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Global weak solution to the viscous two-fluid model with finite energy

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ABSTRACT

In this paper, we prove the existence of global weak solutions to the compressible two-fluid Navier–Stokes equations in three dimensional space. The pressure depends on two different variables from the continuity equations. We develop an argument of variable reduction for the pressure law. This yields to the strong convergence of the densities, and provides the existence of global solutions in time, for the compressible two-fluid Navier–Stokes equations, with large data in three dimensional space.

RÉSUMÉ

Dans cet article, nous prouvons l'existence de solutions globales pour l'équation de Navier–Stokes compressible bifluide en trois dimensions d'espace. La pression dépend de deux quantités transportées par le flot. Nous développons une méthode de réduction de variables pour l'étude de la pression. Nous obtenons ainsi la convergence forte des densités, et l'existence de solutions globales en temps pour le système étudié.

1. Introduction

In this paper, we are considering a viscous compressible two-fluid model with a pressure law in two variables. We show the existence of global weak solutions to the following two-fluid compressible Navier–Stokes system in three dimensional space:

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$$\begin{cases} n_t + \mathrm{div}(nu) = 0, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \mathrm{div}\big[(\rho+n)u\otimes u\big] + \nabla P(n,\rho) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u \quad \text{on } \Omega\times(0,\infty), \end{cases} \tag{1.1}$$

with the initial and boundary conditions

$$n(x,0) = n_0(x),\ \rho(x,0) = \rho_0(x),\ (\rho+n)u(x,0) = M_0(x) \ \text{ for } \ x\in\overline{\Omega}, \tag{1.2}$$

$$u|_{\partial\Omega} = 0 \quad \text{for } t \geq 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $P(n,\rho) = \rho^\gamma + n^\alpha$ denotes the pressure for $\gamma \geq 1$ and $\alpha \geq 1$, u stands for the velocity of fluid, ρ and n are the densities of two fluids, μ and λ are the viscosity coefficients. Here we assume that μ and λ are fixed constants, and

$$\mu > 0, \ \ 2\mu + \lambda \geq 0.$$

The two-fluid model was originally developed by Zuber and Findlay [34], Wallis [32], and Ishii [19,20]. The case $\alpha = 1$ corresponds to the hydrodynamic equations of [6,30]. It was derived in [6,30] as an asymptotic limit of a coupled system of the compressible Navier–Stokes equation with a Vlasov–Fokker–Planck equation. The case $\alpha = 2$ is associated to the compressible Oldroyd-B type model with stress diffusion, see Barrett, Lu, and Suli [1]. The main difference with the classical compressible Navier–Stokes equations is that the pressure law $P(\rho,n) = \rho^\gamma + n^\alpha$ depends on two variables. In this context, the existence of weak solutions to equations (1.1) remained open until now. We refer the reader to [2–4,19,20,32,34] for more physical background and discussion of numerical studies for such mathematical models.

One difficulty dealing with the compressible Navier–Stokes equation is the degeneracy of the system close to the vacuum (when the density is vanishing). The first existence result for the compressible Navier–Stokes equations in one dimensional space was established by Kazhikhov and Shelukhin [22]. This result was restricted to initial densities bounded away from zero. It has been extended by Hoff [14] and Serre [31] to the case of discontinuous initial data, and by Mellet–Vasseur [29] in the case of density dependent viscosity coefficients. For the multidimensional case, the first global existence with small initial data was proved by Matsumura and Nishida [25–27], and later by Hoff [15–17] for discontinuous initial data. Lions, in [23], introduced the concept of renormalized solutions for the compressible Navier–Stokes equations which allows to control the possible oscillations of density. He proved the global existence of 3D solutions for $\gamma \geq \frac{9}{5}$, and large initial values. It was later improved by Jiang and Zhang [21] for spherically symmetric initial data for $\gamma > 1$, and by Feireisl–Novotný–Petzeltová [12] and Feireisl [13] for $\gamma > \frac{3}{2}$, and to Navier–Stokes-Fourier systems. One key ingredient of the theory [23,21,12] is to obtain higher integrability on the density. This is obtained thanks to the elliptic structure on the viscous effective flux, and the specific form of the pressure $P = \rho^\gamma$. Relying on this structure, Lions deduced that the density ρ is uniformly bounded in $L^{\gamma+\frac{2\gamma}{3}-1}$. Note that for $1 \leq \gamma \leq \frac{3}{2}$, the construction of weak solutions for large data remains largely open, see [24]. The primary difficulty is the possible concentration of the convective term in this case. Very recently, Hu [18] studied the concentration phenomenon of the kinetic energy, $\rho|u|^2$, associated to the isentropic compressible Navier–Stokes equations for $1 \leq \gamma \leq \frac{3}{2}$. Finally, let us mention a very promising work of Bresch–Jabin [5]. They developed a new method to obtain compactness on the density. This method is very different from the theory initiated by Lions. It allows already the treatment of non-monotone pressure laws.

The problem becomes even more challenging when the pressure law depends on two variables as follows

$$P(\rho,n) = \rho^\gamma + n^\alpha. \tag{1.4}$$

To the best of our knowledge, the only results on global existence of weak solutions to System (1.1) with large initial data are restricted to the one dimension case, see [9,11] (see also [8,10,33] for smallness assumptions). In [1], Barrett–Lu–Suli established the existence of weak solutions to a compressible Oldroyd-B type model with pressure law $P = \rho^\gamma + n + n^2$, in the two dimensional space, but with an extra diffusion term on the n equation. This provides higher regularity on n due to the parabolic structure. David–Michalek–Mucha–Novotny–Pokorny–Zatorska in [28] constructed a weak solution of the compressible Navier–Stokes system with the nonlinear pressure law

$$P(\rho, s) = \rho^\gamma \mathcal{T}(s), \quad \gamma \geq \frac{9}{5},$$

where s satisfies the entropy equation. Note that the quantity $\theta = (\mathcal{T}(s))^{\frac{1}{\gamma}}$ can be interpreted as a potential temperature, thus the pressure could take the form $P = (\rho\theta)^\gamma = Z^\gamma$. The quantity Z also satisfies the continuity equation. This allowed them to apply the standard technique for the compressible Navier–Stokes equations to this system.

Because the pressure law depends genuinely on two variables, the treatment of the system (1.1) is more involved. At first sight, it seems that more regularity on the densities is required to control the cross products, like $\rho^\gamma n$ and $n^\alpha \rho$. These extra regularity properties are, so far, out of reach, and the classical techniques cannot be applied directly on (1.1).

For any smooth solution of system (1.1), the following energy inequality holds for any time $0 \leq t \leq T$:

$$\frac{d}{dt} \int_\Omega \Big[\frac{(\rho+n)|u|^2}{2} + G_\alpha(n) + \frac{1}{\gamma-1}\rho^\gamma\Big]\, dx + \int_\Omega \Big[\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2\Big]\, dx \leq 0, \tag{1.5}$$

where

$$G_\alpha(n) = \begin{cases} n\ln n - n + 1, & \text{for} \quad \alpha = 1, \\ \frac{n^\alpha}{\alpha-1}, & \text{for} \quad \alpha > 1. \end{cases}$$

As usual, we assume that

$$\int_\Omega \Big[\frac{(\rho_0+n_0)|u_0|^2}{2} + G_\alpha(n_0) + \frac{1}{\gamma-1}\rho_0^\gamma\Big]\, dx < \infty$$

in the whole paper. Thus, we set the following restriction on the initial data

$$\inf_{x\in\Omega} \rho_0 \geq 0, \quad \inf_{x\in\Omega} n_0 \geq 0, \quad \rho_0 \in L^\gamma(\Omega), \quad G_\alpha(n_0) \in L^1(\Omega), \tag{1.6}$$

and

$$\frac{M_0}{\sqrt{\rho_0+n_0}} \in L^2(\Omega) \ \text{ where } \ \frac{M_0}{\sqrt{\rho_0+n_0}} = 0 \ \text{ on } \ \{x \in \Omega | \rho_0(x) + n_0(x) = 0\}. \tag{1.7}$$

The definition of weak solution in the energy space is given in the following sense.

Definition 1.1. We call $(\rho, n, u) : \Omega \times (0, \infty) \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ a global weak solution of (1.1)–(1.3) if for any $0 < T < +\infty$,

- $\rho \in L^\infty\big(0, T; L^\gamma(\Omega)\big)$, $G_\alpha(n) \in L^\infty\big(0, T; L^1(\Omega)\big)$, $\sqrt{\rho+n}u \in L^\infty\big(0, T; L^2(\Omega)\big), u \in L^2\big(0, T; H_0^1(\Omega)\big)$,
- (ρ, n, u) solves the system (1.1) in $\mathcal{D}'(Q_T)$, where $\mathrm{Q_T} = \Omega \times (0, T)$,
- $\big(\rho, n, (\rho+n)u\big)(x, 0) = \big(\rho_0(x), n_0(x), M_0(x)\big)$, for a.e. $x \in \Omega$,
- the energy inequality (1.5) holds in $\mathcal{D}'\big(\mathbb{R}^3 \times (0, T)\big)$,

- $(1.1)_1$ and $(1.1)_2$ hold in $\mathcal{D}'(\mathbb{R}^3 \times (0,T))$ provided ρ, n, u are prolonged to be zero on $\mathbb{R}^3/\Omega$,
- the equation $(1.1)_1$ and $(1.1)_2$ are satisfied in the sense of renormalized solutions, i.e.,

$$\partial_t b(f) + \operatorname{div}\big(b(f)u\big) + [b'(f)f - b(f)]\operatorname{div}u = 0$$

holds in $\mathcal{D}'(Q_T)$, for any $b \in C^1(\mathbb{R})$ such that $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough, where $f = \rho, n$.

The main result of this paper is as follows.

Theorem 1.2. *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain in $\mathbb{R}^3$ of class $C^{2+\nu}$, $\nu > 0$. Let the initial data be under the conditions* (1.6)–(1.7).

- *If*

$$\alpha \geq 1, \quad \gamma > \frac{9}{5},$$

and the initial data additionally satisfies

$$\frac{1}{\mathrm{c}_0}\rho_0 \leq \mathrm{n}_0 \leq \mathrm{c}_0\rho_0 \text{ on } \Omega, \tag{1.8}$$

where $\mathrm{c}_0 \geq 1$ is a constant, then there exists a global weak solution (ρ, n, u) to (1.1)–(1.3).
- *Without restriction* (1.8), *if*

$$\alpha, \gamma > \frac{9}{5} \text{ and } \max\{\frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1\}, \tag{1.9}$$

then there exists a global weak solution (ρ, n, u) to (1.1)–(1.3).

Remark 1.3. The restriction $\gamma > \frac{9}{5}$ provides the L^2-estimate on the density. This is needed to apply the renormalized argument of DiPerna–Lions to the system. The condition (1.8) is propagated in time, and gives that

$$n \leq C\rho$$

for almost every time $t > 0$. This provides extra integrability on n without more assumption on the system than $\alpha \geq 1$. However, without the condition (1.8), the value of α needs to be close enough to the value of γ. This is required to insure the L^2-estimate on n.

The key idea of our proof is to perform a variable reduction in the pressure law. When considering a family of solutions, we decompose the pressure as

$$P_\varepsilon = \rho_\varepsilon^\gamma + n_\varepsilon^\alpha = A^\alpha d_\varepsilon^\gamma + B^\gamma d_\varepsilon^\alpha + \text{remainder},$$

where $d_\varepsilon = \rho_\varepsilon + n_\varepsilon$. The idea is that we can control the oscillations of $A_\varepsilon = n_\varepsilon/d_\varepsilon$ and $B_\varepsilon = \rho_\varepsilon/d_\varepsilon$.

The structure of this paper is as follows. In Section 2, we develop a new tool to handle the compactness on the terms A_ε and B_ε. In section 3, we solve the approximation system using the Galerkin method. In section 4, we study the limits as ϵ goes to zero. The focus of this section is to prove that

$$\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta.$$

Here and always, $\overline{f}$ is the weak limit of f_ϵ. One of the key step is to control the product of $n_\epsilon^\alpha + \rho_\epsilon^\gamma$ and $n_\epsilon + \rho_\epsilon$, we rewrite them as follows

$$\begin{aligned} n_\epsilon^\alpha + \rho_\epsilon^\gamma =& A_\epsilon^\alpha d_\epsilon^\alpha + B_\epsilon^\gamma d_\epsilon^\gamma = A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma + (A_\epsilon^\alpha - A^\alpha) d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma) d_\epsilon^\gamma, \\ n_\epsilon + \rho_\epsilon =& (A_\epsilon + B_\epsilon) d_\epsilon = (A+B) d_\epsilon + (A_\epsilon - A + B_\epsilon - B) d_\epsilon, \end{aligned}$$

where $d_\epsilon = \rho_\epsilon + n_\epsilon$, $d = \rho + n$, $(A_\epsilon, B_\epsilon) = (\frac{n_\epsilon}{d_\epsilon}, \frac{\rho_\epsilon}{d_\epsilon})$ if $d_\epsilon \neq 0$, $(A, B) = (\frac{n}{d}, \frac{\rho}{d})$ if $d \neq 0$, and $0 \le A_\epsilon, B_\epsilon, A, B \le 1$, and $(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon)$, $(Ad, Bd) = (n, \rho)$, (ρ, n) is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology. Here we want to show

$$\Big[(A_\epsilon^\alpha - A^\alpha) d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma) d_\epsilon^\gamma\Big](n_\epsilon + \rho_\epsilon) \to 0, \text{ and } \Big(A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma\Big)(A_\epsilon - A + B_\epsilon - B) d_\epsilon \to 0$$

in some sense as $\epsilon \to 0$. This can be done because ρ_ϵ and n_ϵ are bounded uniformly for ϵ in $L^{\beta+1}(Q_T)$ where $\beta > \max\{\alpha, \gamma, 4\}$, and, thanks to the result of Section 2

$$\lim_{\epsilon \to 0^+} \int_0^T \int_\Omega d_\epsilon |A_\epsilon - A|^s \, dx \, dt = 0, \text{ and } \lim_{\epsilon \to 0^+} \int_0^T \int_\Omega d_\epsilon |B_\epsilon - B|^s \, dx \, dt = 0.$$

In Section 5, we recover the weak solution by letting δ goes to zero. The highlight of this section is to show that

$$\overline{n^\alpha + \rho^\gamma} = n^\alpha + \rho^\gamma,$$

which is similar to the limits in term of ϵ. However, a new difficulty occurs because of $\delta = 0$. We follow [12] and use a cut-off function in the renormalization to show the strong convergence of ρ_δ and n_δ. This can be done using again the variable reduction of Section 2. At this level of approximation, we require $\gamma > \frac{9}{5}$ such that ρ_δ is bounded in $L^{\gamma+\theta_2}(Q_T)$ with $\gamma + \theta_2 > 2$ for some θ_2 satisfying $\theta_2 < \frac{\gamma}{3}$ and $\theta_2 \le \min\left\{1, \frac{2\gamma}{3} - 1\right\}$. In order to guarantee that n_δ is bounded in $L^{q_1}(Q_T)$ for some $q_1 > 2$, we require either $\alpha \in (\frac{9}{5}, \infty) \cap (\gamma - \theta_1, \gamma + \theta_2)$ or $\alpha \in [1, \infty)$ and $\frac{1}{c_0}\rho_0 \le n_0 \le c_0 \rho_0$.

2. An error estimate

Our main goal of this section is to prove the following Theorem 2.2. The proof relies on the DiPerna–Lions theory of the renormalized solutions to the transport equation. This theorem allows us to obtain the weak stability of solutions to (1.1).

Note that if ρ and n are solutions to the two first transport equations of (1.1), then (formally) the ratio ρ/n verifies

$$\frac{\partial}{\partial t}\left(\frac{\rho}{n}\right) + u \cdot \nabla \left(\frac{\rho}{n}\right) = 0. \tag{2.1}$$

The idea is to get some sort of compactness on this quantity. This is reminiscent to the compactness obtained by DiPerna and Lions for transport equation in [7]. They showed that if the quantities (div u) are uniformly bounded in $L^1(L^\infty)$, and initial values of ρ/n are compact in L^p_{loc}, then solutions to (2.1) are compact in $C^0(L^p_{\text{loc}})$. Theorem 2.2 can be seen as an extension of this theorem to possible diffusive versions of (2.1), where the condition on div u is relaxed (but some control on the whole gradient of u is available in $L^2(L^2)$).

We start from the following lemma in this section.

Lemma 2.1. *Let* $\{(g_K, h_K)\}_{K=1}^{\infty}$ *be a sequence with the following properties*

$$(g_K, h_K) \to (g, h) \text{ weakly in } L^p(Q_T) \text{ as } K \to \infty, \tag{2.2}$$

for any given $p > 1$, $g_K, h_K \geq 0$, *and*

$$\lim_{K\to+\infty} \int_0^T \int_\Omega a_K h_K \, dx\, dt \leq \int_0^T \int_\Omega h a_h \, dx\, dt, \tag{2.3}$$

where $a_K = \frac{h_K}{g_K}$ *if* $g_K \neq 0$, $a_h = \frac{h}{g}$ *if* $g \neq 0$, $0 \leq a_K, a_h \leq \mathcal{C}$ *for some positive constant* $\mathcal{C}$ *independent of* K, *and* $a_K g_K = h_K$, $a_h g = h$, *then*

$$\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt = 0. \tag{2.4}$$

In particular,

$$\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^s \, dx\, dt = 0, \tag{2.5}$$

for any $s > 1$.

Proof. Note that

$$\int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt = \int_0^T \int_\Omega a_K h_K \, dx\, dt - 2\int_0^T \int_\Omega h_K a_h \, dx\, dt + \int_0^T \int_\Omega g_K a_h^2 \, dx\, dt,$$

one obtains

$$\begin{aligned}\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt \leq & \int_0^T \int_\Omega h a_h \, dx\, dt - 2\int_0^T \int_\Omega h a_h \, dx\, dt + \int_0^T \int_\Omega h a_h \, dx\, dt \\ =& 0,\end{aligned}$$

where we have used $a_K g_K = h_K$, $a_h g = h$, (2.3) and the weak compactness of g_K and h_K in (2.2). This deduces (2.4).

By the Hölder inequality and (2.4), (2.5) follows for $s \in (1, 2)$. If $s \in [2, \infty)$, note that $(a_K - a_h)$ is bounded in $L^\infty\big(\Omega \times (0, T)\big)$. This allows us to have (2.5). □

The following theorem is our main result of this section.

Theorem 2.2. *Let* $\nu_K \to 0$ *as* $K \to +\infty$, *and* $\nu_K \geq 0$. *If* $\rho_K \geq 0$ *and* $n_K \geq 0$ *are the solutions to*

$$(\rho_K)_t + \operatorname{div}(\rho_K u_K) = \nu_K \Delta\rho_K, \;\; \rho_K|_{t=0} = \rho_0, \;\; \nu_K \frac{\partial \rho_K}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.6}$$

and

$$(n_K)_t + \mathrm{div}(n_K u_K) = \nu_K \Delta n_K, \;\; n_K|_{t=0} = n_0, \;\; \nu_K \frac{\partial n_K}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.7}$$

respectively, with $C_0 \geq 1$ independent of K such that

- $\|(\rho_K, n_K)\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0$, $\sqrt{\nu_K}\|\nabla \rho_K\|_{L^2(0,T;L^2(\Omega))} \leq C_0$, $\sqrt{\nu_K}\|\nabla n_K\|_{L^2(0,T;L^2(\Omega))} \leq C_0$.
- $\|u_K\|_{L^2(0,T;H_0^1(\Omega))} \leq C_0$.
- *For any $K > 0$ and any $t > 0$:*

$$\int_\Omega \frac{b_K^2}{d_K}\, dx \leq \int_\Omega \frac{b_0^2}{d_0}\, dx, \tag{2.8}$$

where $b_K = \rho_K$ or n_K, and $d_K = \rho_K + n_K$.

Then, up to a subsequence, we have

$$n_K \to n, \quad \rho_K \to \rho \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)),$$
$$u_K \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega)),$$

and for any $s > 1$,

$$\lim_{K\to+\infty} \int_0^T \int_\Omega d_K |a_K - a|^s\, dx\, dt = 0, \tag{2.9}$$

where $a_K = \frac{b_K}{d_K}$ if $d_K \neq 0$, $a = \frac{b}{d}$ if $d \neq 0$, and $a_K d_K = b_K$, $ad = b$. Here (b, d) is the weak limit of (b_K, d_K).

Remark 2.3. The proof relies on the DiPerna–Lions renormalized argument for transport equation. The L^2 bounds of the densities ρ_K and n_K make it possible to use this theory for equations (2.6).

Remark 2.4. If $\nu_K > 0$, u_K is smooth enough and ρ_K is bounded by below, then (2.8) is verified. In fact, choosing $\varphi(b_K, d_K) = \frac{b_K^2}{d_K}$, one obtains

$$\frac{\partial \varphi(b_K, d_K)}{\partial t} + \mathrm{div}(\varphi u_K) + [\frac{\partial \varphi}{\partial b_K} b_K + \frac{\partial \varphi}{\partial d_K} d_K - \varphi]\mathrm{div} u_K$$
$$+ \nu_K (\frac{\partial^2 \varphi}{\partial b_K^2} |\nabla b_K|^2 + \frac{\partial^2 \varphi}{\partial d_K^2} |\nabla d_K|^2 + 2\frac{\partial^2 \varphi}{\partial b_K \partial d_K} \nabla b_K \cdot \nabla d_K) - \nu_K \Delta \varphi = 0.$$

Note that

$$\frac{\partial \varphi}{\partial b_K} b_K + \frac{\partial \varphi}{\partial d_K} d_K - \varphi = 0$$

and φ is convex, thus we have

$$\frac{d}{dt} \int_\Omega \varphi(b_K, d_K)\, dx \leq 0.$$

We will rely on the following lemma to show Theorem 2.2.

Lemma 2.5. *Let $\beta : \mathbb{R}^N \to \mathbb{R}$ be a C^1 function with $|\nabla \beta(X)| \in L^\infty(\mathbb{R}^N)$, and $R \in \big(L^2(0,T;L^2(\Omega))\big)^N$, $u \in L^2(0,T;H_0^1(\Omega))$ satisfy*

$$\frac{\partial}{\partial_t} R + \mathrm{div}(u \otimes R) = 0, \quad R|_{t=0} = R_0(x) \tag{2.10}$$

in the distribution sense. Then we have

$$(\beta(R))_t + \mathrm{div}(\beta(R)u) + [\nabla\beta(R) \cdot R - \beta(R)]\mathrm{div}u = 0 \tag{2.11}$$

in the distribution sense. Moreover, if $R \in L^\infty(0,T; L^\gamma(\Omega))$ for $\gamma > 1$, then

$$R \in C([0,T]; L^1(\Omega)),$$

and so

$$\int_\Omega \beta(R)\,dx(t) = \int_\Omega \beta(R_0)\,dx - \int_0^t \int_\Omega [\nabla\beta(R) \cdot R - \beta(R)]\mathrm{div}u\,dx\,dt.$$

Remark 2.6. If $N = 1$, it is the result of Feireisl [13].

To prove Lemma 2.5, we shall rely on the following lemma which was called the commutator lemma.

Lemma 2.7. *[23]. There exists $C > 0$ such that for any $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)$,*

$$\|\eta_\sigma * \mathrm{div}(\rho u) - \mathrm{div}(u(\rho * \eta_\sigma))\|_{L^1(\mathbb{R}^d)} \le C\|u\|_{H^1(R^d)}\|\rho\|_{L^2(\mathbb{R}^d)}.$$

In addition,

$$\eta_\sigma * \mathrm{div}(\rho u) - \mathrm{div}(u(\rho * \eta_\sigma)) \to 0 \text{ in } L^1(\mathbb{R}^d), \text{ as } \sigma \to 0,$$

where $\eta_\sigma = \frac{1}{\sigma^d}\eta(\frac{x}{\sigma})$, and $\eta(x) \ge 0$ is a smooth even function compactly supported in the space ball of radius 1, and with integral equal to 1.

Proof of Lemma 2.5. Here, we devote the proof of Lemma 2.5. The first two steps are similar to the work of [12].

Step 1: Proof of (2.11).

Applying the regularizing operator $f \longmapsto f * \eta_\sigma$ to both sides of (2.10), we obtain

$$(R_\sigma)_t + \mathrm{div}(u \otimes R_\sigma) = S^\sigma, \tag{2.12}$$

almost everywhere on $O \subset \bar{O} \subset (0,T) \times \Omega$ provided $\sigma > 0$ small enough, where

$$S^\sigma = \mathrm{div}(u \otimes R_\sigma) - (\mathrm{div}(u \otimes R))_\sigma,$$

and $f_\sigma(x) = f * \eta_\sigma$. Thanks to Lemma 2.7, we conclude that

$$S^\sigma \to 0 \quad \text{in } L^1(O) \quad \text{as } \sigma \to 0.$$

Equation (2.12) multiplied by $\nabla\beta(R)$, where β is a C^1 function, gives us

$$[\beta(R_\sigma)]_t + \mathrm{div}[\beta(R_\sigma)u] + [\nabla\beta(R_\sigma) \cdot R_\sigma - \beta(R_\sigma)]\mathrm{div}u = \nabla\beta(R_\sigma) \cdot S^\sigma.$$

This yields (2.11) by letting $\sigma \to 0$.

Step 2: Continuity of R in the strong topology.

By (2.11), we have

$$\frac{\partial}{\partial_t}T_K(R)+\operatorname{div}(T_K(R)u)+(\nabla T_K(R)\cdot R-T_K(R))\operatorname{div}u=0 \quad \text{in } \mathfrak{D}'((0,T)\times\Omega), \tag{2.13}$$

where $T_k(R)$ is a cutoff function verifying

$$T_K(R)=\widetilde{T_K}(|R|), \quad \text{and } \widetilde{T_K}(z)=KT(\frac{z}{K}),$$

and $T(z)=z$ for any $z\in[0,1]$, and it is concave on $[0,\infty)$, $T(z)=2$ for any $z\geq 3$, and T is a C^∞ function. We conclude that $T_K(R)$ is bounded in $C([0,T];L^2_{weak}(\Omega))$ due to $R\in L^\infty(0,T;L^2(\Omega))$. Thanks to (2.13), we have

$$T_K(R) \quad \text{belong to } \; C([0,T];L^\gamma_{\text{weak}}(\Omega)) \tag{2.14}$$

for any $K\geq 1$.

Applying the same argument as in step 1 for (2.13), we have

$$\frac{\partial}{\partial_t}[T_K(R)]_\sigma+\operatorname{div}([T_K(R)]_\sigma u)=A^\sigma_K \quad \text{a.e. on } (0,T)\times U, \tag{2.15}$$

where U is a compact subset of Ω. Thanks to Lemma 2.7, A^σ_K is bounded in $L^2(0,T;L^1(\Omega))$. Meanwhile, using $2[T_K(R)]_\sigma$ to multiply (2.15), we have

$$\frac{\partial}{\partial_t}([T_K(R)]_\sigma)^2+\operatorname{div}(([T_K(R)]_\sigma)^2u)+([T_K(R)]_\sigma)^2\operatorname{div}u=2[T_K(R)]_\sigma A^\sigma_K.$$

Thus, for any test function $\eta\in\mathcal{D}(\Omega)$, the family of functions with respect to σ for fixed K

$$t\longmapsto\int_\Omega([T_K(R)]_\sigma)^2(t,x)\eta(x)\,dx,\ \sigma>0 \ \text{ is precompact in } C[0,T].$$

Note that $[T_K(R)]_\sigma\to[T_K(R)]$ in $L^2(\Omega)$ for any $t\in[0,T]$ as $\sigma\to 0$, we obtain

$$t\longmapsto\int_\Omega([T_K(R)])^2(t,x)\eta(x)\,dx \text{ is in } C[0,T]$$

for any fixed $\eta(x)$. Thus, $T_K(R)\in C([0,T];L^2(\Omega))$ for any fixed $K\geq 1$. It allows us to have

$$R\in(C([0,T];L^1(\Omega)))^N,$$

thanks to (2.14).

Step 3: Final inequality.

Taking integration on (2.11) with respect to t, we have

$$\int_\Omega\beta(R(t))\,dx=\int_\Omega\beta(R(s))\,dx-\int_s^t\int_\Omega[\nabla\beta(R)\cdot R-\beta(R)]\operatorname{div}u\,dx\,dt,$$

where $0 < s < t < T$. Thanks to

$$R \in (C([0,T]; L^1(\Omega)))^N.$$

Letting $s \to 0$, thus we have

$$\int_\Omega \beta(R(t))\,dx = \int_\Omega \beta(R_0)\,dx - \int_0^t \int_\Omega [\nabla\beta(R)\cdot R - \beta(R)]\mathrm{div}u\,dx\,dt$$

for any $0 \le t \le T$. $\square$

With above lemmas in hand, we are ready to show Theorem 2.2.

Proof of Theorem 2.2. Up to a subsequence,

$$\rho_K \to \rho,\ n_K \to n \ \text{ weakly in } L^\infty(0,T;L^2(\Omega)), \quad u_K \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{2.16}$$

as $K \to \infty$. Passing to the limit as $K \to \infty$ in (2.6) and (2.7) respectively, we have

$$\rho_t + \mathrm{div}(\rho u) = 0, \qquad \rho|_{t=0} = \rho_0,$$

and

$$n_t + \mathrm{div}(nu) = 0, \qquad n|_{t=0} = n_0.$$

Using Lemma 2.5 with $R = (b,d)$ and $u = u$, $\beta_\sigma(b,d) = \frac{b^2}{d+\sigma}$, note that

$$\nabla\beta_\sigma(R)\cdot R - \beta_\sigma(R) = \sigma\frac{b^2}{(d+\sigma)^2},$$

one obtains

$$\int_\Omega \frac{b(t,x)^2}{d(t,x)+\sigma}\,dx = \int_\Omega \frac{b_0^2}{d_0+\sigma}\,dx - \sigma\int_0^t\int_\Omega \frac{b^2}{(d+\sigma)^2}\mathrm{div}u\,dx\,dt,$$

for almost everywhere $t \in [0,T]$.

Note that

$$\frac{b^2}{d+\sigma} \le \frac{b^2}{d} \quad \text{and} \quad \frac{b_0^2}{d_0+\sigma} \le \frac{b_0^2}{d_0},$$

by the dominated convergence theorem, we obtain the following equality by letting σ goes to zero,

$$\int_\Omega \frac{b(t,x)^2}{d(t,x)}\,dx = \int_\Omega \frac{b_0^2}{d_0}\,dx \tag{2.17}$$

for almost everywhere $t \in [0,T]$.

By (2.8) and (2.17), we find

$$\int_{\Omega} \frac{b_K(t,x)^2}{d_K(t,x)}\,dx = \int_{\Omega} b_K a_K\,dx \le \int_{\Omega} \frac{b(t,x)^2}{d(t,x)}\,dx = \int_{\Omega} ba\,dx \tag{2.18}$$

for almost everywhere $t \in [0,T]$.

Thanks to (2.16), setting $(b_K, d_K) = (h_K, g_K)$ for Lemma 2.1, one obtains (2.9) for any $s > 1$. $\square$

3. Faedo–Galerkin approach

In this section, we construct a global weak solution (ρ, n, u) to the following approximation (3.1)–(3.3) with a finite energy. Motivated by the work of [12], we propose the following approximation system

$$\begin{cases} n_t + \operatorname{div}(nu) = \epsilon\Delta n, \\ \rho_t + \operatorname{div}(\rho u) = \epsilon\Delta\rho, \\ \big[(\rho+n)u\big]_t + \operatorname{div}\big[(\rho+n)u\otimes u\big] + \nabla(n^\alpha + \rho^\gamma) + \delta\nabla(\rho+n)^\beta + \epsilon\nabla u\cdot\nabla(\rho+n) \\ = \mu\Delta u + (\mu+\lambda)\nabla\operatorname{div}u \end{cases} \tag{3.1}$$

on $\Omega \times (0,\infty)$, with initial and boundary condition

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \overline{\Omega}, \tag{3.2}$$

$$\Big(\frac{\partial\rho}{\partial\nu}, \frac{\partial n}{\partial\nu}, u\Big)|_{\partial\Omega} = 0, \tag{3.3}$$

where $\epsilon, \delta > 0$, $\beta > \max\{\alpha, \gamma\}$, $M_{0,\delta} = (\rho_{0,\delta} + n_{0,\delta})u_{0,\delta}$ and $n_{0,\delta}, \rho_{0,\delta} \in C^3(\overline{\Omega})$, $u_{0,\delta} \in C_0^3(\Omega)$ satisfying

$$\begin{cases} 0 < \delta \le \rho_{0,\delta}, n_{0,\delta} \le \delta^{-\frac{1}{2\beta}}, \quad (\frac{\partial n_{0,\delta}}{\partial\nu}, \frac{\partial\rho_{0,\delta}}{\partial\nu})|_{\partial\Omega} = 0, \\ \lim\limits_{\delta\to 0}\big(\|\rho_{0,\delta} - \rho_0\|_{L^\gamma(\Omega)} + \|n_{0,\delta} - n_0\|_{L^\alpha(\Omega)}\big) = 0, \\ u_{0\delta} = \frac{\varphi_\delta}{\sqrt{\rho_{0,\delta}+n_{0,\delta}}}\eta_\delta * (\frac{M_0}{\sqrt{\rho_0+n_0}}), \\ \sqrt{\rho_{0,\delta}+n_{0,\delta}}u_{0,\delta} \to \frac{M_0}{\sqrt{\rho_0+n_0}} \quad \text{in } L^2(\Omega) \text{ as } \delta\to 0, \\ m_{0,\delta} \to M_0 \quad \text{in } L^1(\Omega) \text{ as } \delta \to 0, \\ \frac{1}{c_0}\rho_{0,\delta} \le n_{0,\delta} \le c_0\rho_{0,\delta} \quad \text{if } \frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0, \end{cases} \tag{3.4}$$

where $\delta \in (0,1)$, η is the standard mollifier, $\varphi_\delta \in C_0^\infty(\Omega)$, $0 \le \varphi_\delta \le 1$ on $\overline{\Omega}$ and $\varphi_\delta \equiv 1$ on $\big\{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \delta\big\}$.

We are able to use Faedo–Galerkin approach to construct a global weak solution to (3.1), (3.2) and (3.3). To begin with, we consider a sequence of finite dimensional spaces

$$X_k = [span\{\psi_j\}_{j=1}^k]^3, \quad k \in \{1, 2, 3, \cdots\},$$

where $\{\psi_i\}_{i=1}^\infty$ is the set of the eigenfunctions of the Laplacian:

$$\begin{cases} -\Delta\psi_i = \lambda_i\psi_i \quad \text{on } \Omega, \\ \psi_i|_{\partial\Omega} = 0. \end{cases}$$

For any given $\epsilon, \delta > 0$, we shall look for the approximate solution $u_k \in C([0,T]; X_k)$ (for any fixed $T > 0$) given by the following form:

$$
\begin{aligned}
&\int_\Omega (\rho_k + n_k)u_k(t) \cdot \psi \, dx - \int_\Omega m_{0,\delta} \cdot \psi \, dx = \int_0^t \int_\Omega \left[\mu \Delta u_k + (\mu+\lambda)\nabla \mathrm{div} u_k\right] \cdot \psi \, dx \, ds \\
&- \int_0^t \int_\Omega \Big[\mathrm{div}\big[(\rho_k + n_k)u_k \otimes u_k\big] + \nabla(n_k^\alpha + \rho_k^\gamma) + \delta \nabla(\rho_k + n_k)^\beta + \epsilon \nabla u_k \cdot \nabla(\rho_k + n_k)\Big] \cdot \psi \, dx \, ds
\end{aligned}
\tag{3.5}
$$

for $t \in [0,T]$ and $\psi \in X_k$, where $\rho_k = \rho_k(u_k)$ and $n_k = n_k(u_k)$ satisfying

$$
\begin{cases}
\partial_t n_k + \mathrm{div}(n_k u_k) = \epsilon \Delta n_k, \\
\partial_t \rho_k + \mathrm{div}(\rho_k u_k) = \epsilon \Delta \rho_k, \\
n_k|_{t=0} = n_{0,\delta}, \quad \rho_k|_{t=0} = \rho_{0,\delta}, \\
(\frac{\partial \rho_k}{\partial \nu}, \frac{\partial n_k}{\partial \nu})|_{\partial\Omega} = 0.
\end{cases}
\tag{3.6}
$$

Due to Lemmas 2.1 and 2.2 in [12], the problem (3.5) can be solved on a short time interval $[0, T_k]$ for $T_k \leq T$ by a standard fixed point theorem on the Banach space $C([0,T_k]; X_k)$. To show that $T_k = T$, we need the uniform estimates resulting from the following energy equality

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega \Big[\frac{(\rho_k + n_k)|u_k|^2}{2} + G_\alpha(n_k) + \frac{1}{\gamma - 1}\rho_k^\gamma + \frac{\delta}{\beta - 1}(\rho_k + n_k)^\beta\Big] \, dx \\
&+ \int_\Omega \Big[\mu |\nabla u_k|^2 + (\mu + \lambda)|\mathrm{div} u_k|^2\Big] \, dx \\
&+ \int_\Omega \Big[\epsilon \alpha n_k^{\alpha-2}|\nabla n_k|^2 + \epsilon \gamma \rho_k^{\gamma-2}|\nabla \rho_k|^2 + \epsilon \beta \delta (\rho_k + n_k)^{\beta-2}|\nabla(\rho_k + n_k)|^2\Big] \, dx = 0, \text{ on } (0, T_k),
\end{aligned}
\tag{3.7}
$$

where

$$
G_\alpha(n_k) = \begin{cases}
n_k \ln n_k - n_k + 1, & \text{for} \quad \alpha = 1, \\
\frac{n_k^\alpha}{\alpha - 1}, & \text{for} \quad \alpha > 1.
\end{cases}
$$

This could be done by differentiating (3.5) with respect to time, taking $\psi = u_k(t)$ and using (3.6). We refer the readers to [12] for more details. Thus, we obtain a solution (ρ_k, n_k, u_k) to (3.5)–(3.6) globally in time t.

The next step is to pass the limit of (ρ_k, n_k, u_k) as $k \to \infty$. Following the same arguments of Section 2.3 of [12], energy equality (3.7) gives us the following bounds

$$
0 < \frac{1}{c_k} \leq \rho_k(x,t), n_k(x,t) \leq c_k \text{ for a.e.}(x,t) \in \Omega \times (0,T), \tag{3.8}
$$

$$
\sup_{t \in [0,T]} \|\rho_k(t)\|_{L^\gamma(\Omega)}^\gamma \leq C(\rho_0, n_0, M_0), \tag{3.9}
$$

$$
\sup_{t \in [0,T]} \|n_k(t)\|_{L^\alpha(\Omega)}^\alpha \leq C(\rho_0, n_0, M_0) \quad \text{for } \alpha \geq 1, \tag{3.10}
$$

$$
\delta \sup_{t \in [0,T]} \|\rho_k(t) + n_k(t)\|_{L^\beta(\Omega)}^\beta \leq C(\rho_0, n_0, M_0), \tag{3.11}
$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho_k+n_k}(t)u_k(t)\|^2_{L^2(\Omega)} \le C(\rho_0,n_0,M_0), \tag{3.12}$$

$$\int_0^T \|u_k(t)\|^2_{H^1_0(\Omega)}\,dt \le C(\rho_0,n_0,M_0), \tag{3.13}$$

$$\epsilon\int_0^T \left(\|\nabla\rho_k(t)\|^2_{L^2(\Omega)} + \|\nabla n_k(t)\|^2_{L^2(\Omega)}\right)dt \le C(\beta,\delta,\rho_0,n_0,M_0), \tag{3.14}$$

and

$$\|\rho_k+n_k\|_{L^{\beta+1}(Q_T)} \le C(\epsilon,\beta,\delta,\rho_0,n_0,M_0), \tag{3.15}$$

where $Q_T = \Omega\times(0,T)$ and $\beta\ge 4$.

We are able to control n_k by ρ_k if some additional initial data is given in (3.16).

Lemma 3.1. *If (ρ_k,n_k,u_k) is a solution to (3.5) and (3.6) with the initial data satisfying*

$$\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0 \tag{3.16}$$

on Ω, then the following inequality holds

$$\frac{1}{c_0}\rho_k(x,t) \le n_k(x,t) \le c_0\rho_k(x,t) \tag{3.17}$$

for a.e. $(x,t)\in Q_T$.

Proof. It is easy to check that $n_k - c_0\rho_k$ is a solution of the following parabolic equation

$$\begin{cases} (n_k-c_0\rho_k)_t + \operatorname{div}\big[(n_k-c_0\rho_k)u_k\big] = \epsilon\Delta(n_k-c_0\rho_k), \\ (n_k-c_0\rho_k)|_{t=0} = n_{0,\delta}-c_0\rho_{0,\delta}, \\ \nabla(n_k-c_0\rho_k)\cdot\nu|_{\partial\Omega} = 0. \end{cases}$$

The right inequality of (3.17) can be obtained by applying the maximum principle on it. Similarly, we obtain the left inequality of (3.17). □

If the initial data satisfies (3.16) and with (3.9), (3.10), and (3.17), we have

$$\sup_{t\in[0,T]} \left(\|\rho_k(t)\|^{\alpha_1}_{L^{\alpha_1}(\Omega)} + \|n_k(t)\|^{\alpha_1}_{L^{\alpha_1}(\Omega)}\right) \le C(\rho_0,n_0,M_0), \tag{3.18}$$

where $\alpha_1 = \max\{\alpha,\gamma\}$.

Relying on the above uniform estimates, i.e., (3.9)–(3.17) and (3.18), and the Aubin–Lions lemma, we are able to recover the global solution to the approximation system (3.1)–(3.3) by passing to the limit for (ρ_k,n_k,u_k) as $k\to\infty$. We have the following Proposition on the weak solutions of the approximation (3.1), (3.2) and (3.3).

Proposition 3.2. *Suppose $\beta > \max\{4, \alpha, \gamma\}$. For any given $\epsilon, \delta > 0$, there exists a global weak solution (ρ, n, u) to (3.1), (3.2) and (3.3) such that for any given $T > 0$, the following estimates*

$$\sup_{t\in[0,T]} \|\rho(t)\|^{\gamma}_{L^{\gamma}(\Omega)} \le C(\rho_0, n_0, M_0), \tag{3.19}$$

$$\sup_{t\in[0,T]} \|n(t)\|^{\alpha}_{L^{\alpha}(\Omega)} \le C(\rho_0, n_0, M_0), \tag{3.20}$$

$$\delta \sup_{t\in[0,T]} \|(\rho(t), n(t))\|^{\beta}_{L^{\beta}(\Omega)} \le C(\rho_0, n_0, M_0), \tag{3.21}$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho + n}(t)u(t)\|^{2}_{L^{2}(\Omega)} \le C(\rho_0, n_0, M_0), \tag{3.22}$$

$$\int_0^T \|u(t)\|^{2}_{H^1_0(\Omega)}\, dt \le C(\rho_0, n_0, M_0), \tag{3.23}$$

$$\epsilon \int_0^T \|(\nabla\rho(t), \nabla n(t))\|^{2}_{L^{2}(\Omega)}\, dt \le C(\beta, \delta, \rho_0, n_0, M_0), \tag{3.24}$$

and

$$\|(\rho(t), n(t))\|_{L^{\beta+1}(Q_T)} \le C(\epsilon, \beta, \delta, \rho_0, n_0, M_0) \tag{3.25}$$

hold, where the norm $\|(\cdot,\cdot)\|$ denotes $\|\cdot\| + \|\cdot\|$, and $\rho, n \ge 0$ a.e. on Q_T.

In addition, if the initial data satisfy $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$ on Ω, then

$$\begin{cases} \frac{1}{c_0}\rho \le n \le c_0\rho \quad \text{a.e. on } \Omega\times(0,T), \\ \sup\limits_{t\in[0,T]} \|(\rho, n)(t)\|^{\alpha_1}_{L^{\alpha_1}(\Omega)} \le C(\rho_0, n_0, M_0), \end{cases} \tag{3.26}$$

where $\alpha_1 = \max\{\alpha, \gamma\}$. Finally, there exists $r > 1$ such that $\rho_t, n_t, \nabla^2\rho, \nabla^2 n \in L^r(Q_T)$ and the equations $(3.1)_1$ and $(3.1)_2$ are satisfied a.e. on Q_T.

Remark 3.3. The solution (ρ, n, u) stated in Proposition 3.2 actually depends on ϵ and δ. We omit the dependence in the solution itself for brevity.

4. The vanishing viscosity limit $\epsilon \to 0^+$

The goal of this section is to pass to the limit of $(\rho_\epsilon, n_\epsilon, u_\epsilon)$ as ϵ goes to zero. To vanish ϵ, the uniform estimates are needed. Compared to the work of [12], the pressure law involves two variables, which bring new difficulty-possible oscillation of $\rho^\gamma + n^\alpha$. The uniform estimates resulting from the energy inequality in Proposition 3.2 and Lemma 4.1 are not enough to handle the weak limit of such a pressure. In Section 4.1, we pass to the limits for the weak solution constructed in Proposition 3.2 as ϵ goes to zero by standard compactness argument. In Section 4.2, we will focus on the weak limit of the pressure and the strong convergence of ρ_ϵ and n_ϵ. This section 4.2 is where we are using the new compactness Theorem 2.2. In this section, let C denote a generic positive constant depending on the initial data and δ but independent of ϵ.

4.1. Passing to the limit as $\epsilon \to 0^+$

The uniform estimates resulting from (3.19), (3.20), (3.21), and (3.26) are not enough to obtain the convergence of the pressure term $\rho_\epsilon^\gamma + n_\epsilon^\alpha$. Thus we need to obtain higher integrability estimates of the pressure term uniformly for ϵ.

First, following the same argument in [12], we are able to get the following estimate in Lemma 4.1.

Lemma 4.1. *Let (ρ, n, u) be the solution stated in Proposition 3.2, then*

$$\int_0^T \int_\Omega (n^{\alpha+1} + \rho^{\gamma+1} + \delta\rho^{\beta+1} + \delta n^{\beta+1})\, dx\, dt \le C$$

for $\beta > 4$.

In this step, we fix $\delta > 0$ and shall let $\epsilon \to 0^+$. Then the solution (ρ, n, u) constructed in Proposition 3.2 is naturally dressed in the subscript "ϵ", i.e., $(\rho_\epsilon, n_\epsilon, u_\epsilon)$.

With (3.19)–(3.24), and Lemma 4.1, letting $\epsilon \to 0^+$ (take the subsequence if necessary), we have

$$\begin{cases} (\rho_\epsilon, n_\epsilon) \to (\rho, n) \text{ in } C([0,T]; L^\beta_{weak}(\Omega)) \text{ and weakly in } L^{\beta+1}(Q_T) \text{ as } \epsilon \to 0^+, \\ (\epsilon\Delta\rho_\epsilon, \epsilon\Delta n_\epsilon) \to 0 \text{ weakly in } L^2(0,T; H^{-1}(\Omega)) \text{ as } \epsilon \to 0^+, \\ u_\epsilon \to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon + n_\epsilon)u_\epsilon \to (\rho+n)u \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}) \cap C([0,T]; H^{-1}(\Omega)) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon u_\epsilon, n_\epsilon u_\epsilon) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon + n_\epsilon)u_\epsilon \otimes u_\epsilon \to (\rho+n)u \otimes u \text{ in } \mathcal{D}'(Q_T) \text{ as } \epsilon \to 0^+, \\ n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta \to \overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} \text{ weakly in } L^{\frac{\beta+1}{\beta}}(Q_T) \text{ as } \epsilon \to 0^+, \\ \epsilon\nabla u_\epsilon \cdot \nabla(\rho_\epsilon + n_\epsilon) \to 0 \text{ in } L^1(Q_T) \text{ as } \epsilon \to 0^+, \end{cases} \tag{4.1}$$

and $\rho, n \ge 0$.

By virtue of (3.26) and $(4.1)_1$, if $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$, we have

$$\frac{1}{c_0}\rho_\epsilon(x,t) \le n_\epsilon(x,t) \le c_0\rho_\epsilon(x,t) \text{ and } \frac{1}{c_0}\rho(x,t) \le n(x,t) \le c_0\rho(x,t) \quad \text{for a.e. } (x,t) \in Q_T.$$

With $(4.1)_1$ and $(4.1)_4$, we get

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}).$$

In summary, the limit (ρ, n, u) solves the following system in the sense of distribution on Q_T for any $T > 0$:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \operatorname{div}\big[(\rho+n)u \otimes u\big] + \nabla(\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta}) = \mu\Delta u + (\mu+\lambda)\nabla\operatorname{div} u \end{cases} \tag{4.2}$$

with initial and boundary condition

$$\big(\rho, n, (\rho + n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}), \tag{4.3}$$

$$u|_{\partial\Omega} = 0, \tag{4.4}$$

where $\overline{f(t,x)}$ denotes the weak limit of $f_\epsilon(t,x)$ as $\epsilon \to 0$.

To this end, we have to show that $\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta$, which is a nonlinear two-variable function in term of ρ and n. It seems that the argument in [12] fails here due to the difficulty resulting from the new variable n. New ideas are necessary to handle this weak limit. We are going to focus on this issue next subsection.

4.2. The weak limit of pressure

The main task of this subsection is to handle the possible oscillation for the pressure $n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta$. To achieve this goal, we have to show the strong convergence of ρ_ϵ and n_ϵ. It allows us to have the following Proposition on the weak limit of pressure.

Proposition 4.2.

$$\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta$$

a.e. on Q_T.

To prove this proposition, we shall rely on the following lemmas. The first one is on the effective viscous flux associated with pressure involving two variables. In particular, let

$$\begin{aligned} H_\epsilon &:= n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta - (2\mu+\lambda)\mathrm{div} u_\epsilon, \\ H &:= \overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} - (2\mu+\lambda)\mathrm{div} u, \end{aligned}$$

then we will have the following lemma. The proof is very similar to the work of [12].

Lemma 4.3. *Let $(\rho_\epsilon, n_\epsilon, u_\epsilon)$ be the solution stated in Lemma 3.2, and (ρ, n, u) be the limit in the sense of (4.1), then*

$$\lim_{\epsilon\to 0^+} \int_0^T \psi \int_\Omega \phi H_\epsilon(\rho_\epsilon + n_\epsilon)\, dx\, dt = \int_0^T \psi \int_\Omega \phi H(\rho + n)\, dx\, dt, \tag{4.5}$$

for any $\psi \in C_0^\infty(0,T)$ and $\phi \in C_0^\infty(\Omega)$.

The key idea of proving Proposition 4.2 is to rewrite the terms related pressure as follows

$$\begin{aligned} n_\epsilon^\alpha + \rho_\epsilon^\gamma &= A_\epsilon^\alpha d_\epsilon^\alpha + B_\epsilon^\gamma d_\epsilon^\gamma = A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma + (A_\epsilon^\alpha - A^\alpha) d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma) d_\epsilon^\gamma, \\ n_\epsilon + \rho_\epsilon &= (A_\epsilon + B_\epsilon) d_\epsilon = (A+B) d_\epsilon + (A_\epsilon - A + B_\epsilon - B) d_\epsilon, \end{aligned}$$

where $d_\epsilon = \rho_\epsilon + n_\epsilon$, $d = \rho + n$, $(A_\epsilon, B_\epsilon) = (\frac{n_\epsilon}{d_\epsilon}, \frac{\rho_\epsilon}{d_\epsilon})$ if $d_\epsilon \neq 0$, $(A, B) = (\frac{n}{d}, \frac{\rho}{d})$ if $d \neq 0$, $0 \le A_\epsilon, B_\epsilon, A, B \le 1$, and $(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon), (Ad, Bd) = (n, \rho)$, (ρ, n) is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology. We are able to apply the ideas in [12] to handle the product $A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma$ and $(A+B)d_\epsilon$, because A and B are bounded in $L^\infty(0,T; L^\infty(\Omega))$ and they are viewed as the coefficients. The difficult part is to show that the remainder tends to zero as ε goes to zero. Theorem 2.2 allows us to show the terms $\Big[(A_\epsilon^\alpha - A^\alpha)d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma)d_\epsilon^\gamma\Big](n_\epsilon + \rho_\epsilon)$ and $\Big(A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma\Big)(A_\epsilon - A + B_\epsilon - B)d_\epsilon$ approach to zero as ϵ goes to zero.

We divide the proof of Proposition 4.2 into several steps as follows:

Step 1: Control ρ_ϵ and n_ϵ in $L\log L$.

The current step of our proof is to control ρ_ϵ and n_ϵ in the space of $L\log L$. This helps us to obtain the strong convergence of ρ_ϵ and n_ϵ. We give our control in the following lemma.

Lemma 4.4. *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.2, and (ρ, n) be the limit in the sense of (4.1), then*

$$\begin{aligned}&\int_\Omega \big[\rho_\epsilon \log \rho_\epsilon - \rho\log\rho + n_\epsilon \log n_\epsilon - n\log n\big](t)\,dx\\ &\le \int_0^t\int_\Omega (\rho+n)\mathrm{div}u\,dx\,ds - \int_0^t\int_\Omega (\rho_\epsilon + n_\epsilon)\mathrm{div}u_\epsilon\,dx\,ds\end{aligned} \tag{4.6}$$

for a.e. $t \in (0,T)$.

Proof. Since n_ϵ and ρ_ϵ solve $(3.1)_1$ and $(3.1)_2$ a.e. on Q_T, respectively, we have

$$[\beta_j(f_\epsilon)]_t + \mathrm{div}\big(\beta_j(f_\epsilon)u_\epsilon\big) + \big[\beta_j'(f_\epsilon)f_\epsilon - \beta_j(f_\epsilon)\big]\mathrm{div}u_\epsilon = \epsilon\Delta\beta_j(f_\epsilon) - \epsilon\beta_j''(f_\epsilon)|\nabla f_\epsilon|^2 \text{ on } \mathrm{Q_T}, \tag{4.7}$$

where $f_\epsilon = \rho_\epsilon, n_\epsilon$, and $\beta_j \in C^2[0,\infty)$.

Taking $\beta_j(z) = (z + \frac{1}{j})\log(z+\frac{1}{j})$ in (4.7), and integrating it over $\Omega\times(0,t)$ for $t\in[0,T]$, we have

$$\begin{aligned}&\int_\Omega (f_\epsilon + \frac{1}{j})\log(f_\epsilon + \frac{1}{j})(t)\,dx + \int_0^t\int_\Omega \big[f_\epsilon - \frac{1}{j}\log(f_\epsilon + \frac{1}{j})\big]\mathrm{div}u_\epsilon\,dx\,ds\\ &\le \int_\Omega (f_{0,\epsilon} + \frac{1}{j})\log(f_{0,\epsilon} + \frac{1}{j})\,dx,\end{aligned} \tag{4.8}$$

where we have used the convexity of β_j and the boundary condition (3.3). Letting $j\to\infty$ in (4.8), one obtains

$$\int_\Omega (f_\epsilon \log f_\epsilon)(t)\,dx + \int_0^t\int_\Omega f_\epsilon \mathrm{div}u_\epsilon\,dx\,ds \le \int_\Omega f_{0,\delta}\log f_{0,\delta}\,dx, \tag{4.9}$$

where $f_\epsilon = \rho_\epsilon, n_\epsilon$ and $f_{0,\delta} = \rho_{0,\delta}, n_{0,\delta}$.

Since the limit (n,u) and (ρ,u) solve $(4.2)_1$ and $(4.2)_2$ in the sense of renormalized solutions, we can take $\beta(z) = z\log z$ in accordance with Remark 1.1 in [12] or by approximating the function $z\log z$ by a sequence of such the $\beta(z)$ stated in Lemma 2.5 and then passing to the limit. This allows us to have

$$\int_\Omega (f\log f)(t)\,dx + \int_0^t\int_\Omega f\mathrm{div}u\,dx\,ds = \int_\Omega f_{0,\delta}\log f_{0,\delta}\,dx, \tag{4.10}$$

where $f = \rho, n$ and $f_{0,\delta} = \rho_{0,\delta}, n_{0,\delta}$. Thanks to (4.9) and (4.10), (4.6) follows.

Step 2: Control the right hand side of (4.6)

It is crucial to control the right hand side of (4.6). Thanks to Theorem 2.2, we show the following lemma which can help us to finish this step.

Lemma 4.5. *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.2, and (ρ, n) be the limit in the sense of (4.1), then*

$$\int_0^t \psi \int_\Omega \phi(\rho+n)\overline{n^\alpha+\rho^\gamma}\,dx\,ds \le \int_0^t \psi \int_\Omega \phi\overline{(\rho+n)(n^\alpha+\rho^\gamma)}\,dx\,ds$$

for any $t \in [0,T]$ and any $\psi \in C[0,t]$, $\phi \in C(\overline{\Omega})$ where $\psi, \phi \ge 0$.

Proof. Note that

$$\begin{aligned} n_\epsilon^\alpha + \rho_\epsilon^\gamma &= A_\epsilon^\alpha d_\epsilon^\alpha + B_\epsilon^\gamma d_\epsilon^\gamma = A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma + (A_\epsilon^\alpha - A^\alpha)d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma)d_\epsilon^\gamma,\\ n_\epsilon + \rho_\epsilon &= (A_\epsilon + B_\epsilon)d_\epsilon = (A+B)d_\epsilon + (A_\epsilon - A + B_\epsilon - B)d_\epsilon, \end{aligned}$$

where $d_\epsilon = \rho_\epsilon + n_\epsilon$, $d = \rho + n$, $(A_\epsilon, B_\epsilon) = (\frac{n_\epsilon}{d_\epsilon}, \frac{\rho_\epsilon}{d_\epsilon})$ if $d_\epsilon \neq 0$, $(A, B) = (\frac{n}{d}, \frac{\rho}{d})$ if $d \neq 0$, $0 \le A_\epsilon, B_\epsilon, A, B \le 1$, and $(A_\epsilon d_\epsilon, B_\epsilon d_\epsilon) = (n_\epsilon, \rho_\epsilon), (Ad, Bd) = (n, \rho)$, (ρ, n) is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology.

For any $\psi \in C([0,t])$, $\phi \in C(\overline{\Omega})$ where $\psi, \phi \ge 0$, we have

$$\begin{aligned} &\int_0^t \psi \int_\Omega \phi(n_\epsilon^\alpha + \rho_\epsilon^\gamma)(\rho_\epsilon + n_\epsilon)\,dx\,ds\\ =&\int_0^t \psi \int_\Omega \phi(A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma)(A+B)d_\epsilon\,dx\,ds\\ &+\int_0^t \psi \int_\Omega \phi(A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma)(A_\epsilon - A + B_\epsilon - B)d_\epsilon\,dx\,ds\\ &+\int_0^t \psi \int_\Omega \phi\Big[(A_\epsilon^\alpha - A^\alpha)d_\epsilon^\alpha + (B_\epsilon^\gamma - B^\gamma)d_\epsilon^\gamma\Big](n_\epsilon + \rho_\epsilon)\,dx\,ds\\ =&\sum_{i=1}^3 II_i. \end{aligned} \tag{4.11}$$

For II_2, there exists a positive integer k_0 large enough such that

$$\max\{\frac{k_0\gamma}{k_0-1}, \frac{k_0\alpha}{k_0-1}\} \le \beta \tag{4.12}$$

due to the assumption that $\max\{\alpha, \gamma\} < \beta$.

Using the Hölder inequality, Lemma 4.1 and (4.12), we have

$$|II_2| \le C\left(\int_0^T\int_\Omega d_\epsilon|A_\epsilon - A|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}}\left(\int_0^T\int_\Omega d_\epsilon|A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma|^{\frac{k_0}{k_0-1}}\,dx\,dt\right)^{\frac{k_0-1}{k_0}}$$

$$+ C\left(\int_0^T\int_\Omega d_\epsilon|B_\epsilon - B|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}}\left(\int_0^T\int_\Omega d_\epsilon|A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma|^{\frac{k_0}{k_0-1}}\,dx\,dt\right)^{\frac{k_0-1}{k_0}} \tag{4.13}$$

$$\le C\left(\int_0^T\int_\Omega d_\epsilon|A_\epsilon - A|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}} + C\left(\int_0^T\int_\Omega d_\epsilon|B_\epsilon - B|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}}.$$

Choosing $\nu_k = \epsilon$ for Theorem 2.2, we conclude that

$$\begin{aligned}&\left(\int_0^T\int_\Omega d_\epsilon|A_\epsilon - A|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}} \to 0,\\ &\left(\int_0^T\int_\Omega d_\epsilon|B_\epsilon - B|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}} \to 0\end{aligned} \tag{4.14}$$

as ϵ goes to zero. In fact, $d_\epsilon \in L^\infty(0,T;L^\beta(\Omega))$ for $\beta > 4$, and $u_\epsilon \in L^2(0,T;H_0^1(\Omega))$, and

$$\sqrt{\epsilon}\|\nabla\rho_\epsilon\|_{L^2(0,T;L^2(\Omega))} \le C_0,\ \sqrt{\epsilon}\|\nabla n_\epsilon\|_{L^2(0,T;L^2(\Omega))} \le C_0,$$

and for any $\epsilon > 0$ and any $t > 0$:

$$\int_\Omega \frac{b_\epsilon^2}{d_\epsilon}\,dx \le \int_\Omega \frac{b_0^2}{d_0}\,dx \tag{4.15}$$

where $d_\epsilon = \rho_\epsilon + n_\epsilon$, $b_\epsilon = \rho_\epsilon,\, n_\epsilon$, and (4.15) is obtained in Remark 2.4. Thus, we are able to apply Theorem 2.2 to deduce (4.14). Hence we have $II_2 \to 0$ as $\epsilon \to 0$.

For II_3, there exists a positive integer k_1 large enough such that

$$\begin{aligned}&(\alpha + 1 - \frac{1}{k_1})\frac{k_1}{k_1 - 1} < \beta + 1,\\ &(\gamma + 1 - \frac{1}{k_1})\frac{k_1}{k_1 - 1} < \beta + 1,\end{aligned} \tag{4.16}$$

due to the assumption $\alpha < \beta$. We employ the Hölder inequality to have

$$\begin{aligned}|II_3| \le &C\left(\int_0^T\int_\Omega d_\epsilon^{(\alpha+1-\frac{1}{k_1})\frac{k_1}{k_1-1}}\,dx\,dt\right)^{\frac{k_1-1}{k_1}}\left(\int_0^T\int_\Omega d_\epsilon\left|A_\epsilon^\alpha - A^\alpha\right|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}}\\ &+ C\left(\int_0^T\int_\Omega d_\epsilon^{(\gamma+1-\frac{1}{k_1})\frac{k_1}{k_1-1}}\,dx\,dt\right)^{\frac{k_1-1}{k_1}}\left(\int_0^T\int_\Omega d_\epsilon\left|B_\epsilon^\gamma - B^\gamma\right|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}}\\ \le &C\left(\int_0^T\int_\Omega d_\epsilon\left|A_\epsilon^\alpha - A^\alpha\right|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}} + C\left(\int_0^T\int_\Omega d_\epsilon\left|B_\epsilon^\gamma - B^\gamma\right|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}} \to 0\end{aligned} \tag{4.17}$$

as $\epsilon \to 0^+$, where we have used (3.26), (4.16), Lemma 4.1, and the fact that

$$\begin{aligned}
\int_0^T \int_\Omega d_\epsilon \big|A_\epsilon^\alpha - A^\alpha\big|^{k_1}\,dx\,dt &\le \alpha^{k_1} \int_0^T \int_\Omega d_\epsilon \big(\max\{A_\epsilon, A\}\big)^{\alpha-1} \big|A_\epsilon - A\big|^{k_1}\,dx\,dt \\
&\le C \int_0^T \int_\Omega d_\epsilon \big|A_\epsilon - A\big|^{k_1}\,dx\,dt \to 0, \\
\int_0^T \int_\Omega d_\epsilon \big|B_\epsilon^\gamma - B^\gamma\big|^{k_1}\,dx\,dt &\le \gamma^{k_1} \int_0^T \int_\Omega d_\epsilon \big(\max\{B_\epsilon, B\}\big)^{\gamma-1} \big|B_\epsilon - B\big|^{k_1}\,dx\,dt \\
&\le C \int_0^T \int_\Omega d_\epsilon \big|B_\epsilon - B\big|^{k_1}\,dx\,dt \to 0
\end{aligned} \tag{4.18}$$

as $\epsilon \to 0^+$, due to Theorem 2.2 with $\nu_K = \epsilon$.

By virtue of (4.11), (4.13) and (4.17), one deduces that

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \int_0^t \psi \int_\Omega \phi(n_\epsilon^\alpha + \rho_\epsilon^\gamma)(\rho_\epsilon + n_\epsilon)\,dx\,ds &= \int_0^t \psi \int_\Omega \phi(A+B)(A^\alpha \overline{d^{\alpha+1}} + B^\gamma \overline{d^{\gamma+1}})\,dx\,ds \\
&\ge \int_0^t \psi \int_\Omega \phi(A+B)(A^\alpha \overline{d^\alpha} d + B^\gamma \overline{d^\gamma} d)\,dx\,ds \\
&= \int_0^t \psi \int_\Omega \phi\, d(A^\alpha \overline{d^\alpha} + B^\gamma \overline{d^\gamma})\,dx\,ds
\end{aligned} \tag{4.19}$$

where we have used $A + B = 1$, $\overline{d^{\alpha+1}} \ge \overline{d^\alpha} d$, and $\overline{d^{\gamma+1}} \ge \overline{d^\gamma} d$, because the functions $z \mapsto z^\alpha (\text{or } z^\gamma)$ and $z \mapsto z$ are increasing functions.

On the other hand,

$$\begin{aligned}
&\int_0^t \psi \int_\Omega \phi(\rho+n)\overline{n^\alpha + \rho^\gamma}\,dx\,ds \\
&= \lim_{\epsilon\to 0^+} \int_0^t \psi \int_\Omega \phi(\rho+n)(n_\epsilon^\alpha + \rho_\epsilon^\gamma)\,dx\,ds \\
&= \lim_{\epsilon\to 0^+} \int_0^t \psi \int_\Omega \phi d(A^\alpha d_\epsilon^\alpha + B^\gamma d_\epsilon^\gamma)\,dx\,ds + \lim_{\epsilon\to 0^+} \int_0^t \psi \int_\Omega \phi d d_\epsilon^\alpha (A_\epsilon^\alpha - A^\alpha)\,dx\,ds \\
&\quad + \lim_{\epsilon\to 0^+} \int_0^t \psi \int_\Omega \phi d d_\epsilon^\gamma (B_\epsilon^\gamma - B^\gamma)\,dx\,ds \\
&= \int_0^t \psi \int_\Omega \phi d(A^\alpha \overline{d^\alpha} + B^\gamma \overline{d^\gamma})\,dx\,ds,
\end{aligned} \tag{4.20}$$

thanks to

$$\lim_{\epsilon\to 0^+}\int_0^t \psi \int_\Omega \phi\, dd_\epsilon^\alpha (A_\epsilon^\alpha - A^\alpha)\, dx\, ds \to 0, \quad \text{and} \quad \lim_{\epsilon\to 0^+}\int_0^t \psi \int_\Omega \phi\, dd_\epsilon^\gamma (B_\epsilon^\gamma - B^\gamma)\, dx\, ds \to 0,$$

as $\epsilon \to 0^+$.

By (4.19) and (4.20), we complete the proof of the lemma. □

Since ψ and ϕ are arbitrary, we immediately get

Corollary 4.6. *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.2, and (ρ, n) be the limit in the sense of (4.1), then*

$$(\rho+n)\overline{n^\alpha+\rho^\gamma} \le \overline{(\rho+n)(n^\alpha+\rho^\gamma)} \tag{4.21}$$

a.e. on $\Omega\times(0,T)$.

Now we are ready to control the right hand side of (4.6) in the following lemma.

Lemma 4.7. *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Lemma 3.2, and (ρ, n) be the limit in the sense of (4.1), then*

$$\int_0^t\int_\Omega (\rho+n)\mathrm{div} u\, dx\, ds \le \lim_{\epsilon\to 0^+}\int_0^t\int_\Omega (\rho_\epsilon+n_\epsilon)\mathrm{div} u_\epsilon\, dx\, ds \tag{4.22}$$

for a.e. $t\in(0,T)$.

Proof. For $\psi_j\in C_0^\infty(0,t)$, $\phi_j\in C_0^\infty(\Omega)$ given by

$$\psi_j\in C_0^\infty(0,T), \quad \psi_j(t)\equiv 1 \text{ for any } t\in[\frac{1}{j}, T-\frac{1}{j}],\ 0\le\psi_j\le 1,\ \psi_j\to 1, \tag{4.23}$$

as $j\to\infty$, and

$$\phi_j\in C_0^\infty(\Omega), \quad \phi_j(x)\equiv 1 \text{ for any } x\in\{x\in\Omega \big| \mathrm{dist(x,\partial\Omega)} \ge \frac{1}{\mathrm{j}}\},\ 0\le\phi_j\le 1,\ \phi_j\to 1, \tag{4.24}$$

as $j\to\infty$, respectively, then

$$\begin{aligned}
&\int_0^t\int_\Omega (\rho+n)\mathrm{div} u\, dx\, ds\\
=&\int_0^t \psi_j\int_\Omega \phi_j(\rho+n)\mathrm{div} u\, dx\, ds + \int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div} u\, dx\, ds\\
=&\frac{1}{2\mu+\lambda}\int_0^t \psi_j\int_\Omega \phi_j(\rho+n)\overline{n^\alpha+\rho^\gamma}\, dx\, ds + \frac{1}{2\mu+\lambda}\int_0^t \psi_j\int_\Omega \phi_j(\rho+n)\overline{\delta(\rho+n)^\beta}\, dx\, ds\\
&-\frac{1}{2\mu+\lambda}\int_0^t \psi_j\int_\Omega \phi_j(\rho+n)H\, dx\, ds + \int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div} u\, dx\, ds\\
=&RHS_1+RHS_2+RHS_3+RHS_4,
\end{aligned} \tag{4.25}$$

where we have used

$$(2\mu+\lambda)\mathrm{div}u = \overline{n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta} - H,$$

and

$$\overline{n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta} = \overline{n^\alpha+\rho^\gamma} + \overline{\delta(\rho+n)^\beta}.$$

For RHS_2, we have

$$\begin{aligned} RHS_2 =& \frac{1}{2\mu+\lambda}\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)\overline{\delta(\rho+n)^\beta}\,dx\,ds \\ \le& \frac{1}{2\mu+\lambda}\liminf_{\epsilon\to 0^+}\int_0^t \psi_j \int_\Omega \phi_j \delta(\rho_\epsilon+n_\epsilon)(\rho_\epsilon+n_\epsilon)^\beta\,dx\,ds, \end{aligned} \tag{4.26}$$

because $z\mapsto z$ and $z\mapsto z^\beta$ are increasing functions.

By virtue of (4.25), (4.21), (4.26), and (4.5), we have

$$\begin{aligned} &\int_0^t\int_\Omega (\rho+n)\mathrm{div}u\,dx\,ds \\ \le& \frac{1}{2\mu+\lambda}\lim_{\epsilon\to 0^+}\int_0^t \psi_j\int_\Omega \phi_j(n_\epsilon^\alpha+\rho_\epsilon^\gamma)(\rho_\epsilon+n_\epsilon)\,dx\,ds \\ &+\frac{1}{2\mu+\lambda}\liminf_{\epsilon\to 0^+}\int_0^t \psi_j\int_\Omega \phi_j(\rho_\epsilon+n_\epsilon)\delta(\rho_\epsilon+n_\epsilon)^\beta\,dx\,ds \\ &-\frac{1}{2\mu+\lambda}\lim_{\epsilon\to 0^+}\int_0^t \psi_j\int_\Omega \phi_j(\rho_\epsilon+n_\epsilon)H_\epsilon\,dx\,ds + \int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\,dx\,ds \\ \le& \lim_{\epsilon\to 0^+}\int_0^t\int_\Omega (\rho_\epsilon+n_\epsilon)\mathrm{div}u_\epsilon\,dx\,ds + \lim_{\epsilon\to 0^+}\int_0^t\int_\Omega (\psi_j\phi_j-1)(\rho_\epsilon+n_\epsilon)\mathrm{div}u_\epsilon\,dx\,ds \\ &+\int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\,dx\,ds. \end{aligned} \tag{4.27}$$

Letting $j\to\infty$ in (4.27), we complete the proof of the lemma. □

Step 3: Strong convergence of ρ_ϵ and n_ϵ

The main task is to show the strong convergence of ρ_ϵ and n_ϵ. This yields Proposition 4.2. In particular, with (4.22), letting $\epsilon\to 0^+$ in (4.6), we deduce that

$$\int_\Omega \left[\overline{\rho\log\rho} - \rho\log\rho + \overline{n\log n} - n\log n\right](t)\,dx \le 0.$$

Thanks to the convexity of $z \mapsto z\log z$, we have

$$\overline{\rho\log\rho} \ge \rho\log\rho \quad \text{and} \quad \overline{n\log n} \ge n\log n$$

a.e. on Q_T. This turns out that

$$\int_\Omega \left[\overline{\rho\log\rho} - \rho\log\rho + \overline{n\log n} - n\log n\right](t)\,dx = 0.$$

Hence we get

$$\overline{\rho\log\rho} = \rho\log\rho \quad \text{and} \quad \overline{n\log n} = n\log n$$

a.e. on Q_T, which combined with Lemma 4.1 implies strong convergence of $\rho_\epsilon, n_\epsilon$ in $L^\beta(Q_T)$. Thus we complete the proof.

With Proposition 4.2, we recover a global weak solution to the system (4.2), (4.3) and (4.4) with $\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta}$ replaced by $n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta$.

Proposition 4.8. *Suppose $\beta > \max\{4, \alpha, \gamma\}$. For any given $\delta > 0$, there exists a global weak solution $(\rho_\delta, n_\delta, u_\delta)$ to the following system over $\Omega \times (0, \infty)$:*

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \operatorname{div}\big[(\rho+n)u\otimes u\big] + \nabla(n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta) = \mu\Delta u + (\mu+\lambda)\nabla\operatorname{div}u, \end{cases} \tag{4.28}$$

with initial and boundary condition

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \overline{\Omega}, \tag{4.29}$$

$$u|_{\partial\Omega} = 0 \text{ for } t \ge 0, \tag{4.30}$$

such that for any given $T > 0$, the following estimates

$$\sup_{t\in[0,T]} \|\rho_\delta(t)\|^\gamma_{L^\gamma(\Omega)} \le C(\rho_0, n_0, M_0), \tag{4.31}$$

$$\sup_{t\in[0,T]} \|n_\delta(t)\|^\alpha_{L^\alpha(\Omega)} \le C(\rho_0, n_0, M_0), \tag{4.32}$$

$$\delta \sup_{t\in[0,T]} \|(\rho_\delta(t), n_\delta(t))\|^\beta_{L^\beta(\Omega)} \le C(\rho_0, n_0, M_0), \tag{4.33}$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho_\delta + n_\delta}(t)u_\delta(t)\|^2_{L^2(\Omega)} \le C(\rho_0, n_0, M_0), \tag{4.34}$$

$$\int_0^T \|u_\delta(t)\|^2_{H^1_0(\Omega)}\,dt \le C(\rho_0, n_0, M_0), \tag{4.35}$$

and

$$\|(\rho_\delta(t), n_\delta(t))\|_{L^{\beta+1}(Q_T)} \le C(\beta, \delta, \rho_0, n_0, M_0) \tag{4.36}$$

hold, where the norm $\|(\cdot,\cdot)\|$ *denotes* $\|\cdot\|+\|\cdot\|$. *Besides, if* $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$, *we have*

$$\begin{cases} \frac{1}{c_0}\rho_\delta(x,t) \le n_\delta(x,t) \le c_0\rho_\delta(x,t) \quad \textit{for a.e. } (x,t) \in Q_T, \\ \sup\limits_{t\in[0,T]} \|(\rho_\delta, n_\delta)(t)\|_{L^{\alpha_1}(\Omega)}^{\alpha_1} \le C(\rho_0, n_0, M_0), \end{cases} \tag{4.37}$$

where $\alpha_1 = \max\{\alpha,\gamma\}$.

5. Passing to the limit in the artificial pressure term as $\delta \to 0^+$

In this section, we shall recover the weak solution to (1.1)–(1.3) by passing to the limit of $(\rho_\delta, n_\delta, u_\delta)$ as $\delta \to 0$. Note that the estimate (4.36) depends on δ. Thus to begin with, we have to get the higher integrability estimates of the pressure term uniformly for δ. The new compactness Theorem 2.2 is crucial for the proof of section 5.2.

Let C be a generic constant independent of δ which will be used throughout this section.

5.1. Passing to the limit as $\delta \to 0^+$

We can follow the similar argument as in [12] to have the higher integrability estimates of ρ and n in the following lemma. We only need to modify the proof a little bit on n.

Lemma 5.1. *Let* $(\rho_\delta, n_\delta, u_\delta)$ *be the solution stated in Proposition 4.8, then*

$$\int_0^T\int_\Omega (n_\delta^{\alpha+\theta_1} + \rho_\delta^{\gamma+\theta_2} + \delta n_\delta^{\beta+\theta_1} + \delta\rho_\delta^{\beta+\theta_2})\, dxdt \le C(\theta_1,\theta_2) \tag{5.1}$$

for any positive constants θ_1 *and* θ_2 *satisfying*

$$\begin{cases} \theta_1 < \frac{\alpha}{3} \textit{ and } \theta_1 \le \min\{1, \frac{2\alpha}{3}-1\};\ \theta_2 < \frac{\gamma}{3} \textit{ and } \theta_2 \le \min\{1, \frac{2\gamma}{3}-1\} \quad \text{if } \alpha,\gamma \in (\frac{3}{2},\infty), \\ \theta < \frac{\max\{\alpha,\gamma\}}{3} \textit{ and } \theta \le \min\{1, \frac{2\max\{\alpha,\gamma\}}{3}-1\} \quad \text{if } \gamma \in (\frac{3}{2},\infty),\ \alpha \in [1,\infty),\ \textit{and } \frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0, \end{cases}$$

where $\theta = \theta_1 = \theta_2$.

In view of (5.1) and (4.37), we have the following corollary.

Corollary 5.2. *Let* $(\rho_\delta, n_\delta, u_\delta)$ *be the solution stated in Proposition 4.8, if* $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$, *then*

$$\int_0^T\int_\Omega \rho_\delta^{\max\{\alpha+\theta_1,\gamma+\theta_2\}} + n_\delta^{\max\{\alpha+\theta_1,\gamma+\theta_2\}}\, dx\, dt \le C. \tag{5.2}$$

With (4.31), (4.32), (4.34), (4.35), (4.37), (5.1), and (5.2), letting $\delta \to 0^+$ (take the subsequence if necessary), we have

Case 1. $\alpha, \gamma \in (\frac{9}{5}, \infty)$, and $\max\{\frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1\}$.

$$\begin{cases} \rho_\delta \to \rho \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)) \text{ and weakly in } L^{\gamma+\theta_2}(Q_T) \text{ as } \delta \to 0^+, \\ n_\delta \to n \text{ in } C([0,T]; L^\alpha_{weak}(\Omega)) \text{ and weakly in } L^{\alpha+\theta_1}(Q_T) \text{ as } \delta \to 0^+, \\ u_\delta \to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \to (\rho+n)u \text{ in } C([0,T]; L_{weak}^{\frac{2\min\{\gamma,\alpha\}}{\min\{\gamma,\alpha\}+1}}) \cap C([0,T]; H^{-1}(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta u_\delta, n_\delta u_\delta) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \otimes u_\delta \to (\rho+n)u \otimes u \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ n^\alpha_\delta + \rho^\gamma_\delta \to \overline{n^\alpha + \rho^\gamma} \text{ weakly in } L^{\min\{\frac{\gamma+\theta_2}{\alpha}, \frac{\alpha+\theta_1}{\gamma}\}}(Q_T) \text{ as } \delta \to 0^+, \\ \delta(\rho_\delta + n_\delta)^\beta \to 0 \text{ in } L^1(Q_T) \text{ as } \delta \to 0^+. \end{cases} \tag{5.3}$$

Case 2. $\alpha \in [1, \infty),\ \gamma \in (\frac{9}{5}, \infty)$, and $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$.

$$\begin{cases} (\rho_\delta, n_\delta) \to (\rho, n) \text{ in } C([0,T]; L^{\max\{\gamma,\alpha\}}_{weak}(\Omega)) \text{ and weakly in } L^{\max\{\alpha+\theta_1, \gamma+\theta_2\}}(Q_T) \text{ as } \delta \to 0^+, \\ u_\delta \to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \to (\rho+n)u \text{ in } C([0,T]; L_{weak}^{\frac{2\max\{\gamma,\alpha\}}{\max\{\gamma,\alpha\}+1}}) \cap C([0,T]; H^{-1}(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta u_\delta, n_\delta u_\delta) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \otimes u_\delta \to (\rho+n)u \otimes u \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ n^\alpha_\delta + \rho^\gamma_\delta \to \overline{n^\alpha + \rho^\gamma} \text{ weakly in } L^{\max\{\frac{\gamma+\theta_2}{\alpha}, \frac{\alpha+\theta_1}{\gamma}\}}(Q_T) \text{ as } \delta \to 0^+, \\ \delta(\rho_\delta + n_\delta)^\beta \to 0 \text{ in } L^1(Q_T) \text{ as } \delta \to 0^+, \\ \frac{1}{c_0}\rho_\delta(x,t) \le n_\delta(x,t) \le c_0\rho_\delta(x,t), \text{ for a.e., } (x,t) \in Q_T, \\ \frac{1}{c_0}\rho(x,t) \le n(x,t) \le c_0\rho(x,t), \text{ for a.e., } (x,t) \in Q_T. \end{cases} \tag{5.4}$$

In summary, the limit (ρ, n, u) solves the following system in the sense of distribution over $\Omega \times [0,T]$ for any given $T > 0$:

$$\begin{cases} n_t + \mathrm{div}(nu) = 0, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \mathrm{div}\big[(\rho+n)u \otimes u\big] + \nabla(\overline{\rho^\gamma + n^\alpha}) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u, \end{cases} \tag{5.5}$$

with initial and boundary condition

$$(\rho, n, (\rho+n)u)|_{t=0} = (\rho_0, n_0, M_0) \quad \text{on } \overline{\Omega}, \tag{5.6}$$

$$u|_{\partial\Omega} = 0 \quad \text{for } t \ge 0, \tag{5.7}$$

where the convergence of the approximate initial data in (4.29) is due to (3.4).

To recover a weak solution to (1.1)–(1.3), we only need to show the following claim:

- **Claim.** $\overline{\rho^\gamma + n^\alpha} = \rho^\gamma + n^\alpha$.

5.2. The weak limit of pressure

The objective of this subsection is to show the strong convergence of ρ_δ and n_δ as δ goes to zero. This allows us to prove $\overline{\rho^\gamma + n^\alpha} = \rho^\gamma + n^\alpha$ as $\delta \to 0$. From now, we need that ρ_δ is bounded in $L^q(Q_T)$ for some $q > 2$. By Lemma 5.1, we need the restriction $\gamma > \frac{9}{5}$.

We consider a family of cut-off functions introduced in [12] and references therein, i.e.,

$$T_k(z) = kT(\frac{z}{k}),\ \ z \in \mathbb{R},\ \ k = 1, 2, \cdots$$

where $T \in C^\infty(\mathbb{R})$ satisfying

$$T(z) = z \ for\ z \le 1, \quad T(z) = 2\ for\ z \ge 3. \quad T\ is\ concave.$$

The cut-off functions will be used in particular to handle the cross terms due to the two-variable pressure, see the proof of Lemma 5.5. Since $\rho_\delta \in L^2(Q_T), u_\delta \in L^2(0, T; H_0^1(\Omega))$, Lemma 2.5 suggests that (ρ_δ, u_δ) is a renormalized solution of $(5.5)_2$. Thus we have

$$[T_k(\rho_\delta)]_t + \mathrm{div}[T_k(\rho_\delta)u_\delta] + [T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T). \tag{5.8}$$

For any given k, $T_k(\rho_\delta)$ is bounded in $L^\infty(Q_T)$. Passing to the limit as $\delta \to 0^+$ (taking the subsequence if necessary), we have

$$\begin{aligned}
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ weak* in } L^\infty(Q_T),\\
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0, T]; L^p_{weak}(\Omega)),\ \text{for any } p \in [1, \infty),\\
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0, T]; H^{-1}(\Omega)),\\
&[T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\mathrm{div}u_\delta \to \overline{[T_k'(\rho)\rho - T_k(\rho)]\mathrm{div}u} \text{ weakly in } L^2(Q_T).
\end{aligned}$$

This yields

$$[\overline{T_k(\rho)}]_t + \mathrm{div}[\overline{T_k(\rho)}u] + \overline{[T_k'(\rho)\rho - T_k(\rho)]\mathrm{div}u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)). \tag{5.9}$$

Similarly, we have

$$[T_k(n_\delta)]_t + \mathrm{div}[T_k(n_\delta)u_\delta] + [T_k'(n_\delta)n_\delta - T_k(n_\delta)]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T), \tag{5.10}$$

and

$$[\overline{T_k(n)}]_t + \mathrm{div}[\overline{T_k(n)}u] + \overline{[T_k'(n)n - T_k(n)]\mathrm{div}u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)). \tag{5.11}$$

Denote

$$\begin{aligned}
H_\delta :=& \rho_\delta^\gamma + n_\delta^\alpha - (2\mu + \lambda)\mathrm{div}u_\delta,\\
\overline{H} :=& \overline{\rho^\gamma + n^\alpha} - (2\mu + \lambda)\mathrm{div}u.
\end{aligned}$$

We will have the following Lemma on H_δ and $\overline{H}$.

Lemma 5.3. *Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.8 and (ρ, n, u) be the limit, then*

$$\lim_{\delta\to 0^+}\int_0^T \psi \int_\Omega \phi H_\delta\big[T_k(\rho_\delta)+T_k(n_\delta)\big]\,dx\,dt = \int_0^T \psi \int_\Omega \phi \overline{H}\ \big[\overline{T_k(\rho)}+\overline{T_k(n)}\big]\,dx\,dt, \tag{5.12}$$

for any $\psi \in C_0^\infty(0,T)$ and $\phi \in C_0^\infty(\Omega)$.

Proof. The proof is similar to the work of [12]. □

Now let us focus on the following Proposition.

Proposition 5.4. *For any $\alpha, \gamma > \frac{9}{5}$ and $\max\{\frac{3\gamma}{4}, \gamma-1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma+1, \frac{5\gamma}{3}-1\}$, then*

$$\overline{n^\alpha + \rho^\gamma} = n^\alpha + \rho^\gamma \tag{5.13}$$

a.e. on Q_T. In addition, if the initial data satisfy

$$\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0,$$

then for $\alpha \ge 1,\ \gamma > \frac{9}{5}$, (5.13) holds.

There are two steps to prove it.

Step 1: Study for the weak limit of $\rho_\delta^\gamma + n_\delta^\alpha$

Relying on Theorem 2.2 with $\nu_K = 0$, we are able to show the following lemma. It is crucial to obtain Proposition 5.4.

Lemma 5.5. *Let (ρ_δ, n_δ) be the solutions constructed in Proposition 4.8, and (ρ, n) be the limit, then*

$$\int_0^t \psi \int_\Omega \phi\Big[\overline{T_k(\rho)}+\overline{T_k(n)}\Big]\big(\overline{\rho^\gamma + n^\alpha}\big)\,dx\,ds \le \int_0^t \psi \int_\Omega \phi\overline{\big[T_k(\rho)+T_k(n)\big]\big(\rho^\gamma+n^\alpha\big)}\,dx\,ds, \tag{5.14}$$

for any $t \in [0,T]$ and any $\psi \in C[0,t]$, $\phi \in C(\overline{\Omega})$ where $\psi, \phi \ge 0$, $\alpha, \gamma > \frac{9}{5}$ and

$$\max\{\frac{3\gamma}{4}, \gamma-1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma+1, \frac{5\gamma}{3}-1\}.$$

In additional, if initial data satisfies

$$\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0,$$

then for $\alpha \ge 1,\ \gamma > \frac{9}{5}$, (5.14) holds.

Proof.

$$
\begin{aligned}
&\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi\big[T_k(\rho_\delta)+T_k(n_\delta)\big]\big(\rho_\delta^\gamma+n_\delta^\alpha\big)\,dx\,ds\\
&=\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\gamma\,dx\,ds+\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(\rho_\delta)n_\delta^\alpha\,dx\,ds\\
&\quad+\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(n_\delta)\rho_\delta^\gamma\,dx\,ds+\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(n_\delta)n_\delta^\alpha\,dx\,ds\\
&=\sum_{i=1}^4 IV_i.
\end{aligned}
\tag{5.15}
$$

For IV_1, since $z\mapsto T_k(z)$ and $z\mapsto z^\gamma$ are increasing functions, we have

$$
\begin{aligned}
0&\le\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi\big[T_k(\rho_\delta)-T_k(\rho)\big]\big[\rho_\delta^\gamma-\rho^\gamma\big]\,dx\,ds\\
&=\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)\rho^\gamma}\,dx\,ds-\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)}\rho^\gamma\,dx\,ds-\int_0^t \psi\int_\Omega \phi T_k(\rho)\overline{\rho^\gamma}\,dx\,ds\\
&\quad+\int_0^t \psi\int_\Omega \phi T_k(\rho)\rho^\gamma\,dx\,ds\\
&=\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)\rho^\gamma}\,dx\,ds-\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)}\,\overline{\rho^\gamma}\,dx\,ds\\
&\quad+\int_0^t \psi\int_\Omega \phi\big[\overline{T_k(\rho)}-T_k(\rho)\big]\big(\overline{\rho^\gamma}-\rho^\gamma\big)\,dx\,ds\\
&\le\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)\rho^\gamma}\,dx\,ds-\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)}\,\overline{\rho^\gamma}\,dx\,ds
\end{aligned}
\tag{5.16}
$$

where we have used the fact $\overline{\rho^\gamma}\ge\rho^\gamma$ and $\overline{T_k(\rho)}\le T_k(\rho)$, which could be done by the convexity of $z\mapsto z^\gamma$ and the concavity of $z\mapsto T_k(z)$.

Thanks to (5.16), we have

$$
\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)}\,\overline{\rho^\gamma}\,dx\,ds\le\int_0^t \psi\int_\Omega \phi\overline{T_k(\rho)\rho^\gamma}\,dx\,ds=\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\gamma\,dx\,ds=IV_1.
\tag{5.17}
$$

Similar to (5.17), we have

$$
\int_0^t \psi\int_\Omega \phi\overline{T_k(n)}\,\overline{n^\alpha}\,dx\,ds\le\lim_{\delta\to0^+}\int_0^t \psi\int_\Omega \phi T_k(n_\delta)n_\delta^\alpha\,dx\,ds=IV_4.
\tag{5.18}
$$

For IV_2, we need to discuss the sizes of α and γ in order to guarantee the boundedness of ρ_δ^α and n_δ^γ in $L^q(Q_T)$ for some $q>1$.

Case 1. $\alpha, \gamma > \frac{9}{5}$ and $\max\{\frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1\}$.

In this case, there exist two positive constants $\theta_1 \in (\gamma - \alpha, \min\{\frac{\alpha}{3}, 1, \frac{2\alpha}{3} - 1\})$ and $\theta_2 \in (\alpha - \gamma, \min\{\frac{\gamma}{3}, 1, \frac{2\gamma}{3} - 1\})$, since $\max\{\frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma+1)}{5}\} < \alpha < \min\{\frac{4\gamma}{3}, \gamma + 1, \frac{5\gamma}{3} - 1\}$ implies that

$$\gamma - \alpha < \min\{\frac{\alpha}{3}, 1, \frac{2\alpha}{3} - 1\}, \quad \text{and} \quad \alpha - \gamma < \min\{\frac{\gamma}{3}, 1, \frac{2\gamma}{3} - 1\}.$$

Note that we are able to take θ_1 and θ_2 here the same as those in Lemma 5.1. Then there exists a positive integer k_2 large enough such that

$$\begin{cases} 0 < \frac{\alpha k_2}{k_2 - 1} - \frac{1}{k_2 - 1} < \gamma + \theta_2, \\ 0 < \frac{\alpha k_2}{k_2 - 1} - \frac{1}{k_2 - 1} < \alpha + \theta_1. \end{cases} \tag{5.19}$$

In this case, $d_\delta = \rho_\delta + n_\delta$ is bounded in $L^{\frac{\alpha k_2}{k_2-1} - \frac{1}{k_2-1}}(Q_T)$. Then

$$\begin{aligned} IV_2 =& \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta) d_\delta^\alpha (A_\delta^\alpha - A^\alpha)\, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta) d_\delta^\alpha A^\alpha \, dx\, ds \\ \geq& - 2kC \lim_{\delta \to 0^+} \Big(\int_0^t \int_\Omega d_\delta \big| A_\delta^\alpha - A^\alpha \big|^{k_2} \, dx\, ds \Big)^{\frac{1}{k_2}} \Big(\int_0^t \int_\Omega d_\delta^{\frac{\alpha k_2}{k_2-1} - \frac{1}{k_2-1}} \, dx\, ds \Big)^{\frac{k_2-1}{k_2}} \\ &+ \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi T_k(B d_\delta) d_\delta^\alpha A^\alpha \, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi [T_k(B_\delta d_\delta) - T_k(B d_\delta)] d_\delta^\alpha A^\alpha \, dx\, ds \\ \geq& - 2kC\alpha \lim_{\delta \to 0^+} \Big(\int_0^t \int_\Omega d_\delta \Big| \big(\max\{A_\delta, A\} \big)^{\alpha - 1} \big| A_\delta - A \big| \Big|^{k_2} \, dx\, dt \Big)^{\frac{1}{k_2}} \\ &+ \int_0^t \psi \int_\Omega \phi \overline{T_k(Bd) d^\alpha} A^\alpha \, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi [T_k(B_\delta d_\delta) - T_k(B d_\delta)] d_\delta^\alpha A^\alpha \, dx\, ds \\ \geq& - 2kC\alpha \lim_{\delta \to 0^+} \Big(\int_0^t \int_\Omega d_\delta \big| A_\delta - A \big|^{k_2} \, dx\, dt \Big)^{\frac{1}{k_2}} \\ &+ \int_0^t \psi \int_\Omega \phi \overline{T_k(Bd) d^\alpha} A^\alpha \, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi [T_k(B_\delta d_\delta) - T_k(B d_\delta)] d_\delta^\alpha A^\alpha \, dx\, ds, \end{aligned} \tag{5.20}$$

where $(A_\delta, B_\delta) = (\frac{n_\delta}{d_\delta}, \frac{\rho_\delta}{d_\delta})$ with $d_\delta = \rho_\delta + n_\delta$, $(A, B) = (\frac{n}{d}, \frac{\rho}{d})$ with $d = \rho + n$.

In view of Theorem 2.2 with $\nu_K = 0$, (5.20), and the arguments similar to (5.16), we have

$$\begin{aligned} IV_2 \geq& \int_0^t \psi \int_\Omega \phi \overline{T_k(Bd)}\; \overline{d^\alpha} A^\alpha \, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi [T_k(B_\delta d_\delta) - T_k(B d_\delta)] d_\delta^\alpha A^\alpha \, dx\, ds \\ =& \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\; \overline{d^\alpha} A^\alpha \, dx\, ds + \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi \Big(T_k(B d_\delta) - T_k(\rho_\delta) \Big) \overline{d^\alpha} A^\alpha \, dx\, ds \\ &+ \lim_{\delta \to 0^+} \int_0^t \psi \int_\Omega \phi [T_k(\rho_\delta) - T_k(B d_\delta)] d_\delta^\alpha A^\alpha \, dx\, ds \end{aligned} \tag{5.21}$$

$$
= \int_0^t \psi \int_\Omega \phi\overline{T_k(\rho)}\ \overline{n^\alpha}\,dx\,ds + \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi\Big(T_k(Bd_\delta) - T_k(\rho_\delta)\Big)\ \overline{d^\alpha}A^\alpha\,dx\,ds +
$$

$$
\lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi\overline{T_k(\rho)}\ (d_\delta^\alpha A^\alpha - n_\delta^\alpha)\,dx\,ds + \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi[T_k(\rho_\delta) - T_k(Bd_\delta)]d_\delta^\alpha A^\alpha\,dx\,ds.
$$

In view of Theorem 2.2 with $\nu_K = 0$, in particular, of (2.9), we have

$$
\begin{cases} n_\delta - Ad_\delta \to 0 \quad \text{a.e. in } Q_T, \\ \rho_\delta - Bd_\delta \to 0 \quad \text{a.e. in } Q_T, \end{cases} \tag{5.22}
$$

as $\delta \to 0^+$ (take the subsequence if necessary). $(5.22)_2$ implies that

$$
T_k\big(Bd_\delta\big) - T_k\big(\rho_\delta\big) \to 0 \quad \text{a.e. in } Q_T \tag{5.23}
$$

as $\delta \to 0^+$ (take the subsequence if necessary).

Since $\Big(T_k(Bd_\delta) - T_k(\rho_\delta)\Big)\ \overline{d^\alpha}A^\alpha$, $\overline{T_k(\rho)}\ (d_\delta^\alpha A^\alpha - n_\delta^\alpha)$, and $[T_k(\rho_\delta) - T_k(Bd_\delta)]d_\delta^\alpha A^\alpha$ are bounded uniformly for δ in $L^{\frac{\min\{\alpha+\theta_1,\gamma+\theta_2\}}{\alpha}}(Q_T)$ norm for any fixed $k > 0$, we can use the Egrov theorem to conclude that the last three terms on the right hand side of (5.21) vanish. Then we have

$$
IV_2 \geq \int_0^t \psi \int_\Omega \phi\overline{T_k(\rho)}\ \overline{n^\alpha}\,dx\,ds. \tag{5.24}
$$

Case 2. $\alpha \in [1, \infty)$, $\gamma \in (\frac{9}{5}, \infty)$, and $\frac{1}{c_0}\rho_0 \leq n_0 \leq c_0\rho_0$.

In this case, we have (5.2). Then repeating the corresponding steps in Case 1, we get (5.24).

For IV_3, we have

$$
\begin{aligned}
IV_3 &= \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi T_k(n_\delta)\rho_\delta^\gamma\,dx\,ds \\
&= \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi T_k\big(Ad_\delta\big)\rho_\delta^\gamma\,dx\,ds + \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi\Big(T_k(n_\delta) - T_k\big(Ad_\delta\big)\Big)\rho_\delta^\gamma\,dx\,ds \\
&= \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi T_k\big(Ad_\delta\big)B^\gamma d_\delta^\gamma\,dx\,ds + \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi T_k\big(Ad_\delta\big)\Big(\rho_\delta^\gamma - B^\gamma d_\delta^\gamma\Big)\,dx\,ds \\
&\quad + \lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi\Big(T_k(n_\delta) - T_k\big(Ad_\delta\big)\Big)\rho_\delta^\gamma\,dx\,ds.
\end{aligned} \tag{5.25}
$$

Similar to the proof of (5.17), we have

$$
\begin{aligned}
&\lim_{\delta\to0^+}\int_0^t \psi \int_\Omega \phi T_k\big(Ad_\delta\big)B^\gamma d_\delta^\gamma\,dx\,ds \\
&\geq \int_0^t \psi \int_\Omega \phi\overline{T_k\big(Ad\big)}B^\gamma\overline{d^\gamma}\,dx\,ds
\end{aligned}
$$

$$= \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi\big[T_k(Ad_\delta) - T_k(n_\delta)\big] B^\gamma \overline{d^\gamma}\, dx\, ds + \int_0^t \psi \int_\Omega \phi \overline{T_k(n)} B^\gamma \overline{d^\gamma}\, dx\, ds \tag{5.26}$$

$$= \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi\big[T_k(Ad_\delta) - T_k(n_\delta)\big] B^\gamma \overline{d^\gamma}\, dx\, ds + \int_0^t \psi \int_\Omega \phi \overline{T_k(n)}\,\overline{\rho^\gamma}\, dx\, ds$$

$$+ \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \overline{T_k(n)} \Big(B^\gamma d_\delta^\gamma - \rho_\delta^\gamma\Big)\, dx\, ds.$$

For $\alpha, \gamma > \frac{9}{5}$ and $\alpha \in (\gamma - \theta_1, \gamma + \theta_2)$, we have $\gamma < \alpha + \theta_1$. In this case, d_δ^γ and $\overline{d^\gamma}$ are bounded uniformly for δ in $L^{\frac{\min\{\alpha+\theta_1,\gamma+\theta_2\}}{\gamma}}(Q_T)$ norm. For $\alpha \in [1, \infty)$, $\gamma \in (\frac{9}{5}, \infty)$, and $\frac{1}{c_0}\rho_0 \le n_0 \le c_0\rho_0$, we have (5.2) which implies that d_δ^γ and $\overline{d^\gamma}$ are bounded uniformly for δ in $L^{\frac{\max\{\alpha+\theta_1,\gamma+\theta_2\}}{\gamma}}(Q_T)$ norm. Then using some similar arguments as in the estimates of IV_2, we conclude that the last two terms on the right hand side of (5.25) and the first and the third terms on the right hand side of (5.26) vanish. Thus

$$IV_3 \ge \int_0^t \psi \int_\Omega \phi \overline{T_k(n)}\,\overline{\rho^\gamma}\, dx\, ds. \tag{5.27}$$

(5.15) combined with the estimates of IV_i, $i = 1, 2, 3, 4$, i.e., (5.17), (5.18), (5.24), and (5.27), we have

$$\begin{aligned} &\int_0^t \psi \int_\Omega \phi \Big[\overline{T_k(\rho)} + \overline{T_k(n)}\Big] \big(\overline{\rho^\gamma + n^\alpha}\big)\, dx\, ds \\ &\le \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big(\rho_\delta^\gamma + n_\delta^\alpha\big)\, dx\, ds, \end{aligned} \tag{5.28}$$

where we have used

$$\overline{\rho^\gamma} + \overline{n^\alpha} = \overline{\rho^\gamma + n^\alpha}.$$

(5.28) implies (5.14). The proof of the lemma is complete. □

Since ψ and ϕ are arbitrary, we immediately get

Corollary 5.6. *Let (ρ_δ, n_δ) be the solutions constructed in Proposition 4.8, and (ρ, n) be the limit, then*

$$\Big[\overline{T_k(\rho)} + \overline{T_k(n)}\Big]\big(\overline{\rho^\gamma + n^\alpha}\big) \le \overline{\big[T_k(\rho) + T_k(n)\big]\big(\rho^\gamma + n^\alpha\big)}$$

a.e. on $\Omega \times (0, T)$.

Step 2: Strong convergence of ρ_δ and n_δ

Here, we want to show the strong convergence of ρ_δ and n_δ. This allows us to have Proposition 5.4. As in [12], we define

$$L_k(z) = \begin{cases} z\log z, & 0 \le z \le k, \\ z\log k + z\displaystyle\int_k^z \frac{T_k(s)}{s^2}\,ds, & z \ge k, \end{cases}$$

satisfying

$$L_k(z) = \beta_k z - 2k \ for \ all \ z \ge 3k,$$

where

$$\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2}\,ds + \frac{2}{3}.$$

We denote $b_k(z) := L_k(z) - \beta_k z$ where $b'_k(z) = 0$ for all large z, and

$$b'_k(z)z - b_k(z) = T_k(z). \tag{5.29}$$

Note that $\rho_\delta, n_\delta \in L^2(Q_T)$, $\rho, n \in L^2(Q_T)$, and $u_\delta, u \in L^2(0,T;H_0^1(\Omega))$. By Lemma 2.5, we conclude that (n_δ, u_δ), (ρ_δ, u_δ), (n,u) and (ρ,u) are the renormalized solutions of $(4.28)_i$ and $(5.5)_i$ for $i = 1,2$, respectively. Thus we have

$$\begin{cases} [b_k(f_\delta)]_t + \mathrm{div}\big[b_k(f_\delta)u_\delta\big] + \big[b'_k(f_\delta)f_\delta - b_k(f_\delta)\big]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T), \\ [b_k(f)]_t + \mathrm{div}\big[b_k(f)u\big] + \big[b'_k(f)f - b_k(f)\big]\mathrm{div}u = 0 \quad \text{in } \mathcal{D}'(Q_T), \end{cases}$$

where $f_\delta = \rho_\delta, n_\delta$ and $f = \rho, n$. Thanks to (5.29) and $b_k(z) = L_k(z) - \beta_k z$, we arrive at

$$\begin{cases} [L_k(\rho_\delta) + L_k(n_\delta)]_t + \mathrm{div}\big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta\big] + \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T), \\ [L_k(\rho) + L_k(n)]_t + \mathrm{div}\big[\big(L_k(\rho) + L_k(n)\big)u\big] + \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u = 0 \quad \text{in } \mathcal{D}'(Q_T). \end{cases}$$

This gives

$$\begin{aligned} &[L_k(\rho_\delta) - L_k(\rho) + L_k(n_\delta) - L_k(n)]_t + \mathrm{div}\big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta - \big(L_k(\rho) + L_k(n)\big)u\big] \\ &+ \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u = 0. \end{aligned} \tag{5.30}$$

Taking ϕ_j as the test function of (5.30), and letting $\delta \to \infty$, we have

$$\begin{aligned} &\int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)]\phi_j\,dx \\ &- \lim_{\delta\to 0^+} \int_0^t \int_\Omega \big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta - \big(L_k(\rho) + L_k(n)\big)u\big] \cdot \nabla\phi_j\,dx\,ds \\ &+ \lim_{\delta\to 0^+} \int_0^t \int_\Omega \Big(\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u\Big)\phi_j\,dx\,ds = 0, \end{aligned} \tag{5.31}$$

where

$$\begin{aligned}&\phi_j \in C_0^\infty(\Omega), \quad \phi_j(x) \equiv 1 \text{ for any } x \in \Big\{x \in \Omega \Big| \mathrm{dist(x}, \partial\Omega) \geq \frac{1}{\mathrm{j}}\Big\},\ 0 \leq \phi_j \leq 1, \\ &|\nabla \phi_j| \leq c_0 j,\ \phi_j \to 1 \quad as\ m \to \infty\end{aligned} \tag{5.32}$$

for some positive c_0 independent of j.

Letting $j \to \infty$ in (5.31), we gain

$$\begin{aligned}&\int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)]\, dx \\ &= -\lim_{\delta \to 0^+} \int_0^t \int_\Omega \Big(\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div} u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div} u\Big)\, dx\, ds.\end{aligned} \tag{5.33}$$

In view of Lemma 5.3, we have

$$\begin{aligned}&- \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div} u_\delta\, dx\, ds \\ &= -\frac{1}{2\mu+\lambda} \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[(2\mu+\lambda)\mathrm{div} u_\delta - \rho_\delta^\gamma - n_\delta^\alpha\big]\, dx\, ds \\ &\quad - \frac{1}{2\mu+\lambda} \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[\rho_\delta^\gamma + n_\delta^\alpha\big]\, dx\, ds \\ &= -\frac{1}{2\mu+\lambda} \int_0^t \int_\Omega \psi_j \phi_j \big[\overline{T_k(\rho)} + \overline{T_k(n)}\big]\big[(2\mu+\lambda)\mathrm{div} u - \overline{\rho^\gamma + n^\alpha}\big]\, dx\, ds \\ &\quad - \frac{1}{2\mu+\lambda} \lim_{\delta \to 0^+} \int_0^t \int_\Omega (1 - \psi_j \phi_j)\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[(2\mu+\lambda)\mathrm{div} u_\delta - \rho_\delta^\gamma - n_\delta^\alpha\big]\, dx\, ds \\ &\quad - \frac{1}{2\mu+\lambda} \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[\rho_\delta^\gamma + n_\delta^\alpha\big]\, dx\, ds,\end{aligned} \tag{5.34}$$

where ψ_j and ϕ_j are given by (4.23) and (4.24) respectively. Letting $j \to \infty$ in (5.34), we have

$$\begin{aligned}&- \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div} u_\delta\, dx\, ds \\ &= -\frac{1}{2\mu+\lambda} \int_0^t \int_\Omega \big[\overline{T_k(\rho)} + \overline{T_k(n)}\big]\big[(2\mu+\lambda)\mathrm{div} u - \overline{\rho^\gamma + n^\alpha}\big]\, dx\, ds \\ &\quad - \frac{1}{2\mu+\lambda} \lim_{\delta \to 0^+} \int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[\rho_\delta^\gamma + n_\delta^\alpha\big]\, dx\, ds.\end{aligned} \tag{5.35}$$

In view of (5.33) and (5.35), we have

$$\int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)]\, dx$$

$$
\begin{aligned}
=&\frac{1}{2\mu+\lambda}\int_0^t\int_\Omega \left(\overline{T_k(\rho)}+\overline{T_k(n)}\right)\left(\overline{\rho^\gamma}+\overline{n^\alpha}\right)dx\,ds\\
&-\frac{1}{2\mu+\lambda}\lim_{\delta\to0^+}\int_0^t\int_\Omega \left[T_k(\rho_\delta)+T_k(n_\delta)\right]\left(\rho_\delta^\gamma+n_\delta^\alpha\right)dx\,ds\\
&+\int_0^t\int_\Omega [T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds,
\end{aligned}
$$

with Corollary 5.6, which gives

$$
\int_\Omega [\overline{L_k(\rho)}-L_k(\rho)+\overline{L_k(n)}-L_k(n)]\,dx\le\int_0^t\int_\Omega [T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds. \tag{5.36}
$$

Here we are able to control the right-hand side of (5.36) as in the following lemma.

Lemma 5.7.

$$
\lim_{k\to\infty}\int_0^t\int_\Omega [T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds=0. \tag{5.37}
$$

Proof. Recalling that $T(z)\le z$ for all z, we have

$$
\begin{aligned}
\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_T)}\le&\liminf_{\delta\to0^+}\|T_k(\rho)-T_k(\rho_\delta)\|_{L^2(Q_T)}\\
\le&C\liminf_{\delta\to0^+}\|\rho+\rho_\delta\|_{L^{\gamma+\theta_2}(Q_T)}\\
\le&C,
\end{aligned}
$$

where we have used the Hölder inequality, $\gamma+\theta_2\ge2$, (5.1), (5.3), and (5.4). With the help of this estimate, (5.3), and (5.4), one deduces

$$
\begin{aligned}
&\left|\int_0^t\int_\Omega [T_k(\rho)-\overline{T_k(\rho)}]\mathrm{div}u\,dx\,ds\right|\\
\le&\int_{Q_t\cap\{\rho\ge k\}}|T_k(\rho)-\overline{T_k(\rho)}|\,|\mathrm{div}u|\,dx\,ds+\int_{Q_t\cap\{\rho\le k\}}|T_k(\rho)-\overline{T_k(\rho)}|\,|\mathrm{div}u|\,dx\,ds\\
\le&\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_T)}\|\mathrm{div}u\|_{L^2(Q_t\cap\{\rho\ge k\})}+\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_t\cap\{\rho\le k\})}\|\mathrm{div}u\|_{L^2(Q_T)}\\
\le&C\|\mathrm{div}u\|_{L^2(Q_t\cap\{\rho\ge k\})}+C\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_t\cap\{\rho\le k\})}.
\end{aligned} \tag{5.38}
$$

Note that $T_k(z)=z$ if $z\le k$, we have

$$
\begin{aligned}
\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_t\cap\{\rho\le k\})}=&\|\rho-\overline{T_k(\rho)}\|_{L^2(Q_t\cap\{\rho\le k\})}\\
\le&\liminf_{\delta\to0^+}\|\rho_\delta-T_k(\rho_\delta)\|_{L^2(Q_T)}\\
=&\liminf_{\delta\to0^+}\|\rho_\delta-T_k(\rho_\delta)\|_{L^2(Q_T\cap\{\rho_\delta>k\})}\\
\le&2\liminf_{\delta\to0^+}\|\rho_\delta\|_{L^2(Q_T\cap\{\rho_\delta>k\})}
\end{aligned} \tag{5.39}
$$

$$\leq 2k^{1-\frac{\gamma+\theta_2}{2}} \liminf_{\delta\to 0^+} \|\rho_\delta\|_{L^{\gamma+\theta_2}(Q_T)}^{\frac{\gamma+\theta_2}{2}} \to 0$$

as $k \to \infty$, due to (5.1) and the assumption $\gamma > \frac{9}{5}$ such that $\gamma + \theta_2 > 2$.

By (5.38) and (5.39), we conclude

$$\lim_{k\to\infty} \int_0^t \int_\Omega [T_k(\rho) - \overline{T_k(\rho)}] \mathrm{div} u \, dx \, ds = 0. \tag{5.40}$$

Similarly, we have

$$\lim_{k\to\infty} \int_0^t \int_\Omega [T_k(n) - \overline{T_k(n)}] \mathrm{div} u \, dx \, ds = 0. \tag{5.41}$$

With (5.40) and (5.41), (5.37) follows. □

Note that (5.36) and (5.37), we have

$$\limsup_{k\to\infty} \int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)] \, dx \leq 0. \tag{5.42}$$

By the definition of $L(\cdot)$, it is not difficult to justify that

$$\begin{cases} \lim\limits_{k\to\infty} \left[\|L_k(\rho) - \rho \log \rho\|_{L^1(\Omega)} + \|L_k(n) - n \log n\|_{L^1(\Omega)} \right] = 0, \\ \lim\limits_{k\to\infty} \left[\|\overline{L_k(\rho)} - \overline{\rho \log \rho}\|_{L^1(\Omega)} + \|\overline{L_k(n)} - \overline{n \log n}\|_{L^1(\Omega)} \right] = 0. \end{cases} \tag{5.43}$$

Since $\rho \log \rho \leq \overline{\rho \log \rho}$ and $n \log n \leq \overline{n \log n}$ due to the convexity of $z \mapsto z \log z$, we have

$$0 \leq \int_\Omega [\overline{\rho \log \rho} - \rho \log \rho + \overline{n \log n} - n \log n] \, dx \leq 0, \tag{5.44}$$

where we have used (5.42) and (5.43). Thus we obtain

$$\overline{\rho \log \rho} = \rho \log \rho \quad \text{and} \quad \overline{n \log n} = n \log n.$$

It allows us to have the strong convergence of ρ_δ and n_δ in $L^\gamma(Q_T)$ and in $L^\alpha(Q_T)$ respectively. Therefore we proved (5.13). □

With Proposition 5.4, the proof of Theorem 1.2 can be done.

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