeding the universal BCS limit. The elementary excitations of this superconductor are not Bogoliubov quasiparticles but rather superpositions of doublons and holons, composite excitations that show that the superconducting ground state of the doped Mott insulator inherits the non-Fermi liquid character of the normal state. An unexpected feature of this model is that it exhibits a superconductivity-induced transfer of spectral weight from high to low energies, as seen in the cuprates³, as well as a suppression of the superfluid density relative to that in BCS theory.

Cooper's⁴ demonstration that the normal state of a metal is unstable to a pairing interaction between two electrons above the Fermi surface paved the way to the eventual solution to the problem of superconductivity in elemental metals such as mercury. In modern renormalization group language^{5,6}, the Cooper instability is understood as the only relevant perturbation along a Fermi surface given that all renormalizations due to short-ranged repulsive interactions are benign. The problem of high-temperature superconductivity in copper-oxide ceramics persists because the normal state is a doped Mott insulator, a strongly correlated state of matter in which no organizing principle such as quasiparticles on a Fermi surface can be invoked. The following question thus arises: is there an analogue of Cooper's argument for a doped Mott insulator? Such a demonstration would be non-trivial, as the simplest model relevant to the cuprates—the Hubbard model—is intractable in $d > 1$ (d the spatial dimension). Given this intractability, we seek a simplification that permits a definitive answer to the Cooper problem in the presence of Mottness.

We demonstrate such an instability for the Hatsugai–Kohmoto (HK)^{1,2} model of a doped Mott insulator. The minimum feature of Mottness^{7,8}, thereby setting it apart from a Fermi liquid, is a bifurcation of the spectral weight per momentum state into low- and high-energy components. Such a bifurcation creates a surface of zeros of the single-particle Green function, called a Luttinger surface⁸, and is known to be essential in describing high–low energy mixing in doped Mott systems^{3,7–12}. As the HK model is the

simplest example that captures how quasiparticles (poles of the single-particle Green function) on a Fermi surface are converted to zeros, any superconducting instability found in such a model could ultimately illuminate the solution to the full problem. Recent progress, in fact part of the motivation for this Letter, has been made along these lines in the zero chemical potential limit¹³ in a phenomenological model for the Luttinger surface.

A key experimental signature of superconductivity in the cuprates that any model of pairing in a doped Mott insulator should inform is the in-plane transfer of spectral weight from high to low energies, dubbed the colour change³. Specifically, the integrated weight of the optical conductivity over the lower Hubbard band scales (below 1 eV) increases below the superconducting temperature, whereas the high-energy component (between 1 and 2 eV) decreases. Because the integrated weight determines the number of charge carriers, the colour change indicates that high-energy scales contribute to the superfluid density, in contrast to the standard BCS picture. In addition, the superfluid density is suppressed relative to its value in BCS theory¹⁴. We show that the HK model exhibits both of these features as a direct consequence of Mottness. The latter arises from just the splitting of the spectrum into lower and upper bands, while the first is a consequence of dynamical mixing between the upper and lower Hubbard bands induced by the pairing interaction. As dynamical (hopping-driven) mixing between the upper and lower Hubbard bands is present in the Hubbard model as a result of the non-commutativity^{7,9,10} of the kinetic and potential energy terms, our conclusion that dynamical spectral weight transfer (DSWT) is the mechanism for the colour change transcends the HK model and is a general consequence of Mottness.

Because the HK model is a simplification of the Hubbard model, a loose analogy (based on the all-to-all interactions) with the Sachdev–Ye–Kitaev (SYK)^{15,16} models applied to the strange metal phase of cuprates is appropriate, because, although they do not mirror the physics accurately, they do offer controlled analytics on non-Fermi liquid states. The HK model is probably more powerful in this regard as it actually models a Mott insulator with a Luttinger surface that gives rise to a non-Fermi liquid upon doping. What the HK model lays plain is that DSWT can be separated from the Mott gap. With this in mind, we perform calculations in the non-Fermi liquid state and study the superconducting instability through an exact calculation of the pair-field susceptibility. We then include a weak pairing interaction and explore the nature of the superconducting ground state and its elementary excitations, finding fundamental differences with the BCS ground state that ultimately arise from the non-Fermi liquid physics.

As in the SYK model¹⁵, a key ingredient that makes the HK and Baskaran² models tractable is the presence of all-to-all interactions. In the HK model and the Baskaran model² as well

Department of Physics and Institute of Condensed Matter Theory, University of Illinois at Urbana-Champaign, Urbana, IL, USA.

✉e-mail: dimer@illinois.edu

$$H_{\text{HK}} = -t \sum_{\langle j,l \rangle, \sigma} (c_{j\sigma}^\dagger c_{l\sigma} + \text{h.c.}) - \mu \sum_{j\sigma} c_{j\sigma}^\dagger c_{j\sigma} + \frac{U}{L^d} \sum_{j_1, j_2, j_3, j_4} \delta_{j_1+j_3, j_2+j_4} c_{j_1\uparrow}^\dagger c_{j_2\uparrow}^\dagger c_{j_3\downarrow}^\dagger c_{j_4\downarrow} \quad (1)$$

the interaction term is not random but a constant, U , and, unlike SYK, a hopping term (t) is present between nearest neighbours (j, l) that gives the model dimensionality. An additional feature is the presence of a constraint $j_1 + j_3 = j_2 + j_4$ that the electrons must satisfy for the interaction term U to be felt. Here, μ is the chemical potential and L^d is the number of lattice sites. Although the SYK model is tractable only in the limit of a large number of flavours, the HK model is exactly solvable, as can be seen from Fourier transforming to momentum space:

$$H_{\text{HK}} = \sum_k H_k = \sum_k (\xi_k (n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow} n_{k\downarrow}) \quad (2)$$

Here, the momenta are normalized by the lattice constant a and summed over a square Brillouin zone $[-\pi, \pi]^d$, within which the quasiparticle spectrum $\xi_k = \epsilon_k - \mu$ is set by the dispersion $\epsilon_k = -(W/2d) \sum_{\mu=1}^d \cos k^\mu$ with non-interacting bandwidth $W=4dt$ and offset by a chemical potential μ , $n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma}$ is the fermion number operator for the mode with momentum k and spin $\sigma = \uparrow, \downarrow$. It is clear that the kinetic and potential energy terms commute. Consequently, momentum is a good quantum number, unlike the Hubbard model, and all eigenstates have a fixed unfluctuating occupancy in k -space.

What is surprising about the HK model is that, although the potential and kinetic energy terms commute, a correlated metal-insulator transition still exists¹. Consider the exact single-particle Green function

$$G_{k\sigma}(i\omega_n) \equiv - \int_0^\beta d\tau \langle c_{k\sigma}(\tau) c_{k\sigma}^\dagger(0) \rangle e^{i\omega_n \tau} \quad (3)$$

$$G_{k\sigma}(i\omega_n \rightarrow \omega) = \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{\omega - \xi_k} + \frac{\langle n_{k\bar{\sigma}} \rangle}{\omega - (\xi_k + U)} \quad (4)$$

which we plot in Fig. 1a–c. The two-pole structure is reminiscent of the atomic limit of the Hubbard model, except $i\omega_n$ is now replaced with $i\omega_n - \xi_k$. The corresponding density of states shares a mutual energy region only for $U < W$. Consequently, a gap ($\Delta E = U - W$) appears in the single-particle spectrum for $U > W$, resulting in a Mott insulating state at half-filling (Fig. 1a), with $\langle n_{k\bar{\sigma}} \rangle = 1/2$ for all k . Doping away from half-filling or reducing the interaction strength, $U < W$, eliminates the gap or shifts it away from the chemical potential, leading to a compressible metallic state (Fig. 1b,c). In this metallic state, the momentum occupancy changes discontinuously from $\langle n_k \rangle = \langle n_{k\uparrow} + n_{k\downarrow} \rangle = 1$ for singly occupied momenta in Ω_1 to 2 for the doubly occupied part (Ω_2) and 0 for the empty region (Ω_0).

Spin-rotation invariance of the HK model dictates that, although singly-occupied momenta exist in the metal, they cannot appear as pure states. The metal is the mixed state consisting of a uniform ensemble of spin states:

$$|\Psi_G; \{\sigma_k\}\rangle = \prod_{k \in \Omega_1} c_{k\sigma_k}^\dagger \prod_{k \in \Omega_2} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger |0\rangle \quad (5)$$

which results in a large ground-state degeneracy. Because of the lack of mixing between the two Hubbard bands, excitations in the lower band are created by $\zeta_{k\sigma} = c_{k\sigma}^\dagger (1 - n_{k\bar{\sigma}})$ at energy ξ_k , and those in the upper by $\eta_{k\sigma} = c_{k\sigma}^\dagger n_{k\bar{\sigma}}$ at energy $\xi_k + U$. As a result, excitations in the metal are more akin to doublon and holon composite excitations and hence the metal of the HK model lacks any interpretation in terms of Fermi-liquid quasiparticles. This can be seen directly from the retarded Green function:

$$G_\sigma^R(k, \omega) = \frac{1}{\omega + i0^+ - (\xi_k + U/2) - \frac{(U/2)^2}{\omega + i0^+ - (\xi_k + U/2)}} \quad (6)$$

when the occupancies are equal. Both real and imaginary parts of the self-energy diverge at $\omega=0$ along the surface defined by $\xi_k = -U/2$. Such a divergence indicates that the Green function cannot be derived perturbatively from the non-interacting limit—there is no Fermi liquid.

The two-pole structure determines the sign changes of the Green function and is crucial in calculating the Luttinger count $2 \sum_k \theta(\text{Re} G(k, 0))$. Except for fine-tuned cases, such as at $\mu = U/2$, where there is exact particle-hole symmetry, or when $U=0$ where the model is non-interacting, Luttinger's theorem is violated in the HK model and Hubbard models in similar limits¹⁷. The Luttinger count is decoupled from the true occupancy and can exceed it by a factor of up to 2 (Fig. 1d). This is evident deep in the doped Mott insulating regime (Fig. 1b), where Ω_1 is singly occupied but contributes fully to the Luttinger count when the zeros of the Green function are above the chemical potential. The violation of Luttinger's theorem throughout the phase diagram (Fig. 1d) indicates that the metallic state of the HK model is incompatible with Fermi-liquid theory.

Having established that we have a completely controlled non-Fermi liquid metallic state, we can address the following question: is such a state unstable to pairing? To this end, we append the HK Hamiltonian with an attractive ($g > 0$) pairing interaction:

$$H = H_{\text{HK}} - g H_p, \quad H_p = \frac{1}{L^d} \Delta^\dagger \Delta \quad (7)$$

where $\Delta = \sum_k b_k = \sum_k c_{-k\downarrow} c_{k\uparrow}$ is the s -wave pair creation operator at zero total momentum. Seeking an analogue of Cooper's argument, we first focus on the pair amplitude

$$i\hbar \frac{\partial}{\partial t} \alpha_k(t=0) = i\hbar \frac{\partial}{\partial t} \langle \text{GS}(t) | b_k | \psi(t) \rangle = \langle \text{GS}(t) | [b_k, H] | \psi(t) \rangle \quad (8)$$

where $|\text{GS}\rangle$ is a metallic state in the zero-temperature ensemble described by equation (5), and $|\psi\rangle$ is the state with a single pair given by

$$|\psi\rangle = \sum_{k \in \Omega_0} \alpha_k b_k^\dagger |\text{GS}\rangle \quad (9)$$

For clarity, we take a maximally polarized state for $|\text{GS}\rangle = \left(\prod_{k \in \Omega_2} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger \right) \left(\prod_{k \in \Omega_1} c_{k\uparrow}^\dagger \right) |0\rangle$. (In the Supplementary Information we show that the zero-temperature Gibbs state recovers the same result.) From

$$[b_k, H] = (2\xi_k + U(n_{k\downarrow} + n_{-k\uparrow})) b_k - \frac{g}{L^d} (1 - n_{k\uparrow} - n_{-k\downarrow}) \sum_{k'} b_{k'} \quad (10)$$

the equations of motion take on the form

$$(i\hbar \partial_t - 2\xi_k - U(n_{k\downarrow} + n_{-k\uparrow})) \alpha_k = - \frac{g}{L^d} \langle 1 - n_{k\uparrow} - n_{-k\downarrow} \rangle \sum_{k'} \alpha_{k'} \quad (11)$$

where $\langle \dots \rangle$ denotes an expectation value in the state $|\text{GS}\rangle$. A similar equation can also be derived for the Baskaran¹⁸ model.

We solve this equation in the standard way by letting $\alpha_k(t) = e^{-iEt/\hbar} \alpha_k(t=0)$. Dividing by the coefficient on the left-hand side and performing the sum over momentum, we obtain

$$1 = - \frac{g}{L^d} \sum_{k \in \Omega_0} \frac{\langle 1 - n_{k\uparrow} + n_{-k\downarrow} \rangle}{E - 2\xi_k - U \langle n_{k\downarrow} + n_{-k\uparrow} \rangle} \quad (12)$$

as the familiar criterion for a superconducting instability. In the range of integration, $\langle n_{k\sigma} \rangle = 0$, resulting in the simplified expression

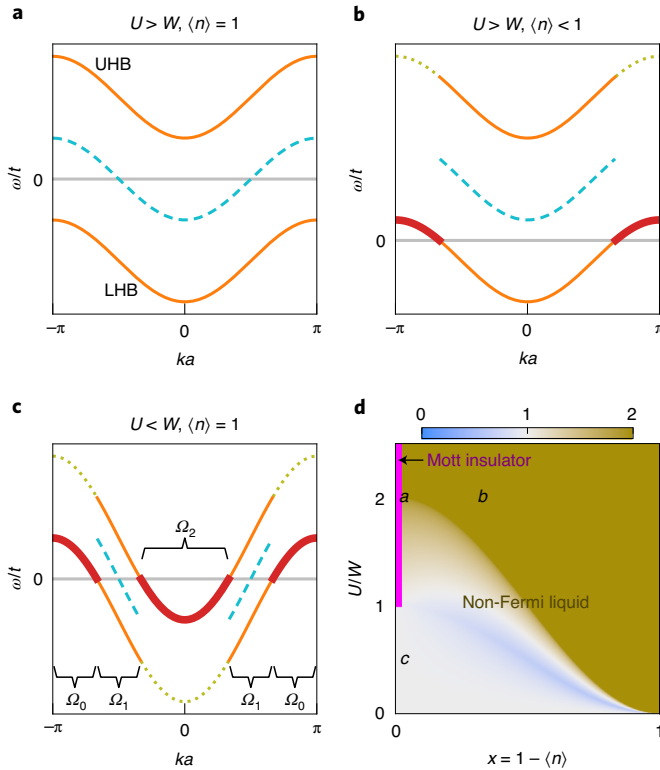


Fig. 1 | Single-particle Green functions and phase diagram of the HK model. **a–c**, Poles and zeros of the single-particle Green function (equation (3)). A one-dimensional (1D) tight-binding dispersion is used for simplicity. Zeros are indicated by dashed blue lines. Poles with weights 0, 0.5 and 1 are indicated by dotted olive, thin orange and bold red lines, respectively. The upper Hubbard band (UHB) and lower Hubbard band (LHB) are labelled in **a**. Regions of occupancy $\langle n \rangle = i$ are labelled as Ω_i for $i = 0, 1, 2$ in **c**, **d**. Ground-state phase diagram of the HK model. The non-Fermi liquid covers the entire diagram except for the half-filled Mott insulator for $U > W$. Colour represents the ratio of Luttinger count to filling; deviation from 1 (white) indicates violation of Luttinger's theorem.

$$1 = -g \int_{\mu}^{W/2} d\epsilon \frac{\rho(\epsilon)}{E - 2\epsilon + 2\mu} \quad (13)$$

having converted the sum to an integral weighted by the density of states $\rho(\epsilon)$ for the band ϵ_k . This is, up to the limits of integration and density of states, exactly of the BCS form. It therefore results in a bound-state energy $E < 0$ for any $g > 0$, which we plot in Fig. 2 for $d = 2$. In the case of half-filled metal ($\mu = U/2$) in one dimension, for example, the binding energy,

$$E_b = -E \sim W(1 - (U/W)^2)e^{-\pi W \sqrt{1 - (U/W)^2}/g} \quad (14)$$

is exponentially small in $1/g$. The full d -dimensional dependence is shown in Supplementary Figs. 1 and 2. Hence, the HK model exhibits an analogue of the instability Cooper⁴ found for a Fermi liquid and answers the question affirmatively asked in ref. 19.

The advantage of the HK model is that we can also compute the pair susceptibility

$$\chi(i\nu_n) \equiv \frac{1}{L^d} \int_0^\beta d\tau e^{i\nu_n \tau} \langle T\Delta(\tau)\Delta^\dagger \rangle_g \quad (15)$$

exactly at all temperatures, although in the following we will emphasize the low-temperature regime $T \ll U, W$. Here $\langle \dots \rangle_g$ (resp. $\langle \dots \rangle_0$)

is an expectation value in the Gibbs state $e^{-\beta H}/Z$ (resp. $e^{-\beta H_{\text{HK}}}/Z_0$), and $\nu_n = 2\pi n/\beta$ is a bosonic Matsubara frequency. As we show in the Supplementary Information, $\chi(i\nu_n)$ is related to the ‘bare’ susceptibility, $\chi_0(i\nu_n)$ at $g = 0$, through the Dyson equation $\chi = \chi_0 + g\chi_0\chi$. Then the fluctuation propagator $L \equiv -g\chi/\chi_0$ satisfies the usual equation²⁰

$$L = -g + g\chi_0 L = \frac{1}{\chi_0 - 1/g} \quad (16)$$

such that $L(\omega = 0)$ diverges when $\chi_0(0)|_{T=T_c} = 1/g$, thereby fixing the critical superconducting temperature T_c . To compute χ_0 , we first simplify

$$\langle T\Delta(\tau)\Delta^\dagger \rangle_0 = \sum_{k,p} \langle Tc_{-k\downarrow}(\tau)c_{k\uparrow}(\tau)c_{p\uparrow}^\dagger c_{-p\downarrow}^\dagger \rangle_0 \quad (17)$$

$$= \sum_k \langle Tc_{-k\downarrow}(\tau)c_{k\uparrow}(\tau)c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle_0 \quad (18)$$

$$\sim \sum_k G_{-k\downarrow}(\tau)G_{k\uparrow}(\tau) \quad (19)$$

up to an unimportant subextensive contribution from coincident terms with $k = -k$, having used that the ensemble consists only of Fock states (guaranteed by the preservation of $n_{k\sigma}$ as a good quantum number in the HK model) in the second line, and in the third line that the Gibbs state $e^{-\beta H_{\text{HK}}}/Z_0$ factorizes in k -space. We note that, despite the appearance of equation (19), we have not utilized Wick's theorem, which does not apply in general to the HK model. Writing out the single-particle Green function

$$-G_{k\sigma}(\tau) = \langle c_{k\sigma}(\tau)c_{k\sigma}^\dagger \rangle_0 = n_{k\sigma}^l f(-\xi_k^l) e^{-\tau\xi_k^l} + (l \rightarrow u) \quad (20)$$

for $\xi_k^l = \xi_k$ and $\xi_k^u = \xi_k + U$, $n_{k\sigma}^u = \langle n_{k\sigma} \rangle_0$ and $n_{k\sigma}^l = 1 - n_{k\sigma}^u$, and $f(\omega)$, the Fermi function at temperature T , we have

$$\chi_0(i\nu_n) = \chi_0^{ll} + \chi_0^{uu} + \chi_0^{lu} + \chi_0^{ul} \quad (21)$$

$$\chi_0^{ab} = \frac{1}{L^d} \sum_k n_{k\uparrow}^a n_{-k\downarrow}^b \frac{f(\omega_k^a) + f(\omega_{-k}^b) - 1}{i\nu_n - \omega_k^a - \omega_{-k}^b} \quad (22)$$

where the superscripts ab may represent ll, uu, lu or ul . Because of the factors $f(\omega_k^l) + f(\omega_{-k}^u) - 1$, the cross terms χ_0^{lu} and χ_0^{ul} (between the lower and upper Hubbard bands) contribute no low-energy spectral weight when $T \ll U$ and are dropped hereafter. Using $\xi_k = \xi_{-k}$ and $\langle n_{k\uparrow} \rangle = \langle n_{k\downarrow} \rangle$ and changing variables, we finally arrive at

$$\chi_0(0) = \int d\omega N'(\omega) \frac{\tanh \frac{\beta\omega}{2}}{2\omega} \quad (23)$$

$$L^d N'(\omega) = \sum_{k \in \Omega_0} \delta(\omega - \xi_k^l) + \sum_{k \in \Omega_2} \delta(\omega - \xi_k^u) + \frac{1}{4} \sum_{k \in \Omega_1} \delta(\omega - \xi_k^l) + \delta(\omega - \xi_k^u) \quad (24)$$

Here, $N'(\omega)$ is an effective density of states, similar to but not equal to the HK model's single-particle density of states $N(\omega) = \frac{1}{L^d} \sum_k \frac{1}{\pi} \text{Im} G(k, \omega + i0^+)$, which has a factor of $\frac{1}{2}$ rather than $\frac{1}{4}$ before the sum over the singly occupied region. By the same manipulations as in the free fermion case, it is clear that $\chi_0(0)$ grows as $\ln \frac{1}{T}$ at low temperature. Hence, for any non-zero pairing strength, g , $\chi(0)$ diverges at the transition temperature $T_c \propto e^{-1/(N'(0)g)}$ (see Supplementary Information). In Fig. 2, the transition temperatures are calculated explicitly for a variety of parameters.

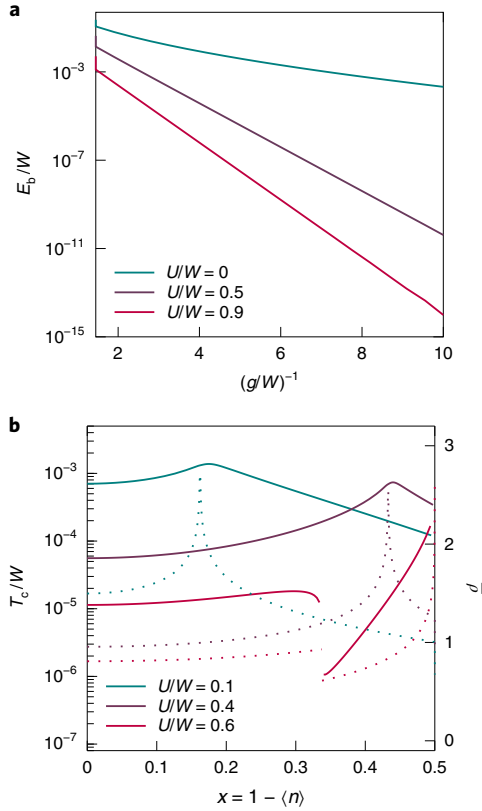


Fig. 2 | Superconducting energy scales in two dimensions. a, Cooper pair binding energy E_b at half-filling ($x=0$). The linear regime $\ln(E_b/W) \sim -(g/W)^{-1}$ is achieved at large values of the inverse pair coupling $(g/W)^{-1}$. **b**, Superconducting temperature T_c (solid) at pair coupling $g/W=0.1$, over a range of hole dopings $x=1-\sum_{k\sigma}\langle n_{k\sigma}\rangle/L^d$ away from half-filling. Its qualitative behaviour under doping is given by the mean density of states $\bar{\rho} = \frac{1}{2}(\rho(\mu) + \rho(\mu-U))$ (dotted) at the same pair coupling.

We now characterize the ground state of the model in the presence of a non-zero pairing interaction. We work with the variational wavefunction

$$|\psi\rangle = \prod_{k>0} (x_k + y_k b_k^\dagger b_{-k}^\dagger + \frac{z_k}{\sqrt{2}} (b_k^\dagger + b_{-k}^\dagger)) |0\rangle \quad (25)$$

normalized by $|x_k|^2 + |y_k|^2 + |z_k|^2 = 1$. This is a generalization of the BCS wavefunction, which corresponds to $x_k = u_k^2$, $y_k = v_k^2$, $z_k = \sqrt{2}u_kv_k$. The utility of this generalized wavefunction is that the state with $x_k=1$ for $k \in \Omega_0$, $y_k=1$ for $k \in \Omega_2$ and $z_k=1$ for $k \in \Omega_1$, is a ground state of the HK model. Minimizing the energy variationally leads to two equations:

$$\xi_k^l x_k z_k = (x_k^2 - z_k^2) \frac{g}{L^d} \sum_{p>0} (x_p z_p + z_p y_p) \quad (26)$$

$$\xi_k^u y_k z_k = (z_k^2 - y_k^2) \frac{g}{L^d} \sum_{p>0} (x_p z_p + z_p y_p) \quad (27)$$

after taking the limit $g \ll U, W$. Details are provided in the Supplementary Information. After a few changes of variables, we obtain a gap equation

$$1 = \frac{g}{2} \int d\omega \frac{N''(\omega)}{\sqrt{\omega^2 + \Delta^2}} \quad (28)$$

$$N''(\omega) = \sum_{k \in \Omega_0} \delta(\omega - \xi_k^l) + \sum_{k \in \Omega_2} \delta(\omega - \xi_k^u) + \sum_{k \in \Omega_1} \delta(\omega - \xi_k^l) + \delta(\omega - \xi_k^u)$$

This is the BCS gap equation except for the effective density of states $N''(\omega)$. The solution has $\Delta \propto e^{-1/(N''(0)g)}$, which is verified in the Supplementary Information for a specific example. Note that $N''(\omega)$ is different from $N'(\omega)$, which controls T_c , as there is no factor of $\frac{1}{4}$ before the sum $\sum_{k \in \Omega_i}$. Because this sum over the singly occupied region affects the low-energy spectra, $N''(0) > N'(0)$ and the superconducting gap-to-transition temperature ratio diverges as $e^{3/(10\rho(\mu)g)}$ for $g \rightarrow 0$, in contrast to the universal BCS result $\frac{2\Delta}{T_c} = 3.53\dots$ for s -wave pairing in the weak coupling limit. Therefore, despite apparent mathematical similarities between pairing in the HK model and BCS pairing of free fermions, the presence of the singly occupied region Ω_1 leads to qualitatively distinct phenomena.

The elementary excitations of a BCS superconductor are Bogoliubov quasiparticles $\gamma_{k\sigma} = u_k c_{k\sigma} - \sigma v_k c_{-k\sigma}^\dagger$. The excitations of the superconducting state of the HK model with pairing cannot be the same. This is for the same reason that $c_{k\sigma}^\dagger$ is not an elementary excitation of the HK model, namely that, in the singly occupied region, both the upper and lower Hubbard bands have non-zero spectral weight. On the other hand, the Green functions for the holon and doublon excitations, $\zeta_{k\sigma}$ and $\eta_{k\sigma}$, have weight only in the lower and upper Hubbard bands, respectively; these composite operators describe the elementary excitations of the HK model. On turning on pairing and entering the superconducting state, the new excitations are given by mixing of the composite operators and their conjugates:

$$\gamma_{k\sigma}^l \propto \sqrt{2} x_k \zeta_{k\sigma}^\dagger - \sigma z_k \zeta_{-k\sigma} \quad (29)$$

$$\gamma_{k\sigma}^u \propto z_k \eta_{k\sigma}^\dagger - \sigma \sqrt{2} y_k \eta_{-k\sigma} \quad (30)$$

It is straightforward to check that

$$\gamma_{k\sigma}^{u/l} |\psi\rangle = 0 \quad (31)$$

$$\langle \psi | \gamma_{k\sigma}^{u/l} H (\gamma_{k\sigma}^{u/l})^\dagger | \psi \rangle = \langle \psi | H | \psi \rangle + E_k^{u/l} \quad (32)$$

where $E_k^{u/l} = \sqrt{\xi_k^{u/l2} + \Delta^2}$. Therefore, $\gamma_{k\sigma}^{u/l}$ are analogous to the Bogoliubov quasiparticle excitations of a BCS superconductor but composed of doublons or holons, indicating that the non-Fermi liquid nature of the metallic state carries over into the superconducting state. To further reveal the role of such excitations, we compute the spectral function in the superconducting state, treating the pairing interaction (g) at the mean-field level and the Mott interaction (U) exactly. Plots for the doped Mott insulator and the half-filled metal are provided in Fig. 3a,b. Here, we see that these excitations are present in the single-particle spectral function at energies $\pm E_k^{u/l}$. Where the Hubbard bands cross the chemical potential ($\omega/t=0$), we observe the superconducting gap and associated back-bending of the dispersions. Therefore, despite the unconventional nature of the excitations of the superconducting state as discussed previously, their spectroscopic signatures appear BCS-like.

From the spectral function, we can compute the integrated weight to see if the colour change³ emerges. Figure 3c attests that the integrated weight of the lower band (w_l) in the superconducting state relative to the weight at $g=0$ is a monotonically increasing function of the pair coupling constant, g . The size of this increase is consistent with the 6% increase seen experimentally³. The origin of this effect in the HK model is simple. The pairing interaction ruins the commutativity of the kinetic and potential energy terms in equation (7). This will result in dynamical mixing between the upper and lower Hubbard bands, which is well documented^{7,9} to increase the spectral weight in the lower band as it increases the number of low-energy degrees of freedom. Conservation of the spectral weight necessitates

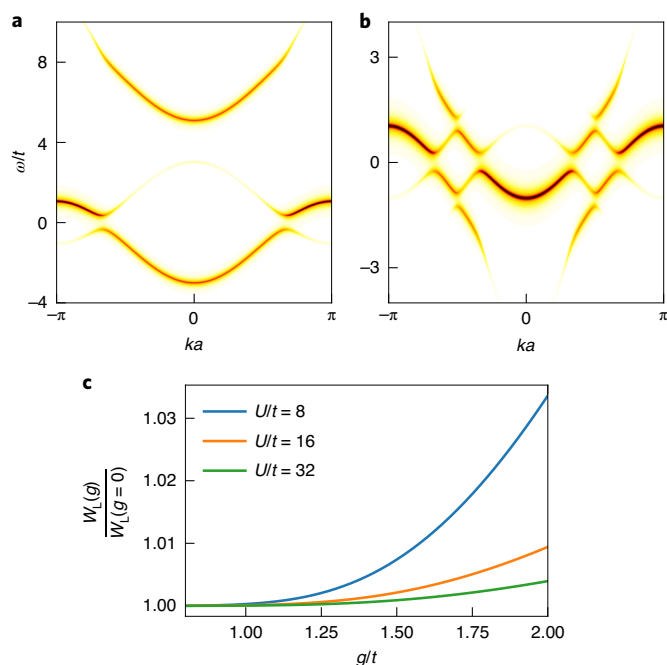


Fig. 3 | Spectral function in the superconducting ground state. **a**, Doped Mott state with parameters $U/t=8$, $g/t=1.75$, $\mu/t=1$. **b**, Half-filled metal with parameters $U/t=2$, $g/t=1$, $\mu/t=1$. Darker colours indicate higher intensity. Details of the calculation are provided in the Supplementary Information. **c**, Integrated spectral weight in the lower Hubbard band ($-4 < \omega/t < 4$), at chemical potential $\mu/t=1$, relative to the value for $g=0$. The increase here is consistent with the experimental trends³.

an equivalent decrease in the high-energy part of the spectrum, as seen experimentally³. Because it is the pairing term that leads to a breakdown of the integrability of the HK model, DSWT in the HK model is a direct result of opening the superconducting gap. As the subsequent curves show in Fig. 3c, the integrated weight falls off as $1/U$, as is expected for dynamical mixing across the Hubbard bands. Such mixing is a general consequence of Mott physics and hence transcends the HK model. Such dynamical mixing can certainly be enhanced by instantaneous pairing interactions as such interactions involve all energy scales.

The consequences of Mottness on the superconducting state extend also to the superfluid stiffness. Again treating the pairing interaction at the mean-field level, we have calculated the superfluid stiffness in the superconducting ground state in Fig. 4. As compared to a regular BCS superconductor, the superfluid stiffness in the HK model is significantly suppressed, particularly near half-filling. This is a direct consequence of the Mott interaction: proximity to the Mott insulator reduces the kinetic energy and hence the effective carrier density, which is an upper bound to the superfluid stiffness. Although models with disorder and fluctuations of the pair-field also lead to a suppression, Mottness supercedes all such effects and hence the effect we have found here is quite general.

Given the recent spate of papers on superconductivity in the absence of quasiparticles^{21–26}, our approach offers a systematic Hamiltonian-based approach to the breakdown of the quasiparticle picture without invoking randomness. As remarked previously, recent work¹³ on the Luttinger surface applies strictly to the insulator where the susceptibility appears to diverge, exhibiting SYK dynamics. Our analysis has revealed here that when both upper and lower Hubbard bands carry spectral weight, the essence of Mottness, the fundamental excitations of either the metallic or superconducting

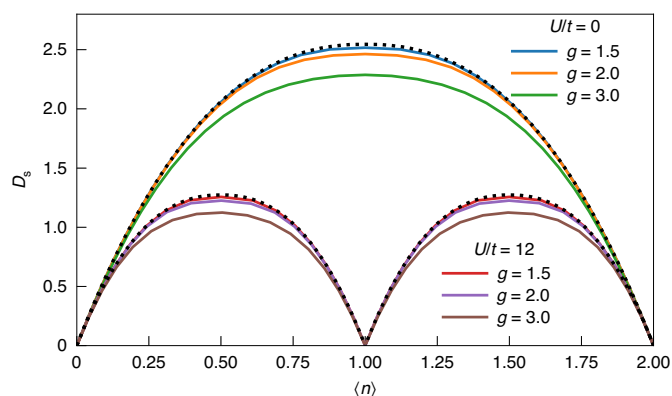


Fig. 4 | Ground-state superfluid stiffness. Solid lines are the superfluid stiffness for various values of interaction strength U/t and pairing strength g/t . Dotted lines are proportional to the kinetic energy when $g=0$, representing an upper bound on the superfluid stiffness³⁰.

states of a doped Mott insulator cannot be described by conventional quasiparticles.

This is a conclusion that applies also to the cuprates, in which the spectral weight of the upper Hubbard band in hole-doped compounds has been observed and compared to calculations of the Hubbard model^{7,27}. Although we do not know precisely the excitations of the strange metal normal state of cuprates²⁸, there is overwhelming evidence that they are not Fermi liquid quasiparticles. These arguments extend this notion to the superconducting state. Our findings for the HK model and the origin of the colour change that should hold for the Hubbard model thus challenge the assumption that the appearance of coherent peaks and back-bending in the spectral function of superconducting cuprates is a signature of regular Bogoliubov quasiparticles. Conversely, they suggest that detailed studies of the superconducting state and its excitations can help unravel the mysteries of the normal state as a general consequence of ultraviolet–infrared mixing²⁹.

Although the HK model is complex enough to capture zeros of the Green function and their associated consequences on the metallic state and on the superconducting instability, it does not support dynamical spectral weight transfer unless pairing is included. A promising line of inquiry would be to see how stable the present results are to such dynamical mixing arising from repulsive interactions. Whether a renormalization principle can be established to show that the excitations on a zero surface are impervious to such mixing remains an open question.

Online content

Any methods, additional references, Nature Research reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at <https://doi.org/10.1038/s41567-020-0988-4>.

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Data availability

All data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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Author contributions

P.W.P. initiated the problem and computed the pair instability. L.Y. computed the susceptibility with the numerics on the binding energy and E.W.H. computed the wave function, elementary excitations and the superfluid density.

Competing interests

The authors declare no competing interests.

Additional information

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Correspondence and requests for materials should be addressed to P.W.P.

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