

On Numerical Examination of Uniform Ensemble Controllability for Linear Ensemble Systems

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Abstract—In this paper, we propose a numerical approach to examine uniform ensemble controllability of linear ensemble systems. We show that the linear ensemble defined on the Banach space of compactly supported continuous functions is uniformly ensemble controllable if the differentiation set associated with the ensemble is dense, and only if the reachable set is dense, in the L^2 -space. We also demonstrate that under certain conditions, L^2 -denseness of the differentiation set is necessary for uniform ensemble controllability of a linear ensemble system. Then, we provide a tractable numerical method to test the denseness of an arbitrary set in Hilbert space with a quantifiable error bound, which informs uniform ensemble controllability. We conduct several numerical experiments to illustrate the efficacy and robustness of the proposed numerical approach.

Index Terms—Computational methods, Large-scale systems, Linear systems

I. INTRODUCTION

THE problem of controlling a population of structurally identical systems using a broadcast signal, known as the ensemble control problem, has recently attracted considerable attention due to its unexplored mathematical structure and intimate relevance to numerous practical applications ranging from quantum physics [1]–[3] and neuroscience [4]–[6] to robotics [7]. In past decades, a research focus in ensemble control theory was placed on investigating fundamental properties of ensemble systems, such as ensemble controllability and ensemble observability [8]–[12]. Various specialized techniques based on polynomial approximation [9], separating points [13], representation theory [14], complex functional analysis [15]–[17], statistical moment-based approaches [18], [19] and convex-geometric approaches [20] were developed to analyze these properties; however, most of these theoretical developments were not suitable or tractable for numerical implementation and may lead to unreliable or even erroneous numerical results.

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The major challenge for numerically examining uniform ensemble controllability arises from the nature of the underlying state space of the ensemble, which is in general a Banach space of compactly supported continuous functions. In a Banach space, the sup-norm of a function cannot be evaluated numerically with a quantifiable error bound through discretizing the support of the function. In this paper, we provide a computationally tractable procedure to study uniform ensemble controllability of linear ensemble systems. Our main idea is to cast the problem of checking uniform ensemble controllability on a Banach space as the problem of verifying the denseness of a set in the L^2 -space, which is tractable by leveraging additional geometric structures inherited in Hilbert space.

This paper is organized as follows. In Section II, we provide preliminaries of ensemble systems and uniform ensemble controllability defined on Banach space. In Section III, we show how the problem of analyzing uniform ensemble controllability can be transformed to the problem of evaluating denseness in the L^2 -space, so that numerical examination of uniform ensemble controllability becomes possible. In Section IV, we present several numerical experiments showing the efficacy and robustness of the proposed numerical method.

II. PRELIMINARIES

In this section, we introduce ensemble systems defined on a Banach space and the notion of uniform ensemble controllability. Specifically, we consider the ensemble of linear systems indexed by the parameter $\beta \in K$ for $K = [\beta_1, \beta_2] \subset \mathbb{R}$, given by

$$\frac{d}{dt}X(t, \beta) = A(\beta)X(t, \beta) + B(\beta)u(t), \quad (1)$$

where $X(t, \cdot) \in C(K, \mathbb{R}^n)$, a continuous function over K , is the state; $A(\cdot) \in C^1(K, \mathbb{R}^{n \times n})$ and $B(\cdot) \in C^1(K, \mathbb{R}^{n \times m})$ are matrices of continuously differentiable functions over K ; $u : [0, T] \rightarrow \mathbb{R}^m$ is the piecewise constant control signal.

Definition 1 (Uniform Ensemble Controllability). The ensemble in (1) is said to be *uniformly ensemble controllable* on $C(K, \mathbb{R}^n)$ if for any $X_0(\cdot), X_F(\cdot) \in C(K, \mathbb{R}^n)$ and $\epsilon > 0$, there exists a piecewise constant control signal $u(t)$ that steers the ensemble from $X_0(\cdot)$ to be within an ϵ -neighborhood of $X_F(\cdot)$ at time T with respect to the sup-norm, i.e., $\sup_{\beta \in K} |X_F(\beta) - X(T, \beta)| < \epsilon$.

Throughout this paper, we denote $\text{span}(S)$ as the *linear span* of a set S of vectors in a vector space, i.e., the set of all

finite linear combinations of vectors in S given by,

$$\text{span}(S) := \left\{ \sum_{i=1}^k \lambda_i v_i : k \in \mathbb{N}_+, v_i \in S, \lambda_i \in \mathbb{R} \right\}.$$

Given two sets S and V of vectors in the same vector space, we denote $S + V$ as the sum of these two sets, namely,

$$S + V := \{s + v : s \in S, v \in V\}.$$

Additionally, we say S is dense in V with respect to a metric d (or d -dense in V as abbreviation), if for any $v \in V$ and $\epsilon > 0$, there exists $s \in S$ such that $d(s, v) < \epsilon$.

Also, we denote $\|\cdot\|_2$ and $\|\cdot\|_\infty$ as the 2-norm and the sup-norm in Euclidean space, respectively, defined by

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}, \quad \|x\|_\infty = \sup_{i \in [1:n]} |x_i|,$$

where x_i is the i^{th} component of $x \in \mathbb{R}^n$ and $[1:n]$ denotes the set $\{1, \dots, n\}$. Given a compact set $K \subset \mathbb{R}$, we denote $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ as the L^2 -norm in $L^2(K, \mathbb{R}^n)$ and the sup-norm in $C(K, \mathbb{R}^n)$, respectively, characterized by

$$\|f\|_{L^2} = \left(\sum_{i=1}^n \int_K |f_i(s)|^2 ds \right)^{\frac{1}{2}},$$

$$\|f\|_{L^\infty} = \sup_{s \in K, i \in [1:n]} |f_i(s)|,$$

where f_i is the i^{th} component of f . The inner product in $L^2(K, \mathbb{R}^n)$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$.

III. METHODOLOGY

In this section, we present the idea of analyzing uniform ensemble controllability in the context of denseness in the L^2 -space, which can be evaluated numerically via orthogonal projections in Hilbert space.

A. From Denseness in Banach Space to Denseness in Hilbert Space

The most straightforward way to study uniform ensemble controllability of the ensemble in (1) is to compare its reachable set with the state space $C(K, \mathbb{R}^n)$. It is known that the ensemble in (1) is uniformly ensemble controllable on $C(K, \mathbb{R}^n)$ if and only if its reachable set, characterized by

$$\mathcal{L} = \text{span} \{A^k(\cdot)B(\cdot) : k = 0, 1, 2, \dots\}, \quad (2)$$

is dense in $C(K, \mathbb{R}^n)$ with respect to the topology induced by the sup-norm. However, it is in general difficult to test numerically whether $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$ since the sup-norm cannot be evaluated with a quantifiable error bound, through discretization over K , which is the primary step when implementing an existing algorithm on a computer for checking uniform ensemble controllability.

To resolve this bottleneck, we devise a tractable framework for studying uniform ensemble controllability numerically. An essential step is to evaluate the denseness of $\mathcal{L}$ in $C(K, \mathbb{R}^n)$ through a Hilbert-space setting. In the following, we provide a necessary and a sufficient condition for checking the denseness

of $\mathcal{L}$ with respect to the sup-norm using the denseness of sets in the L^2 -space. For ease of exposition, we denote $\mathcal{L}_0$ as the set of constant $\mathbb{R}^n$ -valued functions over K .

Lemma 2. *Let $K = [\beta_1, \beta_2] \subset \mathbb{R}$. Consider $\tilde{\mathcal{L}} := \mathcal{L}_0 \oplus \text{span} \{P_k(\cdot) : k = 0, 1, 2, \dots\}$ with $P_k(\cdot) \in C^1(K, \mathbb{R}^n)$ satisfying $P_k(\beta_0) = 0$ for some fixed $\beta_0 \in K$. Then $\tilde{\mathcal{L}}$ is dense in $C(K, \mathbb{R}^n)$ with respect to the sup-norm if $\text{span} \{P'_k(\cdot) : k = 0, 1, 2, \dots\}$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm.*

Proof. We will first prove this lemma for the one-dimensional case, i.e., $n = 1$. Since the space of polynomials is dense in $C(K, \mathbb{R})$ with respect to the sup-norm, to show that $\tilde{\mathcal{L}}$ is dense, it suffices to prove that a basis of all polynomials, e.g., $\{H_\alpha(\beta) := (\beta - \beta_0)^\alpha : \alpha = 0, 1, \dots\}$, is contained in $\text{cl}(\tilde{\mathcal{L}})$, where $\text{cl}(\tilde{\mathcal{L}})$ denotes the closure of $\tilde{\mathcal{L}}$ under the sup-norm. We first note that $\mathbb{1} = H_0 \in \text{cl}(\tilde{\mathcal{L}})$ since $\mathcal{L}_0 \subset \tilde{\mathcal{L}}$. For monomials H_α with $\alpha \geq 1$, we observe that for any $\beta \in K$, $a_0, \dots, a_N \in \mathbb{R}$, it holds that

$$\begin{aligned} & |H_\alpha(\beta) - \sum_{k=0}^N a_k P_k(\beta)| \\ &= \left| \int_{\beta_0}^{\beta} [\alpha(s - \beta_0)^{\alpha-1} - \sum_{k=0}^N a_k P'_k(s)] ds \right| \\ &\leq \int_{\beta_0}^{\beta} |\alpha(s - \beta_0)^{\alpha-1} - \sum_{k=0}^N a_k P'_k(s)| ds \\ &\leq |K|^{\frac{1}{2}} \left(\int_{\beta_0}^{\beta} |\alpha(s - \beta_0)^{\alpha-1} - \sum_{k=0}^N a_k P'_k(s)|^2 ds \right)^{\frac{1}{2}} \quad (3) \end{aligned}$$

$$\leq |K|^{\frac{1}{2}} \cdot \left\| \alpha H_{\alpha-1}(\cdot) - \sum_{k=0}^N a_k P'_k(\cdot) \right\|_{L^2}, \quad (4)$$

where $|K| = \beta_2 - \beta_1$ and (3) comes from Hölder's inequality. Taking supremum of β over K on both sides of (4) yields

$$\begin{aligned} & \|H_\alpha(\cdot) - \sum_{k=0}^N a_k P_k(\cdot)\|_{L^\infty} \\ &\leq |K|^{\frac{1}{2}} \cdot \left\| \alpha H_{\alpha-1}(\cdot) - \sum_{k=0}^N a_k P'_k(\cdot) \right\|_{L^2}. \quad (5) \end{aligned}$$

Now if the span of $\{P'_k(\cdot) : k = 0, 1, 2, \dots\}$ is dense in $L^2(K, \mathbb{R})$ with respect to the L^2 -norm, then the right-hand side of (5) can be made arbitrarily small, which concludes that $\text{cl}(\tilde{\mathcal{L}})$ contains H_α , for any $\alpha \geq 1$.

For the cases with higher dimensions, i.e., $n \geq 2$, following the same logic above, it suffices to replace the term H_α in above equations with $H_\alpha e_i$, where $e_i \in \mathbb{R}^n$ denotes the i^{th} vector in the standard basis of $\mathbb{R}^n$. So for any $\beta \in K$ and $a_0, \dots, a_N \in \mathbb{R}$, we have that

$$\begin{aligned} & \|H_\alpha(\cdot) e_i - \sum_{k=0}^N a_k P_k(\cdot)\|_\infty \\ &\leq |K|^{\frac{1}{2}} \sup_{j \in [1:n]} \left(\int_{\beta_0}^{\beta} |\{\alpha H_{\alpha-1}(s) e_i - \sum_{k=0}^N a_k P'_k(s)\}_j|^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

$$\leq |K|^{\frac{1}{2}} \cdot \left\| \alpha H_{\alpha-1}(\cdot) e_i - \sum_{k=0}^N a_k P'_k(\cdot) \right\|_{L^2}, \quad (6)$$

where $\{\cdot\}_j$ denotes the j^{th} component of an $\mathbb{R}^n$ -valued vector. Hence, taking supremum of β over K on both sides of (6) yields (5) for multi-dimensional case so that the proof is concluded. $\square$

Lemma 3. Let $K = [\beta_1, \beta_2] \subset \mathbb{R}$ and $\mathcal{L}$ be defined as in (2). Then, $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$ with respect to the sup-norm only if it is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm.

Proof. If $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$, then for any $f \in C(K, \mathbb{R}^n)$, there exists a sequence of functions $\{g^m\}_{m=1}^\infty \subset \mathcal{L}$ such that $\|f - g^m\|_{L^\infty} \rightarrow 0$ as $m \rightarrow \infty$. Hence, for the L^2 -norm of $f - g^m$, we have:

$$\begin{aligned} \|f - g^m\|_{L^2} &= \left(\sum_{j=1}^n \int_K |f_j(\beta) - g_j^m(\beta)|^2 d\beta \right)^{1/2} \\ &\leq \sqrt{n|K|} \|f - g^m\|_{L^\infty} \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, which suggests that $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$ with respect to the L^2 -norm. Therefore, considering the fact that $C(K, \mathbb{R}^n)$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm, we conclude that $\mathcal{L}$ is L^2 -dense in $L^2(K, \mathbb{R}^n)$. $\square$

Following Lemma 2, we introduce the notion of *differentiation set*, which helps verify uniform-denseness of $\mathcal{L}$, and thus examine uniform ensemble controllability of the ensemble in (1).

Definition 4 (Differentiation Set). Given the ensemble in (1), the set

$$\mathcal{L}' := \text{span} \left\{ \frac{d}{d\beta} (A^k b_j) : j = 1, \dots, m; k = 1, 2, \dots \right\} \quad (7)$$

is called the *differentiation set* associated with the ensemble in (1), where $b_j(\beta) \in \mathbb{R}^n$ denotes the j^{th} column of $B(\beta)$.

Remark 1. Note that compared to the reachable set $\mathcal{L}$ in (2), the index k in (7) is taken from 1 instead of 0.

Now let us take a look back on the linear ensemble in (1). When its drift and control vector fields satisfy $A(\beta_0) = 0$ for a fixed $\beta_0 \in K$ and $B(\beta_0)$ has full row-rank, respectively, we can define $P_{kj}(\cdot) := A^k(\cdot) b_j(\cdot)$ so that $P_{kj}(\beta_0) = 0$ for any $j = 1, \dots, m; k = 1, 2, \dots$. It then follows from Lemma 2 that this ensemble is uniformly ensemble controllable on $C(K, \mathbb{R}^n)$ if its differentiation set is dense in $L^2(K, \mathbb{R}^n)$. On the other hand, by the contraposition of Lemma 3, the ensemble is not uniform ensemble controllable on $C(K, \mathbb{R}^n)$ if $\mathcal{L}$ is not L^2 -dense in $L^2(K, \mathbb{R}^n)$. In summary, we have the following theorem establishing a connection between the uniform ensemble controllability of a linear ensemble and L^2 -denseness of its reachable set and differentiation set.

Theorem 5. Consider the ensemble of linear systems in (1) with its reachable set $\mathcal{L}$ defined as in (2) and its differentiation set $\mathcal{L}'$ defined as in (7). Assume that there exists $\beta_0 \in K$ such that $A(\beta_0) = 0$. Then, the ensemble in (1) is uniformly ensemble controllable on $C(K, \mathbb{R}^n)$

- (i) if $\mathcal{L}'$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm and $\text{rank } B(\beta_0) = n$;
- (ii) only if $\mathcal{L}$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm.

Proof. Since condition (ii) is straightforward to obtain from Lemma 3, we focus on proving condition (i) of this theorem.

Let $\mathcal{L}_0$ be the set of constant $\mathbb{R}^n$ -valued functions over K , and $\mathcal{C}_0 \subset C(K, \mathbb{R}^n)$ denote the set of continuous functions over K that vanish at β_0 , i.e., $\mathcal{C}_0 := \{h \in C(K, \mathbb{R}^n) : h(\beta_0) = 0\}$. If $\mathcal{L}'$ is dense in $L^2(K, \mathbb{R}^n)$, then as a result of Lemma 2, it holds that the set $\tilde{\mathcal{L}} = \mathcal{L}_0 \oplus \mathcal{L}_1$, where $\mathcal{L}_1 := \text{span}\{A^k b_j : j = 1, \dots, m; k = 1, 2, \dots\}$, is dense in $C(K, \mathbb{R}^n)$. We observe that all elements in $\tilde{\mathcal{L}}$ that vanish at $\beta_0 \in K$ must lie in $\mathcal{L}_1$. Now, if $\text{rank } B(\beta_0) = n$, there exists $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ such that $f(\beta_0) = \sum_{j=1}^m \gamma_j b_j(\beta_0)$. Then, we may approximate $\hat{f}(\cdot) := f(\cdot) - \sum_{j=1}^m \gamma_j b_j(\cdot) \in \mathcal{C}_0$ using elements in $\mathcal{L}_1$. Namely, for any $\epsilon > 0$, there exists $N > 0, c_{ij} \in \mathbb{R}$ such that

$$\left\| \hat{f} - \sum_{i=1}^N \sum_{j=1}^m c_{ij} A^i b_j \right\|_{L^\infty} < \epsilon. \quad (8)$$

Substituting $\hat{f} = f - \sum_{j=1}^m \gamma_j b_j$ into (8) yields $\|f - \sum_{j=1}^m \gamma_j b_j - \sum_{i=1}^N \sum_{j=1}^m c_{ij} A^i b_j\|_{L^\infty} < \epsilon$, which implies that f can be uniformly approximated by $\sum_{j=1}^m \gamma_j b_j + \sum_{i=1}^N \sum_{j=1}^m c_{ij} A^i b_j \in \mathcal{L}$, and hence concludes the proof. $\square$

Next, we demonstrate that under certain conditions, L^2 -denseness of $\mathcal{L}'$ is also necessary for the uniform-denseness of $\mathcal{L}$. Therefore, examining uniform controllability of a linear ensemble is fully equivalent to checking L^2 -denseness of its differentiation set.

Theorem 6. Consider the ensemble of linear systems in (1) with its reachable set $\mathcal{L}$ defined as in (2). Assume that A' is non-singular, A' commutes with A , i.e., $A'A = AA'$, and b is constant over K . Then, $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$ with respect to the sup-norm only if $\mathcal{L}'$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm.

Proof. Since $C(K, \mathbb{R}^n)$ is dense in $L^2(K, \mathbb{R}^n)$ with respect to the L^2 -norm, for any $g \in L^2(K, \mathbb{R}^n)$, there exists $f \in C(K, \mathbb{R}^n)$ such that $\|g - f\|_{L^2} < \epsilon$. Then, we prove that f can be approximated using elements in $\mathcal{L}'$ with respect to L^2 -norm. Since A' is non-singular over K , it suffices to show $A'f$ can be approximated using elements in $\mathcal{L}'$ under the L^2 -norm. This is because for any $h \in \mathcal{L}'$, it holds that

$$\begin{aligned} \|(A')^{-1}h - f\|_{L^2} &= \|(A')^{-1}(h - A'f)\|_{L^2} \\ &\leq \left(\int_K \|(A')^{-1}(\beta)\|_2^2 \cdot \|(h - A'f)(\beta)\|_2^2 d\beta \right)^{\frac{1}{2}} \\ &\leq \left(\sup_{\beta \in K} \|(A'(\beta))^{-1}\|_2 \right) \cdot \|h - A'f\|_{L^2}, \end{aligned}$$

where $\|(A'(\beta))^{-1}\|_2$ denotes the matrix 2-norm of $(A'(\beta))^{-1}$, and the supremum is finite since A' is non-singular over K . If $\mathcal{L}$ is dense in $C(K, \mathbb{R}^n)$ with respect to the sup-norm, then for any $f \in C(K, \mathbb{R}^n)$ and $\epsilon > 0$, there exist $N > 0, c_{kj} \in \mathbb{R}$ such that $\|f - \sum_{k=0}^N \sum_{j=1}^m c_{kj} A^k b_j\|_{L^\infty} < \epsilon$. Since

A' commutes with A and b is constant over K , we have $\frac{d}{d\beta}(A^k b_j) = \sum_{j=1}^k A^{k-j} A' A^{j-1} b_j = k A' A^{k-1} b_j$. Hence, it holds that

$$\begin{aligned} \|A'(f - \sum_{k=0}^N \sum_{j=1}^m c_{kj} A^k b_j)\|_{L^\infty} &= \|A'f - \sum_{k,j} c_{kj} A' A^k b_j\|_{L^\infty} \\ &= \|A'f - \sum_{k=0}^N \sum_{j=1}^m \frac{c_{kj}}{k+1} \frac{d}{d\beta}(A^{k+1} b_j)\|_{L^\infty} < \epsilon M, \end{aligned}$$

where $M := \sup_{\beta \in K} \|A'(\beta)\|_\infty$ and $\|A'(\beta)\|_\infty$ denotes the matrix infinity-norm of $A'(\beta)$. Therefore, we have

$$\begin{aligned} \|A'f - \sum_{k=0}^N \sum_{j=1}^m \frac{c_k}{k+1} \frac{d}{d\beta}(A^{k+1} b_j)\|_{L^2} \\ \leq \left(n \int_K \|A'f - \sum_{k,j} \frac{c_k}{k+1} \frac{d}{d\beta}(A^{k+1} b_j)\|_{L^\infty}^2 d\beta \right)^{\frac{1}{2}} \\ \leq \epsilon M \sqrt{n|K|}, \end{aligned}$$

which implies that $A'f$ can be approximated by elements in $\mathcal{L}'$ in the L^2 -sense. $\square$

Remark 2. It is worthwhile to mention that, unlike Theorem 5, the above theorem does not require A to vanish somewhere on K . Whether the converse of Theorem 5(i) holds true without imposing additional assumptions remains an open problem to our best knowledge.

B. Numerical Examination of Denseness in Hilbert Space

So far, we have shown that the examination of uniform ensemble controllability of the ensemble in (1) can be achieved through checking the denseness of its reachable set and its differentiation set in $L^2(K, \mathbb{R}^n)$. This result makes it possible to numerically evaluate uniform ensemble controllability by leveraging the geometric structures inherited in a Hilbert space.

Lemma 7. *Let $\{\phi_i\}_{i=1}^\infty$ be a basis of $L^2(K, \mathbb{R}^n)$ and $\mathcal{S} := \text{span}\{P_k, k = 0, 1, \dots\}$ with $P_k \in L^2(K, \mathbb{R}^n)$. Then, $\mathcal{S}$ is dense in $L^2(K, \mathbb{R}^n)$ if and only if*

$$\lim_{k \rightarrow \infty} \|(I_d - Q_k^*(Q_k^* Q_k)^{-1} Q_k) \phi_i\|_2 = 0,$$

for all $i = 1, 2, \dots$, where $Q_k, k = 1, 2, \dots$, is a linear functional defined by

$$Q_k : L^2(K, \mathbb{R}^n) \rightarrow \mathbb{R}^k, \quad \phi_i \mapsto \begin{pmatrix} \langle P_1, \phi_i \rangle_{L^2} \\ \vdots \\ \langle P_k, \phi_i \rangle_{L^2} \end{pmatrix} \quad (9)$$

where Q_k^* is the adjoint operator of Q_k ; and I_d is the identity operator in $L^2(K, \mathbb{R}^n)$.

Proof. Let us consider a finite truncation of $\mathcal{S}$, denoted as $\mathcal{S}_k = \text{span}\{P_j, j = 0, 1, \dots, k\}$. It is known that the distance between ϕ_i and $\mathcal{S}_k$, denoted as $r_k(\phi_i)$ can be explicitly written as

$$\begin{aligned} r_k(\phi_i) &:= \min_{v \in \mathcal{S}_k} \|\phi_i - v\|_{L^2} \\ &= \|(I_d - Q_k^*(Q_k^* Q_k)^{-1} Q_k) \phi_i\|_2, \end{aligned}$$

where Q_k is defined as in (9). Therefore, $\phi_i \in \text{cl}(\mathcal{S})$ if and only if $\lim_{k \rightarrow \infty} r_k(\phi_i) = 0$, which concludes the proof. $\square$

Following Theorems 5, 6, and Lemma 7, a pseudo-code describing the procedure to test uniform ensemble controllability of linear ensembles with continuously differentiable drift and control vector fields $A(\cdot)$ and $B(\cdot)$ satisfying $A(\beta_0) = 0$ and $\text{rank } B(\beta_0) = n$ for some $\beta_0 \in K$, respectively, is presented in Algorithm 1, where $A^\top$ denotes the transpose of the matrix A .

Algorithm 1 Numerical test of uniform ensemble controllability

function IS_ENSEMBLE_CONTROLLABLE($A(\cdot), B(\cdot)$)

Initialize: $\{\phi_i\}_{i=1}^\infty$ as basis of $L^2(K, \mathbb{R}^n)$,

ϵ as error tolerance,

N_1, N_2 as positive integers.

for $i \leftarrow 1, 2, \dots, m$ **do**

Assign $b_i(\cdot)$ to be the i^{th} column of $B(\cdot)$.

Compute $A_i(\cdot) = [b_i(\cdot), A(\cdot)b_i(\cdot), \dots, A^{N_1}(\cdot)b_i(\cdot)]$ symbolically.

end for

Compute $L_1(\beta) = [A_1(\beta) \ \cdots \ A_m(\beta)]$,
and $L_2(\beta) = \frac{d}{d\beta} L_1(\beta)$.

Compute $M_1 = \int_K L_1^\top(\beta) L_1(\beta) d\beta$,
and $M_2 = \int_K L_2^\top(\beta) L_2(\beta) d\beta$.

for $i \leftarrow 1, 2, \dots, N_2$ **do:**

Compute $c_1 = M_1^{-1} \int_K L_1^\top(\beta) \phi_i(\beta) d\beta$,
and $c_2 = M_2^{-1} \int_K L_2^\top(\beta) \phi_i(\beta) d\beta$.

Compute the residuals as

$$\begin{aligned} r_1(\phi_i) &= \|\int_K [\phi_i(\beta) - L_1(\beta)c_1] d\beta\|_2, \\ r_2(\phi_i) &= \|\int_K [\phi_i(\beta) - L_2(\beta)c_2] d\beta\|_2. \end{aligned}$$

if $r_1(\phi_i) > \epsilon$ **then**

return False.

end if

if $r_2(\phi_i) > \epsilon$ **then**

if A' is non-singular, A' commutes with A ,
and b is constant **then**

return False.

else

return Unknown.

end if

end if

end for

return True.

end function

In Algorithm 1, we set an error tolerance $\epsilon > 0$; then truncate $\mathcal{L}$ and $L^2(K, \mathbb{R}^n)$ into $\mathcal{L}_{N_1} := \text{span}\{A^k b_j : k = 0, 1, \dots, N_1; j = 1, \dots, m\}$ and $\text{span}\{\phi_i : i = 1, \dots, N_2\}$, respectively. Although we can approximate the reachable set $\mathcal{L}$ using its truncation $\mathcal{L}_{N_1}$ arbitrarily well with sufficiently large N_1 , it remains to discuss the error of the ensemble controllability analysis due to the finite truncation of reachable set.

Let us define $\mathcal{L}_0 = \emptyset$ and $\mathcal{L}_k := \text{span}\{A^l b_j : l = 0, 1, \dots, k; j = 1, \dots, m\}$. Then, by the definition of $\mathcal{L}_k, k = 1, 2, \dots$, we have $\mathcal{L}_k \subset \mathcal{L}_{k+1}, k = 0, 1, 2, \dots$. Therefore, we

can decompose $\mathcal{L}_{k+1}$ into two subspaces that are orthogonal to each other, denoted as $\mathcal{L}_{k+1} = \mathcal{L}_k \oplus \mathcal{W}_k$, where $\oplus$ denotes the direct-sum of two sets. By iteratively applying this orthogonal decomposition, we have

$$\begin{aligned}\mathcal{L}_k &= \mathcal{L}_{k-1} \oplus \mathcal{W}_{k-1} = \mathcal{L}_{k-2} \oplus \mathcal{W}_{k-2} \oplus \mathcal{W}_{k-1} = \dots \\ &= \mathcal{L}_0 \oplus \mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{k-1} = \mathcal{W}_0 \oplus \dots \oplus \mathcal{W}_{k-1},\end{aligned}\quad (10)$$

for all $k = 2, 3, \dots$

Now given a fixed $f \in L^2(K, \mathbb{R}^n)$, for every $\epsilon > 0$, there exists $N > 0$ s.t. $\|P_{\mathcal{L}}f - P_{\mathcal{L}_N}f\|_{L^2} < \epsilon$. We write $P_{\mathcal{L}_N}f = \sum_{i=0}^N \sum_{j=1}^m g_{ij}w_{ij}$, where $w_{ij}, j = 1, \dots, m$ is a set of orthonormal basis of $\mathcal{W}_i$ defined in (10). Then, the error between projecting f onto the truncated reachable set $\mathcal{L}_k$ and onto the full reachable set $\mathcal{L}$ is $\|P_{\mathcal{L}}f - P_{\mathcal{L}_k}f\|_{L^2} \leq \|P_{\mathcal{L}_N}f - P_{\mathcal{L}_k}f\|_{L^2} + \|P_{\mathcal{L}}f - P_{\mathcal{L}_N}f\|_{L^2} < \sum_{i=k+1}^N \sum_{j=1}^m g_{ij}^2 + \epsilon$, which implies that the projection error resulting from the finite truncation of the reachable set can be bounded above by g_{ij} 's. Following the same analysis above, we can also bound the error of projecting f onto finite truncation of the differentiation set. Therefore, Algorithm 1 presents a tractable numerical approach to study uniform ensemble controllability of systems defined on a Banach space.

IV. NUMERICAL EXPERIMENTS

In this section, we present several numerical experiments demonstrating the efficacy of the proposed numerical approach. We choose the basis of $L^2(K, \mathbb{R}^2)$, denoted as $\phi_i, i = 1, 2, \dots, N_2$, given by $\phi_{2i-1} = (0, x^i)^\top$ and $\phi_{2i} = (x^i, 0)^\top$. If not specifically mentioned, we select the parameters in Algorithm 1 as $\epsilon = 10^{-3}$, $N_2 = 20$, and vary N_1 from 1 to 20.

We first present an example, in which the ensemble system is theoretically identified to be uniformly ensemble controllable. We verify this result using our proposed numerical approach.

Example 1. Consider an ensemble of harmonic oscillators indexed by $\beta \in [-1, 1]$, given by

$$\begin{aligned}\frac{d}{dt}X(t, \beta) &= \beta AX(t, \beta) + Bu(t) \\ &= \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X(t, \beta) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t).\end{aligned}\quad (11)$$

This system is uniformly ensemble controllable because $\text{rank}(A) = \text{rank}(B) = 2$ [21].

Figure 1 shows the heatmap describing the residuals of projecting each basis function ϕ_i onto the truncated differentiation set of the ensemble in (11). It can be observed from this figure that as N_1 increases, the residuals for each basis function decrease and they are below the error tolerance $\epsilon = 10^{-3}$. Therefore, by condition (i) of Theorem 5, we conclude that the ensemble in (11) is uniformly ensemble controllable on $C([-1, 1], \mathbb{R}^2)$, up to an ϵ -tolerance.

In the following examples, we use the developed numerical approach to study two linear ensembles, for which theoretical verification of uniform ensemble controllability is difficult.

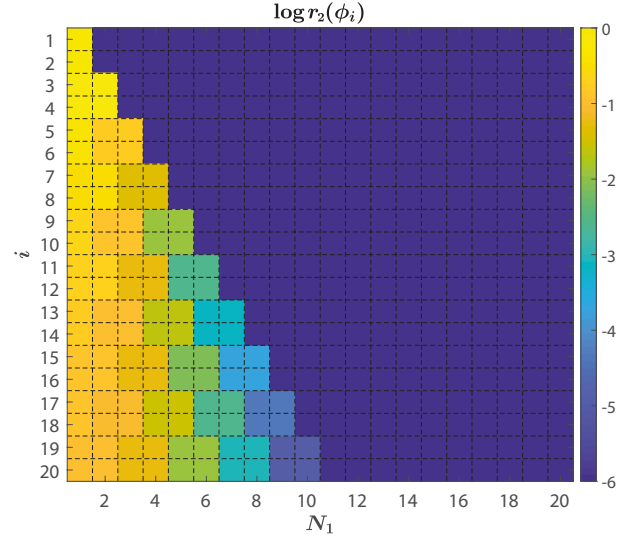


Fig. 1: Results of Example 1. All basis functions yield a residual smaller than $\epsilon = 10^{-3}$ when $N_1 = 20$. Hence, the ensemble in (11) is uniformly ensemble controllable on $C([-1, 1], \mathbb{R}^2)$.

Example 2. Consider an ensemble of linear systems indexed by $\beta \in [-1, 1]$, given by

$$\frac{d}{dt}X(t, \beta) = \theta(\beta) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X(t, \beta) + \begin{bmatrix} 1 & 0 \\ 0 & \beta + 1 \end{bmatrix} u(t), \quad (12)$$

where $\theta(\beta) = \beta(\beta^2 - 0.25)$. Applying the developed numerical algorithm, in Figure 2 we display the heatmap describing the residuals of projecting each basis function ϕ_i onto the truncated reachable set $\mathcal{L}$. Since increasing the number of basis elements does not yield a decreased residual to be within the ϵ -threshold, we learn that $\mathcal{L}$ is not L^2 -dense in $L^2(K, \mathbb{R}^n)$. We also observe that θ vanishes at $\beta = 0, \pm 0.5$, at which the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \beta + 1 \end{bmatrix}$ is of full-rank. Therefore, Theorem 5 applies. By the contraposition of condition (ii), we conclude that the ensemble in (12) is not uniformly ensemble controllable on $C([-1, 1], \mathbb{R}^2)$.

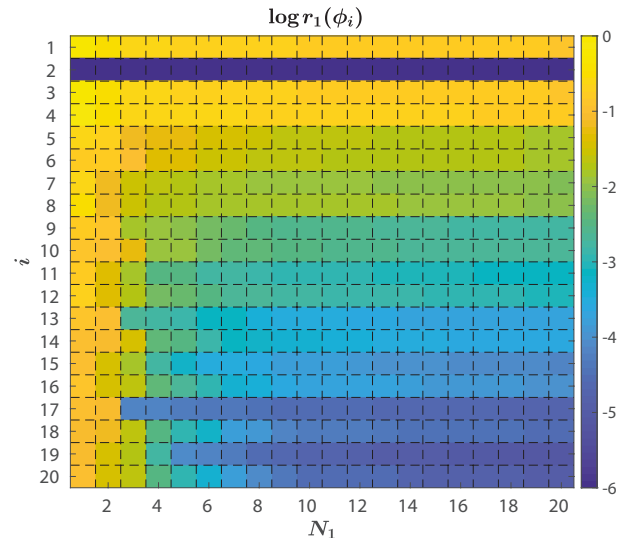


Fig. 2: Results of Example 2. Since $r_1(\phi_1)$ does not decrease to be within the threshold, we conclude that the ensemble in (12) is not uniformly ensemble controllable on $C([-1, 1], \mathbb{R}^2)$.

Example 3. Consider an ensemble of linear systems indexed by $\beta \in [-1, 1]$ given by,

$$\frac{d}{dt}X(t, \beta) = \begin{bmatrix} \alpha(\beta) & -\omega(\beta) \\ \omega(\beta) & \alpha(\beta) \end{bmatrix} X(t, \beta) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t), \quad (13)$$

where $\alpha(\beta) = \beta^3 - 0.25\beta + 1$ and $\omega(\beta) = \beta^4 - 0.89\beta^2 + 1.16$. If we let $A(\beta) = \begin{bmatrix} \alpha(\beta) & -\omega(\beta) \\ \omega(\beta) & \alpha(\beta) \end{bmatrix}$, then $A'(\beta) = \begin{bmatrix} 3\beta^2 - 0.25 & -(4\beta^3 - 1.78\beta) \\ (4\beta^3 - 1.78\beta) & 3\beta^2 - 0.25 \end{bmatrix}$. Although A does not vanish on $[-1, 1]$ so that Theorem 5 no longer applies, it is not hard to verify that A' is non-singular on $[-1, 1]$ and A' commutes with A . Hence we can apply Theorem 6 to detect uniform ensemble uncontrollability of the ensemble in (13) through Hilbert space settings.

Figure 3 illustrates the heatmap describing the residuals of projecting each basis function ϕ_i onto the truncated differentiation sets of the ensemble in (13). Since increasing the number of basis functions does not yield a decreased residual to be within the ϵ -threshold, this differentiation set is not dense in $L^2([-1, 1], \mathbb{R}^2)$. Therefore, by the contraposition of Theorem 6, the ensemble in (13) is not uniformly ensemble controllable.

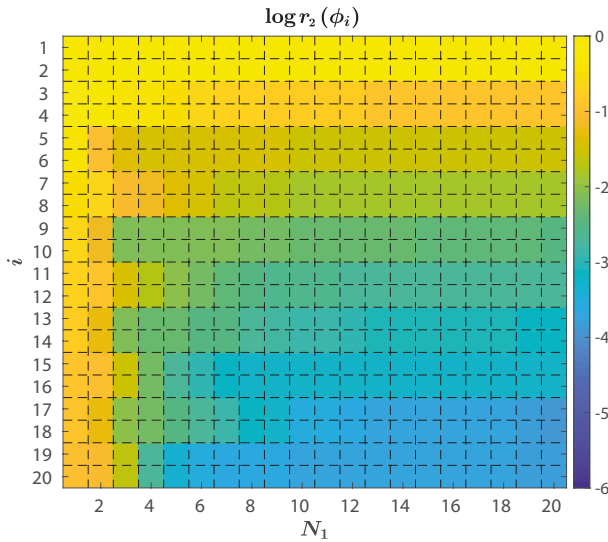


Fig. 3: Results of Example 3. Since several basis functions, e.g., ϕ_1, ϕ_2 , do not yield a residual smaller than ϵ , we conclude that the ensemble in (13) is not uniformly ensemble controllable on $C([-1, 1], \mathbb{R}^2)$.

V. CONCLUSION

In this paper, we study uniform ensemble controllability of a linear ensemble system defined on Banach space from a numerical perspective. We show that such a linear ensemble is uniformly ensemble controllable if its differentiation set is dense, and only if its reachable set is dense, in the L^2 -space. We further show that under certain conditions, uniform ensemble controllability of a linear ensemble implies L^2 -denseness of its differentiation set as well. These results enable a numerical examination of uniform ensemble controllability through projecting basis functions of a Hilbert space onto the reachable set and differentiation set of the ensemble system. We further provide theoretical analysis on the error bound of

the proposed numerical approach. Our method is applicable to linear ensemble systems that may not be fully analyzed by existing theoretical approaches.

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