Distributed Stabilization of Two Interdependent Markov Jump Linear Systems With Partial Information

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***Abstract*—In this letter, we study the stabilization of two interdependent Markov jump linear systems (MJLSs) with partial information, where the interdependency arises as the transition of the mode of one system depends on the states of the other system. First, we formulate a framework for the two interdependent MJLSs to capture the interactions between various entities in the system, where the modes of the system cannot be observed directly. Instead, a signal which contains information of the modes can be observed. Then, depending on the scope of the available system state information (global or local), we design centralized and distributed controllers, respectively, that can stochastically stabilize the overall interdependent MJLS. In addition, we derive the sufficient stabilization conditions for the system under both types of information structure. Finally, we use a numerical example to illustrate the effectiveness of the designed controllers.**

***Index Terms*—Interdependent systems, Markov jump linear systems, distributed stabilization, partial information.**

I. Introduction

DYNAMIC systems subject to random abrupt changes in their structures and parameters can be modeled by stochastic jump systems. Particularly, when the random jump process is described by a Markovian process with given transition rates, the system is categorized into the class of Markov jump systems. Extensive research and investigations have been done on the stability analysis and (optimal) control design of Markov jump linear systems (MJLSs) [1]–[5]. Two common features of the system models studied in these literature are: (i) the state transition rate matrix is time-invariant; (ii) the Markov parameters of the transition matrix can be directly observed.

However, in many applications, the transition rate matrix of a system can depend on the system state. For example, the failure probability of a wind turbine is related to its use time, level of wear, stress, and stiffness on the blades [6]. The general Markov jump system models considered in [1], [7] are not directly applicable to these scenarios. Moreover, the modes of the system often cannot be observed, as seen in robot navigation problems, machine maintenance, and planning under uncertainty [8]–[10]. In these applications, the modes can only be inferred from the emitted distorted signals. To address these problems, [11] has modeled the system as a state-dependent MJLS with partial information, in which the transition rate matrix is time-varying due to the evolution of the dynamical system, and the controller has only partial information of the system modes.

With the emerging advanced information and communication technologies (ICTs), the real-world systems are becoming more complex. One main characteristic of these modern control systems is that they are interdependent, creating system-of-systems [12]–[14]. In many examples, we see that the state/condition of one system will have an impact on the operation of other systems. This particular structure of interdependencies can lead to cascading failures among various entities in homogeneous and heterogeneous networks [15]–[18]. One specific example is the interdependent communication and power systems, which can be seen as a class of cyber-physical systems (CPS). The power system operator leverages the communication for real-time control of the grids, while the communication system consumes energy from the power system. A point of failure in one system will propagate to the other one due to their interdependencies. We can use the Markov jump mode to represent a failure mode of an infrastructure, where the transition rate is influenced by the state of another connected system. Hence, the cascading failures can be modeled through the interdependencies across mode and state. For clarity, Fig. 1 depicts a framework of two coupled MJLSs. Note that the traditional single Markov jump system is not sufficient to capture these interdependent features in the networked systems. Therefore, we establish an interdependent

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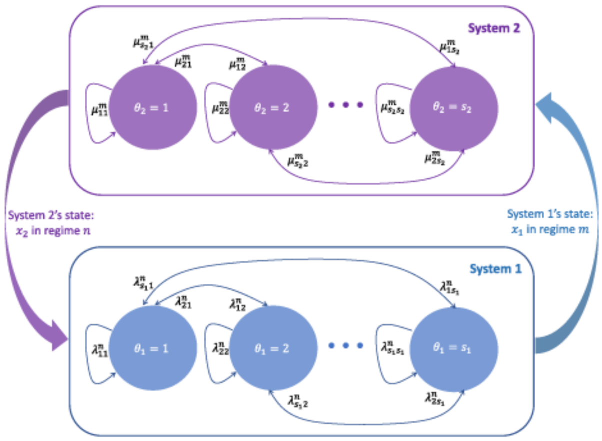


Fig. 1. Two interdependent Markov jump systems model. The operational mode of one system is influenced by the state of the other system.

MJLS framework in this letter, as shown in Fig. 1, to better understand the interdependencies between different systems and also design controllers for the complex systems.

In this letter, we first derive its stability criterion and design stochastic stabilizing controllers by regarding the multiple MJLSs as an integrated system. In addition, to preserve the distributed nature of various coupled jump systems, we design the distributed stabilizing controllers for each individual system.

The main contributions of this letter are summarized as the following.

1) We establish an interdependent MJLS model with partial information to capture the interactions and couplings, where Markov modes are state-dependent and partially observable.
2) We derive a sufficient stabilization condition using linear matrix inequality (LMI), and design stochastic stable controllers for the integrated MJLS with partial information of the modes.
3) To reduce the complexity of controller design, we design distributed stabilizing controllers for each individual system which ensure the stability of the integrated system-of-systems.

The rest of this letter is organized as follows. In Section II, we describe the interdependent MJLS framework under partial information and the corresponding integrated system. Section III studies the stabilization problem from an integrated system perspective. Section IV investigates the distributed stabilizing controller design. Numerical examples are given in Section V to validate the effectiveness of the designed controllers, and finally Section VI concludes this letter and discusses possible directions of future work.

II. Interdependent MJLSs and the Integrated System Model

In this section, we present the interdependent MJLS framework as illustrated in Fig. 1. Specifically, we consider a model of two coupled MJLSs (namely, System 1 and System 2):

$$\dot{x}_k(t) = A_{k,\theta_k(t)}x_k(t) + B_{k,\theta_k(t)}u_k(t) + D_{k,\theta_k(t)}w_k(t),\ \ k = 1, 2, \tag{1}$$

where $x_k(t) \in \mathbb{R}^{N_{k,x}}$; $x_k(t_0)$ is a fixed (known) initial state of the physical plant at starting time t_0; $u_k(t) \in \mathbb{R}^{N_{k,u}}$ is the control input; $w_k(t) \in \mathbb{R}^{N_{k,w}}$ is the deterministic disturbance; and all these quantities lie at the physical and control layers of the system. Note that $N_{k,x}$, $N_{k,u}$, and $N_{k,w}$, $k = 1, 2$, are all positive integers. Furthermore, the system mode of System k, $\theta_k(t) \in \mathbb{R}$, is a Markov jump process with right-continuous sample paths and initial distribution $\pi_{k,0}$. The possible values of $\theta_k(t)$ are assumed to be in the finite set $\mathscr{S}_k := \{1, 2, \dots, |\mathscr{S}_k|\}$. Moreover, $A_{k,\theta_k(t)}$, $B_{k,\theta_k(t)}$, and $D_{k,\theta_k(t)}$, $k = 1, 2$, are system matrices of appropriate dimensions whose entries are continuous functions of time t. We assume that the system disturbance $w_k(t)$ satisfies $\int_{t_0}^{\infty} w_k(t)^{\mathrm{T}} w_k(t)dt < \infty,\ k = 1, 2.$

The MJLSs in (1) are interdependent in the sense that the transition rate matrix of the system mode of one MJLS is dependent on the state of the other MJLS. Without loss of generality, we consider the interdependency in a chain structure. Based on the interdependent structure of two MJLSs, we have

$$\begin{aligned}\Pr[\theta_1(t+\Delta) &= j_1 | \theta_1(t) = i_1, x_2(t) \in \mathscr{C}_2^{m_2}]\\ &= \begin{cases} \lambda_{i_1 j_1}^{m_2}\Delta + o(\Delta) & \text{if } i_1 \neq j_1,\\ 1 + \lambda_{i_1 j_1}^{m_2}\Delta + o(\Delta) & \text{otherwise,}\end{cases}\end{aligned} \tag{2}$$

and

$$\begin{aligned}\Pr[\theta_2(t+\Delta) &= j_2 | \theta_2(t) = i_2, x_1(t) \in \mathscr{C}_1^{m_1}]\\ &= \begin{cases} \mu_{i_2 j_2}^{m_1}\Delta + o(\Delta) & \text{if } i_2 \neq j_2,\\ 1 + \mu_{i_2 j_2}^{m_1}\Delta + o(\Delta) & \text{otherwise,}\end{cases}\end{aligned} \tag{3}$$

where $\mathscr{C}_k^1, \mathscr{C}_k^2, \dots, \mathscr{C}_k^{M_k}$, $k = 1, 2$, are nonempty and disjoint sets, and $\cup_{m_k \in \mathscr{M}_k} \mathscr{C}_k^{m_k}$ denotes the space containing all the possible states of $x_k(t)$, where $\mathscr{M}_k := \{1, 2, \dots, M_k\}$. The transition rates for the Markov jump process, $\theta_1(t)$ and $\theta_2(t)$, are denoted by $\{\lambda_{i_1 j_1}^{m_2}\}_{i_1, j_1 \in \mathscr{S}_1}$ and $\{\mu_{i_2 j_2}^{m_1}\}_{i_2, j_2 \in \mathscr{S}_2}$, respectively. The finite partition of the state space in (2) and (3) is motivated by the interdependent critical infrastructure applications. For example, in the coupled power and communication systems, communication delay will impact the power system operation. Power system operates under different conditions (e.g., efficient, delay-tolerant, conservative) depending on the significance of the delay (e.g., minimal, intermediate, enormous).

In the focused scenario, for each MJLS, the system mode $\theta_k(t)$ cannot be directly observed. Instead, a signal $\hat{\theta}_k(t)$ is observed. At time t, given $x_k(t) \in \mathscr{C}_k^{m_k}$ and $\theta_k(t) = i_k$, the observation probabilities are assumed to be the following conditional probabilities:

$$\Pr\left[\hat{\theta}_k(t) = \hat{i}_k | \theta_k(t) = i_k, x_k(t) \in \mathscr{C}_k^{m_k}\right] = \alpha_{\hat{i}_k i_k}^{k, m_k}, \quad k = 1, 2,$$

where $\hat{\theta}_k(t) \in \hat{\mathscr{S}}_k$ is the observation of System k, and $\hat{\mathscr{S}}_k$ denotes the set that contains all the possible observations of System k.

The following assumptions are made to hold throughout this letter.

Assumption 1: For each MJLS, the observation does not influence the transition of the system mode, i.e., for all $\Delta > 0$, $\theta_k(t+\Delta) \in \mathscr{S}_k$, $\theta_k(t) \in \mathscr{S}_k$, $x_k(t) \in \mathscr{C}_k^{m_k}$, $m_k \in \mathscr{M}_k$, and

$\hat{\theta}_k(t) \in \hat{\mathscr{S}}_k,\ k = 1, 2,$

$$\Pr\Big[\theta_1(t+\Delta)|\hat{\theta}_1(t), \theta_1(t), x_2(t)\Big] = \Pr[\theta_1(t+\Delta)|\theta_1(t), x_2(t)],$$
$$\Pr\Big[\theta_2(t+\Delta)|\hat{\theta}_2(t), \theta_2(t), x_1(t)\Big] = \Pr[\theta_2(t+\Delta)|\theta_2(t), x_1(t)],$$
$$\Pr\Big[\{\theta_k(t+\Delta)\}_{k=1,2}|\{\hat{\theta}_k(t), \theta_k(t), x_k(t)\}_{k=1,2}\Big] = \Pr\big[\{\theta_k(t+\Delta)\}_{k=1,2}|\{\theta_k(t), x_k(t)\}_{k=1,2}\big].$$

Assumption 2: For each MJLS, the set of observations is the same as the set of system modes, i.e.,

$$\mathscr{S}_k = \hat{\mathscr{S}}_k, \quad k = 1, 2.$$

The two interdependent MJLSs can be jointly represented as an integrated system, which is given as follows (where we omit the time index for notational clarity):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{1,\theta_1} & 0\\ 0 & A_{2,\theta_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_{1,\theta_1} & 0\\ 0 & B_{2,\theta_2}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}D_{1,\theta_1} & 0\\ 0 & D_{2,\theta_2}\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}. \tag{4}$$

The integrated system (4) can be rewritten more compactly as

$$\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + D_{\theta(t)}w(t), \tag{5}$$

where $x := [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, $u := [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}]^{\mathrm{T}}$, $w := [w_1^{\mathrm{T}}, w_2^{\mathrm{T}}]^{\mathrm{T}}$, and θ denotes the mode of the integrated system determined by θ_1 and θ_2. Besides,

$$A_\theta := \begin{bmatrix}A_{1,\theta_1} & 0\\ 0 & A_{2,\theta_2}\end{bmatrix}, \text{ and } \mathrm{B}_\theta := \begin{bmatrix}B_{1,\theta_1} & 0\\ 0 & B_{2,\theta_2}\end{bmatrix}.$$

In Section III, we focus on designing the stabilizing controller of the integrated system (4) in a centralized fashion. These results will facilitate the distributed stabilizing controller design of system (1) in Section IV.

III. Stochastic Stability Analysis and Control of the Integrated MJLS

In this section, we analyze the stability of the integrated system (5) and derive its state-feedback stabilizing control. Note that there exists exact correspondences between (1) and (5) in terms of system parameters including transition probabilities and observation probabilities. To ease the presentation, we redefine the notations of critical variables succinctly for the integrated system (5).

Recall that the integrated system mode θ is determined by $\theta_1 \in \mathscr{S}_1$ and $\theta_2 \in \mathscr{S}_2$. Then, we define the finite set $\mathscr{S} := \{1, 2, \dots, |\mathscr{S}_1||\mathscr{S}_2|\}$, which contains all the possible system modes $\theta(t)$ of (5). Furthermore, let $\mathscr{C}^1, \mathscr{C}^2, \dots, \mathscr{C}^M$ be nonempty and disjoint sets, and $\cup_{m\in\mathscr{M}}\mathscr{C}^m$ denotes the space containing all the possible states of $x(t)$, where $\mathscr{M} := \{1, 2, \dots, M\}$. As $x(t)$ contains both subsystems' states, its partition into M sets is based on the corresponding partitions of $x_1(t)$ and $x_2(t)$, and thus $M = M_1M_2$. Similar to (2) and (3), the transition probabilities of system mode $\theta(t)$ in (5) are given by

$$\Pr[\theta(t+\Delta) = j|\theta(t) = i, x(t) \in \mathscr{C}^m] = \begin{cases}\gamma_{ij}^m\Delta + o(\Delta) & \text{if } i \neq j,\\ 1 + \gamma_{ij}^m\Delta + o(\Delta) & \text{otherwise,}\end{cases} \tag{6}$$

where the transition rates for the Markov jump process $\theta(t)$ are denoted by $\{\gamma_{ij}^m\}_{i,j\in\mathscr{S}}$.

Let $\hat{\theta}(t)$ denote the observation of the integrated system determined by $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$. At time t, given the state $x(t) \in \mathscr{C}^m$ and the system mode $\theta(t) = i \in \mathscr{S}$, we denote the probability of observing $\hat{\theta}(t) = \hat{i} \in \hat{\mathscr{S}}$ by $\alpha_{i\hat{i}}^m$, i.e.,

$$\Pr\Big[\hat{\theta}(t) = \hat{i}\Big|\theta(t) = i, x(t) \in \mathscr{C}^m\Big] = \alpha_{i\hat{i}}^m, \tag{7}$$

where $\hat{\mathscr{S}}$ is the finite set that contains all the observations.

For each $m \in \mathscr{M}$, we define the following notations which will facilitate the controller design. Due to Assumption 2, $\hat{\mathscr{S}} = \mathscr{S}$, and $[\alpha_{i\hat{i}}^m]_{i,\hat{i}\in\mathscr{S}}$ is a square matrix. If $[\alpha_{i\hat{i}}^m]_{i,\hat{i}\in\mathscr{S}}$ is invertible, define

$$[\beta_{\hat{i}i}^m]_{\hat{i},i\in\mathscr{S}} = \Big([\alpha_{i\hat{i}}^m]_{i,\hat{i}\in\mathscr{S}}\Big)^{-1}. \tag{8}$$

Otherwise,

$$[\beta_{\hat{i}i}^m]_{\hat{i},i\in\mathscr{S}} = \Big([\alpha_{i\hat{i}}^m]_{i,\hat{i}\in\mathscr{S}}\Big)^{\dagger}, \tag{9}$$

where $(\cdot)^\dagger$ stands for the pseudo-inverse of a matrix. In (8) (resp. (9)), $\beta_{\hat{i}i}^m$ is the $(\hat{i}, i)$-th entry of the inverse (resp. pseudo-inverse) of the observation probability matrix formed by $\alpha_{i\hat{i}}^m$.

Next, the definition of stochastic stability of a system is given as follows.

Definition 1: The equilibrium point (i.e., origin) of system (1) is stochastically stable if for arbitrary $x(t_0) \in \mathbb{R}^{N_x}$, and $\theta(t_0) \in \mathscr{S}$,

$$\mathbb{E}\Big[\int_{t_0}^{\infty}|x(t)|^2dt\Big] < \infty.$$

In this section, we aim to design controllers such that system (5) is stochastically stable. As the system mode $\theta(t)$ cannot be observed directly, the control inputs can only be designed based on $\hat{\theta}(t)$ and $x(t)$. When $x(t) \in \mathscr{C}^m$ and $\hat{\theta}(t) = \hat{i}$, the control input is designed to take the following state-feedback linear form:

$$u(t) = G_{\hat{\theta}(t)}^m x(t). \tag{10}$$

As shown above, the control gain is dependent on the observation $\hat{\theta}(t)$ and the system state $x(t)$. We can rewrite the closed-loop system under control (10) as

$$\dot{x} = A_{\theta\hat{\theta}}^m x + D_\theta w,$$

where $A_{\theta\hat{\theta}}^m = A_\theta + B_\theta G_{\hat{\theta}}^m$.

Before deriving the stochastic stability criterion of the targeted system, we present Dynkin's formula in the following lemma.

Lemma 1 [11]: Let a random process $(x(t), \theta(t))$ be a Markov process, and its stopping times are denoted by $\tau_0, \tau_1, \dots,$ at step $0, 1, \dots,$, respectively. For Lyapunov function $V(x(t), \theta(t))$, the Dynkin's formula admits the following form:

$$\mathbb{E}[V(x(t), \theta(t))|x(t_0), \theta(t_0)] - V(x(t_0), \theta(t_0)) = \sum_{l=0}^{l^*}\mathbb{E}\Big[\int_{t\wedge\tau_l}^{t\wedge\tau_{l+1}}\mathscr{L}V(x(\upsilon), \theta(\upsilon))d\upsilon|x(t\wedge\tau_l), \theta(t\wedge\tau_l)\Big], \tag{11}$$

where $\tau_0 = 0$, $l = 0, 1, \ldots, l^*$, $l^* \in [0, \infty]$, $\tau_{l^*} \leq \infty$, and $\mathscr{L}V(x(t), \theta(t))$ is the infinitesimal generator given by

$$\mathscr{L}V(x(t), \theta(t)) = \lim_{\Delta \to 0} \frac{1}{\Delta}\Big\{\mathbb{E}[V(x(t+\Delta), \theta(t+\Delta))|x(t), \theta(t)] - V(x(t), \theta(t))\Big\}.$$

Specifically, we choose the Lyapunov function in the following quadratic form:

$$V(x(t), \theta(t)) = x^{\mathrm{T}}(t)P_{\theta(t)}x(t), \tag{12}$$

where $P_{\theta(t)}$ is a symmetric positive definite matrix.

Lemma 2: Assume that $x(t) \in \mathscr{C}^m$, $\theta(t) = i \in \mathscr{S}$, and $\hat{\theta}(t) = \hat{i} \in \mathscr{S}$. Then, the infinitesimal generator of V is equal to

$$\mathscr{L}V(x(t), \theta(t)) = x^{\mathrm{T}}(t)\left(P_i\bar{A}_i^m + \bar{A}_i^{m\mathrm{T}}P_i + \sum_{j\in\mathscr{S}} \gamma_{ij}^m P_j\right)x(t) + 2x^{\mathrm{T}}(t)P_iD_iw(t),$$

where $\bar{A}_i^m = \sum_{\hat{i}\in\mathscr{S}} \alpha_{i\hat{i}}^m A_{\hat{i}}^m$.

Proof: Due to page limit, the proof can be found in the full paper [20]. ■

The following theorem gives a sufficient condition that ensures the stochastic stability of the integrated MJLS under partial information.

Theorem 1: System (5) can be stochastically stabilized if there exist positive definite matrices X_i, Y_i^m, for all $i \in \mathscr{S}$, $m \in \mathscr{M}$, and $\kappa_i > 0$, satisfying

$$X_iA_i^{\mathrm{T}} + Y_i^{m\mathrm{T}}B_i^{\mathrm{T}} + A_iX_i + B_iY_i^m + \gamma_{ii}^m X_i + X_i\left(\sum_{j\in\mathscr{S}/\{i\}} \gamma_{ij}^m X_j^{-1}\right)X_i + \frac{1}{\kappa_i}D_i^{\mathrm{T}}D_i < 0. \tag{13}$$

By Schur complement lemma [19], (13) is equivalent to

$$\begin{bmatrix} \mathscr{E}_i^m & \Lambda_i^m \\ \star & -\mathscr{X}_i \end{bmatrix} < 0, \tag{14}$$

where

$$\mathscr{E}_i^m := X_i\bar{A}_i^{m\mathrm{T}} + Y_i^{m\mathrm{T}}B_i^{\mathrm{T}} + \bar{A}_i^mX_i + B_iY_i^m + \gamma_{ii}^mX_i + (1/\kappa_i)D_i^{\mathrm{T}}D_i,$$
$$\Lambda_i^m := [\sqrt{\gamma_{i1}^m}X_i, \ldots, \sqrt{\gamma_{i(i-1)}^m}X_i, \sqrt{\gamma_{i(i+1)}^m}X_i, \ldots, \sqrt{\gamma_{i|\mathscr{S}|}^m}X_i],$$
$$\mathscr{X}_i := \mathrm{diag}\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{|\mathscr{S}|}\}.$$

The control gain is given by $G_{\hat{i}}^m = \sum_{i\in\mathscr{S}} \beta_{\hat{i}i}^m Y_i^m X_i^{-1}$.

Proof: Due to page limit, the proof can be found in the full paper [20]. ■

In the case of full information where system's mode $\theta(t)$ is observable, we immediately have the following proposition.

Proposition 1: System (5) can be stochastically stabilized if there exist positive definite matrices X_i, Y_i, for all $i \in \mathscr{S}$, $m \in \mathscr{M}$, and $\kappa_i > 0$, satisfying

$$X_iA_i^{\mathrm{T}} + Y_i^{\mathrm{T}}B_i^{\mathrm{T}} + A_iX_i + B_iY_i + \gamma_{ii}^m X_i + X_i\left(\sum_{j\in\mathscr{S}/\{i\}} \gamma_{ij}^m X_j^{-1}\right)X_i + \frac{1}{\kappa_i}D_i^{\mathrm{T}}D_i < 0. \tag{15}$$

By Schur complement lemma [19], (15) is equivalent to

$$\begin{bmatrix} \mathscr{E}_i^m & \Lambda_i^m \\ \star & -\mathscr{X}_i \end{bmatrix} < 0, \tag{16}$$

where

$$\mathscr{E}_i^m := X_iA_i^{\mathrm{T}} + Y_i^{\mathrm{T}}B_i^{\mathrm{T}} + A_iX_i + B_iY_i^m + \gamma_{ii}^mX_i + (1/\kappa_i)D_i^{\mathrm{T}}D_i,$$
$$\Lambda_i^m := [\sqrt{\gamma_{i1}^m}X_i, \ldots, \sqrt{\gamma_{i(i-1)}^m}X_i, \sqrt{\gamma_{i(i+1)}^m}X_i, \ldots, \sqrt{\gamma_{i|\mathscr{S}|}^m}X_i],$$
$$\mathscr{X}_i := \mathrm{diag}\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{|\mathscr{S}|}\}.$$

Then, when the system mode is i, the control gain of the system is given by $\tilde{G}_i = Y_iX_i^{-1}$.

Sketch of Proof: Note that in the fully observable case, for all $m \in \mathscr{M}$, we obtain

$$\alpha_{\hat{i}\hat{i}}^m = \begin{cases} 1 & \text{when } i = \hat{i}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the proposition is an immediate result from Theorem 1.

IV. Distributed Stabilization of the Interdependent MJLSs

In this section, we focus on (1) which includes two interdependent MJLSs. In Section III, we have studied the stability of the integrated MJLS which requires to know global system's state information. However, due to the distributed structure and different types of the jump systems, obtaining the overall system's information is not always possible/convenient. Thus, to enable the distributed control of the interdependent Markov jump systems, we aim to investigate the criterion that leads to the stochastic stability of each individual system in this section.

Similar to (8) and (9), for $k = 1, 2$, when $[\alpha_{i_k\hat{i}_k}^{k,m_k}]_{i_k,\hat{i}_k\in\mathscr{S}_k}$ is invertible, we define

$$[\beta_{\hat{i}_ki_k}^{k,m_k}]_{\hat{i}_k,i_k\in\mathscr{S}_k} = \left([\alpha_{i_k\hat{i}_k}^{k,m_k}]_{i_k,\hat{i}_k\in\mathscr{S}_k}\right)^{-1}.$$

Otherwise,

$$[\beta_{\hat{i}_ki_k}^{k,m_k}]_{\hat{i}_k,i_k\in\mathscr{S}_k} = \left([\alpha_{i_k\hat{i}_k}^{k,m_k}]_{i_k,\hat{i}_k\in\mathscr{S}_k}\right)^{\dagger}.$$

Furthermore, similar to (10), when $x_1(t) \in \mathscr{C}_1^{m_1}$, $x_2(t) \in \mathscr{C}_2^{m_2}$, the controllers for the two interdependent MJLSs are given by the following state-feedback form:

$$u_k(t) = G_{k,\hat{\theta}_k(t)}^{m_1,m_2}x_k(t), \quad k = 1, 2. \tag{17}$$

That is, the control gain of System k is dependent on the observation $\hat{\theta}_k(t)$ and the state pair $(x_1(t), x_2(t))$.

Before proceeding to the main result of this section, we give the following corollary, which presents how the individual stabilizing control of each system can lead to a stable integrated system.

Corollary 1: The stochastic stability of both MJLSs ensures a stochastically stable integrated system. In addition, for $x_1 \in \mathscr{C}_1^{m_1}$, $m_1 \in \mathscr{M}_1$, and $x_2 \in \mathscr{C}_2^{m_2}$, $m_2 \in \mathscr{M}_2$, the stabilizing control gains $G_{2,\hat{i}_2}^{m_1,m_2}$ and $G_{2,\hat{i}_2}^{m_1,m_2}$, for all $\hat{i}_1 \in \mathscr{S}_1$ and $\hat{i}_2 \in \mathscr{S}_2$, of individual System 1 and System 2 lead to a stable integrated interdependent MJLS (4).

Proof: Due to page limit, the proof can be found in the full paper [20]. ∎

The following theorem provides sufficient conditions for the integrated MJLS under the stabilizing controllers designed in a distributed fashion.

Theorem 2: The integrated MJLS is stochastically stabilized if there exist positive definite matrices $X_{k,i_k} > 0$, $Y_{k,i_k}^{m_1,m_2} > 0$, for all $i_k \in \mathscr{S}_k$, $m_k \in \mathscr{M}_k$, and $\kappa_{k,i_k} > 0$, $k = 1, 2$, satisfying

$$X_{1,i_1}A_{1,i_1}^{\mathrm{T}} + Y_{1,i_1}^{m_1,m_2\mathrm{T}}B_{1,i_1}^{\mathrm{T}} + A_{1,i_1}X_{1,i_1} + B_{1,i_1}Y_{1,i_1}^{m_1,m_2} + \lambda_{i_1i_1}^{m_2}X_{1,i_1} + X_{1,i_1}\left(\sum_{j_1\in\mathscr{S}_1/\{i_1\}}\lambda_{i_1j_1}^{m_2}(X_{1,j_1})^{-1}\right)X_{1,i_1} + \frac{1}{\kappa_{1,i_1}}D_{1,i_1}^{\mathrm{T}}D_{1,i_1} < 0,$$

$$X_{2,i_2}A_{2,i_2}^{\mathrm{T}} + Y_{2,i_2}^{m_1,m_2\mathrm{T}}B_{2,i_2}^{\mathrm{T}} + A_{2,i_2}X_{2,i_2} + B_{2,i_2}Y_{2,i_2}^{m_1,m_2} + \mu_{i_2i_2}^{m_1}X_{2,i_2} + X_{2,i_2}\left(\sum_{j_2\in\mathscr{S}_2/\{i_2\}}\mu_{i_2j_2}^{m_1}(X_{2,j_2})^{-1}\right)X_{2,i_2} + \frac{1}{\kappa_{2,i_2}}D_{2,i_2}^{\mathrm{T}}D_{2,i_2} < 0,$$

which is equivalent to

$$\begin{bmatrix} \mathscr{E}_{1,i_1}^{m_1,m_2} & \Lambda_{1,i_1}^{m_2} \\ \star & -\mathscr{X}_{1,i_1} \end{bmatrix} < 0, \quad \text{and} \quad \begin{bmatrix} \mathscr{E}_{2,i_2}^{m_1,m_2} & \Lambda_{2,i_2}^{m_1} \\ \star & -\mathscr{X}_{2,i_2} \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\mathscr{E}_{1,i_1}^{m_1,m_2} &:= X_{1,i_1}A_{1,i_1}^{\mathrm{T}} + Y_{1,i_1}^{m_1,m_2\mathrm{T}}B_{1,i_1}^{\mathrm{T}} + A_{1,i_1}X_{1,i_1} \\
&\quad + B_{1,i_1}Y_{1,i_1}^{m_1,m_2} + \lambda_{i_1i_1}^{m_2}X_{1,i_1} + (1/\kappa_{1,i_1})D_{1,i_1}^{\mathrm{T}}D_{1,i_1}, \\
\mathscr{E}_{2,i_2}^{m_1,m_2} &:= X_{2,i_2}A_{2,i_2}^{\mathrm{T}} + Y_{2,i_2}^{m_1,m_2\mathrm{T}}B_{2,i_2}^{\mathrm{T}} + A_{2,i_2}X_{2,i_2} \\
&\quad + B_{2,i_2}Y_{2,i_2}^{m_1,m_2} + \mu_{i_2i_2}^{m_1}X_{2,i_2} + (1/\kappa_{2,i_2})D_{2,i_2}^{\mathrm{T}}D_{2,i_2}, \\
\Lambda_{1,i_1}^{m_2} &:= [\sqrt{\lambda_{i_11}^{m_2}}X_{1,i_1}, \ldots, \sqrt{\lambda_{i_1(i_1-1)}^{m_2}}X_{1,i_1}, \\
&\quad \sqrt{\lambda_{i_1(i_1+1)}^{m_2}}X_{1,i_1}, \ldots, \sqrt{\lambda_{i_1|\mathscr{S}_1|}^{m_2}}X_{1,i_1}], \\
\Lambda_{2,i_2}^{m_1} &:= [\sqrt{\mu_{i_21}^{m_1}}X_{2,i_2}, \ldots, \sqrt{\mu_{i_2(i_2-1)}^{m_1}}X_{2,i_2}, \\
&\quad \sqrt{\mu_{i_2(i_2+1)}^{m_1}}X_{2,i_2}, \ldots, \sqrt{\mu_{i_2|\mathscr{S}_2|}^{m_1}}X_{2,i_2}], \\
\mathscr{X}_{1,i_1} &:= \mathrm{diag}\{X_{1,1}, \ldots, X_{1,i_1-1}, X_{1,i_1+1}, \ldots, X_{1,|\mathscr{S}_1|}\}, \\
\mathscr{X}_{2,i_2} &:= \mathrm{diag}\{X_{2,1}, \ldots, X_{2,i_2-1}, X_{2,i_2+1}, \ldots, X_{2,|\mathscr{S}_2|}\}.
\end{aligned}$$

Moreover, the control gain for System k is

$$G_{k,\hat{i}_k}^{m_1,m_2} = \sum_{i_k\in\mathscr{S}_k}\beta_{\hat{i}_ki_k}^{k,m_k}Y_{k,i_k}^{m_1,m_2}(X_{k,i_k})^{-1}, \quad k = 1, 2,$$

for all $\hat{i}_k \in \mathscr{S}_k$.

Proof: The proof follows immediately from Theorem 1 and Corollary 1. ∎

Remark: By comparing the designed stabilizing controllers in Sections III and IV, we can find that the number of controllers is different in these two scenarios. Specifically, it requires $M_1M_2|\mathscr{S}_1||\mathscr{S}_2|$ controllers using the centralized design method (Section III), while the distributed one reduces it to $M_1M_2(|\mathscr{S}_1| + |\mathscr{S}_2|)$ (Section IV), which simplifies the complexity of the control design.

V. Numerical Experiments

In this section, we present a numerical example to illustrate the obtained analytical results. The parameters of the system are $\theta_1 \in \mathscr{S}_1 = \{1, 2\}$ and $\theta_2 \in \mathscr{S}_2 = \{1, 2, 3\}$. The system matrices of the independent MJLSs are given as follows:

$$A_{1,1} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}, A_{1,2} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix},$$

$$B_{1,1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B_{1,2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, A_{2,1} = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 2 & 6 \\ -9 & 0 & 2 \end{bmatrix},$$

$$A_{2,2} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 5 & 6 & 3 \end{bmatrix}, A_{2,3} = \begin{bmatrix} 4 & -1 & 8 \\ 5 & 8 & 0 \\ -1 & 7 & 5 \end{bmatrix},$$

$$B_{2,1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, B_{2,2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, B_{2,3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

In addition, the transition rate matrices are

$$\lambda^1 = \begin{bmatrix} -0.6 & 0.6 \\ -0.4 & 0.4 \end{bmatrix}, \lambda^2 = \begin{bmatrix} -0.2 & 0.2 \\ -0.8 & 0.8 \end{bmatrix}, \lambda^3 = \begin{bmatrix} -0.5 & 0.5 \\ -1.2 & 1.2 \end{bmatrix},$$

$$\mu^1 = \begin{bmatrix} -0.8 & 0.2 & 0.6 \\ 0.2 & -0.9 & 0.7 \\ 0.5 & 0.4 & -0.9 \end{bmatrix}, \mu^2 = \begin{bmatrix} -0.4 & 0.2 & 0.2 \\ 0.2 & -0.5 & 0.4 \\ 0.5 & 0.6 & -1.1 \end{bmatrix}.$$

Specifically, λ^1, λ^2 and λ^3 are transition rate matrices of System 1 under the condition that $x_2 \in \mathscr{C}_2^1 = \{x_2 : |x_2|^2 < 5\}$, $x_2 \in \mathscr{C}_2^2 = \{x_2 : 5 \leq |x_2|^2 \leq 10\}$, and $x_2 \in \mathscr{C}_2^3 = \{x_2 : |x_2|^2 > 10\}$, respectively. Similarly, μ^1 and μ^2 are transition rate matrices of System 2 under the condition that $x_1 \in \mathscr{C}_1^1 = \{x_1 : |x_1|^2 < 10\}$, and, $x_1 \in \mathscr{C}_1^2 = \{x_1 : |x_1|^2 \geq 10\}$, respectively.

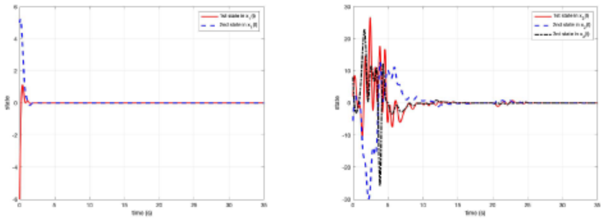
Moreover, the observation matrices of System 1 and System 2 are given by $P^{m_1} = [\alpha_{i_1\hat{i}_1}^{1,m_1}]_{i_1,\hat{i}_1\in\mathscr{S}_1}$, $m_1 = 1, 2$. and $Q^{m_2} = [\alpha_{i_2\hat{i}_2}^{2,m_2}]_{i_2,\hat{i}_2\in\mathscr{S}_2}$, $m_2 = 1, 2, 3$, respectively, with matrices taking the following forms:

$$P^1 = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}, P^2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix},$$

$$Q^1 = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, Q^2 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix},$$

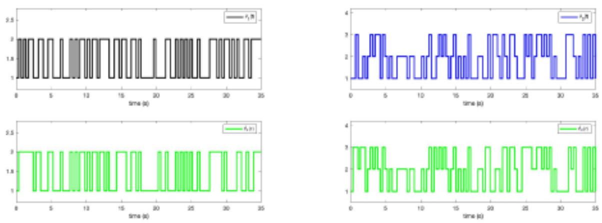
$$Q^3 = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}.$$

With the designed controllers based on solving LMIs in Theorem 2, Fig. 2 shows the state trajectories of the interdependent systems with the initial conditions $x_1(0) = [-6, 5]^{\mathrm{T}}$, and $x_2(0) = [2, -5.5, 8]^{\mathrm{T}}$. Fig. 3 depicts the sampled Markov chains of the underlying parameters $\theta_1(t)$ and $\theta_2(t)$, and their observations $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$, respectively. For comparison, Fig. 4 illustrates the results with the control designed under complete observations. With a perfect knowledge on the system mode, the state trajectories in Fig. 4 are relatively smoother and reach the steady state faster than those in Fig. 2. However, the advantage of the designed distributed control strategy lies in the fact that, though the systems' modes are



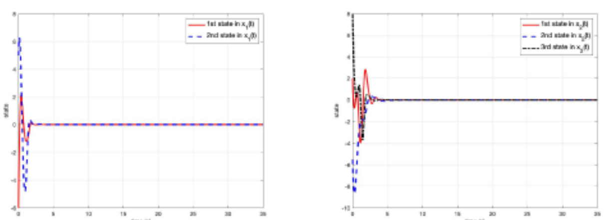
(a) System 1's States (b) System 2's States

Fig. 2. (a) and (b) show the stabilized state trajectories of System 1 and System 2, respectively, with the control designed under partial observation.



(a) System 1's Mode (b) System 2's Mode

Fig. 3. The sampled Markov chains of System 1 and System 2, respectively.



(a) System 1's States (b) System 2's States

Fig. 4. (a) and (b) show the stabilized state trajectories of System 1 and System 2, respectively, with the control designed under full observation.

not directly observable, it can still stabilize the interdependent MJLSs with satisfactory performance as shown in Fig. 2.

VI. Conclusion

In this letter, we have studied the interdependent multiple MJLSs. We have designed distributed stabilizing controllers for each MJLS with partial information, which only require the system state information and indirect observations of the local mode. In addition, these designed controllers can stabilize the integrated Markov jump system. The distributed feature of these controllers reduces the information exchange and communication costs among different Markov jump systems. The future work would extend the stabilizing control design to an optimal control framework considering state and control costs for the coupled MJLSs under incomplete information.

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