GLOBAL SMOOTH SOLUTIONS FOR 1D BAROTROPIC NAVIER-STOKES EQUATIONS WITH A LARGE CLASS OF DEGENERATE VISCOSITIES

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Abstract. We prove the global existence and uniqueness of smooth solutions to the one-dimensional barotropic Navier-Stokes system with degenerate viscosity $\mu(\rho) = \rho^\alpha$. We establish that the smooth solutions have possibly two different far-fields, and the initial density remains positive globally in time, for the initial data satisfying the same conditions. In addition, our result works for any $\alpha > 0$, i.e., for a large class of degenerate viscosities. In particular, our models include the viscous shallow water equations. This extends the result of Constantin-Drivas-Nguyen-Pasqualotto [5, Theorem 1.6] (on the case of periodic domain) to the case where smooth solutions connect possibly two different limits at the infinity on the whole space.

Contents

1. Introduction

We consider the one-dimensional barotropic Navier-Stokes system in the Eulerian coordinates:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + p(\rho)_x = (\mu(\rho) u_x)_x, \end{cases} \tag{1.1}$$

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where the pressure $p(\rho)$ follows the case of a polytropic perfect gas, i.e.,

$$p(\rho) = \rho^{\gamma}, \quad \gamma > 1, \tag{1.2}$$

with γ the adiabatic constant. Here, μ denotes the viscosity coefficient given by

$$\mu(\rho) = \rho^{\alpha}. \tag{1.3}$$

Notice that if $\alpha > 0$, $\mu(\rho)$ degenerates near the vacuum, i.e., near $\rho = 0$. Very often, the viscosity coefficient is assumed to be constant, i.e., $\alpha = 0$. However, in the physical context the viscosity of a gas depends on the temperature (see Chapman and Cowling [4]). In the barotropic case, the viscosity depends directly on the density. In general, the viscosity is expected to degenerate on the vacuum as a power of the density as in (1.3).

There are many results on the existence of solutions to the compressible Navier-Stokes equations with the constant viscosity for the one-dimensional case. The existence of weak solutions was first established by Kazhikhov and Shelukhin [13] for smooth enough initial data close to the equilibrium bounded away from zero. The case of discontinuous data but still bounded away from zero was addressed by Shelukhin [17, 18, 20] and then by Serre [16] and Hoff [8]. First result for vanishing initial density was obtained by Shelukhin [19]. Hoff [9] proved the existence of global weak solutions with large discontinuous initial data, possibly having different limits at the infinity. There, he also proved that the vacuum cannot form in finite time. The issues on regularity and uniqueness of solutions was first studied by Solonnikov [21] for smooth initial data and for small time. However, the regularity may blow-up as the solution gets close to vacuum. Hoff and Smoller [10] show that any weak solution of the one-dimensional Navier-Stokes equations do not have vacuum states for every time, provided that no vacuum states initially exist.

Concerning the 1D existence theory for the degenerate case (1.1), Mellet-Vasseur [15] proved the global existence and uniqueness of strong solutions with large initial data having possibly different limits at the infinity without no vacuum states in the case of $\alpha < 1/2$ and $\gamma > 1$. To control the L^{∞}-norm of $1/\rho$ globally in time, they used the relative entropy inequality based on the Bresch-Desjardins entropy, which was derived in [1] for the multi-dimensional Korteweg system of equations (for the case of $\alpha = 1$ and with an additional capillary term) and later generalized in [3]. In the one-dimensional case, a similar inequality was introduced earlier by Vaigant [22] for flows with constant viscosity.

The result of Mellet-Vasseur [15] was extended by Haspot [7] to the case of $\alpha \in (1/2, 1]$. Recently, Constantin-Drivas-Nguyen-Pasqualotto [5, Theorem 1.6] extended it to the case of $\alpha \geq 0$ and $\gamma \in [\alpha, \alpha + 1]$ with $\gamma > 1$, but they dealt with it on the periodic domain, and with an additional technical condition (see (1.6)).

In this article, we aim to extend the result [5, Theorem 1.6] to the case where smooth solutions have possibly different limits at the infinity on the whole space. This extended result is motivated by the recent works [11, 12] of the authors on the contraction property, up to a time-dependent shift, for large perturbations of viscous shocks (connecting two different end states at $x = \pm\infty$) for the one-dimensional barotropic Navier-Stokes system with degenerate viscosity. In [11, 12], solutions of the Navier-Stokes system need to be regular for the existence of the time-dependent shift.

1.1. **Main results.** We study global existence of smooth solutions to (1.1) with initial data having possibly two different limits $(\rho_{\pm}, u_{\pm})$ at $x = \pm\infty$, where $\rho_{\pm} > 0$. For that, we let $\bar{\rho}$

and $\bar{u}$ be smooth monotone functions such that

$$\bar{\rho}(x) = \rho_\pm > 0 \quad \text{and} \quad \bar{u}(x) = u_\pm, \quad \text{when } \pm x \geq 1. \tag{1.4}$$

Theorem 1.1. *Assume $\gamma > 1, \alpha > 0$, and $\gamma \in [\alpha, \alpha+1]$. Let ρ_0 and u_0 be the initial data such that*

$$\begin{aligned} &\rho_0 - \bar{\rho} \in H^k(\mathbb{R}), \qquad u_0 - \bar{u} \in H^k(\mathbb{R}), \qquad \textit{for some integer } k \geq 4, \\ &0 < \underline{\kappa}_0 \leq \rho_0(x) \leq \overline{\kappa}_0, \quad \forall x \in \mathbb{R}, \qquad \textit{for some constants } \underline{\kappa}_0, \overline{\kappa}_0, \end{aligned} \tag{1.5}$$

and

$$\partial_x u_0(x) \leq \rho_0(x)^{\gamma-\alpha}, \qquad \forall x \in \mathbb{R}, \tag{1.6}$$

where $\bar{\rho}$ and $\bar{u}$ are the smooth monotone functions satisfying (1.4).
Then there exists a global-in-time unique smooth solution (ρ, u) of (1.1)-(1.3) *such that for any $T > 0$,*

$$\begin{aligned} &\rho - \bar{\rho} \in L^\infty(0, T; H^k(\mathbb{R})) \\ &u - \bar{u} \in L^\infty(0, T; H^k(\mathbb{R})) \cap L^2(0, T; H^{k+1}(\mathbb{R})). \end{aligned}$$

Moreover, there exists constants $\underline{\kappa}(T)$ and $\overline{\kappa}(T)$ such that

$$\underline{\kappa}(T) \leq \rho(t, x) \leq \overline{\kappa}(T), \qquad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Remark 1.1. *Note that the system* (1.1) *is equivalent to the one in the mass Lagrangian coordinates for the regularity in Theorem 1.1. Therefore, the above result provides a class of global-in-time solutions smooth enough, in which the authors proved the contraction property* [11, 12] *for viscous shocks of the barotropic Navier-Stokes system in the mass Lagrangian coordinates, with any large initial data satisfying* (1.5) *and* (1.6).

Remark 1.2. *Note from the assumption on α and γ that Theorem 1.1 also holds for the viscous shallow water equations (i.e., $\gamma = 2$, $\alpha = 1$). We refer to Gerbeau-Perthame* [6] *for a derivation of the viscous shallow water equations from the incompressible Navier-Stokes equations with free boundary.*

Remark 1.3. *The initial assumptions on* (1.6) *and $k \geq 4$ in* (1.5) *are the same conditions as in* [5, Theorem 1.5]*, which is used to control the active potential* (2.9) *defined by the density and the velocity (see Lemma 2.2).*

Remark 1.4. *In* [12]*, the authors showed some stability property of entropy shock of the Euler system as the inviscid case $\nu = 0$ of the Navier-Stokes system:*

$$\begin{cases} \rho^\nu_t + (\rho^\nu u^\nu)_x = 0, \\ (\rho^\nu u^\nu)_t + (\rho^\nu (u^\nu)^2)_x + p(\rho^\nu)_x = \nu(\mu(\rho^\nu) u_x)_x. \end{cases} \tag{1.7}$$

There, the proof is based on stability for viscous shock of (1.7)*, uniform with respect to ν. This theory is to substitute the notion of inviscid limit of Navier-Stokes system for the notion of weak solution of the Euler system. More specifically, for any initial data (ρ^0, u^0) for the inviscid dynamics, consider $\mathcal{F}_{(\rho^0,u^0)}$ the set of inviscid limits ($\nu \to 0$) of solutions for* (1.7) *with suitable initial values (ρ^ν_0, u^ν_0) converging to (ρ^0, u^0). This set can be seen as a generalization of the set of entropy solutions to the Euler system with the initial data (ρ^0, u^0). In* [12]*, it was proved that the entropy shocks are stable in this class $\mathcal{F}_{(\rho^0,u^0)}$. However, the existence of the class $\mathcal{F}_{(\rho^0,u^0)}$ is subject to the existence of solutions to the Navier-Stokes system* (1.7) *for any fixed $\nu > 0$. This requirement is achieved by*

Theorem 1.1. Note that, for the initial value (ρ_0^ν, u_0^ν) *of* (1.7)*, the technical condition* (1.6) *corresponds to* $\partial_x u_0^\nu(x) \le \nu^{-1}\rho_0^\nu(x)^{\gamma-\alpha}$*, which is not restrictive in the limit process* $\nu \to 0$*.*

2. Proof of Theorem 1.1

2.1. **Idea of Proof.** Since we are looking for solutions converging to possibly two different limits $(\rho_\pm, u_\pm)$ at $x = \pm\infty$, we do not expect that solutions are integrable. Thus, as a starting point, we may take advantage of the existence result [15], for solutions (ρ, u) to satisfy $\rho - \bar\rho, u - \bar u \in L^\infty(0, T; L^2(\mathbb{R}))$. However, since the result [15] require the assumption $\alpha < 1/2$ while we consider any $\alpha > 0$, we may perturb the viscosity coefficient (1.3) by adding $\varepsilon\rho^{1/4}$ with small parameter ε as in (2.4), under which we ensure the global existence of strong solution $(\rho_\varepsilon, u_\varepsilon)$ satisfying the H^1-spatial regularity and the positive lower-bound of the density (see (2.7) and (2.8)).
To remove the ε-dependence of the approximate viscosity μ_ε as in (2.21), we may first show that the lower bound of the density ρ_ε is independent of ε as in Proposition 2.2. For that, we basically use the idea in [5] on the analysis for the time-evolution of the active potential (see Lemma 2.2). To perform the analysis, we need at least H^4-spatial regularity of $(\rho_\varepsilon, u_\varepsilon)$, which requires the initial condition (1.5).

2.2. **Approximate viscosity.** As mentioned above, we first recall the existence result in [15] as follows:

Proposition 2.1. [15] *Let* ρ_0 *and* u_0 *be the initial data such that*

$$0 < \underline{\kappa}_0 \le \rho_0(x) \le \overline{\kappa}_0, \quad \rho_0 - \bar\rho \in H^1(\mathbb{R}), \quad u_0 - \bar u \in H^1(\mathbb{R}), \tag{2.1}$$

for some constants $\underline{\kappa}_0, \overline{\kappa}_0$*. Let* $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that for some constants* $C > 0$ *and* $q \in [0, 1/2)$*,*

$$\nu(y) \ge \begin{cases} Cy^q & \forall y \le 1 \\ C & \forall y \ge 1, \end{cases} \tag{2.2}$$

and

$$\nu(y) \le C + Cy^\gamma \qquad \forall y \ge 0. \tag{2.3}$$

Then there exists a global-in-time unique strong solution (ρ, u) *of* (1.1)-(1.2) *with* $\mu = \nu$ *such that the following holds:*
For any $T > 0$*, there exist positive constants* $\underline{\beta}(T)$ *and* $\overline{\beta}(T)$ *such that*

$$\begin{aligned} &\rho - \bar\rho \in L^\infty(0, T; H^1(\mathbb{R})), \\ &u - \bar u \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})), \\ &\underline{\beta}(T) \le \rho(t, x) \le \overline{\beta}(T), \qquad \forall (t, x) \in [0, T] \times \mathbb{R}. \end{aligned}$$

To use Proposition 2.1, we consider an approximate viscosity coefficient μ_ε defined by perturbing the viscosity μ in (1.3) as follows: For any $0 < \varepsilon < 1$,

$$\mu_\varepsilon(\rho) := \max\left(\mu(\rho), \varepsilon\rho^{\alpha_*}\right), \quad \forall \rho \ge 0, \quad \text{where } \alpha_* := \frac{1}{2}\min\left(\alpha, \frac{1}{2}\right). \tag{2.4}$$

Since

$$\mu_\varepsilon(\rho) \geq \begin{cases} \varepsilon\rho^{1/4} & \forall\rho \leq 1 \\ \varepsilon & \forall\rho \geq 1, \end{cases}$$

and it follows from $\gamma \geq \alpha$ that

$$\mu_\varepsilon(\rho) \leq 1 + \rho^\gamma \qquad \forall\rho \geq 0, \tag{2.5}$$

μ_ε satisfies the assumptions (2.2) and (2.3). Therefore, for the initial datum (ρ_0, u_0) satisfying (1.5), Proposition 2.1 implies that there exists a global-in-time unique strong solution $(\rho_\varepsilon, u_\varepsilon)$ of (1.1)-(1.2) with $\mu = \mu_\varepsilon$, i.e.,

$$\begin{cases} \partial_t\rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) + \partial_x p(\rho_\varepsilon) = \partial_x(\mu_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon) \\ (\rho_\varepsilon, u_\varepsilon)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{2.6}$$

such that the following holds: For any $T > 0$, there exist positive constants $\underline{\kappa}_\varepsilon(T)$, $\overline{\kappa}_\varepsilon(T)$ and $C = C(T, \varepsilon, \underline{\kappa}_0, \overline{\kappa}_0)$ such that

$$\|\rho_\varepsilon - \bar\rho\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|u_\varepsilon - \bar u\|_{L^\infty(0,T;H^1(\mathbb{R}))} + \|u_\varepsilon - \bar u\|_{L^2(0,T;H^2(\mathbb{R}))} \leq C, \tag{2.7}$$

and

$$\underline{\kappa}_\varepsilon(T) \leq \rho_\varepsilon(t,x) \leq \overline{\kappa}_\varepsilon(T), \qquad \forall(t,x) \in (0,T)\times\mathbb{R}. \tag{2.8}$$

2.3. **Higher Sobolev regularity.** For the system (2.6), we consider the active potential

$$w_\varepsilon := -p(\rho_\varepsilon) + \mu_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon. \tag{2.9}$$

This is the potential in the momentum equation of (2.6). Indeed, its gradient is the force:

$$\rho_\varepsilon(\partial_t u_\varepsilon + u_\varepsilon\partial_x u_\varepsilon) = \partial_x w_\varepsilon.$$

Then it follows from [5, Proposition 3.1] that w_ε satisfies a forced quadratic heat equation with linear drift:

(2.10)

$$\begin{aligned}\partial_t w_\varepsilon = \frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}\partial_x^2 w_\varepsilon - \left(u_\varepsilon + \mu_\varepsilon(\rho_\varepsilon)\frac{\partial_x\rho_\varepsilon}{\rho_\varepsilon^2}\right)\partial_x w_\varepsilon + \left(\rho_\varepsilon\frac{p'(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)} - 2p(\rho_\varepsilon)\frac{\rho_\varepsilon\mu_\varepsilon'(\rho_\varepsilon) + \mu_\varepsilon(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)^2}\right)w_\varepsilon \\ - \frac{\rho_\varepsilon\mu_\varepsilon'(\rho_\varepsilon) + \mu_\varepsilon(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)^2}w_\varepsilon^2 + \left(\rho_\varepsilon\frac{p'(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)} - p(\rho_\varepsilon)\frac{\rho_\varepsilon\mu_\varepsilon'(\rho_\varepsilon) + \mu_\varepsilon(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)^2}\right)p(\rho_\varepsilon).\end{aligned}$$

Note that the new viscosity coefficient $\mu_\varepsilon(\rho_\varepsilon)/\rho_\varepsilon$ of the parabolic equation (2.10) on w_ε is less degenerate than the viscosity coefficient $\mu_\varepsilon(\rho_\varepsilon)$ of the momentum equation in (2.6). Through the coupled system of (2.10) and the continuity equation $(2.6)_1$, we obtain the higher Sobolev regularity of ρ_ε and w_ε as long as ρ_ε is positive (that is guaranteed by (2.8)) as follows:

Lemma 2.1. *Let γ, α be any real numbers. Assume that the initial data ρ_0 and u_0 satisfy*

$$\begin{aligned} &\rho_0 - \bar\rho \in H^k(\mathbb{R}), \quad u_0 - \bar u \in H^k(\mathbb{R}), \quad \textit{for some integer } k \geq 2, \\ &0 < \underline{\kappa}_0 \leq \rho_0(x) \leq \overline{\kappa}_0, \quad \forall x \in \mathbb{R}, \end{aligned} \tag{2.11}$$

for some constants $\underline{\kappa}_0, \overline{\kappa}_0$. Then, there exists a global-in-time unique smooth solution $(\rho_\varepsilon, u_\varepsilon)$ of (2.6) such that the following holds: For any $T > 0$, there exists positive constants $\underline{\kappa}_\varepsilon(T)$,

$\overline{\kappa}_\varepsilon(T)$ and $C = C(T,\gamma,\alpha,k,\varepsilon,\underline{\kappa}_0,\overline{\kappa}_0)$ such that (2.7), (2.8) and

$$\|\partial_x^k \rho_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x^{k-1} w_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x^k w_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))} + \|\partial_x^k u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x^{k+1} u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))} \le C.$$

This follows straightforwardly from [5, Lemma 4.2 and 4.3] when $\|w_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}$ is bounded. However, for the density having two different limits at the infinity, we do not have a L^2-bound on $w_\varepsilon(t,x)$ for each t. Therefore, we may prove Lemma 2.1 without using a L^2-bound on w_ε. Although we need a slight modification of the proof in [5], we present details of the proof in Appendix A for the sake of completeness and the justification on uniformity of the high Sobolev norms in Proposition 2.4.

2.4. Uniform lower bound for the density.

Lemma 2.2. *Assume the same hypotheses as in Theorem 1.1. Then, for any $T>0$, there exist positive constants C_γ and ε_γ such that*

$$w_\varepsilon(t,x) \le C_\gamma \varepsilon^\theta, \qquad \forall \varepsilon \le \varepsilon_\gamma, \quad \forall t \le T, \quad \forall x \in \mathbb{R},$$

where θ is the positive constant as follows:

$$\theta := \frac{\gamma}{\alpha-\alpha_*}, \qquad \text{where } \alpha_* \text{ is the constant as in (2.4).} \tag{2.12}$$

Proof. First of all, using Lemma 2.1 with $k\ge 4$, together with (2.6) and (2.9), we have

$$\rho_\varepsilon, u_\varepsilon, w_\varepsilon \in C^1([0,T]\times\mathbb{R}).$$

Then, note from (2.9), (2.4), (1.2), (1.3) and the initial condition (1.6) that

$$w_\varepsilon(0,x) = -p(\rho_0) + \max\left(\mu(\rho_0), \varepsilon\rho_0^{\alpha_*}\right)\partial_x u_0 \le -\rho_0^\gamma + \max\left(\rho_0^\alpha, \varepsilon\rho_0^{\alpha_*}\right)\rho_0^{\gamma-\alpha}.$$

Since, for all $x\in\mathbb{R}$,

$$\begin{aligned} w_\varepsilon(0,x) &\le \left(-\rho_0^\gamma + \rho_0^\alpha\rho_0^{\gamma-\alpha}\right)\mathbf{1}_{\{\rho_0^\alpha > \varepsilon\rho_0^{\alpha_*}\}} + \left(-\rho_0^\gamma + \varepsilon\rho_0^{\alpha_*}\rho_0^{\gamma-\alpha}\right)\mathbf{1}_{\{\rho_0^\alpha \le \varepsilon\rho_0^{\alpha_*}\}} \\ &\le \varepsilon\rho_0^{\gamma-(\alpha-\alpha_*)}\mathbf{1}_{\{\rho_0^\alpha \le \varepsilon\rho_0^{\alpha_*}\}} \le \varepsilon^{\frac{\gamma}{\alpha-\alpha_*}}, \end{aligned}$$

we have

$$w_\varepsilon(0,x) \le \varepsilon^\theta, \quad \forall x\in\mathbb{R}.$$

Since $w_\varepsilon \in C([0,T]\times\mathbb{R})$, if there exists a point $(t_0,x_0)\in(0,T]\times\mathbb{R}$ such that $w_\varepsilon(t_0,x_0) > \varepsilon^\theta$, then there exists $t_1\ge 0$ such that

$$\sup_{x\in\mathbb{R}} w_\varepsilon(t,x) \le \varepsilon^\theta \quad \forall t\in[0,t_1], \tag{2.13}$$

and

$$\sup_{x\in\mathbb{R}} w_\varepsilon(t,x) > \varepsilon^\theta \quad \forall t\in(t_1,t_0].$$

Let

$$t_2 := \sup\left\{ t\in(t_1,T] \;\middle|\; \sup_{x\in\mathbb{R}} w_\varepsilon(t,x) > \varepsilon^\theta \right\}.$$

Then,

$$\sup_{x\in\mathbb{R}} w_\varepsilon(t,x) \ge \varepsilon^\theta \quad \forall t\in[t_1,t_2].$$

Thus, using the fact that for each $t \le T$,

$$w_\varepsilon(t,x) \to -p(\rho_\pm) \le 0 \quad \text{as} \quad x \to \pm\infty,$$

we can define the function

$$w_M(t) := \max_{x\in\mathbb{R}} w_\varepsilon(t,x),$$

which is Lipschitz continuous, and differentiable almost everywhere on $[t_1,t_2]$ thanks to the regularity $w_\varepsilon \in C^1([0,T]\times\mathbb{R})$. Moreover, for each $t \in [t_1,t_2]$, there exists x_t such that

$$w_M(t) = w_\varepsilon(t,x_t).$$

Then $w_M'(t) = (\partial_t w_\varepsilon)(t,x_t)$ for a.e. $t \in (t_1,t_2)$, since

$$\begin{aligned}
w_M'(t) &= \lim_{h\to 0+} \frac{w_\varepsilon(t+h,x_{t+h}) - w_\varepsilon(t,x_t)}{h} \\
&\ge \lim_{h\to 0+} \frac{w_\varepsilon(t+h,x_t) - w_\varepsilon(t,x_t)}{h} = \partial_t w_\varepsilon(t,x_t), \\
w_M'(t) &= \lim_{h\to 0+} \frac{w_\varepsilon(t,x_t) - w_\varepsilon(t-h,x_{t-h})}{h} \\
&\le \lim_{h\to 0+} \frac{w_\varepsilon(t,x_t) - w_\varepsilon(t-h,x_t)}{h} = \partial_t w_\varepsilon(t,x_t).
\end{aligned}$$

Using this together with $\partial_x^2 w_\varepsilon(t,x_t) \le 0$, $\partial_x w_\varepsilon(t,x_t) = 0$ and $\rho_\varepsilon \mu_\varepsilon'(\rho_\varepsilon) \ge 0$, we have from (2.10) that

$$w_M'(t) \le J_1(t) w_M(t) + J_2(t), \quad t \in (t_1,t_2),$$

where (putting $\rho_M(t) := \rho_\varepsilon(t,x_t)$)

$$\begin{aligned}
J_1(t) &:= \frac{\rho_M^\gamma}{\mu_\varepsilon(\rho_M)^2}\left(\gamma\mu_\varepsilon(\rho_M) - 2\left(\rho_M \mu_\varepsilon'(\rho_M) + \mu_\varepsilon(\rho_M)\right)\right), \\
J_2(t) &:= \frac{\rho_M^{2\gamma}}{\mu_\varepsilon(\rho_M)^2}\left(\gamma\mu_\varepsilon(\rho_M) - \left(\rho_M \mu_\varepsilon'(\rho_M) + \mu_\varepsilon(\rho_M)\right)\right).
\end{aligned}$$

Since $\gamma \le \alpha + 1$, we have

$$\begin{aligned}
J_1(t) &= \frac{\rho_M^\gamma}{\mu_\varepsilon(\rho_M)^2}\left((\gamma - 2(\alpha+1))\rho_M^\alpha \mathbf{1}_{\{\rho_M^\alpha > \varepsilon\rho_M^{\alpha_*}\}} + \varepsilon\left(\gamma - 2(\alpha_*+1)\right)\rho_M^{\alpha_*}\mathbf{1}_{\{\rho_M^\alpha \le \varepsilon\rho_M^{\alpha_*}\}}\right) \\
&\le \frac{\rho_M^\gamma}{\mu_\varepsilon(\rho_M)^2}\varepsilon\left|\gamma - 2(\alpha_*+1)\right|\rho_M^{\alpha_*}\mathbf{1}_{\{\rho_M^\alpha \le \varepsilon\rho_M^{\alpha_*}\}}.
\end{aligned}$$

Moreover, using $\mu_\varepsilon(\rho_M) \ge \varepsilon\rho_M^{\alpha_*}$ and $\mu_\varepsilon(\rho_M) \ge \rho_M^\alpha$ by the definition, we have

$$J_1(t) \le \left|\gamma - 2(\alpha_*+1)\right|\rho_M^{\gamma-\alpha}\mathbf{1}_{\{\rho_M^\alpha \le \varepsilon\rho_M^{\alpha_*}\}} \le \left|\gamma - 2(\alpha_*+1)\right|\varepsilon^{\frac{\gamma-\alpha}{\alpha-\alpha_*}}.$$

Likewise, we have

$$\begin{aligned}
J_2(t) &= \frac{\rho_M^{2\gamma}}{\mu_\varepsilon(\rho_M)^2}\left((\gamma - (\alpha+1))\rho_M^\alpha \mathbf{1}_{\{\rho_M^\alpha > \varepsilon\rho_M^{\alpha_*}\}} + \varepsilon\left(\gamma - (\alpha_*+1)\right)\rho_M^{\alpha_*}\mathbf{1}_{\{\rho_M^\alpha \le \varepsilon\rho_M^{\alpha_*}\}}\right) \\
&\le \frac{\rho_M^{2\gamma}}{\mu_\varepsilon(\rho_M)^2}\varepsilon\left|\gamma - (\alpha_*+1)\right|\rho_M^{\alpha_*}\mathbf{1}_{\{\rho_M^\alpha \le \varepsilon\rho_M^{\alpha_*}\}} \\
&\le \left|\gamma - (\alpha_*+1)\right|\varepsilon^{\frac{2\gamma-\alpha}{\alpha-\alpha_*}}.
\end{aligned}$$

The above estimates and (2.13) imply that for any $t \in [t_1, t_2]$ and $\varepsilon \in (0,1)$,

$$
\begin{aligned}
w_M(t) &\le w_M(t_1)\exp\left(\int_{t_1}^t J_1(s)ds\right) + \int_{t_1}^t J_2(s)\exp\left(\int_s^t J_1(\tau)d\tau\right)ds \\
&\le \exp\left(T\left|\gamma - 2(\alpha_*+1)\right|\right)\left(\varepsilon^\theta + \varepsilon^{\frac{2\gamma-\alpha}{\alpha-\alpha_*}} T\left|\gamma-(\alpha_*+1)\right|\right),
\end{aligned} \tag{2.14}
$$

If $\gamma > \alpha$, it follows from (2.14) that for all ε satisfying

$$
\varepsilon \le \left(\frac{1}{1+T\left|\gamma-(\alpha_*+1)\right|}\right)^{\frac{\alpha-\alpha_*}{\gamma-\alpha}},
$$

the following holds:

$$
w_M(t) \le 2\exp\left(T\left|\gamma-2(\alpha_*+1)\right|\right)\varepsilon^\theta, \quad \forall t \in [t_1, t_2].
$$

If $\gamma = \alpha$, since $\theta = \frac{2\gamma-\alpha}{\alpha-\alpha_*}$, it follows from (2.14) that

$$
w_M(t) \le 2\left(1+T\left|\gamma-(\alpha_*+1)\right|\right)\exp\left(T\left|\gamma-2(\alpha_*+1)\right|\right)\varepsilon^\theta, \quad \forall \varepsilon \le 1, \quad \forall t \in [t_1, t_2].
$$

Therefore, the above estimates together with (2.13) yield that

$$
\sup_{x\in\mathbb{R}} w_\varepsilon(t,x) \le C_\gamma \varepsilon^\theta, \qquad \forall \varepsilon \le \varepsilon_\gamma, \quad \forall t \in [0, t_2],
$$

where C_γ is the constants as in (2.12).
If $t_2 < T$, then the definition of t_2 implies

$$
\sup_{x\in\mathbb{R}} w_\varepsilon(t,x) \le \varepsilon^\theta, \quad \forall t \in (t_2, T].
$$

Hence we complete the proof. □

Proposition 2.2. *Assume the same hypotheses as in Theorem 1.1. Then, for any $T > 0$, there exist positive constants $\underline{\kappa}(T) = \underline{\kappa}(T)(\gamma, \alpha, \underline{\kappa}_0)$ and $\delta_1 = \delta_1(T, \gamma, \alpha, \underline{\kappa}_0)$ (independent of ε) such that*

$$
\rho_\varepsilon(t,x) \ge \underline{\kappa}(T), \qquad \forall t \le T, \quad \forall x \in \mathbb{R}, \quad \forall \varepsilon \le \delta_1.
$$

Proof. Let

$$
q(\gamma) := \begin{cases} \theta & \text{if } \gamma > \alpha, \\ 1 & \text{if } \gamma = \alpha, \end{cases} \qquad \text{where } \theta = \frac{\gamma}{\alpha-\alpha_*} \text{ as in Lemma 2.2.}
$$

We first choose a constant $\delta_1 > 0$ such that

$$
\delta_1 := \begin{cases} \min\left(\varepsilon_\gamma, \left(\frac{\underline{\kappa}_0}{4}\right)^{\alpha-\alpha_*}, \left(\frac{2^\alpha-1}{\alpha(2^\gamma+C_\gamma)T}\right)^{\frac{\gamma}{q(\gamma)(\gamma-\alpha)}}\right) & \text{if } \gamma > \alpha, \\ \min\left(\varepsilon_\gamma, \left(\frac{\underline{\kappa}_0}{4}\right)^{\alpha}, \left(C_\gamma^{-1}(2^\alpha-1)e^{-\alpha T}\right)^{\frac{\alpha-\alpha_*}{\alpha_*}}\right) & \text{if } \gamma = \alpha, \end{cases} \tag{2.15}
$$

where $\underline{\kappa}_0$ is the constant as in (1.5), and $\varepsilon_\gamma, C_\gamma$ are the constants as in Lemma 2.2.

Then, since

$$
\delta_1 \le \begin{cases} \left(\frac{\underline{\kappa}_0}{4}\right)^{\alpha-\alpha_*} & \text{if } \gamma > \alpha, \\ \left(\frac{\underline{\kappa}_0}{4}\right)^{\alpha} & \text{if } \gamma = \alpha, \end{cases}
$$

we have $2\delta_1^{q(\gamma)/\gamma} < \underline{\kappa}_0$ for any $\gamma \ge \alpha$.
Therefore, it follows from the initial condition of (1.5) that

$$
\inf_{x\in\mathbb{R}} \rho_0(x) \ge 2\delta_1^{q(\gamma)/\gamma}.
$$

For any fixed $\varepsilon \le \delta_1$, since $\rho_\varepsilon \in C([0,T]\times\mathbb{R})$, if there exists a point $(t_0,x_0)\in(0,T]\times\mathbb{R}$ such that $\rho_\varepsilon(t_0,x_0) < 2\delta_1^{q(\gamma)/\gamma}$, then there exists $t_1 \ge 0$ such that

$$\inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) \ge 2\delta_1^{q(\gamma)/\gamma} \quad \forall t\in[0,t_1], \tag{2.16}$$

$$\inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) < 2\delta_1^{q(\gamma)/\gamma} \quad \forall t\in(t_1,t_0].$$

Then,

$$\inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) \le 2\delta_1^{q(\gamma)/\gamma} \quad \forall t\in[t_1,t_2], \tag{2.17}$$

where

$$t_2 := \sup\left\{ t\in(t_1,T] \mid \inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) < 2\delta_1^{q(\gamma)/\gamma} \right\}.$$

Thus, using $2\delta_1^{q(\gamma)/\gamma} < \underline{\kappa}_0 \le \min(\rho_-,\rho_+)$ together with the fact that for each $t\le T$,

$$\rho_\varepsilon(t,x)\to\rho_\pm \quad \text{as} \quad x\to\pm\infty,$$

we define the function

$$\rho_m(t) := \min_{x\in\mathbb{R}} \rho_\varepsilon(t,x),$$

which is Lipschitz continuous, and differentiable almost everywhere on $[t_1,t_2]$ thanks to the regularity $\rho_\varepsilon\in C^1([0,T]\times\mathbb{R})$. So, let y_t be a minimizer for $\rho_m(t)=\rho_\varepsilon(t,y_t)$. Since $\rho_m'(t) = (\partial_t\rho_\varepsilon)(t,y_t)$ for a.e. $t\in(t_1,t_2)$, and $\partial_x\rho_\varepsilon(t,y_t)=0$, we have from the continuity equation of (2.6) that

$$\rho_m'(t) = -\rho_m(t)\partial_x u_\varepsilon(y_t), \quad t\in(t_1,t_2).$$

Then, using (2.9), Lemma 2.2 with $\varepsilon\le\delta_1\le\varepsilon_\gamma$, and $\mu_\varepsilon(\rho_m)\ge\rho_m^\alpha$, we have

$$\rho_m'(t) = -\rho_m(t)\frac{p(\rho_m)+w_\varepsilon(y_t)}{\mu_\varepsilon(\rho_m)} \ge -\rho_m^{1+\gamma-\alpha} - C_\gamma\delta_1^\theta\rho_m^{1-\alpha}, \qquad t\in(t_1,t_2). \tag{2.18}$$

Case of $\gamma>\alpha$) Using (2.17) together with $q(\gamma)=\theta$, we have

$$\rho_m' \ge -(2^\gamma + C_\gamma)\delta_1^\theta\rho_m^{1-\alpha},$$

which yields

$$(\rho_m^\alpha)' \ge -\alpha(2^\gamma+C_\gamma)\delta_1^\theta, \quad t\in(t_1,t_2).$$

Thus, using (2.16), we have

$$\rho_m^\alpha(t) \ge \rho_m^\alpha(t_1) - \alpha(2^\gamma+C_\gamma)\delta_1^\theta T \ge \left(2\delta_1^{q(\gamma)/\gamma}\right)^\alpha - \alpha(2^\gamma+C_\gamma)\delta_1^\theta T, \quad \forall t\in[t_1,t_2].$$

Since $q(\gamma)=\theta$ when $\gamma>\alpha$, and

$$\delta_1 \le \left(\frac{2^\alpha-1}{\alpha(2^\gamma+C_\gamma)T}\right)^{\frac{\gamma}{q(\gamma)(\gamma-\alpha)}},$$

we have

$$\rho_m^\alpha(t) \ge \left(\delta_1^{q(\gamma)/\gamma}\right)^\alpha, \quad \forall t\in[t_1,t_2].$$

Therefore, this together with (2.16) and the definition of t_2 implies

$$\inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) \ge \delta_1^{q(\gamma)/\gamma} \quad \forall t\in[0,T].$$

Case of $\gamma=\alpha$) First, it follows from (2.18) with $\gamma=\alpha$ that

$$\rho_m' \geq -\rho_m - C_\gamma \delta_1^\theta \rho_m^{1-\alpha}, \qquad t\in(t_1,t_2).$$

Then, since

$$(\rho_m^\alpha)' \geq -\alpha\rho_m^\alpha - \alpha C_\gamma \delta_1^\theta, \qquad t\in(t_1,t_2),$$

we have

$$\rho_m^\alpha(t) \geq \rho_m^\alpha(t_1)e^{-\alpha(t-t_1)} - \alpha C_\gamma\delta_1^\theta \int_{t_1}^t e^{-\alpha(t-s)}ds,$$

which together with (2.16) yields

$$\rho_m^\alpha(t) \geq \left(2\delta_1^{q(\gamma)/\gamma}\right)^\alpha e^{-\alpha T} - C_\gamma\delta_1^\theta, \qquad \forall t\in[t_1,t_2].$$

Since $q(\gamma)/\gamma = 1/\alpha$ and $\theta=\alpha/(\alpha-\alpha_*)$ when $\gamma=\alpha$, if needed, taking δ_1 again such that

$$\delta_1 \leq \left(C_\gamma^{-1}(2^\alpha-1)e^{-\alpha T}\right)^{\frac{\alpha-\alpha_*}{\alpha_*}},$$

we have

$$\rho_m^\alpha(t) \geq e^{-\alpha T}\delta_1, \quad \forall t\in[t_1,t_2].$$

Therefore, this together with (2.16) and the definition of t_2 implies

$$\inf_{x\in\mathbb{R}} \rho_\varepsilon(t,x) \geq e^{-T}\delta_1^{1/\alpha} = e^{-T}\delta_1^{q(\gamma)/\gamma} \quad \forall t\in[0,T].$$

Hence we complete the proof. □

2.5. **Uniform bounds for the solutions** $(\rho_\varepsilon,u_\varepsilon)$**.** Thanks to Proposition 2.2, we first have the uniform upper bound for the density as follows:

Proposition 2.3. *Under the same hypotheses as in Theorem 1.1, there exists a positive constant $\overline{\kappa}(T)$ (independent of ε) such that*

$$\rho_\varepsilon(t,x) \leq \overline{\kappa}(T), \qquad \forall t\leq T, \quad \forall x\in\mathbb{R}, \quad \forall\varepsilon\leq\delta_1,$$

where δ_1 is the constant as in Proposition 2.2.

For the proof of Proposition 2.3, we refer to the proof of [15, Proposition 4.5], in which the uniform estimates (2.19) and (2.20) are crucially used to get the uniform upper bound $\overline{\kappa}(T)$ of the density: One estimate is on the uniform lower bound of the viscosity μ_ε as

$$\mu_\varepsilon(\rho_\varepsilon) \geq \rho_\varepsilon^\alpha \geq \underline{\kappa}(T)^\alpha, \qquad \forall t\leq T, \quad \forall x\in\mathbb{R}, \quad \forall\varepsilon\leq\delta_1. \tag{2.19}$$

The others are the estimates [15, Lemmas 3.1 and 3.2] on the relative entropy related to the Bresch-Desjardins entropy (see [1, 2, 3]) as follows:

$$\begin{aligned} &\sup_{0\leq t\leq T}\int_{\mathbb{R}} \Big(\rho_\varepsilon\,|u_\varepsilon-\bar{u}|^2 + p(\rho_\varepsilon|\bar{\rho})\Big)\,dx + \int_0^T\int_{\mathbb{R}} \mu_\varepsilon(\rho_\varepsilon)|\partial_x u_\varepsilon|^2 dxdt \leq K,\\ &\sup_{0\leq t\leq T}\int_{\mathbb{R}} \Big(\rho_\varepsilon\,|(u_\varepsilon-\bar{u}) + \partial_x(\varphi(\rho_\varepsilon))|^2 + p(\rho_\varepsilon|\bar{\rho})\Big)\,dx \leq K, \end{aligned} \tag{2.20}$$

where $\varphi'(\rho_\varepsilon) := \mu_\varepsilon(\rho_\varepsilon)/\rho_\varepsilon^2$, and the above constant K is independent of ε thanks to (2.5). Indeed, it follows from [15, Lemmas 3.1 and 3.2] that the constant K depends only on $T,\gamma,(\bar{\rho},\bar{u}),(\rho_0,u_0)$, and the constants appearing in (2.3).

Propositions 2.2 and 2.3 together with the above estimates (2.19)-(2.20) imply the following uniform estimates on the Sobolev norms of the solutions $(\rho_\varepsilon,u_\varepsilon)$:

Proposition 2.4. *Under the same hypotheses as in Theorem 1.1, there exists a constant C (independent of ε) such that*

$$\|\rho_\varepsilon - \bar{\rho}\|_{L^\infty(0,T;H^k(\mathbb{R}))} + \|u_\varepsilon - \bar{u}\|_{L^\infty(0,T;H^k(\mathbb{R}))} + \|u_\varepsilon - \bar{u}\|_{L^2(0,T;H^{k+1}(\mathbb{R}))} \leq C.$$

For the proof of proposition 2.4, we first refer to the proof of [15, Proposition 4.6 and 4.7], from which the constant in (2.7) does not depend on ε anymore. Then, from the proof of Lemma 2.1, we deduce that the constant C in Lemma 2.1 is independent of ε. Therefore, we have Proposition 2.4

2.6. **Conclusion.** We have shown that for any $\varepsilon \leq \delta_1$, the system (2.6) has the unique smooth solution $(\rho_\varepsilon, u_\varepsilon)$ such that Propositions 2.2, 2.3 and 2.4 hold.
We now take δ_T as

$$\delta_T = \min\left(\underline{\kappa}(T)^{\alpha-\alpha_*}, \delta_1\right),$$

where the constants $\underline{\kappa}(T)$ and δ_1 are as in Proposition 2.2.
Then, since Proposition 2.2 implies that for all $\varepsilon < \delta_T$,

$$\varepsilon\rho_\varepsilon^{\alpha_*} < \delta_T\rho_\varepsilon^{\alpha_*} \leq \underline{\kappa}(T)^{\alpha-\alpha_*}\rho_\varepsilon^{\alpha_*} \leq \rho_\varepsilon^{\alpha}, \qquad \forall t \leq T, \quad \forall x \in \mathbb{R},$$

it follows from the definition (1.3) that

$$\mu_\varepsilon(\rho_\varepsilon) = \mu(\rho_\varepsilon), \qquad \forall\varepsilon < \delta_T, \quad \forall t \leq T, \quad \forall x \in \mathbb{R}. \tag{2.21}$$

Recall that the approximate system (2.6) represents the system (1.1) with μ_ε instead of μ. Therefore, for any $T > 0$, and any ε with $\varepsilon < \delta_T$, $(\rho_\varepsilon, u_\varepsilon)$ is the unique smooth solution of (1.1) with the initial datum (ρ_0, u_0) such that Propositions 2.2, 2.3 and 2.4 hold.
Hence we complete the proof.

Appendix A. Proof of Lemma 2.1

Let $(\rho_\varepsilon, u_\varepsilon)$ be the global strong solution to (2.6) such that (2.7) and (2.8) hold.
Once the desired estimates for $k = 2$ are obtained, the remaining part proceeds by induction in k, which follows the same proof of [5, Lemma 4.3]. Therefore, we here present the proof only when $k = 2$, based on the proof of [5, Lemma 4.2].

First of all, since $\partial_x u_\varepsilon \in L^2(0,T;L^\infty(\mathbb{R}))$ by (2.7), using (2.7) and (2.8), we have

$$\begin{aligned} &w_\varepsilon \in L^2(0,T;L^\infty(\mathbb{R})), \\ &\partial_x w_\varepsilon = -p'(\rho_\varepsilon)\partial_x\rho_\varepsilon + \mu_\varepsilon'(\rho_\varepsilon)\partial_x\rho_\varepsilon\partial_x u_\varepsilon + \mu_\varepsilon(\rho_\varepsilon)\partial_x^2 u_\varepsilon \in L^2(0,T;L^2(\mathbb{R})). \end{aligned} \tag{A.1}$$

Step 1) Differentiating the equation (2.10) in space, multiplying the resulting equation by $\partial_x w_\varepsilon$ and integrating by parts, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \frac{|\partial_x w_\varepsilon|^2}{2} dx = &-\int_{\mathbb{R}} \frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}|\partial_x^2 w_\varepsilon|^2 dx + \int_{\mathbb{R}} \left(u_\varepsilon + \frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\partial_x\rho_\varepsilon\right)\partial_x w_\varepsilon \partial_x^2 w_\varepsilon dx \\ &+ \int_{\mathbb{R}} f_1(\rho_\varepsilon)|\partial_x w_\varepsilon|^2 dx + \int_{\mathbb{R}} f_1'(\rho_\varepsilon)\partial_x\rho_\varepsilon w_\varepsilon \partial_x w_\varepsilon dx - 2\int_{\mathbb{R}} f_2(\rho_\varepsilon) w_\varepsilon |\partial_x w_\varepsilon|^2 dx \\ &- \int_{\mathbb{R}} f_2'(\rho_\varepsilon)\partial_x\rho_\varepsilon w_\varepsilon^2 \partial_x w_\varepsilon dx + \int_{\mathbb{R}} f_3'(\rho_\varepsilon)\partial_x\rho_\varepsilon\partial_x w_\varepsilon dx \\ =: &-\int_{\mathbb{R}} \frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}|\partial_x^2 w_\varepsilon|^2 dx + \sum_{j=1}^{6} I_j. \end{aligned}$$

where

$$f_1(\rho) := \rho\frac{p'(\rho)}{\mu_\varepsilon(\rho)} - 2p(\rho)\frac{\rho\mu_\varepsilon'(\rho)+\mu_\varepsilon(\rho)}{\mu_\varepsilon(\rho)^2},$$
$$f_2(\rho) := \frac{\rho\mu_\varepsilon'(\rho)+\mu_\varepsilon(\rho)}{\mu_\varepsilon(\rho)^2},$$
$$f_3(\rho) := \left(\rho\frac{p'(\rho)}{\mu_\varepsilon(\rho)} - p(\rho)\frac{\rho\mu_\varepsilon'(\rho)+\mu_\varepsilon(\rho)}{\mu_\varepsilon(\rho)^2}\right)p(\rho).$$

Since, thanks to (2.8), $L^\infty([0,T]\times\mathbb{R})$-norms of ρ_ε to some power are all bounded, there exists a positive constant $C_1 = C_1(\underline{\kappa}_\varepsilon(T), \overline{\kappa}_\varepsilon(T))$ such that

$$-\int_{\mathbb{R}} \frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon}|\partial_x^2 w_\varepsilon|^2 dx \le -C_1\int_{\mathbb{R}} |\partial_x^2 w_\varepsilon|^2 dx,$$

and

$$\left\|\frac{\mu_\varepsilon(\rho_\varepsilon)}{\rho_\varepsilon^2}\right\|_{L^\infty([0,T]\times\mathbb{R})} + \sum_{j=1}^3 \left(\|f_j(\rho_\varepsilon)\|_{L^\infty([0,T]\times\mathbb{R})} + \|f_j'(\rho_\varepsilon)\|_{L^\infty([0,T]\times\mathbb{R})}\right) \le C_1.$$

Thus, the above terms I_j can be controlled as follows:

$$\begin{aligned}|I_1| &\le \|u_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})} + C_1\|\partial_x\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})}\\ &\le \frac{C_1}{2}\|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})}^2 + C\left(\|u_\varepsilon\|_{L^\infty(\mathbb{R})}^2 + \|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 + \|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2\right)\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2,\end{aligned}$$

$$\begin{aligned}|I_2| &\le C_1\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2,\\ |I_3| &\le C_1\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}\|w_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})} \le C_1\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}\left(\|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2 + \|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2\right),\\ |I_4| &\le 2C_1\|w_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2,\\ |I_5| &\le C_1\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}\|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}\\ &\le C_1\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}\left(\|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2 + \|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2\right),\\ |I_6| &\le C_1\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 + C_1\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Moreover, since it follows from (2.7) and $\bar\rho \in L^\infty(\mathbb{R})$ that

$$\partial_x\rho_\varepsilon \in L^\infty(0,T;L^2(\mathbb{R})) \quad \text{and} \quad u_\varepsilon \in L^\infty(0,T;L^\infty(\mathbb{R})), \tag{A.2}$$

we have

(A.3)
$$\frac{d}{dt}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2 + C_1\|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})}^2 \le C\left(1 + \|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 + \|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2\right)\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})}^2 + F,$$

where

$$F = C\left(1 + \|w_\varepsilon\|_{L^\infty(\mathbb{R})}^2\right).$$

Note from (A.1) that $F \in L^1((0,T))$.

Step 2) We next estimate $\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}$, to control $\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2$ in (A.3).

Differentiating the continuity equation of (2.6) twice in space, and multiplying the resulting equation by $\partial_x^2\rho_\varepsilon$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\frac{|\partial_x^2\rho_\varepsilon|^2}{2}dx &= -\int_{\mathbb{R}}\partial_x^2(u_\varepsilon\partial_x\rho_\varepsilon)\partial_x^2\rho_\varepsilon dx - \int_{\mathbb{R}}\partial_x^2(\rho_\varepsilon\partial_x u_\varepsilon)\partial_x^2\rho_\varepsilon dx\\ &= -\int_{\mathbb{R}} u_\varepsilon\partial_x\left(\frac{|\partial_x^2\rho_\varepsilon|^2}{2}\right)dx - \int_{\mathbb{R}}\underbrace{\left(\partial_x^2(u_\varepsilon\partial_x\rho_\varepsilon) - u_\varepsilon\partial_x^2\partial_x\rho_\varepsilon\right)}_{=:J_1}\partial_x^2\rho_\varepsilon dx\\ &\quad -\int_{\mathbb{R}}\rho_\varepsilon\partial_x^3 u_\varepsilon\partial_x^2\rho_\varepsilon dx - \int_{\mathbb{R}}\underbrace{\left(\partial_x^2(\rho_\varepsilon\partial_x u_\varepsilon) - \rho_\varepsilon\partial_x^3 u_\varepsilon\right)}_{=:J_2}\partial_x^2\rho_\varepsilon dx.\end{aligned}$$

Using the commutator estimates [14, Lemma 3.4] and the Sobolev embedding, we have

$$\begin{aligned}\|J_1\|_{L^2(\mathbb{R})} &\le C\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x\rho_\varepsilon\|_{L^\infty(\mathbb{R})} + C\|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\\ &\le C\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x\rho_\varepsilon\|_{H^1(\mathbb{R})} + C\|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})},\\ \|J_2\|_{L^2(\mathbb{R})} &\le C\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})} + C\|\partial_x\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}\\ &\le C\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})} + C\|\partial_x\rho_\varepsilon\|_{H^1(\mathbb{R})}\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\frac{|\partial_x^2\rho_\varepsilon|^2}{2}dx &\le \frac{1}{2}\|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 + \|\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x^3 u_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\\ &\quad + C\left(\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\right)\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}.\end{aligned}$$

Moreover, using (2.8), (A.2) and the Sobolev embedding, we have

$$\begin{aligned}\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 &\le C\left(\|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})} + \|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})}^2\right)\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2\\ &\quad + C\|\partial_x^3 u_\varepsilon\|_{L^2(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})} + C.\end{aligned}\tag{A.4}$$

To estimate $\|\partial_x^3 u_\varepsilon\|_{L^2(\mathbb{R})}$ in (A.4), we use the definition (2.9) of w_ε as follows:

$$\partial_x u_\varepsilon = g(\rho_\varepsilon)w_\varepsilon + h(\rho_\varepsilon), \quad \text{where } g(\rho_\varepsilon) := \frac{1}{\mu_\varepsilon(\rho_\varepsilon)},\ h(\rho_\varepsilon) := \frac{p(\rho_\varepsilon)}{\mu_\varepsilon(\rho_\varepsilon)}.\tag{A.5}$$

Since

$$\begin{aligned}\partial_x^3 u_\varepsilon &= g''(\rho_\varepsilon)|\partial_x\rho_\varepsilon|^2 w_\varepsilon + g'(\rho_\varepsilon)\partial_x^2\rho_\varepsilon w_\varepsilon + 2g'(\rho_\varepsilon)\partial_x\rho_\varepsilon\partial_x w_\varepsilon + g(\rho_\varepsilon)\partial_x^2 w_\varepsilon\\ &\quad + h''(\rho_\varepsilon)|\partial_x\rho_\varepsilon|^2 + h'(\rho_\varepsilon)\partial_x^2\rho_\varepsilon,\end{aligned}$$

we use (2.8) to have

$$\begin{aligned}\|\partial_x^3 u_\varepsilon\|_{L^2(\mathbb{R})} &\le C\Big(\left(\|w_\varepsilon\|_{L^\infty(\mathbb{R})} + 1\right)\|\partial_x\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x\rho_\varepsilon\|_{L^2(\mathbb{R})} + \|w_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\\ &\quad + \|\partial_x\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})} + \|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})} + \|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}\Big).\end{aligned}\tag{A.6}$$

Combining this with (A.4), and using (A.2) and the Sobolev embedding, we have

$$\frac{d}{dt}\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 \le \frac{C_1}{2}\|\partial_x^2 w_\varepsilon\|_{L^2(\mathbb{R})}^2 + G_1\|\partial_x^2\rho_\varepsilon\|_{L^2(\mathbb{R})}^2 + G_2,\tag{A.7}$$

where

$$G_1 := C\left(\|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})} + \|\partial_x^2 u_\varepsilon\|^2_{L^2(\mathbb{R})} + \|w_\varepsilon\|_{L^\infty(\mathbb{R})} + \|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})} + 1\right),$$
$$G_2 := C\left(\|w_\varepsilon\|^2_{L^\infty(\mathbb{R})} + \|\partial_x w_\varepsilon\|^2_{L^2(\mathbb{R})} + 1\right).$$

Note that $G_1, G_2 \in L^1((0,T))$ by (2.7) and (A.1).
Step 3) Adding (A.3) to (A.7), we have

$$\frac{d}{dt}\left(\|\partial_x w_\varepsilon\|^2_{L^2(\mathbb{R})} + \|\partial_x^2 \rho_\varepsilon\|^2_{L^2(\mathbb{R})}\right) + \frac{C_1}{2}\|\partial_x^2 w_\varepsilon\|^2_{L^2(\mathbb{R})}$$
$$\leq H\left(\|\partial_x w_\varepsilon\|^2_{L^2(\mathbb{R})} + \|\partial_x^2 \rho_\varepsilon\|^2_{L^2(\mathbb{R})}\right) + F + G_2,$$

where

$$H := C\left(1 + \|\partial_x w_\varepsilon\|^2_{L^2(\mathbb{R})} + \|w_\varepsilon\|^2_{L^\infty(\mathbb{R})} + \|\partial_x u_\varepsilon\|_{H^1(\mathbb{R})} + \|\partial_x^2 u_\varepsilon\|^2_{L^2(\mathbb{R})}\right).$$

Since $H, F, G_2 \in L^1((0,T))$, and it follows from (2.9) and (2.11) that

$$\|\partial_x w_\varepsilon(0)\|_{L^2(\mathbb{R})} \leq C(\underline{\kappa}_0, \overline{\kappa}_0)\left(\|\partial_x \rho_0\|_{L^2(\mathbb{R})} + \|\partial_x \rho_0\|_{L^2(\mathbb{R})}\|\partial_x u_0\|_{L^2(\mathbb{R})} + \|\partial_x^2 u_0\|_{L^2(\mathbb{R})}\right),$$

Grönwall lemma implies that

$$\|\partial_x^2 \rho_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x w_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x^2 w_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C, \tag{A.8}$$

where the constant $C > 0$ depends on T and the bounds of (2.7), (2.8) and (2.11).
This now together with (A.1), (A.2) and (A.6) imply the bound for $\partial_x^3 u_\varepsilon$:

$$\|\partial_x^3 u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C.$$

Moreover, differentiating the both sides of (A.5) in x, and using (2.8), we have

$$\|\partial_x^2 u_\varepsilon\|_{L^2(\mathbb{R})} \leq C\Big(\|\partial_x \rho_\varepsilon\|_{L^2(\mathbb{R})}\|w_\varepsilon\|_{L^\infty(\mathbb{R})} + \|\partial_x w_\varepsilon\|_{L^2(\mathbb{R})} + \|\partial_x \rho_\varepsilon\|_{L^2(\mathbb{R})}\Big).$$

Therefore, we use (2.7), (2.8) and (A.8) to have

$$\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C.$$

Indeed, since it follows from (2.7) and (2.8) that

$$w_\varepsilon = -p(\rho_\varepsilon) + \mu_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon \in L^\infty((0,T)\times\mathbb{R}) + L^\infty(0,T;L^2(\mathbb{R})),$$

we use (A.8) to have

$$|w_\varepsilon(x)| \leq \frac{1}{2}\int_{x-1}^{x+1}(|p(\rho_\varepsilon)| + |\mu_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon|)dy + \frac{1}{2}\int_{x-1}^{x+1}\int_y^x |\partial_z w_\varepsilon| dz dy$$
$$\leq \|p(\rho_\varepsilon)\|_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{\sqrt{2}}\|\mu_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \sqrt{2}\|\partial_x w_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))},$$

which gives $\|w_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C$.
Hence we complete the proof.

References

[1] D. Bresch and B. Desjardins. Sur un modèle de Saint-Venant visqueux et sa limite quasi-géostrophique. *C. R. Math. Acad. Sci. Paris*, 335:1079–1084, 2002.

[2] D. Bresch and B. Desjardins. Existence of global weak solutions for 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238:211–223, 2003.

[3] D. Bresch and B. Desjardins. Some diffusive capillary models of Korteweg type. *C. R. Math. Acad. Sci. Paris, Section Mécanique*, 332:881–886, 2004.

[4] S. Chapman and T.G. Cowling. The mathematical theory of non-uniform gases. *Cambridge University Press, London*, 3rd ed., 1970.

[5] P. Constantin, T. D. Drivas, H. Q. Nguyen, and F. Pasqualotto. Compressible fluids and active potentials,. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire, To appear*, 2019.

[6] J.-F. Gerbeau and B. Perthame. Derivation of viscous saint-venant system for laminar shallow water; numerical validation,. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):89–102, 2018.

[7] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D,. *Mathematische Nachrichten*, 291:2188–2203, 2018.

[8] D. Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Soc*, 303:169–181, 1987.

[9] D. Hoff. Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states. *Z. Angew. Math. Phys.*, 49:774–785, 1987.

[10] D. Hoff and J. Smoller. Non-formation of vacuum states for compressible Navier-Stokes equations. *Comm. Math. Phys.*, 216:255–276, 2001.

[11] M.-J. Kang and A. Vasseur. Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system. *J. Eur. Math. Soc. (JEMS), To appear. https://arxiv.org/pdf/1712.07348.pdf.*

[12] M.-J. Kang and A. Vasseur. Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems. *https://arxiv.org/pdf/1902.01792.pdf.*

[13] A. V. Kazhikhov and V. V. Shelukhin. Unique global solution with respect to time of initial- boundary value problems for one-dimensional equations of a viscous gas,. *Prikl. Mat. Meh.*, 41:282–291, 1977.

[14] A. Majda and A. Bertozzi. *Vorticity and Incompressible Flow.* Cambridge Univ. Press, 2002.

[15] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[16] D. Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris Sér. I Math.*, 303:639–642, 1986.

[17] V. V. Shelukhin. Motion with a contact discontinuity in a viscous heat conducting gas,. *Dinamika Sploshn. Sredy*, 57:131–152, 1982.

[18] V. V. Shelukhin. Evolution of a contact discontinuity in the barotropic flow of a viscous gas,. *Prikl. Mat. Mekh.*, 47:870–872, 1983.

[19] V. V. Shelukhin. Boundary value problems for equations of a barotropic viscous gas with nonnegative initial density. *Dinamika Sploshn. Sredy*, 74:108–125, 1986.

[20] V.V. Shelukhin. On the structure of generalized solutions of the one-dimensional equations of a polytropic viscous gas,. *J. Appl. Math. Mech., 48(1984), 665–672; translated from Prikl. Mat. Mekh. 48(1984), no. 6, 912–920.*

[21] V. A. Solonnikov. The solvability of the initial-boundary value problem for the equations of motion of a viscous compressible fluid. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.*, 59:128–142, 1976.

[22] V. A. Vaigant. Nonhomogeneous boundary value problems for equations of a viscous heat- conducting gas. *Dinamika Sploshn. Sredy*, 97:3–21, 1990.

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