CONTRACTION PROPERTY FOR LARGE PERTURBATIONS OF SHOCKS OF THE BAROTROPIC NAVIER-STOKES SYSTEM

MOON-JIN KANG AND ALEXIS F. VASSEUR

Abstract. This paper is dedicated to the construction of a pseudo-norm, for which small shock profiles of the barotropic Navier-Stokes equations have a contraction property. This contraction property holds in the class of any large solutions to the barotropic Navier-Stokes equations. It implies a stability condition which is independent of the strength of the viscosity. The proof is based on the relative entropy method, and is reminiscent to the notion of a-contraction first introduced by the authors in the hyperbolic case.

Contents

Date: January 9, 2019.

1991 *Mathematics Subject Classification.* 76N15, 35B35, 35Q30.

Key words and phrases. Contraction, Shock, Compressible Navier-Stokes, Stability, Relative entropy, Conservation law.

Acknowledgment. The first author was partially supported by the NRF-2017R1C1B5076510. The second author was partially supported by the NSF grant: DMS 1614918.

1. Introduction

In this article, we consider the one-dimensional barotropic Navier-Stokes equations in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = \left(\frac{\mu(v)}{v} u_x\right)_x, \end{cases} \tag{1.1}$$

where v denotes the specific volume, u is the fluid velocity, and $p(v)$ is the pressure law. We consider the case of polytropic perfect gas where the pressure verifies

$$p(v) = v^{-\gamma}, \quad \gamma > 1, \tag{1.2}$$

with γ the adiabatic constant. The quantity $\mu(v) = bv^{-\alpha}$ is the viscosity coefficient. Notice that if $\alpha > 0$, $\mu(v)$ degenerates near the vacuum, i.e., near $v = +\infty$. Very often, the viscosity coefficient is assumed to be constant, i.e., $\alpha = 0$. However, in the physical context the viscosity of a gas depends on the temperature (see Chapman and Cowling [6]). In the barotropic case, the temperature depends directly on the density ($\rho = 1/v$). The viscosity is expected to degenerate near the vacuum as a power of the density, which is translated into $\mu(v) = bv^{-\alpha}$ in terms of v with $\alpha > 0$. Global strong solutions of the system (1.1) can be constructed for a large family of initial data without vacuum. These solutions are also unique (see Constantin-Drivas-Nguyen-Pasqualotto [8], Haspot [14] and [28]). For simplification, we will restrict in this paper to the case where $\alpha = \gamma$.

The system (1.1) admits viscous shock waves connecting two end states (v_-, u_-) and (v_+, u_+), provided the two end states satisfy the Rankine-Hugoniot condition and the Lax entropy condition (see Matsumura and Wang [27]):

$$\begin{aligned} &\exists\, \sigma \quad \text{s.t.} \quad \begin{cases} -\sigma(v_+ - v_-) - (u_+ - u_-) = 0, \\ -\sigma(u_+ - u_-) + p(v_+) - p(v_-) = 0, \end{cases} \\ &\text{and either } v_- > v_+ \text{ and } u_- > u_+ \text{ or } v_- < v_+ \text{ and } u_- > u_+ \text{ holds.} \end{aligned} \tag{1.3}$$

In other words, for given constant states (v_-, u_-) and (v_+, u_+) satisfying (1.3), there exists a viscous shock wave $(\tilde{v}, \tilde{u})(x - \sigma t)$ as a solution of

$$\begin{cases} -\sigma\tilde{v}' - \tilde{u}' = 0, \\ -\sigma\tilde{u}' + p(\tilde{v})' = \left(\frac{\mu(\tilde{v})}{\tilde{v}}\tilde{u}'\right)' \\ \lim_{\xi\to\pm\infty}(\tilde{v}, \tilde{u})(\xi) = (v_\pm, u_\pm). \end{cases} \tag{1.4}$$

Here, if $v_- > v_+$, the solution of (1.4) is a 1-shock wave with velocity $\sigma = -\sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}$, whereas if $v_- < v_+$, that is a 2-shock wave with $\sigma = \sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}$.

The stability of the viscous shock waves for the compressible Navier-Stokes system is a very important issue in both mathematical and physical viewpoints. In the case of constant viscosity ($\alpha = 0$), Matsumura-Nishihara [26] showed the time-asymptotic stability for small initial perturbations with integral zero. Later on, the assumption on integral zero was removed by Mascia-Zumbrun [25] and Liu-Zeng [24]. We also refer to Barker-Humpherys-Laffite-Rudd-Zumbrun [2, 15] and the references therein for the spectral stability of small perturbations of large shocks. For the system (1.1) with degenerate viscosity ($\alpha > 0$), Matsumura-Wang [27] showed the asymptotic stability for small initial perturbations with integral zero under the assumption $\alpha \geq \frac{1}{2}(\gamma - 1)$. This assumption was recently removed by the second author and Yao [36].

To the best of our knowledge, up to now, there were no result on stability, independent of the size of the perturbation, for viscous shocks of compressible Navier-Stokes system.

The main contribution of this article is to show the existence of a contraction property for viscous shocks, up to a shift, for any possibly large perturbations, in the case of the Navier-Stokes system (1.1) with $\alpha = \gamma$ (see Theorem 1.1).

This result reaches a new milestone in the study of contractions of shock waves of conservation laws based on the relative entropy. In the inviscid case, the L^2 contraction of shocks was first obtained by Leger [22] for scalar conservation laws (see also Adimurthi, Goshal, and Veerappa Gowda [1] for contraction in the L^p norm). In [29], it was shown that this property is not true, for most systems, when considering homogenous norms. However it is true, at least for extremal shocks, if we consider an adapted non-homogenous pseudo-norm [23, 34]. This was theorized with the notion of a-contraction in [19]. There, the case of intermediate shocks was also considered. This situation is more delicate. The contraction works for some systems, as the Euler system with energy [31, 30], and can fail for others [16]. In the viscous case, based on the L^2 norm a first result was obtained for viscous shocks in the case of the viscous Burgers equation [20] (see also [17]). Our paper can be seen as a generalization of this result in the system case. Of course, the system case is far more involved. Especially, since these results are independent of the size of the perturbations, by rescaling the equation, they are valid uniformly in the vanishing viscosity limit. Because of the negative result of [29] for the Euler system, the result cannot be true for the Navier-Stokes equations when considering a homogenous pseudo-norm. This difficulty is compounded with the degenerate parabolic structure of Navier-Stokes, where the equation on v is purely hyperbolic.
We also mention a first attempt to extend the theory to the multi-variables setting in the scalar case [21], and the application of the method for the study of asymptotic limits [7, 35].

In an analytical viewpoint, handling the contraction property of the viscous shocks is pretty different from the inviscid situation. The main difficulty is due to the balance between the hyperbolic and parabolic terms.

1.1. **Main result.** We first introduce a relative functional $E(\cdot|\cdot)$ defined as follows:

$$\text{for any functions } v_1, u_1, v_2, u_2,$$
$$E((v_1,u_1)|(v_2,u_2)) := \frac{1}{2}\big(u_1 + p(v_1)_x - u_2 - p(v_2)_x\big)^2 + Q(v_1|v_2), \tag{1.5}$$

where $Q(v_1|v_2) := Q(v_1) - Q(v_2) - Q'(v_2)(v_1 - v_2)$ is a relative functional associated with the strictly convex function $Q(v) := \frac{v^{-\gamma+1}}{\gamma-1}$. The functional E is associated to the BD entropy (see Bresch-Desjardins [3, 4, 5]). Since $Q(v_1|v_2)$ is positive definite, (1.5) is also positive definite, that is, for any functions (v_1, u_1) and (v_2, u_2) we have $E((v_1,u_1)|(v_2,u_2)) \geq 0$, and

$$E((v_1,u_1)|(v_2,u_2)) = 0 \text{ a.e.} \quad \Leftrightarrow \quad (v_1,u_1) = (v_2,u_2) \text{ a.e.}$$

Our main result shows a contraction property measured by the relative functional (1.5). Our result is stated for the system (1.1) with the viscosity $\mu(v) = \gamma v^{-\gamma}$, i.e., the exponent α is identical to the adiabatic constant γ. A new approach developed in this paper can be applied to the case of more general viscosity (see [18]).

Theorem 1.1. *Consider the system* (1.1)-(1.2) *with the viscosity* $\mu(v) = \gamma v^{-\gamma}$, $\gamma > 1$. *For a given constant state* $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$, *there exists constants* $\varepsilon_0, \delta_0 > 0$ *such that the following is true.*

For any $\varepsilon < \varepsilon_0$, $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, *and any* $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$ *satisfying* (1.3) *with* $|p(v_-) - p(v_+)| = \varepsilon$, *there exists a smooth monotone function* $a : \mathbb{R} \to \mathbb{R}^+$ *with* $\lim_{x\to\pm\infty} a(x) = 1 + a_\pm$ *for some constants* a_-, a_+ *with* $|a_+ - a_-| = \lambda$ *such that the following holds.*
Let $\tilde{U} := (\tilde{v}, \tilde{u})$ *be the viscous shock connecting* (v_-, u_-) *and* (v_+, u_+) *as a solution of* (1.4). *For any solution* $U := (v, u)$ *to* (1.1) *with initial data* $U_0 := (v_0, u_0)$ *satisfying* $\int_{-\infty}^{\infty} E(U_0|\tilde{U})dx < \infty$, *there exists a shift* $X \in W^{1,1}_{loc}(\mathbb{R}^+)$ *such that*

$$\frac{d}{dt}\int_{-\infty}^{\infty} a(x)E\big(U(t, x + X(t))|\tilde{U}(x)\big)dx \le 0, \tag{1.6}$$

and

$$\begin{aligned} &|\dot{X}(t)| \le \frac{1}{\varepsilon^2}(1 + f(t)), \quad t > 0, \\ &\textit{for some positive function } f \textit{ satisfying} \quad \|f\|_{L^1(0,\infty)} \le \frac{2\lambda}{\delta_0\varepsilon}\int_{-\infty}^{\infty} E(U_0|\tilde{U})dx. \end{aligned} \tag{1.7}$$

Remark 1.1. *Theorem 1.1 provides a contraction property for viscous shocks with suitably small amplitude parametrized by* $\varepsilon = |p(v_-) - p(v_+)|$. *This smallness together with* (1.3) *implies* $|v_- - v_+| = \mathcal{O}(\varepsilon)$ *and* $|u_- - u_+| = \mathcal{O}(\varepsilon)$. *However, for such a fixed small shock, the contraction holds for any weak solutions to (1.1), without any smallness condition imposed on* U_0. *This is important to study the inviscid limit problem* ($\nu \to 0$) *of:*

$$\begin{cases} v^\nu_t - u^\nu_x = 0 \\ u^\nu_t + p(v^\nu)_x = \nu\Big(\frac{\mu(v^\nu)}{v^\nu}u^\nu_x\Big)_x. \end{cases} \tag{1.8}$$

By rescaling the result of Theorem 1.1 as $(t, x) \to (t/\nu, x/\nu)$ *we obtain the exact same theorem for the system (1.8). Therefore we obtain a stability result on viscous shocks of fixed strength which is independent of the strength of the viscosity* ν *(see* [18]*).*

Remark 1.2. *The contraction property is non-homogenous in* x, *as measured by the function* $x \to a(x)$. *This is consistant with the hyperbolic case (with* $\nu = 0$*). In the hyperbolic case, it was shown in* [29] *that a homogenous contraction cannot hold for the full Euler system. However, the contraction property is true if we consider a non-homogenous pseudo-distance* [34] *providing the so-called a-contraction* [19]. *Our main result shows that the non-homogeneity of the pseudo-distance can be chosen of a similar size as the strength of the shock (as measured by the quantity* λ*).*

1.2. **Transformation of the system** (1.1)**.** We first introduce a new effective velocity $h := u + p(v)_x$. The system (1.1) with $\mu(v) = \gamma v^{-\gamma}$ is then transformed into

$$\begin{cases} v_t - h_x = -(p(v))_{xx} \\ h_t + p(v)_x = 0. \end{cases} \tag{1.9}$$

Notice that the above system has a parabolic regularization on the specific volume, contrary to the regularization on the velocity for the original system (1.1). This is better for our analysis, since the hyperbolic part of the system is linear in u (or h) but nonlinear in v (via the pressure). This effective velocity was first introduced by Shelukhin [32] for $\alpha = 0$, and in the general case (in Eulerian coordinates) by Bresch-Desjardins [3, 4, 5], and Haspot [12, 11, 14]. It was also used in [36].

As mentioned in Theorem 1.1, we consider shock waves with suitably small amplitude ε. For that, let $(\tilde{v}_\varepsilon, \tilde{u}_\varepsilon)(x - \sigma_\varepsilon t)$ denote a shock wave with amplitude $|p(v_-) - p(v_+)| = \varepsilon$ as a solution of (1.4) with $\mu(v)=\gamma v^{-\gamma}$. Then, setting $\tilde h_\varepsilon := \tilde u_\varepsilon + (p(\tilde v_\varepsilon))_x$, the shock wave $(\tilde v_\varepsilon, \tilde h_\varepsilon)(x-\sigma_\varepsilon t)$ satisfies

$$\begin{cases} -\sigma_\varepsilon \tilde v_\varepsilon' - \tilde h_\varepsilon' = -(p(\tilde v_\varepsilon))'' \\ -\sigma_\varepsilon \tilde h_\varepsilon' + p(\tilde v_\varepsilon)' = 0 \\ \lim_{\xi\to\pm\infty} (\tilde v_\varepsilon, \tilde h_\varepsilon)(\xi) = (v_\pm, u_\pm). \end{cases} \tag{1.10}$$

For simplification of our analysis, we rewrite (1.9) into the following system, based on the change of variable $(t,x)\mapsto (t, \xi = x-\sigma_\varepsilon t)$:

$$\begin{cases} v_t - \sigma_\varepsilon v_\xi - h_\xi = -(p(v))_{\xi\xi} \\ h_t - \sigma_\varepsilon h_\xi + p(v)_\xi = 0 \\ v|_{t=0} = v_0, \quad h|_{t=0} = u_0. \end{cases} \tag{1.11}$$

Remark 1.3. *In* (1.11)*, the dissipation is in v and has the specific form $(-p(v))_{\xi\xi}$, whose structure is due to the fact that $\alpha=\gamma$. This simplifies our analysis a lot, since we consider the entropy $Q(v)$ with $Q'(v) = -p(v)$.*

Theorem 1.1 is a direct consequence of the following theorem on the contraction of shocks for the system (1.9). To measure the contraction, we use the relative entropy associated to the entropy of (1.9) as

$$\eta((v_1,h_1)|(v_2,h_2)) := \frac{|h_1-h_2|^2}{2} + Q(v_1|v_2),$$

where $Q(v_1|v_2) := Q(v_1) - Q(v_2) - Q'(v_2)(v_1-v_2)$ and $Q(v) := \frac{v^{-\gamma+1}}{\gamma-1}$.

Theorem 1.2. *For a given constant state $(v_-,u_-)\in\mathbb{R}^+\times\mathbb{R}$, there exist constants $\varepsilon_0, \delta_0 > 0$ such that the following holds.*
For any $\varepsilon<\varepsilon_0$, $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, and any $(v_+,u_+)\in\mathbb{R}^+\times\mathbb{R}$ satisfying (1.3) *with $|p(v_-) - p(v_+)| = \varepsilon$, there exists a smooth monotone function $a:\mathbb{R}\to\mathbb{R}^+$ with $\lim_{x\to\pm\infty} a(x) = 1 + a_\pm$ for some constants a_-, a_+ with $|a_- - a_+| = \lambda$ such that the following holds.*
Let $\tilde U_\varepsilon := (\tilde v_\varepsilon, \tilde h_\varepsilon)$ be a viscous shock connecting (v_-,u_-) and (v_+,u_+) as a solution of (1.10)*. For any solution $U := (v,h)$ to* (1.11) *with initial data $U_0 := (v_0,u_0)$ satisfying $\int_{-\infty}^\infty \eta(U_0|\tilde U_\varepsilon)dx < \infty$, there exists a shift function $X\in W^{1,1}_{loc}(\mathbb{R}^+)$ such that*

$$\frac{d}{dt}\int_{-\infty}^{\infty} a(\xi)\eta\big(U(t,\xi+X(t))|\tilde U_\varepsilon(\xi)\big)d\xi \le 0, \tag{1.12}$$

and

$$\begin{aligned} &|\dot X(t)| \le \frac{1}{\varepsilon^2}(1+f(t)), \quad t>0, \\ &\text{for some positive function } f \text{ satisfying } \quad \|f\|_{L^1(0,\infty)} \le \frac{2\lambda}{\delta_0\varepsilon}\int_{-\infty}^{\infty}\eta(U_0|\tilde U_\varepsilon)d\xi. \end{aligned} \tag{1.13}$$

Notice that it is enough to prove Theorem 1.2 for 1-shocks. Indeed, the result for 2-shocks is obtained by the change of variables $x\to -x$, $u\to -u$, $\sigma_\varepsilon\to -\sigma_\varepsilon$.
Therefore, from now on, we consider a 1-shock $(\tilde v_\varepsilon, \tilde h_\varepsilon)$, i.e., $v_- > v_+$, $u_- > u_+$, and

$$\sigma_\varepsilon = -\sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}. \tag{1.14}$$

Notations • Throughout the paper, C denotes a positive constant which may change from line to line, but which stays independent on ε (the shock strength) and λ (the total

variation of the function a). The paper will consider two smallness conditions, one on ε, and the other on ε/λ. In the argument, ε will be far smaller than ε/λ .
• To avoid confusion, for any function F of x, we denote: $F'(v) = \frac{d}{dv}F(v)$, $F(v)' = \frac{d}{dx}F(v)$.

1.3. **Ideas of the proof.** In all the computations $\varepsilon > 0$ is the size of the fixed shock. We remind the reader that the perturbation $U_0 - \tilde{U}_\varepsilon = (v_0 - \tilde{v}_\varepsilon, h_0 - \tilde{h}_\varepsilon)$ can be unconditionally big. The non-homogeneity of the semi-norm comes through the function a. This function is decreasing in the case of a 1-shock, and increasing in the case of a 2-shock. The strength of this non-homogeneity is measured by the number $\lambda > 0$, which is the difference between the values of a at $-\infty$ and $+\infty$ (see (2.23)). Typically, λ is small, but it can be far bigger than ε. Actually, in the analysis, we will consider some smallness on both ε and ε/λ, ε being much smaller that ε/λ. Note that the velocity of the shock σ_ε has the same sign as a', so the quantity $\sigma_\varepsilon a'$ is positive. The relative entropy computation (see Lemma 2.3) gives that

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} a(\xi)\eta\big(U(t,\xi+X(t))|\tilde{U}_\varepsilon(\xi)\big)d\xi\\ = \dot{X}(t)Y(U(t,\cdot+X(t))) + \mathcal{B}(U(t,\cdot+X(t))) - \mathcal{G}(U(t,\cdot+X(t))).\end{aligned}$$

The functional $\mathcal{G}(U)$ is non-negative (good term) and can be split into three terms (see (3.47)):

$$\mathcal{G}(U) = \mathcal{G}_1(U) + \mathcal{G}_2(U) + \mathcal{D}(U),$$

where only $\mathcal{G}_1(U)$ depends on h. The term $\mathcal{D}(U)$ corresponds to the diffusive term (which depends on v only, thanks to the transformation of the system). We are able to write this decomposition such that the functional $\mathcal{B}(U)$ (bad terms) depends only on v. This is the main reason why we can consider a degenerate diffusion (the viscosity in u only is replaced by a diffusion in v only, after transformation of the system). The fact that the hyperbolic flux in the Navier-Stokes equations is only linear in h plays a particular role for this matter: the corresponding relative flux then vanishes.

Because of the relative entropy structure, the quantity $\mathcal{G}(U)$ and $\mathcal{B}(U)$ are quadratic when the perturbation is small. However, we have no uniform control on the size of $U(t,\cdot)$, therefore we will have also to carefully estimate what happens for large value of $U(t,x)$.

The shift $X(t)$ introduces the term $\dot{X}(t)Y(U)$. The key idea of the technique, is to take advantage of this term when $Y(U(t,\cdot))$ is not two small, by compensating all the other terms via the choice of the velocity of the shift (see (3.2)). Specifically, we ensure algebraically that the contraction holds as long as $|Y(U(t))| \geq \varepsilon^2$. The rest of the analysis is to ensure that when $|Y(U(t))| \leq \varepsilon^2$, the contraction still holds.

The condition $|Y(U(t))| \leq \varepsilon^2$ ensures a smallness condition that we want to fully exploit. This is where the non-homogeneity of the semi-norm is crucial. In the case where the function a is constant, $Y(U)$ is a linear functional in U. The smallness of $Y(U)$ gives only that a certain weighted mean value of U is almost null. However, when a is decreasing, $Y(U)$ becomes convex. The smallness $Y(U(t)) \leq \varepsilon^2$ implies, for this fixed time t (See Lemma 3.2 with (2.25) and (2.1)):

$$\int_{\mathbb{R}} \varepsilon e^{-C\varepsilon|\xi|} Q(v(t,\xi+X(t))|\tilde{v}_\varepsilon(\xi))\,d\xi \leq C\left(\frac{\varepsilon}{\lambda}\right)^2. \tag{1.15}$$

This gives a control in L^2 for moderate values of v, and in L^1 for big values of v, in the layer region ($|\xi - X(t)| \lesssim 1/\varepsilon$).

The problem now looks, at first glance, as a typical problem of stability with a smallness condition. There are, however, two major difficulties: We have some smallness only in v, for a very weak norm, and only localized in the layer region. More importantly, the smallness is measured with respect to the smallness of the shock. It basically says that, considering only the moderate values of v: the perturbation is not bigger than ε/λ (which is still very big with respect to the size of the shock ε). Actually, as we will see later, it is not possible to consider only the linearized problem: Third order terms appear in the expansion using the smallness condition (the energy method involving the linearization would have only second order term in ε).

In the argument, for the values of t such that $|Y(U(t))| \leq \varepsilon^2$, we construct the shift as a solution to the ODE: $\dot{X}(t) = -Y(U(t, \cdot + X(t)))/\varepsilon^4$. From this point, we forget that $U = U(t, \xi)$ is a solution to (1.11) and $X(t)$ is the shift. That is, we leave out $X(t)$ and the t-variable of U. Then we show that for any function U satisfying $Y(U) \leq \varepsilon^2$, we have

$$-\frac{1}{\varepsilon^4} Y^2(U) + |\mathcal{B}(U)| - \mathcal{G}(U) \leq 0. \tag{1.16}$$

This is the main Proposition 3.1 (actually, the proposition is slightly stronger to ensure the control of the shift). This implies clearly the contraction. There are several steps to prove this proposition.

Step 1: Using the smallness condition, we show that if the good diffusive term verifies

$$\mathcal{D}(U) \geq \frac{\varepsilon^2}{\lambda},$$

then (1.16) holds true. Note that if the values of v were bounded from above and bounded away from 0, we could control $\mathcal{B}(U)$ from (1.15), since both expressions would be quadratic in $v - \tilde{v}_\varepsilon$. The main difficulty in this step is to obtain the control where the values of v are small. Indeed for such small v, the worst term in $\mathcal{B}(U)$ behaves like $p(v)^2 = 1/v^{2\gamma}$, while $Q(v|\tilde{v}_\varepsilon)$ behaves like $1/v^{\gamma-1}$. So we need to use a little bit of $\mathcal{D}(U)$ as a Poincaré type inequality (Remember that $a \geq 1 - \lambda > 0$) from:

$$\mathcal{D}(U) = \int_{\mathbb{R}} a|\partial_\xi (p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi$$

(See (3.60) from Lemma 3.4). We can now restrict ourselves to the case where both $|Y(U)| \leq \varepsilon^2$, and $\mathcal{D}(U) \leq \varepsilon^2/\lambda$.

Step 2: To be able to perform an expansion in ε later, we want to show that it is enough to consider only values of v such that $v - \tilde{v}_\varepsilon$ is bounded (smaller than a δ small enough, but not dependent on ε nor on ε/λ). We need also use only the part $Y_g(v)$ of $Y(U)$ which contains only terms in v (and not the terms in h). We do not have enough estimates on U to show that U is uniformly bounded on $\mathbb{R}$. But we can show that the big values of $|v - \tilde{v}_\varepsilon|$ (which can occur only for big values of ξ) do not change much the estimate (see Section 3.6). It involves a careful study of the contribution of the tails ($U(\xi)$ for $|\xi| \geq 1/\varepsilon$). This is the only part where $\mathcal{G}_1$ is used in order to control $Y_b(U) = Y(U) - Y_g(v)$, the part of $Y(U)$ which depends also on h (see Lemma 3.4). More precisely, this step shows that it is enough to prove that for any functions v such that $|v - \tilde{v}_\varepsilon| \leq \delta$ and $|Y_g(v)| \leq \varepsilon^2/\lambda$, we have

$$-\frac{1}{\varepsilon\delta}|Y_g(v)|^2 + (1+\delta)|\mathcal{B}(v)| - (1-\delta)\mathcal{G}_2(v) - (1-\delta)\mathcal{D}(v) \leq 0.$$

All the terms in this inequality depends on U only through v. Therefore, with a slight abuse of notations, we will write these functions as functions of v. This corresponds to Proposition 3.4. The δ terms are still needed because we lose a bit when truncating the tails, to obtain (1.16). The terms depending on h are not present anymore. So it is now an estimate on scalar functions v. The good term in $Y_g(v)$ involves a smaller power of $1/\varepsilon$, since we had to control the corresponding $Y_b(U)$ with the same power of $1/\varepsilon$.

Step 3: To show Proposition 3.4, we now perform a expansion in ε uniformly in v (but for a fixed δ). Note that the expansion has to be performed up to the third order. Indeed, because of the function a, terms involving the function a or the functions a' do not have the same power in ε/λ. Interestingly, the term $\mathcal{G}_2(v)$ cancels exactly the term of order λ/ε of $\mathcal{B}(v)$. This step shows that, thanks to some rescaling, it is enough to prove that for any $W \in L^2(0,1)$:

$$
\begin{aligned}
-\frac{1}{\delta}\left(\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right)^2 + (1+\delta)\int_0^1 W^2\,dy \\
+\frac{2}{3}\int_0^1 W^3\,dy + \delta\int_0^1 |W|^3\,dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2\,dy \le 0.
\end{aligned}
$$

We need to show this for some $\delta > 0$ possibly very small. So it looks very similar to a nonlinear Poincaré inequality with constraint. The constraint (the term in $1/\delta$) came from the term with $Y_g(v)$ through the asymptotic. This result on W is the Proposition 3.3.

Step 4: To prove Proposition 3.3 , we first reduce the problem to the minimization problem for a polynomial of two variables with a constraint. For this we use two lemmas. Lemma 2.8 provides a sharp L^∞ control using the dissipation term. Lemma 2.9 is a well known sharp Poincaré inequality that was already used in [20]. This reduces the problem to a minimization of a polynomial with variables:

$$
Z_1 = \int_0^1 W(y)\,dy, \qquad Z_2 = \left(\int_0^1 (W - Z_1)^2\,dy\right)^{1/2}.
$$

Because of the constraint, we can reduce this minimization problem to the minimization problem of a polynomial of only one variable (see Lemma 2.7).

It is easier to present the proofs of the propositions and lemmas in reverse. Therefore the rest of the paper is as follows. Section 2 is dedicated to the proofs of preliminaries. It includes some useful estimates on small shock waves, the computation of the time derivative of the relative entropy, the construction of the function a, some global estimates on the relative quantities (for small or big values of v), and the minimization problem for the polynomial functional with one variable. Section 3 is dedicated to the proof of the main Theorem. First we give the construction of the shift, and state the main Proposition 3.1, and then show how the Proposition implies the Theorem. To prove Proposition 3.1, we first show the minimization problem with two variables, then the nonlinear Poincaré type of inequality, and continue backward up to the general situation where we have only the constraint on $Y(U)$.

The range of ε will be reduced from one Lemma to the next, with the same notation on the restriction ε_0. The restriction on ε/λ is more subtle. To ensure that there is no loop in the argument, we will carefully track the smallness needed on this quantity from one lemma to the next. The smallness on ε/λ will be denoted with δ notations. The results in the preliminaries will consider a generic smallness δ_*. they can be safely replace by the same constant δ_* (taking the smallest of all). However, the constant δ_3 will play a crucial role to control the strength of the typical perturbations. Later on, constants will be build that may blow up when δ_3 is very small. It will be important to make sure that δ_3 can be fixed before hand. The following restrictions on ε/λ are less sensitive. Therefore we will just reduced them from one lemma to the next keeping the generic notation δ_0.

2. Preliminaries

2.1. **Small shock waves.**

In this subsection, we present useful properties of the 1-shock waves $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ with small amplitude ε. In the sequel, without loss of generality, we consider the 1-shock wave $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ satisfying $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$. Notice that the estimates in the following lemma also hold for $\tilde{h}_\varepsilon$ since we have $\tilde{h}'_\varepsilon = \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\tilde{v}'_\varepsilon$ and $C^{-1} \leq \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \leq C$. But, since the below estimates for $\tilde{v}_\varepsilon$ are enough in our analysis, we give the estimates only for $\tilde{v}_\varepsilon$.

Lemma 2.1. *We fix $v_- > 0$ and $h_- \in \mathbb{R}$. Then there exists $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$ the following is true. Let $\tilde{v}_\varepsilon$ be the 1-shock wave with amplitude $|p(v_-) - p(v_+)| = \varepsilon$ and such that $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$. Then, there exist constants $C, C_1, C_2 > 0$ such that*

$$-C^{-1}\varepsilon^2 e^{-C_1\varepsilon|\xi|} \leq \tilde{v}'_\varepsilon(\xi) \leq -C\varepsilon^2 e^{-C_2\varepsilon|\xi|}, \quad \forall \xi \in \mathbb{R}. \tag{2.1}$$

Therefore, as a consequence, we have

$$\inf_{[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]} |v'_\varepsilon| \geq C\varepsilon^2. \tag{2.2}$$

Proof. We multiply the first equation of (1.10) by σ_ε and eliminate the dependence on h_ε using the second equation. After integration in ξ, we find:

$$\sigma_\varepsilon(p(\tilde{v}_\varepsilon))' = \sigma_\varepsilon^2(\tilde{v}_\varepsilon - v_+) + p(\tilde{v}_\varepsilon) - p(v_+). \tag{2.3}$$

Dividing by $\tilde{v}_\varepsilon - v_+$ and using (1.14) we get

$$\frac{\sigma_\varepsilon(p(\tilde{v}_\varepsilon))'}{\tilde{v}_\varepsilon - v_+} = -\frac{p(v_-) - p(v_+)}{v_- - v_+} + \frac{p(\tilde{v}_\varepsilon) - p(v_+)}{\tilde{v}_\varepsilon - v_+}.$$

Consider the smooth function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\varphi(v) := \frac{p(v) - p(v_+)}{v - v_+}.$$

Then, the above equality can be written as

$$\frac{\sigma_\varepsilon(p(\tilde{v}_\varepsilon))'}{\tilde{v}_\varepsilon - v_+} = \varphi(\tilde{v}_\varepsilon) - \varphi(v_-). \tag{2.4}$$

To estimate the above r.h.s., we apply the Taylor theorem to the function φ about v_-, so that for any $v \in \mathbb{R}^+$ with $|v - v_-| < \frac{v_-}{2}$, there exists a constant $C > 0$ (depending only on v_-) such that

$$|\varphi(v) - \varphi(v_-) - \varphi'(v_-)(v - v_-)| \leq C(v - v_-)^2. \tag{2.5}$$

It can be shown that (see [27])

$$\tilde{v}'_\varepsilon < 0, \quad \text{and} \quad v_+ < \tilde{v}_\varepsilon < v_-. \tag{2.6}$$

Therefore, for ε_0 small enough:

$$0 \leq v_- - \tilde{v}_\varepsilon \leq v_- - v_+ \leq C\varepsilon < \frac{v_-}{2}.$$

Using (2.5) with $v = \tilde{v}_\varepsilon$, we have

$$|\varphi(\tilde{v}_\varepsilon) - \varphi(v_-) - \varphi'(v_-)(\tilde{v}_\varepsilon - v_-)| \le C\varepsilon(v_- - \tilde{v}_\varepsilon).$$

Moreover, since

$$\varphi'(v_-) = \frac{p'(v_-)(v_- - v_+) - \big(p(v_-) - p(v_+)\big)}{(v_- - v_+)^2} = \frac{p''(v_*)}{2}, \quad \text{for some } v_* \in (v_+, v_-),$$

we take ε_0 small enough such that $p''(v_-) \ge \varphi'(v_-) \ge p''(v_-)/2 > 0$.
Thus, for ε_0 small enough, we have

$$2p''(v_-)(\tilde{v}_\varepsilon - v_-) \le \varphi(\tilde{v}_\varepsilon) - \varphi(v_-) \le \frac{p''(v_-)}{8}(\tilde{v}_\varepsilon - v_-).$$

Then, it follows from (2.4) that

$$2p''(v_-)(\tilde{v}_\varepsilon - v_-)(\tilde{v}_\varepsilon - v_+) \le \sigma_\varepsilon(p(\tilde{v}_\varepsilon))' \le \frac{p''(v_-)}{8}(\tilde{v}_\varepsilon - v_-)(\tilde{v}_\varepsilon - v_+).$$

Since

$$-\sqrt{-p'(v_-/2)} \le \sigma_\varepsilon \le -\sqrt{-p'(v_-)} \quad \text{and} \quad p'(v_-/2) \le p'(\tilde{v}_\varepsilon) \le p'(v_-) < 0, \tag{2.7}$$

the quantity $\sigma_\varepsilon p'(\tilde{v}_\varepsilon)$ is bounded from below and above uniformly in ε.
Therefore

$$C^{-1}(\tilde{v}_\varepsilon - v_-)(\tilde{v}_\varepsilon - v_+) \le \tilde{v}_\varepsilon' \le C(\tilde{v}_\varepsilon - v_-)(\tilde{v}_\varepsilon - v_+). \tag{2.8}$$

To prove the estimate (2.1), we first observe that $\tilde{v}_\varepsilon' < 0$ and $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$ imply

$$\begin{aligned} &\xi \le 0 \;\Rightarrow\; v_- - v_+ \ge \tilde{v}_\varepsilon(\xi) - v_+ \ge \tilde{v}_\varepsilon(0) - v_+ = \frac{v_- - v_+}{2}, \\ &\xi \ge 0 \;\Rightarrow\; v_- - v_+ \ge v_- - \tilde{v}_\varepsilon(\xi) \ge v_- - \tilde{v}_\varepsilon(0) = \frac{v_- - v_+}{2}. \end{aligned} \tag{2.9}$$

Then, using (2.8) and (2.9) with $|v_- - v_+| \le C\varepsilon$, we have

$$\begin{aligned} &\xi \le 0 \;\Rightarrow\; -C^{-1}\varepsilon(v_- - \tilde{v}_\varepsilon) \le \tilde{v}_\varepsilon' \le -C\varepsilon(v_- - \tilde{v}_\varepsilon), \\ &\xi \ge 0 \;\Rightarrow\; -C^{-1}\varepsilon(\tilde{v}_\varepsilon - v_+) \le \tilde{v}_\varepsilon' \le -C\varepsilon(\tilde{v}_\varepsilon - v_+). \end{aligned}$$

Thus,

$$\begin{aligned} &\xi \le 0 \;\Rightarrow\; -C^{-1}\varepsilon(v_- - \tilde{v}_\varepsilon) \ge (v_- - \tilde{v}_\varepsilon)' \ge -C\varepsilon(v_- - \tilde{v}_\varepsilon), \\ &\xi \ge 0 \;\Rightarrow\; -C^{-1}\varepsilon(\tilde{v}_\varepsilon - v_+) \le (\tilde{v}_\varepsilon - v_+)' \le -C\varepsilon(\tilde{v}_\varepsilon - v_+). \end{aligned}$$

These together with $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$ imply

$$\begin{aligned} &\xi \le 0 \;\Rightarrow\; C^{-1}\varepsilon e^{-C_2\varepsilon|\xi|} \le v_- - \tilde{v}_\varepsilon \le C\varepsilon e^{-C_1\varepsilon|\xi|}, \\ &\xi \ge 0 \;\Rightarrow\; C^{-1}\varepsilon e^{-C_2\varepsilon\xi} \le \tilde{v}_\varepsilon - v_+ \le C\varepsilon e^{-C_1\varepsilon\xi}. \end{aligned}$$

Finally, applying the above estimate together with $|\tilde{v}_\varepsilon - v_\pm| \le C\varepsilon$ to (2.8), gives (2.1).
Estimate (2.2) follows directly from the upper bound on $\tilde{v}_\varepsilon'(\xi)$ in (2.1). □

We finish this subsection with an estimate based on the inverse of the pressure function.

Lemma 2.2. *Let us fixed $p_- > 0$. Then, there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $p_+, p > 0$ such that $0 < \varepsilon = p_+ - p_- \le \varepsilon_0$, $p_- \le p \le p_+$, and v, v_-, v_+ such that $p(v) = p, p(v_\pm) = p_\pm$, we have*

$$\left|\frac{v - v_-}{p - p_-} + \frac{v - v_+}{p_+ - p} + \frac{1}{2}\frac{p''(v_-)}{p'(v_-)^2}(v_- - v_+)\right| \le C\varepsilon^2.$$

Proof. Consider the function $v(p) = p^{-1/\gamma}$. Then, using a Taylor expansion at p_-, we find that there exists ε_0 such that for any $|p - p_-| \le \varepsilon_0$ and $|p - p_+| \le \varepsilon_0$ we have

$$\left|v - v_- - \frac{dv}{dp}(p_-)(p - p_-) - \frac{1}{2}\frac{d^2v}{dp^2}(p_-)(p - p_-)^2\right| \le C|p - p_-|^3, \tag{2.10}$$

$$\left|v - v_+ - \frac{dv}{dp}(p_+)(p - p_+) - \frac{1}{2}\frac{d^2v}{dp^2}(p_+)(p - p_+)^2\right| \le C|p - p_+|^3. \tag{2.11}$$

Since

$$\frac{d^2v}{dp^2} = \frac{d}{dp}\left(\frac{1}{p'(v)}\right) = -\frac{p''(v)}{p'(v)^2}\frac{dv}{dp},$$

we get

$$\begin{aligned}
&\left|\frac{1}{2}\frac{p''(v_-)}{p'(v_-)^2}(v_- - v_+) + \frac{1}{2}\frac{d^2v}{dp^2}(p_-)(p_- - p_+)\right| \\
&\qquad \le \frac{p''(v_-)}{2p'(v_-)^2}\left|v_+ - v_- - \frac{dv}{dp}(p_-)(p_+ - p_-)\right| \le C\varepsilon^2.
\end{aligned} \tag{2.12}$$

Since

$$\begin{aligned}
&\left|\frac{1}{2}\frac{d^2v}{dp^2}(p_+)(p - p_+) - \frac{1}{2}\frac{d^2v}{dp^2}(p_-)(p - p_-) + \frac{1}{2}\frac{d^2v}{dp^2}(p_-)(p_+ - p_-)\right| \\
&\qquad = \frac{1}{2}\left|\left(\frac{d^2v}{dp^2}(p_+) - \frac{d^2v}{dp^2}(p_-)\right)(p - p_+)\right| \le C\varepsilon^2.
\end{aligned} \tag{2.13}$$

dividing (2.10) by $p - p_-$, (2.11) by $p_+ - p$, and adding both terms together with the terms estimated in (2.12) and (2.13), we obtain

$$\begin{aligned}
&\left|\frac{v - v_-}{p - p_-} + \frac{v - v_+}{p_+ - p} + \frac{1}{2}\frac{p''(v_-)}{p'(v_-)^2}(v_- - v_+)\right. \\
&\qquad \left. - \left(\frac{dv}{dp}(p_-) - \frac{dv}{dp}(p_+) - \frac{d^2v}{dp^2}(p_-)(p_- - p_+)\right)\right| \le C\varepsilon^2.
\end{aligned}$$

This gives the result, since the second line term is itself of order ε^2. ☐

2.2. **Relative entropy method.** Our analysis is based on the relative entropy. The method is purely nonlinear, and allows to handle rough and large perturbations. The relative entropy method was first introduced by Dafermos [9] and Diperna [10] to prove the L^2 stability and uniqueness of Lipschitz solutions to the hyperbolic conservation laws endowed with a convex entropy.

To use the relative entropy method, we rewrite (1.11) into the following general system of viscous conservation laws:

$$\partial_t U + \partial_\xi A(U) = \begin{pmatrix} -\partial_{\xi\xi} p(v) \\ 0 \end{pmatrix}, \tag{2.14}$$

where

$$U := \begin{pmatrix} v \\ h \end{pmatrix}, \quad A(U) := \begin{pmatrix} -\sigma_\varepsilon v - h \\ -\sigma_\varepsilon h + p(v) \end{pmatrix}.$$

The system (2.14) has a convex entropy $\eta(U) := \frac{h^2}{2} + Q(v)$, where $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$, i.e., $Q'(v) = -p(v)$.
Using the derivative of the entropy as

$$\nabla\eta(U) = \begin{pmatrix} -p(v) \\ h \end{pmatrix}, \tag{2.15}$$

the above system (2.14) can be rewritten as

$$\partial_t U + \partial_\xi A(U) = \partial_\xi \Big(M \partial_\xi \nabla\eta(U) \Big), \tag{2.16}$$

where $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and (1.10) can be rewritten as

$$\partial_\xi A(\tilde{U}_\varepsilon) = \partial_\xi \Big(M \partial_\xi \nabla\eta(\tilde{U}_\varepsilon) \Big). \tag{2.17}$$

Consider the relative entropy function defined by

$$\eta(U|V) = \eta(U) - \eta(V) - \nabla\eta(V) \cdot (U - V),$$

and the relative flux defined by

$$A(U|V) = A(U) - A(V) - \nabla A(V)(U - V).$$

Let $G(\cdot;\cdot)$ be the flux of the relative entropy defined by

$$G(U;V) = G(U) - G(V) - \nabla\eta(V)(A(U) - A(V)),$$

where G is the entropy flux of η, i.e., $\partial_i G(U) = \sum_{k=1}^2 \partial_k \eta(U) \partial_i A_k(U), \quad 1 \le i \le 2.$
Then, for our system (2.14), we have

$$\begin{aligned} &\eta(U|\tilde{U}_\varepsilon) = \frac{|h - \tilde{h}_\varepsilon|^2}{2} + Q(v|\tilde{v}_\varepsilon), \\ &A(U|\tilde{U}_\varepsilon) = \begin{pmatrix} 0 \\ p(v|\tilde{v}_\varepsilon) \end{pmatrix}, \\ &G(U;\tilde{U}_\varepsilon) = (p(v) - p(\tilde{v}_\varepsilon))(h - \tilde{h}_\varepsilon) - \sigma_\varepsilon \eta(U|\tilde{U}_\varepsilon), \end{aligned} \tag{2.18}$$

Note that the relative pressure is defined as

$$p(v|w) = p(v) - p(w) - p'(w)(v - w). \tag{2.19}$$

We consider a weighted relative entropy between the solution U of (2.16) and the viscous shock $\tilde{U}_\varepsilon := \begin{pmatrix} \tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon \end{pmatrix}$ in (1.10) up to a shift $X(t)$:

$$a(\xi)\eta\big(U(t, \xi + X(t))|\tilde{U}_\varepsilon(\xi)\big),$$

where a is a smooth weight function.
The following Lemma provides a quadratic structure on $\frac{d}{dt}\int_{\mathbb{R}} a(\xi)\eta\big(U(t,\xi+X(t))|\tilde{U}_\varepsilon(\xi)\big)d\xi$.
We introduce the following notation: for any function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and the shift $X(t)$,

$$f^{\pm X}(t, \xi) := f(t, \xi \pm X(t)).$$

We also introduce the functional space

$$\mathcal{H} := \{(v, h) \in \mathbb{R}^+ \times \mathbb{R} \mid v^{-1}, v, h \in L^\infty(\mathbb{R}),\ \partial_\xi\big(p(v) - p(\tilde{v}_\varepsilon)\big) \in L^2(\mathbb{R})\},$$

on which the below functionals $Y, \mathcal{B}, \mathcal{G}$ in (2.21) are well-defined.

In this paper we assume that the solution lies in $C(0,T;\mathcal{H})$ for any $T>0$.

Remark 2.1. *The recent result of Constantin-Drivas-Nguyen-Pasqualotto* [8] *provides the global existence and uniqueness of smooth solutions to* (1.1) *for the case of* $\alpha > 1/2$ *and periodic boundary condition. Note that the system* (1.1) *is equivalent to the one in the Eulerian coordinates for smooth solutions. More precisely, it follows from* [8, Theorem 1.5 and Remark 1.6] *that* (1.1) *on the torus* $\mathbb{T}$ *admits a unique smooth solution* v, u *such that for any* $T>0$, $0 < C(T)^{-1} \le v \le C(T)$, $\partial_x v \in L^\infty(\mathbb{T})$ *and* $u \in C(0,T;H^k(\mathbb{T}))$ *in the case of* $\gamma = \alpha > 1$, *as long as the initial datum satisfies* $v_0, u_0 \in H^k$ *and* $\partial_x u_0 \le 1$ *for* $k \ge 4$. *As a consequence, since* $h = u + p'(v)\partial_x v$, *this result guarantees the existence of solutions* v, h *in* $C(0,T;\mathcal{H})$ *on the torus. For an extension of their result for the case where solutions connecting two different states on the whole space as a perturbation of a shock, we leave it as a future work. Let us refer to the previous result* [13] *of Haspot (see also* [28]*) for existence of solutions connecting two different states on the whole space in the case of* $\alpha \le 1$. *Note however that our result needs the case of* $\alpha > 1$.

Lemma 2.3. *Let* $a : \mathbb{R} \to \mathbb{R}^+$ *be a smooth bounded function such that* a', a'' *are integrable. Let* X *be a differentiable function, and* $\tilde{U}_\varepsilon := \binom{\tilde{v}_\varepsilon}{\tilde{h}_\varepsilon}$ *be the viscous shock in* (1.10). *For any solution* $U \in \mathcal{H}$ *to* (2.16), *we have*

$$\frac{d}{dt}\int_{\mathbb{R}} a(\xi)\eta(U^X(t,\xi)|\tilde{U}_\varepsilon(\xi))d\xi = \dot{X}(t)Y(U^X) + \mathcal{B}(U^X) - \mathcal{G}(U^X), \tag{2.20}$$

where

(2.21)
$$\begin{aligned}
Y(U) &:= -\int_{\mathbb{R}} a'\eta(U|\tilde{U}_\varepsilon)d\xi + \int_{\mathbb{R}} a\left(\partial_\xi \nabla\eta(\tilde{U}_\varepsilon)\right)\cdot(U-\tilde{U}_\varepsilon)d\xi,\\
\mathcal{B}(U) &:= \frac{1}{2\sigma_\varepsilon}\int_{\mathbb{R}} a'|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi + \sigma_\varepsilon\int_{\mathbb{R}} a\partial_\xi\tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon)d\xi + \frac{1}{2}\int_{\mathbb{R}} a''|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi,\\
\mathcal{G}(U) &:= \frac{\sigma_\varepsilon}{2}\int_{\mathbb{R}} a'\Big(h-\tilde{h}_\varepsilon - \frac{p(v)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big)^2 d\xi + \sigma_\varepsilon\int_{\mathbb{R}} a'Q(v|\tilde{v}_\varepsilon)d\xi\\
&\qquad + \int_{\mathbb{R}} a|\partial_\xi(p(v)-p(\tilde{v}_\varepsilon))|^2 d\xi.
\end{aligned}$$

Proof. To derive the desired structure, we use here a change of variable $\xi \mapsto \xi - X(t)$ as

$$\int_{\mathbb{R}} a(\xi)\eta(U^X(t,\xi)|\tilde{U}_\varepsilon(\xi))d\xi = \int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}_\varepsilon^{-X}(\xi))d\xi. \tag{2.22}$$

Then, by a straightforward computation together with [33, Lemma 4] and the identity $G(U;V) = G(U|V) - \nabla\eta(V)A(U|V)$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a^{-X}(\xi)\eta(U(t,\xi)|\tilde{U}_\varepsilon^{-X}(\xi))d\xi\\
&= -\dot{X}\int_{\mathbb{R}} a'^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi + \int_{\mathbb{R}} a^{-X}\Big[\Big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\Big(-\partial_\xi A(U) + \partial_\xi\Big(M\partial_\xi\nabla\eta(U)\Big)\Big)\\
&\qquad - \nabla^2\eta(\tilde{U}_\varepsilon^{-X})(U-\tilde{U}_\varepsilon^{-X})\Big(-\dot{X}\partial_\xi\tilde{U}_\varepsilon^{-X} - \partial_\xi A(\tilde{U}_\varepsilon^{-X}) + \partial_\xi\Big(M\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\Big)\Big]d\xi\\
&\quad = \dot{X}\Big(-\int_{\mathbb{R}} a'^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi + \int_{\mathbb{R}} a^{-X}\Big(\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\cdot(U-\tilde{U}_\varepsilon^{-X})\Big) + I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= -\int_{\mathbb{R}} a^{-X}\partial_\xi G(U;\tilde{U}_\varepsilon^{-X})d\xi,\\
I_2 &:= -\int_{\mathbb{R}} a^{-X}\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})A(U|\tilde{U}_\varepsilon^{-X})d\xi,\\
I_3 &:= \int_{\mathbb{R}} a^{-X}\Big(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)\partial_\xi\Big(M\partial_\xi(\nabla\eta(U) - \nabla\eta(\tilde{U}_\varepsilon^{-X}))\Big)d\xi\\
I_4 &:= \int_{\mathbb{R}} a^{-X}(\nabla\eta)(U|\tilde{U}_\varepsilon^{-X})\partial_\xi\Big(M\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})\Big)d\xi.
\end{aligned}
$$

Using (2.18) and (2.15), we have

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}} a'^{-X}G(U;\tilde{U}_\varepsilon^{-X})d\xi = \int_{\mathbb{R}} a'^{-X}\Big(\big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)(h - \tilde{h}_\varepsilon^{-X}) - \sigma_\varepsilon\eta(U|\tilde{U}_\varepsilon^{-X})\Big)d\xi,\\
I_2 &= -\int_{\mathbb{R}} a^{-X}\partial_\xi\tilde{h}_\varepsilon^{-X}p(v|\tilde{v}_\varepsilon^{-X})d\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}} a^{-X}\big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)\partial_{\xi\xi}\big(p(v) - p(\tilde{v}_\varepsilon^{-X})\big)d\xi\\
&= -\int_{\mathbb{R}} a^{-X}|\partial_\xi(p(v) - p(\tilde{v}_\varepsilon^{-X}))|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}} a''^{-X}|p(v) - p(\tilde{v}_\varepsilon^{-X})|^2 d\xi.
\end{aligned}
$$

Since it follows from (2.17) and (2.15) that

$$
I_4 = \int_{\mathbb{R}} a^{-X}(\nabla\eta)(U|\tilde{U}_\varepsilon^{-X})\partial_\xi A(\tilde{U}_\varepsilon^{-X})d\xi = \int_{\mathbb{R}} a^{-X}p(v|\tilde{v}_\varepsilon^{-X})\Big(\partial_\xi\tilde{h}_\varepsilon^{-X} + \sigma_\varepsilon\partial_\xi\tilde{v}_\varepsilon^{-X}\Big)d\xi,
$$

we have some cancellation

$$
I_2 + I_4 = \sigma_\varepsilon\int_{\mathbb{R}} a^{-X}\partial_\xi\tilde{v}_\varepsilon^{-X}p(v|\tilde{v}_\varepsilon^{-X})d\xi.
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi\\
&= \dot{X}\Big(-\int_{\mathbb{R}} a'^{-X}\eta(U|\tilde{U}_\varepsilon^{-X})d\xi + \int_{\mathbb{R}} a^{-X}\partial_\xi\nabla\eta(\tilde{U}_\varepsilon^{-X})(U-\tilde{U}_\varepsilon^{-X})d\xi\Big)\\
&\quad+\int_{\mathbb{R}} a'^{-X}\Big(\big(p(v)-p(\tilde{v}_\varepsilon^{-X})\big)(h-\tilde{h}_\varepsilon^{-X})-\sigma_\varepsilon\eta(U|\tilde{U}_\varepsilon^{-X})\Big)d\xi\\
&\quad+\sigma_\varepsilon\int_{\mathbb{R}} a^{-X}\partial_\xi\tilde{v}_\varepsilon^{-X}p(v|\tilde{v}_\varepsilon^{-X})d\xi+\frac{1}{2}\int_{\mathbb{R}} a''^{-X}|p(v)-p(\tilde{v}_\varepsilon^{-X})|^2d\xi\\
&\qquad\qquad-\int_{\mathbb{R}} a^{-X}|\partial_\xi(p(v)-p(\tilde{v}_\varepsilon^{-X}))|^2d\xi.
\end{aligned}
$$

Again, we use a change of variable $\xi\mapsto\xi+X(t)$ to have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a\eta(U^X|\tilde{U}_\varepsilon)d\xi\\
&= \dot{X}\Big(-\int_{\mathbb{R}} a'\eta(U^X|\tilde{U}_\varepsilon)d\xi+\int_{\mathbb{R}} a\partial_\xi\nabla\eta(\tilde{U}_\varepsilon)(U^X-\tilde{U}_\varepsilon)d\xi\Big)\\
&\quad+\int_{\mathbb{R}} a'\Big(\underbrace{\big(p(v^X)-p(\tilde{v}_\varepsilon)\big)(h^X-\tilde{h}_\varepsilon)-\sigma_\varepsilon\eta(U^X|\tilde{U}_\varepsilon)}_{=:I}\Big)d\xi\\
&\quad+\sigma_\varepsilon\int_{\mathbb{R}} a\partial_\xi\tilde{v}_\varepsilon p(v^X|\tilde{v}_\varepsilon)d\xi+\frac{1}{2}\int_{\mathbb{R}} a''|p(v^X)-p(\tilde{v}_\varepsilon)|^2d\xi-\int_{\mathbb{R}} a|\partial_\xi(p(v^X)-p(\tilde{v}_\varepsilon))|^2d\xi.
\end{aligned}
$$

To extract a quadratic term on $p(v^X)-p(\tilde{v}_\varepsilon)$ from the above hyperbolic part, we rewrite I as

$$
\begin{aligned}
I &= (p(v^X)-p(\tilde{v}_\varepsilon))(h^X-\tilde{h}_\varepsilon)-\sigma_\varepsilon\frac{|h^X-\tilde{h}_\varepsilon|^2}{2}-\sigma_\varepsilon Q(v^X|\tilde{v}_\varepsilon)\\
&= \frac{|p(v^X)-p(\tilde{v}_\varepsilon)|^2}{2\sigma_\varepsilon}-\frac{\sigma_\varepsilon}{2}\Big(h^X-\tilde{h}_\varepsilon-\frac{p(v^X)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big)^2-\sigma_\varepsilon Q(v^X|\tilde{v}_\varepsilon).
\end{aligned}
$$

Hence we have the desired representation (2.20)-(2.21). □

Remark 2.2. *Notice that since $\sigma_\varepsilon, a'<0$, the three terms in $\mathcal{G}$ are non-negative. Therefore, $\mathcal{G}$ consists of good terms, while $\mathcal{B}$ consists of bad terms.*

2.3. **Construction of the weight function.** We define the weight function a by

$$
a(\xi)=1-\lambda\frac{p(\tilde{v}_\varepsilon(\xi))-p(v_-)}{[p]}, \tag{2.23}
$$

where $[p]:=p(v_+)-p(v_-)$.We briefly present some useful properties on the weight a. First of all, the weight function a is positive and decreasing, and satisfies $1-\lambda\le a\le 1$. Since $[p]=\varepsilon$, $p'(v_-/2)\le p'(\tilde{v}_\varepsilon)\le p'(v_-)$ and

$$
a'=-\lambda\frac{\partial_\xi p(\tilde{v}_\varepsilon)}{[p]}, \tag{2.24}
$$

we have

$$
|a'|\sim\frac{\lambda}{\varepsilon}|\tilde{v}_\varepsilon'|. \tag{2.25}
$$

For $a''=-\lambda\frac{\partial_{\xi\xi}p(\tilde v_\varepsilon)}{[p]}$, we use the following relation from (1.10):

$$\partial_{\xi\xi}p(\tilde v_\varepsilon)=\sigma_\varepsilon\partial_\xi\tilde v_\varepsilon+\partial_\xi\tilde h_\varepsilon=\Big(\frac{\sigma_\varepsilon^2}{p'(\tilde v_\varepsilon)}+1\Big)\frac{\partial_\xi p(\tilde v_\varepsilon)}{\sigma_\varepsilon}. \tag{2.26}$$

Notice that $|v_- - v_+|=C\varepsilon$ and (1.14) together with the Taylor theorem imply

$$\sigma_\varepsilon=-\sqrt{-p'(v_-)}+\mathcal{O}(\varepsilon). \tag{2.27}$$

Moreover, since $p'(\tilde v_\varepsilon)^{-1}=p'(v_-)^{-1}+\mathcal{O}(\varepsilon)$, we have

$$|\partial_{\xi\xi}p(\tilde v_\varepsilon)|\le C\varepsilon|\partial_\xi p(\tilde v_\varepsilon)|. \tag{2.28}$$

Thus,

$$|a''|\lesssim\lambda|\tilde v_\varepsilon'|.$$

which together with (2.25) implies

$$|a''|\lesssim\varepsilon|a'|. \tag{2.29}$$

Remark 2.3. *The definition* (2.23) *can be more generally written by*

$$a(\xi)=1-\lambda\frac{\int_{-\infty}^{\xi}|\partial_s\nabla\eta(\tilde U_\varepsilon(s))|ds}{\int_{-\infty}^{\infty}|\partial_s\nabla\eta(\tilde U_\varepsilon(s))|ds}. \tag{2.30}$$

Indeed, since it follows from (1.10) *that* $p(\tilde v_\varepsilon)'=\sigma_\varepsilon\tilde h_\varepsilon'$, *we find that*

$$|\partial_\xi\nabla\eta(\tilde U_\varepsilon(\xi))|=\left|\partial_\xi\begin{pmatrix}-p(\tilde v_\varepsilon(\xi))\\ \tilde h_\varepsilon(\xi)\end{pmatrix}\right|=|\partial_\xi p(\tilde v_\varepsilon(\xi))||(-1,\sigma_\varepsilon^{-1})|.$$

Moreover, since $\partial_\xi p(\tilde v_\varepsilon(\xi))>0$,

$$|\partial_\xi\nabla\eta(\tilde U_\varepsilon(\xi))|=\partial_\xi p(\tilde v_\varepsilon(\xi))|(-1,\sigma_\varepsilon^{-1})|, \tag{2.31}$$

which implies (2.23).

2.4. **Global and local estimates on the relative quantities.** We here present useful inequalities on Q and p that are crucially for the proof of Theorem 1.2.

2.4.1. *Global inequalities on Q and p.* Lemma 2.4 provides some global inequalities on the relative function $Q(\cdot|\cdot)$ corresponding to the convex function $Q(v)=\frac{v^{-\gamma+1}}{\gamma-1}$, $v>0$, $\gamma>1$.

Lemma 2.4. *For given constants $\gamma>1$, and $v_->0$ There exists constants $c_1,c_2>0$ such that the following inequalities hold.*
1) For any $w\in(0,v_-)$,

$$\begin{aligned}&Q(v|w)\ge c_1|v-w|^2,\quad \textit{for all } 0<v\le 3v_-,\\ &Q(v|w)\ge c_2|v-w|,\quad \textit{for all } v\ge 3v_-.\end{aligned} \tag{2.32}$$

2) Moreover if $0<w\le u\le v$ or $0<v\le u\le w$ then

$$Q(v|w)\ge Q(u|w), \tag{2.33}$$

and for any $\delta_*>0$ there exists a constant $C>0$ such that if, in addition, $v_->w>v_--\delta_*/2$ and $|w-u|>\delta_*$, we have

$$Q(v|w)-Q(u|w)\ge C|u-v|. \tag{2.34}$$

Proof. • *proof of* (2.32) : We denote $v^* = 3v_-$. First, for the case of $v \geq v^*$, we rewrite $Q(v|w)$ as

$$Q(v|w) = \int_0^1 \Big(Q'(w + t(v-w)) - Q'(w)\Big) dt(v-w) \tag{2.35}$$

Since $w < v_- < v^* \leq v$ and Q' is increasing, we have

$$Q'(w + t(v-w)) \geq Q'(w + t(v^* - v_-)).$$

Thus,

$$\begin{aligned} Q(v|w) &\geq \int_0^1 \Big(Q'(w + t(v^* - v_-)) - Q'(w)\Big) dt(v-w) \\ &= \int_0^1 \int_0^1 Q''(w + st(v^* - v_-)) t ds dt (v^* - v_-)(v-w). \end{aligned}$$

Moreover, since Q'' is decreasing, we have

$$Q(v|w) \geq \int_0^1 \int_0^1 Q''(v_- + st(v^* - v_-)) t ds dt (v^* - v_-)(v-w),$$

which provides the second inequality in (2.32).

On the other hand, for the case of $v \leq v^*$, we use

$$Q(v|w) = (v-w)^2 \int_0^1 \int_0^1 Q''(w + st(v-w)) t ds dt.$$

Observe that for all $v \leq v^*$,

$$Q''(w + st(v-w)) = \gamma(stv + (1-st)w)^{-\gamma-1} \geq \gamma(stv^* + (1-st)v^*)^{-\gamma-1} = \gamma\Big(\frac{1}{v^*}\Big)^{\gamma+1},$$

where we have used $w < v^*$.
Therefore, we have

$$Q(v|w) \geq \frac{\gamma}{2}\Big(\frac{1}{v^*}\Big)^{\gamma+1}(v-w)^2.$$

• *proof of* (2.34) : Note that $z \mapsto Q(z|y)$ is convex so $\partial_z Q(z|y)$ is increasing in z and zero at $z = y$. Therefore $z \mapsto Q(z|y)$ is increasing in $|z-y|$, which implies

$$Q(v|w) \geq Q(u|w).$$

Moreover, if $v_- > w > v_- - \delta_*/2$ and $|w-u| > \delta_*$, using the facts that Q' is increasing and

$$\begin{aligned} Q(v|w) - Q(u|w) &= Q(v) - Q(u) - Q'(w)(v-u) \\ &= \int_u^v [Q'(y) - Q'(w)]\, dy, \end{aligned}$$

we have the following:
If $w < u < v$, then

$$Q(v|w) - Q(u|w) \geq [Q'(v_- + \delta_*/2) - Q'(v_-)](v-u).$$

If $v < u < w$, then

$$Q(v|w) - Q(u|w) \geq [Q'(v_- - \delta_*/2) - Q'(v_- - \delta_*)](u-v).$$

Hence we have (2.34). □

The following lemma provides some global inequalities on the pressure $p(v) = v^{-\gamma}$, $v > 0$, $\gamma > 1$, and on the associated relative function $p(\cdot|\cdot)$.

Lemma 2.5. *For given constants $\gamma > 1$, and $v_- > 0$, there exist constants $c_3, C > 0$ such that the following inequalities hold.*
1)For any $w \in (v_-/4, v_-)$

$$|p(v) - p(w)| \le c_3|v - w|, \quad \textit{for all } v \ge v_-/2, \tag{2.36}$$

$$p(v|w) \le C|v - w|^2, \quad \textit{for all } v \ge v_-/2. \tag{2.37}$$

2) For any $w \in (v_-/4, v_-)$, and all $v > 0$

$$p(v|w) \le C(|v - w| + |p(v) - p(w)|). \tag{2.38}$$

Proof. • *proof of* (2.36) : Since $v, w \ge v_-/2$, (2.36) follows from using the mean value theorem:

$$|p(v) - p(w)| \le |p'(v_-/2)||v - w|.$$

• *proof of* (2.37) : Since $v, w \ge v_-/2$, (2.36) follows from

$$\begin{aligned} p(v|w) &= (v - w)^2 \int_0^1 \int_0^1 p''(stv + (1 - st)w))tdsdt \\ &\le (v - w)^2 \int_0^1 \int_0^1 p''(v_-/2)tdsdt = \frac{p''(v_-/2)}{2}(v - w)^2. \end{aligned}$$

• *proof of* (2.38) : For every $v > v_-/2$

$$0 \le p(v|w) = p(v) - p(w) - p'(w)(v - w) \le 2|p'(v_-/2)||v - w|.$$

And for every $v \le v_-/2$:

$$|p(v) - p(w)| = \int_v^w |p'(y)|\, dy \ge |p'(w)||v - w| \ge |p'(v_-)||v - w|.$$

Hence

$$0 \le p(v|w) = p(v) - p(w) - p'(w)(v - w) \le (1 + |p'(v_-/2)|^{-1})|p(v) - p(w)|.$$

□

2.4.2. *Local inequalities on Q and p.* We present now some local estimates on $p(v|w)$ and $Q(v|w)$ for $|v-w| \ll 1$, based on Taylor expansions. The specific coefficients of the estimates will be crucially used in our local analysis on a suitably small truncation $|p(v)-p(\tilde{v}_\varepsilon)| \ll 1$.

Lemma 2.6. *For given constants $\gamma > 1$ and $v_- > 0$ there exist positive constants C and δ_* such that for any $0 < \delta < \delta_*$, the following is true.*
1) For any $(v, w) \in \mathbb{R}_+^2$ satisfying $|p(v) - p(w)| < \delta$, and $|p(w) - p(v_-)| < \delta$ the following estimates (2.39)-(2.41) *hold:*

$$p(v|w) \le \left(\frac{\gamma + 1}{2\gamma}\frac{1}{p(w)} + C\delta\right)|p(v) - p(w)|^2, \tag{2.39}$$

$$Q(v|w) \ge \frac{p(w)^{-\frac{1}{\gamma}-1}}{2\gamma}|p(v) - p(w)|^2 - \frac{1 + \gamma}{3\gamma^2}p(w)^{-\frac{1}{\gamma}-2}(p(v) - p(w))^3, \tag{2.40}$$

$$Q(v|w) \le \left(\frac{p(w)^{-\frac{1}{\gamma}-1}}{2\gamma} + C\delta\right)|p(v) - p(w)|^2. \tag{2.41}$$

2) For any $(v,w)\in\mathbb{R}^2_+$ *such that* $|p(w)-p(v_-)|\le\delta$*, and satisfying either* $Q(v|w)<\delta$ *or* $|p(v)-p(w)|<\delta$*,*

$$|p(v)-p(w)|^2\le CQ(v|w). \tag{2.42}$$

Proof. We consider $\delta_*\le p(v_-)/4$.

• **proof of** (2.39) From the hypothesis, we have both $|p(v)-p(v_-)|\le p(v_-)/2$ and $|p(w)-p(v_-)|\le p(v_-)/2$. First, we rewrite $p(v|w)$ in terms of $p(v)-p(w)$ as

$$\begin{aligned}p(v|w)&=p(v)-p(w)+\gamma w^{-\gamma-1}(v-w)\\&=p(v)-p(w)+\gamma p(w)^{\frac{\gamma+1}{\gamma}}\big(p(v)^{-\frac{1}{\gamma}}-p(w)^{-\frac{1}{\gamma}}\big).\end{aligned}$$

Setting $F_1(p):=p-\tilde p+\gamma\tilde p^{\frac{\gamma+1}{\gamma}}(p^{-\frac{1}{\gamma}}-\tilde p^{-\frac{1}{\gamma}})$ where $p:=p(v)$, $\tilde p:=p(w)$, we apply the Taylor theorem to F_1 about $\tilde p$. That is, using

$$F_1'(p)=1-\tilde p^{\frac{\gamma+1}{\gamma}}p^{-\frac{\gamma+1}{\gamma}},\quad F_1''(p)=\frac{\gamma+1}{\gamma}\tilde p^{\frac{\gamma+1}{\gamma}}p^{-\frac{2\gamma+1}{\gamma}},$$

since $F_1(\tilde p)=0, F_1'(\tilde p)=0$, and $F_1''(\tilde p)=\frac{\gamma+1}{\gamma\tilde p}$, we have

$$p(v|w)=F_1(p)=\frac{\gamma+1}{\gamma\tilde p}\frac{|p-\tilde p|^2}{2}+\frac{F_1'''(p_*)}{6}|p-\tilde p|^3,$$

where p_* lies between p and $\tilde p$. Therefore $\frac{p(v_-)}{2}<p_*<2p(v_-)$. Taking $\delta\le\delta_*$, we have

$$p(v|w)\le\frac{\gamma+1}{\gamma\tilde p}\frac{|p-\tilde p|^2}{2}+C\delta|p-\tilde p|^2.$$

Therefore, we have (2.39).

• **proof of** (2.40) **and** (2.41) Likewise, since

$$\begin{aligned}Q(v|w)&=Q(v)-Q(w)+p(w)(v-w)\\&=\frac{p(v)^{\frac{\gamma-1}{\gamma}}}{\gamma-1}-\frac{p(w)^{\frac{\gamma-1}{\gamma}}}{\gamma-1}+p(w)(p(v)^{-\frac{1}{\gamma}}-p(w)^{-\frac{1}{\gamma}}).\end{aligned}$$

setting $F_2(p):=\frac{p^{\frac{\gamma-1}{\gamma}}}{\gamma-1}-\frac{\tilde p^{\frac{\gamma-1}{\gamma}}}{\gamma-1}+\tilde p(p^{-\frac{1}{\gamma}}-\tilde p^{-\frac{1}{\gamma}})$ where $p:=p(v)$, $\tilde p:=p(w)$, we apply the Taylor theorem to F_2 about $\tilde p$. That is, using

$$\begin{aligned}&F_2'(p)=\frac{1}{\gamma}p^{-\frac{1}{\gamma}}\Big(1-\tilde p p^{-1}\Big),\quad F_2''(p)=-\frac{1}{\gamma^2}p^{-\frac{1+\gamma}{\gamma}}\Big(1-(1+\gamma)\tilde p p^{-1}\Big),\\&F_2'''(p)=\frac{1+\gamma}{\gamma^3}p^{-\frac{1+2\gamma}{\gamma}}\Big(1-(1+2\gamma)\tilde p p^{-1}\Big),\\&F_2''''(p)=-\frac{(1+\gamma)(1+2\gamma)}{\gamma^4}p^{-\frac{1+3\gamma}{\gamma}}\Big(1-(1+3\gamma)\tilde p p^{-1}\Big),\end{aligned}$$

and then

$$\begin{aligned}&F_2(\tilde p)=0,\quad F_2'(\tilde p)=0,\quad F_2''(\tilde p)=\frac{1}{\gamma}\tilde p^{-\frac{1}{\gamma}-1},\\&F_2'''(\tilde p)=-\frac{2(1+\gamma)}{\gamma^2}\tilde p^{-\frac{1}{\gamma}-2},\quad F_2''''(\tilde p)=\frac{3(1+\gamma)(1+2\gamma)}{\gamma^3}\tilde p^{-\frac{1+3\gamma}{\gamma}},\end{aligned}$$

we have

$$Q(v|w) = F_2''(\tilde p)\frac{(p-\tilde p)^2}{2} + F_2'''(\tilde p)\frac{(p-\tilde p)^3}{6} + F_2''''(\tilde p)\frac{(p-\tilde p)^4}{24} + F_2^{(5)}(p_*)\frac{(p-\tilde p)^5}{5!}.$$

Since $F_2''''(\tilde p) \geq \frac{3(1+\gamma)(1+2\gamma)}{\gamma^3}[p(v-)/2]^{-\frac{1+3\gamma}{\gamma}} > 0$, taking δ_* smaller if needed, we have for every $\delta < \delta_*$

$$Q(v|w) \geq F_2''(\tilde p)\frac{|p-\tilde p|^2}{2} + F_2'''(\tilde p)\frac{(p-\tilde p)^3}{6},$$

which completes (2.40). The estimate (2.41) follows by considering the 2nd order Taylor polynomial as done in (2.39).

• **proof of** (2.42) Since it follows from (2.32) that $\min\{c_1|v-w|^2, c_2|v-w|\} \leq Q(v|w)$, if $Q(v|w) < \delta < \delta_* \ll 1$, then $|v-w| \ll 1$ and thus $\frac{v_-}{2} < v < 2v_-$ and $c_1|v-w|^2 \leq Q(v|w)$. Therefore,

$$|p(v)-p(w)|^2 \leq |p'(\frac{v_-}{2})|^2|v-w|^2 \leq c_1^{-1}|p'(\frac{v_-}{2})|^2 Q(v|w). \tag{2.43}$$

If $|p(v)-p(w)| < \delta$, then it follows from (2.41) that

$$Q(v|w) \leq C|p(v)-p(w)|^2 < \delta,$$

which gives (2.43). □

2.5. **Some functional inequalities.** We state in this section some standard functional inequalities. Some of the proofs will be postponed to the appendix. The first result is a simple inequality on a specific polynomial functional.

Lemma 2.7. *For all $x \in [-2, 0)$,*

$$2x - 2x^2 - \frac{4}{3}x^3 + \frac{4\theta}{3}\Big(-x^2 - 2x\Big)^{3/2} < 0,$$

where $\theta = \sqrt{5 - \frac{\pi^2}{3}}$.

The proof of Lemma 2.7 is given in Appendix A.

The second result is a sharp point-wise estimate.

Lemma 2.8. *Let $f \in C^1(0,1)$. Then, for all $x \in [0,1)$,*

$$\Big|f(x) - \int_0^1 f(y)dy\Big| \leq \sqrt{L(x) + L(1-x)}\sqrt{\int_0^1 y(1-y)|f'|^2 dy},$$

where $L(x) := -x - \ln(1-x)$. Moreover

$$\Big(\int_0^1 (L(y) + L(1-y))^2 dy\Big)^{1/2} = \sqrt{5 - \frac{\pi^2}{3}} = \theta.$$

Proof. First, since

$$f(x) - \int_0^1 f(y)dy = \int_0^1\int_y^x f'(z)dzdy = \int_0^x\int_y^x f'(z)dzdy + \int_x^1\int_y^x f'(z)dzdy,$$

we have

$$\begin{aligned}\Big|f(x)-\int_0^1 f(y)dy\Big| &\le \int_0^x\int_y^x |f'(z)|dzdy + \int_x^1\int_x^y |f'(z)|dzdy\\ &\le \underbrace{\Big(\int_0^x\int_y^x \frac{1}{1-z}dzdy\Big)^{\frac12}\Big(\int_0^x\int_y^x (1-z)|f'(z)|^2 dzdy\Big)^{\frac12}}_{=:I_1}\\ &\quad+\underbrace{\Big(\int_x^1\int_x^y \frac{1}{z}dzdy\Big)^{\frac12}\Big(\int_x^1\int_x^y z|f'(z)|^2 dzdy\Big)^{\frac12}}_{=:I_2}.\end{aligned}$$

Using Fubini's theorem as $\int_0^x\int_y^x g dzdy = \int_0^x\int_0^z g dydz$, we have

$$\begin{aligned}I_1 &= \Big(\int_0^x \frac{z}{1-z}dz\Big)^{\frac12}\Big(\int_0^x z(1-z)|f'(z)|^2 dz\Big)^{\frac12}\\ &= \big(-x-\ln(1-x)\big)^{\frac12}\Big(\int_0^x z(1-z)|f'(z)|^2 dz\Big)^{\frac12},\end{aligned}$$

and likewise,

$$\begin{aligned}I_2 &= \Big(\int_x^1 \frac{1-z}{z}dz\Big)^{\frac12}\Big(\int_x^1 z(1-z)|f'(z)|^2 dz\Big)^{\frac12}\\ &= \big(-(1-x)-\ln x\big)^{\frac12}\Big(\int_x^1 z(1-z)|f'(z)|^2 dz\Big)^{\frac12}.\end{aligned}$$

Let $L(x) := -x-\ln(1-x)$ and

$$X := \int_0^x z(1-z)|f'(z)|^2 dz, \quad D := \int_0^1 z(1-z)|f'(z)|^2 dz.$$

Then,

$$I_1 + I_2 = \sqrt{L(x)}\sqrt{X} + \sqrt{L(1-x)}\sqrt{D-X}$$

If we consider a function $F(X) := \sqrt{L(x)}\sqrt{X} + \sqrt{L(1-x)}\sqrt{D-X}$ for $X \in [0, D]$, we see that the function F has a maximum at $\bar{X} := \frac{L(x)D}{L(x)+L(1-x)}$.
Thus, we have

$$I_1 + I_2 \le F(\bar{X}) = \sqrt{L(x)+L(1-x)}\sqrt{D},$$

which completes the desired inequality. We now compute the value of θ. We have

$$\begin{aligned}\int_0^1 \big(L(x)+L(1-x)\big)^2 dx &= \int_0^1 \big(1+\ln(1-x)+\ln x\big)^2 dx\\ &= 1 + \int_0^1 \big(\ln(1-x)\big)^2 dx + \int_0^1 \big(\ln x\big)^2 dx + 2\int_0^1 \ln(1-x)\, dx\\ &\quad + 2\int_0^1 \ln x\, dx + 2\int_0^1 \ln(1-x)\ln x\, dx.\end{aligned}$$

Since $\int_0^1 \ln(1-x)\,dx = \int_0^1 \ln x\,dx = -1$, we have

$$\int_0^1 \big(\ln(1-x)\big)^2 dx = \int_0^1 \big(\ln x\big)^2 dx = \Big[x\big(\ln x\big)^2\Big]_0^1 - 2\int_0^1 \ln x\,dx = -2\int_0^1 \ln x\,dx = 2.$$

Thus,

$$\int_0^1 \big(L(x)+L(1-x)\big)^2 dx = 1 + 2\int_0^1 \ln(1-x)\ln x\,dx.$$

To compute the last integral, we find

$$\begin{aligned}\int_0^1 \ln(1-x)\ln x\,dx &= \Big[\Big(x\ln(1-x) - x - \ln(1-x)\Big)\ln x\Big]_0^1 \\ &\qquad - \int_0^1 \frac{x\ln(1-x) - x - \ln(1-x)}{x}dx \\ &= -\int_0^1 \ln(1-x)dx + 1 + \int_0^1 \frac{\ln(1-x)}{x}\,dx = 2 + \int_0^1 \frac{\ln(1-x)}{x}\,dx.\end{aligned}$$

Since

$$\int \frac{\ln(1-x)}{x}\,dx = -\sum_{n=1}^{\infty}\frac{x^n}{n^2}, \quad |x|\le 1,$$

we have

$$\int_0^1 \frac{\ln(1-x)}{x}dx = -\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{6}.$$

This gives the result. $\square$

Lemma 2.9. *For any $f:[0,1]\to\mathbb{R}$ satisfying $\int_0^1 y(1-y)|f'|^2dy<\infty$,*

$$\int_0^1 \Big|f - \int_0^1 f dy\Big|^2 dy \le \frac{1}{2}\int_0^1 y(1-y)|f'|^2 dy. \tag{2.44}$$

The proof of this lemma is given in Appendix B.

3. Proof of Theorem 1.2

3.1. **Construction of the shift X and the main proposition.** For any fixed $\varepsilon>0$, we consider a continuous function Φ_ε defined by

$$\Phi_\varepsilon(y) = \begin{cases} \frac{1}{\varepsilon^2}, & \text{if } y\le -\varepsilon^2, \\ -\frac{1}{\varepsilon^4}y, & \text{if } |y|\le\varepsilon^2, \\ -\frac{1}{\varepsilon^2}, & \text{if } y\ge\varepsilon^2. \end{cases} \tag{3.1}$$

We define a shift function $X(t)$ as a solution of the nonlinear ODE:

$$\begin{cases} \dot X(t) = \Phi_\varepsilon(Y(U^X))\Big(2|\mathcal{B}(U^X)|+1\Big), \\ X(0)=0, \end{cases} \tag{3.2}$$

where Y and $\mathcal{B}$ are as in (2.21). Therefore, for the solution $U\in C(0,T;\mathcal{H})$, the shift X exists and is unique at least locally by the Cauchy-Lipschitz theorem. Indeed, since $\tilde v_\varepsilon', \tilde h_\varepsilon', a', a''$ are bounded smooth and integrable, using $U\in C(0,T;\mathcal{H})$ together with the change of variables $\xi \mapsto \xi - X(t)$ as in (2.22), we find that the right-hand side of the ODE (3.2) is uniformly Lipschitz continuous in X, and is continuous in t.

Moreover, the global-in-time existence and uniqueness of the shift are verified by the a priori estimate (3.8).

The main proposition as a corner stone of proof of the Theorem is the following.

Proposition 3.1. *There exist $\varepsilon_0, \delta_0 > 0$, such that for any $\varepsilon < \varepsilon_0$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$, the following is true.*
For any $U \in \mathcal{H} \cap \{U \mid |Y(U)| \le \varepsilon^2\}$,

$$\mathcal{R}(U) := -\frac{1}{\varepsilon^4}Y^2(U) + (1 + \delta_0(\varepsilon/\lambda))|\mathcal{B}(U)| - \mathcal{G}(U) \le 0. \tag{3.3}$$

Most of the rest of the paper will be dedicated to the proof of this result. We will first show how this proposition implies Theorem 1.2.

3.2. **Proof of Theorem 1.2 from Proposition 3.1.** Based on (2.20) and (3.2), to get the contraction estimate, it is enough to prove that for almost every time $t > 0$

$$\Phi_\varepsilon(Y(U^X))\Big(2|\mathcal{B}(U^X)| + 1\Big)Y(U^X) + \mathcal{B}(U^X) - \mathcal{G}(U^X) \le 0. \tag{3.4}$$

For every $U \in \mathcal{H}$ we define

$$\mathcal{F}(U) := \Phi_\varepsilon(Y(U))\Big(2|\mathcal{B}(U)| + 1\Big)Y(U) + |\mathcal{B}(U)| - \mathcal{G}(U). \tag{3.5}$$

From (3.1), we have

$$\Phi_\varepsilon(Y)\Big(2|\mathcal{B}| + 1\Big)Y \le \begin{cases} -2|\mathcal{B}|, & \text{if } |Y| \ge \varepsilon^2, \\ -\frac{1}{\varepsilon^4}Y^2, & \text{if } |Y| \le \varepsilon^2. \end{cases} \tag{3.6}$$

Hence, for all $U \in \mathcal{H}$ satisfying $|Y(U)| \ge \varepsilon^2$, we have

$$\mathcal{F}(U) \le -|\mathcal{B}(U)| - \mathcal{G}(U) \le 0.$$

Using both (3.6) and Proposition 3.1, we find that for all $U \in \mathcal{H}$ satisfying $|Y(U)| \le \varepsilon^2$,

$$\mathcal{F}(U) \le -\delta_0\left(\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U)| \le 0.$$

Since $\delta_0 < 1/2$, these two estimates show that for every $U \in \mathcal{H}$ we have

$$\mathcal{F}(U) \le -\delta_0\left(\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U)|.$$

For every fixed $t > 0$, using this estimate with $U = U^X(t, \cdot)$, together with (2.20), and (3.4) gives

$$\frac{d}{dt}\int_{\mathbb{R}} a\eta(U^X|\tilde{U}_\varepsilon)d\xi \le \mathcal{F}(U^X) \le -\delta_0\left(\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U^X)|. \tag{3.7}$$

Thus, $\frac{d}{dt}\int_{\mathbb{R}} a\eta(U^X|\tilde{U}_\varepsilon)d\xi \le 0$, which completes (1.12).
Moreover, since it follows from (3.7) that

$$\delta_0\left(\frac{\varepsilon}{\lambda}\right)\int_0^\infty |\mathcal{B}(U^X)|dt \le \int_{\mathbb{R}} \eta(U_0|\tilde{U}_\varepsilon)d\xi < \infty \quad \text{by the initial condition,}$$

using (3.2) and $\|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \le \frac{1}{\varepsilon^2}$ by (3.1), we have

$$|\dot{X}| \le \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon^2}|\mathcal{B}|, \quad \|\mathcal{B}\|_{L^1(0,\infty)} \le \frac{1}{\delta_0}\frac{\lambda}{\varepsilon}\int_{\mathbb{R}} \eta(U_0|\tilde{U}_\varepsilon)d\xi. \tag{3.8}$$

This provides the global-in-time estimate (1.13), thus $X \in W^{1,1}_{loc}(\mathbb{R}^+)$. This completes the proof of Theorem 1.2.

The rest of the paper is dedicated to the proof of Proposition 3.1.

3.3. **An estimate on specific polynomials.** Let $\theta := \sqrt{5-\frac{\pi^2}{3}}$, and $\delta > 0$ be any constant. We consider the following polynomial functionals:

$$\begin{aligned} E(Z_1,Z_2) &:= Z_1^2+Z_2^2+2Z_1, \\ P_\delta(Z_1,Z_2) &:= (1+\delta)(Z_1^2+Z_2^2)+2Z_1Z_2^2+\frac{2}{3}Z_1^3+6\delta(|Z_1|Z_2^2+|Z_1|^3) \\ &\qquad -2\left(1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta Z_2\right)Z_2^2. \end{aligned}$$

This section is dedicated to the proof of the following proposition.

Proposition 3.2. *There exist $\delta_0, \delta_1 > 0$ such that for any $0<\delta<\delta_0$, the following is true. If $(Z_1,Z_2)\in\mathbb{R}^2$ satisfies $|E(Z_1,Z_2)|\le\delta_1$, then*

$$P_\delta(Z_1,Z_2) - |E(Z_1,Z_2)|^2 \le 0. \tag{3.9}$$

This proposition will be used when a smallness condition on the perturbation, due to the shift, will be available. It should be noticed that the expansion leading to this polynomial is not merely a linearization. We end up with a polynomial P_δ which is of order 3.

Proof. We split the proof into three steps.

Step 1. For $r>0$, we denote $B_r(0)$ the open ball centered at the origin with radius r. We show the following claim: There exist $r>0$ and $\delta_0>0$ such that for any $\delta\le\delta_0$,

$$P_\delta(Z_1,Z_2) - |E(Z_1,Z_2)|^2 \le 0, \text{ whenever } (Z_1,Z_2)\in B_r(0). \tag{3.10}$$

To prove the claim, notice first that $|Z_1|, |Z_2| \le r$ on $B_r(0)$. So we have

$$|2Z_1|^2 = (E-(Z_1^2+Z_2^2))^2 \le 2|E|^2 + 2|Z_1^2+Z_2^2|^2 \le 2|E|^2 + 2r^2(Z_1^2+Z_2^2),$$

which implies

$$-|E|^2 \le -2Z_1^2 + r^2(Z_1^2+Z_2^2).$$

Thus, for any $(Z_1,Z_2)\in B_r(0)$,

$$\begin{aligned} P_\delta - |E|^2 &\le -2Z_1^2 + (1+\delta)\Big(Z_1^2+Z_2^2+\frac{r^2}{1+\delta}(Z_1^2+Z_2^2)+\frac{(2+6\delta)r}{1+\delta}Z_2^2+\frac{((2/3)+6\delta)r}{1+\delta}Z_1^2\Big) \\ &\qquad -2\left(1-\delta-\frac{2}{3}(1+\delta)\theta r\right)Z_2^2. \end{aligned}$$

Taking δ_0 and r small enough, we can ensure that for any $\delta<\delta_0$,

$$P_\delta(Z_1,Z_2) - |E|^2 \le 0, \qquad \text{on } B_r(0).$$

This proves the claim (3.10).

Step 2. The second step is dedicated to the proof of the following claim. There exists $\delta_0>0$ (possibly smaller that in the step 1), and $\delta_1>0$ such that for any $0<\delta\le\delta_0$ we have :

$$P_\delta(Z_1,Z_2) < 0, \qquad \text{whenever } |E(Z_1,Z_2)|\le\delta_1, \qquad \text{and } (Z_1,Z_2)\notin B_r(0). \tag{3.11}$$

To show this claim, we first observe the limiting case: if $\delta = 0$ and $E(Z_1, Z_2) = 0$, we have

$$P_0(Z_1, Z_2) = 2Z_1 - 2Z_1^2 - \frac{4}{3}Z_1^3 + \frac{4\theta}{3}\Big(- Z_1^2 - 2Z_1\Big)^{3/2}, \quad Z_1^2 + Z_2^2 + 2Z_1 = 0.$$

Since $(Z_1+1)^2 + Z_2^2 = 1$ by $E = 0$, we have $-2 \le Z_1 \le 0$. Then by the algebraic inequality in Lemma 2.7, we have

$$P_0(Z_1, Z_2) < 0, \quad Z_1^2 + Z_2^2 + 2Z_1 = 0,\ Z_1 \neq 0.$$

Since P_0 is continuous, it attains its maximum $-c < 0$ on the compact set $\{E(Z_1, Z_2) = 0\} \setminus B_r(0)$. In addition, P_0 is uniformly continuous on the compact set $\{|E(Z_1, Z_2)| \le 1\} \setminus B_r(0)$, so there exist $0 < \delta_1 < 1$ such that

$$P_0(Z_1, Z_2) < -c/2, \quad \text{whenever } |E(Z_1, Z_2)| \le \delta_1, \text{ and } (Z_1, Z_2) \notin B_r(0).$$

Taking δ_0 small enough we still have for $\delta < \delta_0$:

$$P_\delta(Z_1, Z_2) < 0, \quad \text{whenever } |E(Z_1, Z_2)| \le \delta_1, \text{ and } (Z_1, Z_2) \notin B_r(0).$$

This proves the claim (3.11).

Step 3. The proved claims (3.10) and (3.11) together give the Proposition 3.2. □

3.4. **A nonlinear Poincaré type inequality.** For any $\delta > 0$, and any function $W \in L^2(0,1)$ such that $\sqrt{y(1-y)}\partial_y W \in L^2(0,1)$, we define

$$\begin{aligned} \mathcal{R}_\delta(W) = & -\frac{1}{\delta}\left(\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right)^2 + (1+\delta)\int_0^1 W^2\,dy \\ & + \frac{2}{3}\int_0^1 W^3\,dy + \delta\int_0^1 |W|^3\,dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2\,dy. \end{aligned}$$

This section is dedicated to the proof of the following proposition.

Proposition 3.3. *For a given $C_1 > 0$, there exists $\delta_2 > 0$, such that for any $\delta < \delta_2$ the following is true.*
For any $W \in L^2(0,1)$ such that $\sqrt{y(1-y)}\partial_y W \in L^2(0,1)$, if $\int_0^1 |W(y)|^2\,dy \le C_1$, then

$$\mathcal{R}_\delta(W) \le 0. \tag{3.12}$$

Note that the constant C_1 may not be small. Therefore we cannot discard the cubic term in $\mathcal{R}_\delta(W)$.

Proof. Let $\overline{W} = \int_0^1 W dy$. We first separate the first cubic term in $\mathcal{R}_\delta$ into the three parts:

$$\begin{aligned} \int_0^1 W^3 dy &= \int_0^1 \Big((W - \overline{W}) + \overline{W}\Big)^3 dy \\ &= \int_0^1 (W - \overline{W})^3 dy + 3\overline{W}\int_0^1 (W - \overline{W})^2 dy + \int_0^1 \overline{W}^3 dy \\ &= \int_0^1 (W - \overline{W})^3 dy + 2\overline{W}\int_0^1 (W - \overline{W})^2 dy + \overline{W}\int_0^1 W^2 dy. \end{aligned} \tag{3.13}$$

Thus, we have

(3.14)

$$\mathcal{R}_\delta(W) = -\frac{1}{\delta}\left(\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right)^2 + (1+\delta)\int_0^1 W^2 dy + \frac{4}{3}\overline{W}\int_0^1 (W-\overline{W})^2 dy$$
$$+\frac{2}{3}\overline{W}\int_0^1 W^2 dy + \frac{2}{3}\int_0^1 (W-\overline{W})^3 dy + \delta\int_0^1 |W|^3 dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2 dy.$$

Let

$$Z_1 := \overline{W}, \quad Z_2 := \Big(\int_0^1 (W-\overline{W})^2 dy\Big)^{\frac{1}{2}}, \quad E(Z_1,Z_2) = Z_1^2 + Z_2^2 + 2Z_1.$$

In what follows, we rewrite $\mathcal{R}_\delta$ in terms of the new variables Z_1 and Z_2.
Since

$$\int_0^1 W^2 dy = Z_1^2 + Z_2^2,$$

and

$$\begin{aligned}\int_0^1 |W|^3 dy &\le \int_0^1 \Big(|W-\overline{W}| + |\overline{W}|\Big)^3 dy\\ &\le \int_0^1 |W-\overline{W}|^3 dy + 3|\overline{W}|\int_0^1 |W-\overline{W}|^2 dy + 3|\overline{W}|^2\int_0^1 |W-\overline{W}| dy + |\overline{W}|^3\\ &\le \int_0^1 |W-\overline{W}|^3 dy + 3|Z_1|Z_2^2 + 3|Z_1|^{3/2}|Z_1|^{1/2}Z_2 + |Z_1|^3\\ &\le \int_0^1 |W-\overline{W}|^3 dy + 6|Z_1|Z_2^2 + 4|Z_1|^3,\end{aligned}$$

we have

(3.15)

$$\mathcal{R}_\delta = -\frac{1}{\delta}|E(Z_1,Z_2)|^2 + (1+\delta)(Z_1^2+Z_2^2) + 2Z_1Z_2^2 + \frac{2}{3}Z_1^3 + 6\delta(|Z_1|Z_2^2 + |Z_1|^3) + \mathcal{P},$$

where

$$\mathcal{P} := \Big(\frac{2}{3}+\delta\Big)\int_0^1 |W-\overline{W}|^3 dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2 dy. \tag{3.16}$$

For the cubic term in $\mathcal{P}$, we use Lemma 2.8 to estimate

$$\begin{aligned}&\int_0^1 \Big|W - \int_0^1 W\Big|^3 dy\\ &\le \int_0^1 z(1-z)|\partial_z W|^2 dz \int_0^1 |L(y)+L(1-y)|\Big|W-\int_0^1 W\Big| dy\\ &\le \int_0^1 z(1-z)|\partial_z W|^2 dz\Big(\int_0^1 (L(y)+L(1-y))^2 dy\Big)^{1/2}\Big(\int_0^1 \Big|W-\int_0^1 W\Big|^2 dy\Big)^{1/2}\\ &= \theta Z_2\int_0^1 y(1-y)|\partial_y W|^2 dy.\end{aligned} \tag{3.17}$$

Thus, we have

$$\mathcal{P} \le -\left(1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta Z_2\right)\int_0^1 y(1-y)|\partial_y W|^2 dy.$$

Since $(Z_1+1)^2+Z_2^2=1+E(Z_1,Z_2)$, we have

$$Z_2 \le \sqrt{1+|E(Z_1,Z_2)|}.$$

Since $\frac{2}{3}\theta = \frac{2}{3}\sqrt{5-\frac{\pi^2}{3}} \approx 0.88 < 1$, there exists a positive constant $\delta_\theta < 1$ such that

$$\frac{2}{3}\theta\sqrt{1+\delta_\theta} < 1.$$

Then, we take $\delta_2 < 1$ such that $\forall \delta < \delta_2$,

$$1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta\sqrt{1+\delta_\theta} > 0.$$

We consider now two cases, whether $|E(Z_1,Z_2)| \le \min\{\delta_\theta,\delta_1\}$, or $|E(Z_1,Z_2)| \ge \min\{\delta_\theta,\delta_1\}$, where δ_1 is the constant as in Proposition 3.2.

Case 1: Assume that

$$|E(Z_1,Z_2)| \le \min\{\delta_\theta,\delta_1\}. \tag{3.18}$$

Then we find that $\forall \delta < \delta_2$,

$$\begin{aligned} 1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta Z_2 \quad &\ge \quad 1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta\sqrt{1+\min\{\delta_\theta,\delta_1\}} \\ &\ge \quad 1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta\sqrt{1+\delta_\theta} > 0. \end{aligned}$$

Therefore, we use the weighted Poincaré inequality (2.44) to have

$$\mathcal{P} \le -2\left(1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta Z_2\right) Z_2^2.$$

Therefore, we have

$$\begin{aligned} \mathcal{R}_\delta \le & -\frac{1}{\delta}|E(Z_1,Z_2)|^2 + (1+\delta)(Z_1^2+Z_2^2) + 2Z_1Z_2^2 + \frac{2}{3}Z_1^3 + 6\delta(|Z_1|Z_2^2+|Z_1|^3) \\ & - 2\left(1-\delta-\Big(\frac{2}{3}+\delta\Big)\theta Z_2\right) Z_2^2 \\ = & -\frac{1}{\delta}|E(Z_1,Z_2)|^2 + P_\delta(Z_1,Z_2). \end{aligned}$$

Hence, taking $\delta_2 < \min\{\delta_0, 1\}$ where δ_0 is the constant as in Proposition 3.2, and using Proposition 3.2 with (3.18), we have $\mathcal{R}_\delta \le 0$ for all $\delta < \delta_2$ under the assumption (3.18).

Case 2. Assume now that

$$|E(Z_1,Z_2)| \ge \min\{\delta_\theta,\delta_1\}.$$

We now use the assumption

$$\int_0^1 |W(y)|^2\, dy \le C_1,$$

from which, all bad terms except for $\int_0^1 (W-\overline{W})^3 dy$ in (3.14) are bounded by some constant $\tilde{C}_1$ depending on C_1. Therefore, we have

$$\mathcal{R}_\delta \le -\frac{1}{\delta}\min\{\delta_\theta,\delta_1\}^2 + \tilde{C}_1 + \frac{2}{3}(1+\delta)\int_0^1 (W-\overline{W})^3 dy - (1-\delta)\int_0^1 y(1-y)|\partial_y W|^2 dy.$$

For the remaining cubic term, we use Lemma 2.8 to have

$$\int_0^1 (W-\overline{W})^3 dy$$
$$\leq \Big(\int_0^1 y(1-y)|\partial_y W_1|^2 dy\Big)^{3/4} \int_0^1 \Big|L(y)+L(1-y)\Big|^{3/4}\Big|W_1 - \int_0^1 W_1\Big|^{3/2} dy$$
$$\leq \Big(\int_0^1 y(1-y)|\partial_y W_1|^2 dy\Big)^{3/4} \Big(\int_0^1 \Big|L(y)+L(1-y)\Big|^3 dy\Big)^{1/4}\Big(\int_0^1 \Big|W_1 - \int_0^1 W_1\Big|^2 dy\Big)^{3/4}.$$

Then, using Young's inequality, we have

$$\frac{2}{3}\int_0^1 (W-\overline{W})^3 dy \leq \frac{1}{2}\int_0^1 y(1-y)|\partial_y W_1|^2 dy + C\Big(\int_0^1 \Big|W_1 - \int_0^1 W_1\Big|^2 dy\Big)^3$$
$$\leq \frac{1}{2}\int_0^1 y(1-y)|\partial_y W_1|^2 dy + \tilde{C}_1,$$

Therefore,

$$\mathcal{R}_\delta \leq -\frac{1}{\delta}\min\{\delta_\theta,\delta_1\}^2 + 2\tilde{C}_1.$$

Hence, choosing $\delta_2 < \min\{\delta_0, 1\}$ small enough such that $-\frac{1}{\delta_2}\min\{\delta_\theta,\delta_1\}^2 + 2\tilde{C}_1 < 0$, we have $\mathcal{R}_\delta < 0$.

This completes the proof of Proposition 3.2. □

3.5. **Expansion in the size of the shock.** We define the following functionals:

$$Y_g(v) := -\frac{1}{2\sigma_\varepsilon^2}\int_{\mathbb{R}} a'|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi - \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\mathbb{R}} a\partial_\xi p(\tilde{v}_\varepsilon)(v-\tilde{v}_\varepsilon) d\xi$$
$$+\frac{1}{\sigma_\varepsilon}\int_{\mathbb{R}} a\partial_\xi \tilde{h}_\varepsilon\big(p(v)-p(\tilde{v}_\varepsilon)\big) d\xi,$$
$$\mathcal{B}_1(v) := \sigma_\varepsilon \int_{\mathbb{R}} a\partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi,$$
$$\mathcal{B}_2(v) := \frac{1}{2\sigma_\varepsilon}\int_{\mathbb{R}} a'|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}} a''|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi,$$
$$\mathcal{G}_2(v) := \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi,$$
$$\mathcal{D}(v) := \int_{\mathbb{R}} a|\partial_\xi(p(v)-p(\tilde{v}_\varepsilon))|^2 d\xi.$$

Note that all these quantities depend only on v (not on h). This section is dedicated to the proof of the following proposition.

Proposition 3.4. *For any $C_2 > 0$, there exist $\varepsilon_0, \delta_3 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $\lambda, \delta \in (0, \delta_3)$ such that $\varepsilon \leq \lambda$, the following is true.*
For any function $v : \mathbb{R} \to \mathbb{R}^+$ such that $\mathcal{D}(v) + \mathcal{G}_2(v)$is finite, if

$$|Y_g(v)| \leq C_2 \frac{\varepsilon^2}{\lambda}, \qquad \|p(v)-p(\tilde{v}_\varepsilon)\|_{L^\infty(\mathbb{R})} \leq \delta_3, \tag{3.19}$$

then

$$\mathcal{R}_{\varepsilon,\delta}(v) := -\frac{1}{\varepsilon\delta}|Y_g(v)|^2 + (1+\delta)|\mathcal{B}_1(v)| + \left(1+\delta\left(\frac{\varepsilon}{\lambda}\right)\right)|\mathcal{B}_2(v)| - \left(1-\delta\left(\frac{\varepsilon}{\lambda}\right)\right)\mathcal{G}_2(v) - (1-\delta)\mathcal{D}(v) \le 0. \tag{3.20}$$

This proposition shows that we can afford an error of order 1 on $\mathcal{D}(v)$ and $\mathcal{B}_1(v)$ (up to δ), but only of order ε/λ on $\mathcal{G}_2(v)$ and $\mathcal{B}_2(v)$.

Proof. We first impose on $(\delta_3, \varepsilon_0)$ that

$$\delta_3 \le \min(\delta_*, 1/2), \qquad \varepsilon_0 \le \min(\delta_*, p(v_-)/2),$$

where δ_* is defined by the Lemma 2.6. That way, the function a is positive, the function $p(\tilde{v}_\varepsilon)$ is uniformly bounded, and we can apply the results of Lemma 2.6 on v and $w = \tilde{v}_\varepsilon$.

To simplify the notations, we set $\sigma = \sqrt{-p'(v_-)} > 0$. This is a fixed quantity which does not depend on ε and λ. Note that from (2.27) we have

$$|\sigma + \sigma_\varepsilon| \le C\varepsilon. \tag{3.21}$$

But, since $|\tilde{v}_\varepsilon - v_-| \le C\varepsilon$, and $\sigma^2 = -p'(v_-) = \gamma p(v_-)^{\frac{1}{\gamma}+1}$ we have actually:

$$\sup_{\xi\in\mathbb{R}} |\sigma^2 + p'(\tilde{v}_\varepsilon(\xi))| \le C\varepsilon, \qquad \text{and} \quad \sup_{\xi\in\mathbb{R}} \left|\frac{1}{\sigma^2} - \frac{p(\tilde{v}_\varepsilon(\xi))^{-\frac{1}{\gamma}-1}}{\gamma}\right| \le C\varepsilon. \tag{3.22}$$

We now rewrite the above functionals $Y_g, \mathcal{B}, \mathcal{G}_2, \mathcal{D}$ w.r.t. the following variables

$$w := p(v) - p(\tilde{v}_\varepsilon), \quad y := \frac{p(\tilde{v}_\varepsilon(\xi)) - p(v_-)}{[p]}. \tag{3.23}$$

Notice that since $p(\tilde{v}_\varepsilon(\xi))$ is increasing in ξ, we use a change of variable $\xi \in \mathbb{R} \mapsto y \in [0,1]$. Then it follows from (2.23) that $a = 1 - \lambda y$ and

$$a'(\xi) = -\lambda \frac{p(\tilde{v}_\varepsilon)'}{[p]}, \qquad \frac{dy}{d\xi} = \frac{p(\tilde{v}_\varepsilon)'}{[p]}, \qquad |a-1| \le \delta_3. \tag{3.24}$$

- **Change of variable for Y_g:** We decompose the Y_g term as follows.

$$Y_g = \underbrace{-\frac{1}{2\sigma_\varepsilon^2}\int_{\mathbb{R}} a'|p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi}_{=:Y_1} \underbrace{- \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi}_{=:Y_2} \underbrace{- \int_{\mathbb{R}} a\partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi}_{=:Y_3}$$
$$\underbrace{+\frac{1}{\sigma_\varepsilon}\int_{\mathbb{R}} a\partial_\xi \tilde{h}_\varepsilon \big(p(v) - p(\tilde{v}_\varepsilon)\big) d\xi}_{=:Y_4}.$$

Using (3.24), we have

$$Y_1 = \frac{\lambda}{2\sigma_\varepsilon^2}\int_0^1 w^2 dy.$$

Using (3.21), we get

$$\left|Y_1 - \frac{\lambda}{2\sigma^2}\int_0^1 w^2 dy\right| \le C\varepsilon_0 \lambda \int_0^1 w^2 dy. \tag{3.25}$$

Using (2.41) and (2.40) from Lemma 2.6, and $\|p(v)-p(\tilde v_\varepsilon)\|_{L^\infty(\mathbb{R})}\le\delta_3$ we find

$$\left|Y_2-\frac{\lambda}{2\gamma}\int_0^1 p(\tilde v_\varepsilon)^{-\frac{1}{\gamma}-1}w^2dy\right|\le C\delta_3\lambda\int_0^1 w^2dy.$$

Moreover, using (3.22), we find

$$\left|Y_2-\frac{\lambda}{2\sigma^2}\int_0^1 w^2dy\right|\le C\lambda(\varepsilon_0+\delta_3)\int_0^1 w^2dy. \tag{3.26}$$

For Y_3, we first write

$$v-\tilde v_\varepsilon=p(v)^{-\frac{1}{\gamma}}-p(\tilde v_\varepsilon)^{-\frac{1}{\gamma}}.$$

Using the Taylor expansion, we find that uniformly in ξ and ε:

$$\left|(v-\tilde v_\varepsilon)+\frac{p(\tilde v_\varepsilon)^{-\frac{1}{\gamma}-1}}{\gamma}\big(p(v)-p(\tilde v_\varepsilon)\big)\right|\le C|p(v)-p(\tilde v_\varepsilon)|^2\le C\delta_3|p(v)-p(\tilde v_\varepsilon)|.$$

Using (3.22), we get

$$\left|(v-\tilde v_\varepsilon)+\frac{1}{\sigma^2}\big(p(v)-p(\tilde v_\varepsilon)\big)\right|\le C(\varepsilon_0+\delta_3)|p(v)-p(\tilde v_\varepsilon)|.$$

Then, using $\partial_\xi p(\tilde v_\varepsilon)=\varepsilon\frac{dy}{d\xi}$ (since $[p]=\varepsilon$), and $|a-1|\le\delta_3$, we have

$$\left|Y_3-\frac{\varepsilon}{\sigma^2}\int_0^1 w\,dy\right|\le C\varepsilon(\varepsilon_0+\delta_3)\int_0^1|w|dy. \tag{3.27}$$

Using $\partial_\xi\tilde h_\varepsilon=\frac{\partial_\xi p(\tilde v_\varepsilon)}{\sigma_\varepsilon}$, we have

$$Y_4=\frac{\varepsilon}{\sigma_\varepsilon^2}\int_0^1(1-\lambda y)wdy,$$

and so

$$\left|Y_4-\frac{\varepsilon}{\sigma^2}\int_0^1 wdy\right|\le C\varepsilon(\delta_3+\varepsilon_0)\int_0^1|w|dy. \tag{3.28}$$

We combine all the terms of Y_g, and write the result on the renormalized quantity:

$$W:=\frac{\lambda}{\varepsilon}w. \tag{3.29}$$

From (3.25), (3.26), (3.27), and (3.28), we obtain:

$$\left|\sigma^2\frac{\lambda}{\varepsilon^2}Y_g-\int_0^1 W^2\,dy-2\int_0^1 W\,dy\right|\le C(\varepsilon_0+\delta_3)\left(\int_0^1 W^2\,dy+\int_0^1|W|\,dy\right). \tag{3.30}$$

- **Change of variable for $\mathcal{B}_1$ and $\mathcal{B}_2$:** We decompose the $\mathcal{B}_2$ term as follows.

$$\mathcal{B}_2=\underbrace{\frac{1}{2\sigma_\varepsilon}\int_{\mathbb{R}}a'|p(v)-p(\tilde v_\varepsilon)|^2d\xi}_{=:\mathcal{B}_{21}}+\underbrace{\frac{1}{2}\int_{\mathbb{R}}a''|p(v)-p(\tilde v_\varepsilon)|^2d\xi}_{=:\mathcal{B}_{22}}.$$

We first have

$$\mathcal{B}_{21}=-\frac{\lambda}{2\sigma_\varepsilon}\int_0^1 w^2dy=\frac{\lambda}{2|\sigma_\varepsilon|}\int_0^1 w^2dy.$$

So

$$\left|\mathcal{B}_{21} - \frac{\lambda}{2\sigma}\int_0^1 w^2 dy\right| \le \lambda\varepsilon \int_0^1 w^2 dy \le \varepsilon\delta_3 \int_0^1 w^2 dy.$$

Using (2.28), we get

$$|\mathcal{B}_{22}| \le C\varepsilon\lambda \int_0^1 w^2 dy \le C\varepsilon\delta_3 \int_0^1 w^2 dy.$$

So, finally:

$$\left|\mathcal{B}_2 - \frac{\lambda}{2\sigma}\int_0^1 w^2 dy\right| \le C\varepsilon\delta_3 \int_0^1 w^2 dy. \tag{3.31}$$

For $\mathcal{B}_1$, using $\partial_\xi \tilde{v}_\varepsilon = \frac{\partial_\xi p(\tilde{v}_\varepsilon)}{p'(\tilde{v}_\varepsilon)}$, we first have

$$\mathcal{B}_1 = \sigma_\varepsilon [p] \int_0^1 (1-\lambda y)\frac{1}{p'(\tilde{v}_\varepsilon)} p(v|\tilde{v}_\varepsilon) dy.$$

Then, using $[p] = \varepsilon$, (2.39), $\lambda \le \delta_3$, and (3.22), we have

$$\begin{aligned}
|\mathcal{B}_1| &\le \varepsilon|\sigma_\varepsilon| \int_0^1 \Big(\frac{\gamma+1}{2\gamma}|p'(\tilde{v}_\varepsilon)|^{-1} p(\tilde{v}_\varepsilon)^{-1} + C\delta_3\Big)(1-\lambda y) w^2\, dy \\
&\le \varepsilon|\sigma_\varepsilon| \int_0^1 \Big(\frac{\gamma+1}{2\gamma}|p'(\tilde{v}_\varepsilon)|^{-1} p(\tilde{v}_\varepsilon)^{-1} + C\delta_3\Big) w^2\, dy \\
&\le \varepsilon|\sigma_\varepsilon| \Big(\frac{\gamma+1}{2\gamma}|p'(v_-)|^{-1} p(v_-)^{-1} + C(\varepsilon_0+\delta_3)\Big) \int_0^1 w^2\, dy
\end{aligned}$$

Therefore

$$|\mathcal{B}_1| \le \varepsilon \frac{\gamma+1}{2\gamma\sigma p(v_-)}(1 + C(\varepsilon_0+\delta_3)) \int_0^1 w^2\, dy. \tag{3.32}$$

Note that the right hand side of (3.31) is small compared to $\mathcal{B}_1$. But the main part of $\mathcal{B}_2$ is big compared to $\mathcal{B}_1$ (as λ/ε). It will be compensated with the first order term in $\mathcal{G}_2$. We denote

$$\alpha_\gamma = \frac{\gamma\sigma p(v_-)}{\gamma+1}.$$

This number depends only on v_- and γ, but not on ε nor λ. We gather all the terms of $\mathcal{B}_1$ and $\mathcal{B}_2$, and write the result on the renormalized quantity (3.29). Thanks to (3.31) and (3.32) we find

$$2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3}|\mathcal{B}_2| \le \left(\frac{\alpha_\gamma}{\sigma}\left(\frac{\lambda}{\varepsilon}\right) + C(\varepsilon_0+\delta_3)\right) \int_0^1 W^2\, dy, \tag{3.33}$$

$$2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3}|\mathcal{B}_1| \le (1 + C(\varepsilon_0+\delta_3)) \int_0^1 W^2\, dy. \tag{3.34}$$

• **Change of variable for $\mathcal{G}_2$:** We use (3.24), (2.40), and (3.22) to get

$$\begin{aligned}\mathcal{G}_2 &= -\sigma_\varepsilon\lambda\int_0^1 Q(v|\tilde{v}_\varepsilon)dy\\ &\geq -\frac{\sigma_\varepsilon\lambda}{2\gamma}\int_0^1 p(\tilde{v}_\varepsilon)^{-\frac{1}{\gamma}-1}w^2dy + \sigma_\varepsilon\lambda\frac{1+\gamma}{3\gamma^2}\int_0^1 p(\tilde{v}_\varepsilon)^{-\frac{1}{\gamma}-2}w^3dy\\ &\geq \left(\frac{\lambda}{2\sigma} - C\varepsilon\delta_3\right)\int_0^1 w^2dy - \frac{\lambda}{3\alpha_\gamma}\int_0^1 w^3dy - C\frac{\varepsilon_0\lambda}{\alpha_\gamma}\int_0^1 |w|^3dy.\end{aligned}$$

When renormalizing with (3.29), we obtain:

$$-2\alpha_\gamma\frac{\lambda^2}{\varepsilon^3}\mathcal{G}_2 \leq \left(-\frac{\alpha_\gamma}{\sigma}\left(\frac{\lambda}{\varepsilon}\right) + C\delta_3\right)\int_0^1 W^2\,dy + \frac{2}{3}\int_0^1 W^3\,dy + C\varepsilon_0\int_0^1 |W|^3\,dy. \tag{3.35}$$

Note that the very first term in the inequality (3.35) will exactly cancel the divergent term of $\mathcal{B}_2$. This is why an expansion to the order three is needed.

• **Change of variable on $\mathcal{D}$:** To deal with the diffusion term $\mathcal{D}$, we first need a uniform in y estimate on $\frac{dy}{d\xi}$. This is provided by the following lemma.

Lemma 3.1. *There exists a constant $C > 0$ such that for any $\varepsilon \leq \varepsilon_0$, and any $y \in [0,1]$:*

$$\left|\frac{dy/d\xi}{y(1-y)} - \frac{\varepsilon}{2\alpha_\gamma}\right| \leq C\varepsilon^2.$$

Proof. From (1.10) we have

$$p(\tilde{v}_\varepsilon)' = \sigma_\varepsilon(\tilde{v}_\varepsilon - v_-) + \frac{p(\tilde{v}_\varepsilon) - p(v_-)}{\sigma_\varepsilon},$$

therefore

$$\varepsilon\frac{dy}{d\xi} = p(\tilde{v}_\varepsilon)' = \frac{1}{\sigma_\varepsilon}\Big(\sigma_\varepsilon^2(\tilde{v}_\varepsilon - v_-) + p(\tilde{v}_\varepsilon) - p(v_-)\Big),$$

with

$$\sigma_\varepsilon^2 = \frac{p(v_+) - p(v_-)}{v_- - v_+}.$$

Plugging the expression of σ_ε^2 into the one of $\varepsilon\frac{dy}{d\xi}$ and writing the result with respect to differences of values of functions at $\tilde{v}_\varepsilon$ and at the end points $v_\pm$, we find

$$\begin{aligned}\varepsilon\frac{dy}{d\xi} &= \frac{1}{\sigma_\varepsilon(v_- - v_+)}\Big((p(v_+) - p(v_-))(\tilde{v}_\varepsilon - v_-) + (p(\tilde{v}_\varepsilon) - p(v_-))(v_- - v_+)\Big)\\ &= \frac{1}{\sigma_\varepsilon(v_- - v_+)}\Big((p(v_+) - p(\tilde{v}_\varepsilon))(\tilde{v}_\varepsilon - v_-) + (p(\tilde{v}_\varepsilon) - p(v_-))(\tilde{v}_\varepsilon - v_-)\\ &\qquad + (p(\tilde{v}_\varepsilon) - p(v_-))(v_- - \tilde{v}_\varepsilon) + (p(\tilde{v}_\varepsilon) - p(v_-))(\tilde{v}_\varepsilon - v_+)\Big)\\ &= \frac{1}{\sigma_\varepsilon(v_- - v_+)}\Big((p(v_+) - p(\tilde{v}_\varepsilon))(\tilde{v}_\varepsilon - v_-) + (p(\tilde{v}_\varepsilon) - p(v_-))(\tilde{v}_\varepsilon - v_+)\Big).\end{aligned}$$

Hence

$$\varepsilon\frac{dy}{d\xi} = \frac{(p(v_+) - p(\tilde{v}_\varepsilon))(p(\tilde{v}_\varepsilon) - p(v_-))}{\sigma_\varepsilon(v_- - v_+)}\left(\frac{\tilde{v}_\varepsilon - v_-}{p(\tilde{v}_\varepsilon) - p(v_-)} + \frac{\tilde{v}_\varepsilon - v_+}{p(v_+) - p(\tilde{v}_\varepsilon)}\right).$$

Then, using

$$y=\frac{p(\tilde{v}_\varepsilon)-p(v_-)}{\varepsilon},\qquad 1-y=\frac{p(v_+)-p(\tilde{v}_\varepsilon)}{\varepsilon},$$

we have

$$\frac{dy/d\xi}{y(1-y)}=\frac{\varepsilon}{\sigma_\varepsilon(v_--v_+)}\left(\frac{\tilde{v}_\varepsilon-v_-}{p(\tilde{v}_\varepsilon)-p(v_-)}+\frac{\tilde{v}_\varepsilon-v_+}{p(v_+)-p(\tilde{v}_\varepsilon)}\right).$$

Then

$$\begin{aligned}&\left|\frac{dy/d\xi}{y(1-y)}-\varepsilon\frac{p''(v_-)}{2p'(v_-)^2\sigma}\right|\\&\quad\le\underbrace{\left|\frac{dy/d\xi}{y(1-y)}-\varepsilon\frac{p''(v_-)}{2p'(v_-)^2\sigma_\varepsilon}\right|}_{=:I_1}+\underbrace{\varepsilon\frac{p''(v_-)}{2p'(v_-)^2}\left|\frac{1}{\sigma_\varepsilon}+\frac{1}{\sigma}\right|}_{=:I_2}.\end{aligned}$$

We use Lemma 2.2 to have

$$I_1=\frac{\varepsilon}{|\sigma_\varepsilon|(v_--v_+)}\left|\frac{\tilde{v}_\varepsilon-v_-}{p(\tilde{v}_\varepsilon)-p(v_-)}+\frac{\tilde{v}_\varepsilon-v_+}{p(v_+)-p(\tilde{v}_\varepsilon)}+\frac{p''(v_-)}{2p'(v_-)^2}(v_--v_+)\right|\le C\varepsilon^2.$$

Since it follows from (3.21) that $I_2\le C\varepsilon^2$, we get

$$\left|\frac{dy/d\xi}{y(1-y)}-\varepsilon\frac{p''(v_-)}{2p'(v_-)^2\sigma}\right|\le C\varepsilon^2.$$

Since $p(v)=v^{-\gamma}$, we have

$$\frac{p''(v_-)}{p'(v_-)^2\sigma}=\frac{\gamma+1}{\gamma\sigma p(v_-)}=\frac{1}{\alpha_\gamma}.$$

This ends the proof of the lemma. □

The diffusion term $\mathcal{D}$ is as follows:

$$\mathcal{D}=\int_0^1(1-\lambda y)|\partial_y w|^2\Big(\frac{dy}{d\xi}\Big)dy. \tag{3.36}$$

Thanks to the last lemma, we have

$$\begin{aligned}\mathcal{D}&\ge(1-\lambda)\int_0^1|\partial_y w|^2\Big(\frac{dy}{d\xi}\Big)dy\\&\ge(1-\lambda)(\varepsilon/(2\alpha_\gamma)-C\varepsilon^2)\int_0^1 y(1-y)|\partial_y w|^2dy\\&\ge\frac{\varepsilon}{2\alpha_\gamma}(1-C(\delta_3+\varepsilon_0))\int_0^1 y(1-y)|\partial_y w|^2dy.\end{aligned}$$

After normalization, we obtain:

$$-2\alpha_\gamma\frac{\lambda^2}{\varepsilon^3}\mathcal{D}\le-(1-C(\varepsilon_0+\delta_3))\int_0^1 y(1-y)|\partial_y W|^2dy. \tag{3.37}$$

- **Control on W:** Using (3.19) and (3.30), we find that

$$\int_0^1 W^2\,dy-2\left|\int_0^1 W\,dy\right|\le C+C(\varepsilon_0+\delta_3)\left(\int_0^1 W^2\,dy+\int_0^1|W|\,dy\right).$$

But

$$\left|\int_0^1 W\,dy\right|\le\int_0^1|W|\,dy\le\frac{1}{8}\int_0^1 W^2\,dy+8.$$

Hence

$$\int_0^1 W^2\,dy \le 2\left|\int_0^1 W\,dy\right| + C + C(\varepsilon_0+\delta_3)\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right) \le C + 24 + \frac{1}{2}\int_0^1 W^2\,dy,$$

if ε_0 and δ_3 are chosen small enough. Hence there exists a constant $C_1 > 0$ depending on C_2, but not on ε nor ε/λ, such that

$$\int_0^1 W^2\,dy \le C_1. \tag{3.38}$$

Note that we cannot expect any smallness on this constant.

- **Control on the $|Y_g|^2$ term:** We have

$$-2\alpha_\gamma\left(\frac{\lambda^2}{\varepsilon^3}\right)\frac{|Y_g|^2}{\varepsilon\delta_3} = -\frac{2\alpha_\gamma}{\delta_3\sigma^4}\left|\frac{\sigma^2\lambda}{\varepsilon^2}Y_g\right|^2.$$

For any $a, b \in \mathbb{R}$, we have

$$b^2 - a^2 = -(b-a)^2 + 2b(b-a) = -(b-a)^2 + 2\frac{b}{\sqrt{2}}\sqrt{2}(b-a) \le (b-a)^2 + \frac{b^2}{2}.$$

So

$$-a^2 \le -\frac{b^2}{2} + |b-a|^2.$$

Using this inequality with

$$a = \frac{\sigma^2\lambda}{\varepsilon^2}Y_g, \qquad b = \int_0^1 W^2\,dy + 2\int_0^1 W\,dy,$$

and using (3.30), we find

$$\begin{aligned}-2\alpha_\gamma\left(\frac{\lambda^2}{\varepsilon^3}\right)\frac{|Y_g|^2}{\varepsilon\delta_3} &\le -\frac{\alpha_\gamma}{\delta_3\sigma^4}\left|\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right|^2 \\ &\quad + \frac{C}{\delta_3}(\varepsilon_0+\delta_3)^2\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right)^2.\end{aligned}$$

Using (3.38), we have

$$\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right)^2 \le \left(\int_0^1 W^2\,dy + \sqrt{\int_0^1 |W|^2\,dy}\right)^2 \le C\int_0^1 W^2\,dy.$$

So, restricting ε_0 such that $\varepsilon_0 \le \delta_3$, we have

$$-2\alpha_\gamma\left(\frac{\lambda^2}{\varepsilon^3}\right)\frac{|Y_g|^2}{\varepsilon\delta_3} \le -\frac{\alpha_\gamma}{\delta_3\sigma^4}\left|\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right|^2 + C\delta_3\int_0^1 W^2\,dy. \tag{3.39}$$

- **Conclusion:** For any $\delta < \delta_3$, we have

$$\begin{aligned}\mathcal{R}_{\varepsilon,\delta}(v) &\le -\frac{1}{\varepsilon\delta_3}|Y_g(v)|^2 + (1+\delta_3)|\mathcal{B}_1(v)| \\ &\quad + \left(1+\delta_3\left(\frac{\varepsilon}{\lambda}\right)\right)|\mathcal{B}_2(v)| - \left(1-\delta_3\left(\frac{\varepsilon}{\lambda}\right)\right)\mathcal{G}_2(v) - (1-\delta_3)\mathcal{D}(v).\end{aligned}$$

Multiplying (3.37) by $(1-\delta_3)$, (3.35) by $1-\delta_3(\varepsilon/\lambda)$, (3.34) by $1+\delta_3$, (3.33) by $1+\delta_3(\varepsilon/\lambda)$, and summing all these terms together with (3.39), we find (remember that $\varepsilon_0 \le \delta_3$ and $\varepsilon/\lambda \le 1$):

$$
\begin{aligned}
&2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon,\delta}(v) \\
&\le -\frac{1}{C_\gamma \delta_3}\left(\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right)^2 + (1+C\delta_3)\int_0^1 W^2\,dy \\
&\quad + \frac{2}{3}\int_0^1 W^3\,dy + C\delta_3 \int_0^1 |W|^3\,dy - (1-C\delta_3)\int_0^1 y(1-y)|\partial_y W|^2\,dy.
\end{aligned}
$$

Let us fix the value of the δ_2 of Proposition 3.3 corresponding to the constant $C_1 = C$ of (3.38).

Consider $\bar{\delta} = \max(C_\gamma, C)\delta_3$, and choose δ_3 small enough, such that $\bar{\delta}$ is smaller than δ_2. Then we have

$$
\begin{aligned}
&2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon,\delta}(v) \\
&\le -\frac{1}{\delta_2}\left(\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right)^2 + (1+\delta_2)\int_0^1 W^2\,dy \\
&\quad + \frac{2}{3}\int_0^1 W^3\,dy + \delta_2 \int_0^1 |W|^3\,dy - (1-\delta_2)\int_0^1 y(1-y)|\partial_y W|^2\,dy = R_{\delta_2}(W).
\end{aligned}
$$

Then from Proposition 3.3, we have $R_{\delta_2}(W) \le 0$. Therefore $\mathcal{R}_{\varepsilon,\delta}(v) \le 0$, for any $\lambda, \delta \le \delta_3$, $\varepsilon \le \varepsilon_0$ with $\varepsilon \le \lambda$, and any v such that $\mathcal{D}(v) + \mathcal{G}_2(v)$ is finite, and verifying (3.19). □

3.6. **Truncation of the big values of** $|p(v) - p(\tilde{v}_\varepsilon)|$**.** In order to use Proposition 3.4 for proof of Proposition 3.1, we need to show that the values for $p(v)$ such that $|p(v)-p(\tilde{v}_\varepsilon)| \ge \delta_3$ have a small effect. However, the value of δ_3 is itself conditioned to the constant C_2 in the proposition. Therefore, we need first to find a uniform bound on Y_g which is not yet conditioned on the level of truncation k.

We consider a truncation on $|p(v) - p(\tilde{v}_\varepsilon)|$ with a constant $k > 0$. Later we will consider the case $k = \delta_3$ as in Proposition 3.4. But for now, we consider the general case k to estimate the constant C_2. For that, let ψ_k be a continuous function defined by

$$\psi_k(y) = \inf\left(k, \sup(-k, y)\right). \tag{3.40}$$

We then define the function $\bar{v}_k$ uniquely (since the function p is one to one) as

$$p(\bar{v}_k) - p(\tilde{v}_\varepsilon) = \psi_k(p(v) - p(\tilde{v}_\varepsilon)).$$

We have the following lemma.

Lemma 3.2. *For a fixed $v_- > 0$, $u_- \in \mathbb{R}$, there exists $C_2, k_0, \varepsilon_0, \delta_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$, $\varepsilon/\lambda \le \delta_0$ with $\lambda < 1/2$, the following is true whenever $|Y(U)| \le \varepsilon^2$:*

$$\int_{\mathbb{R}} |a'||h - \tilde{h}_\varepsilon|^2\,d\xi + \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon)\,d\xi \le C\frac{\varepsilon^2}{\lambda}, \tag{3.41}$$

$$|Y_g(\bar{v}_k)| \le C_2 \frac{\varepsilon^2}{\lambda}, \qquad \text{for every } k \le k_0. \tag{3.42}$$

Proof. • *Proof of* (3.41)*:* We first use (2.32) to estimate

$$\int_{\mathbb{R}} |a'|\eta(U|\tilde{U}_\varepsilon)d\xi \geq \int_{\mathbb{R}} |a'|\frac{|h-\tilde{h}_\varepsilon|^2}{2}d\xi + c_1\int_{v\leq 3v_-}|a'||v-\tilde{v}_\varepsilon|^2 + c_2\int_{v>3v_-}|a'||v-\tilde{v}_\varepsilon|. \tag{3.43}$$

On the other hand, using $\int_{\mathbb{R}} a'\eta(U|\tilde{U}_\varepsilon)d\xi = -Y + \int_{\mathbb{R}} a\partial_\xi\nabla\eta(\tilde{U}_\varepsilon)(U-\tilde{U}_\varepsilon)d\xi$ in (2.21), and $|Y|\leq\varepsilon^2$, we have

$$\int_{\mathbb{R}} |a'|\eta(U|\tilde{U}_\varepsilon)d\xi \leq \varepsilon^2 + \int_{\mathbb{R}} |\partial_\xi\nabla\eta(\tilde{U}_\varepsilon)||U-\tilde{U}_\varepsilon|d\xi.$$

Then, since $|\partial_\xi\nabla\eta(\tilde{U}_\varepsilon)|\leq C|\partial_\xi p(\tilde{v}_\varepsilon)| = C\frac{\varepsilon}{\lambda}|a'|$ by (2.31), we have

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|\eta(U|\tilde{U}_\varepsilon)d\xi\\
&\quad\leq \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{\mathbb{R}}|a'||U-\tilde{U}_\varepsilon|d\xi\\
&\quad\leq \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{v>3v_-}|a'||v-\tilde{v}_\varepsilon|d\xi\\
&\qquad + C\frac{\varepsilon}{\lambda}\Big(\int_{v\leq 3v_-}|a'||v-\tilde{v}_\varepsilon|^2d\xi + \int_{\mathbb{R}}|a'||h-\tilde{h}_\varepsilon|^2d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}}|a'|d\xi\Big)^{1/2}.
\end{aligned}$$

Since it follows from (2.1) that

$$\int_{\mathbb{R}}|a'|d\xi = \frac{\lambda}{\varepsilon}\int_{\mathbb{R}}|\partial_\xi p(\tilde{v}_\varepsilon)|d\xi \leq \frac{\lambda}{\varepsilon}|p'(v_+)|\int_{\mathbb{R}}|\tilde{v}_\varepsilon'|d\xi \leq C\lambda,$$

using Young's inequality, we get that

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|\eta(U|\tilde{U}_\varepsilon)d\xi\\
&\quad\leq \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{v>3v_-}|a'||v-\tilde{v}_\varepsilon|d\xi + \frac{c_1}{2}\int_{v\leq 3v_-}|a'||v-\tilde{v}_\varepsilon|^2d\xi + \frac{1}{2}\int_{\mathbb{R}}|a'||h-\tilde{h}_\varepsilon|^2d\xi + C\frac{\varepsilon^2}{\lambda}.
\end{aligned}\tag{3.44}$$

Now, taking $\delta_0\leq 1/2$ such that $\frac{\varepsilon}{\lambda}<\delta_0\leq 1/2$, and then combining the two estimates (3.43) and (3.44) together with $\varepsilon^2 < C\frac{\varepsilon^2}{\lambda}$, we have

$$\int_{\mathbb{R}} |a'|\frac{|h-\tilde{h}_\varepsilon|^2}{2}d\xi + \int_{v\leq 3v_-}|a'||v-\tilde{v}_\varepsilon|^2 + \int_{v>3v_-}|a'||v-\tilde{v}_\varepsilon| \leq C\frac{\varepsilon^2}{\lambda}. \tag{3.45}$$

Applying the above estimate to (3.44), we complete (3.41).

- *Proof of* (3.42)*:* First of all, we have

$$
\begin{aligned}
|Y_g(\bar v_k)| &= \bigg| -\frac{1}{2\sigma_\varepsilon^2}\int_{\mathbb R} a'|p(\bar v_k)-p(\tilde v_\varepsilon)|^2 d\xi - \int_{\mathbb R} a' Q(\bar v_k|\tilde v_\varepsilon) d\xi \\
&\qquad - \int_{\mathbb R} a\partial_\xi p(\tilde v_\varepsilon)(\bar v_k-\tilde v_\varepsilon) d\xi + \frac{1}{\sigma_\varepsilon}\int_{\mathbb R} a\partial_\xi \tilde h_\varepsilon\big(p(\bar v_k)-p(\tilde v_\varepsilon)\big) d\xi \bigg| \\
&\le \underbrace{C\int_{\mathbb R} |a'||p(\bar v_k)-p(\tilde v_\varepsilon)|^2\, d\xi}_{=:I_1} + \int_{\mathbb R} |a'| Q(\bar v_k|\tilde v_\varepsilon)\, d\xi \\
&\qquad + \underbrace{C\int_{\mathbb R} \frac{\varepsilon}{\lambda}|a'|\Big(|\bar v_k-\tilde v_\varepsilon| + |p(\bar v_k)-p(\tilde v_\varepsilon)|\Big) d\xi}_{=:I_2}.
\end{aligned}
$$

Let us fix $k_0 = \delta_*/2$ of Lemma 2.6. Then, for any $k \le k_0$, we have $|p(\bar v_k)-p(\tilde v_\varepsilon)| \le k \le \frac{\delta_*}{2}$. Thus using (2.42) with $\varepsilon_0 \ll 1$, we have

$$
I_1 \le C\int_{\mathbb R} |a'| Q(\bar v_k|\tilde v_\varepsilon)\, d\xi.
$$

Using (2.32) and (2.42), we have

$$
\begin{aligned}
I_2 &\le \sqrt{\int_{\mathbb R}\Big(\frac{\varepsilon}{\lambda}\Big)^2 |a'|\, d\xi}\sqrt{\int_{\mathbb R} |a'|\Big(|\bar v_k-\tilde v_\varepsilon|^2 + |p(\bar v_k)-p(\tilde v_\varepsilon)|^2\Big)\, d\xi} \\
&\le C\sqrt{\frac{\varepsilon^2}{\lambda}}\sqrt{\int_{\mathbb R} |a'| Q(\bar v_k|\tilde v_\varepsilon)\, d\xi}.
\end{aligned}
$$

Notice that since the definition of $\bar v_k$ implies either $\tilde v_\varepsilon \le \bar v_k \le v$ or $v \le \bar v_k \le \tilde v_\varepsilon$, it follows from (2.33) that

$$
Q(v|\tilde v_\varepsilon) \ge Q(\bar v_k|\tilde v_\varepsilon).
$$

Therefore, using (3.41), there exists a constant $C_2 > 0$ such that

$$
|Y_g(\bar v_k)| \le C\int_{\mathbb R} |a'| Q(v|\tilde v_\varepsilon)\, d\xi + C\sqrt{\frac{\varepsilon^2}{\lambda}}\sqrt{\int_{\mathbb R} |a'| Q(v|\tilde v_\varepsilon)\, d\xi} \le C_2\frac{\varepsilon^2}{\lambda}.
$$

□

We now fix the constant δ_3 of Proposition 3.4 associated to the constant C_2 of Lemma 3.2. Without loss of generality, we can assume that $\delta_3 < k_0$ (since Proposition 3.4 is valid for any smaller δ_3). From now on, we set

$$
\bar v := \bar v_{\delta_3}, \qquad \bar U := (\bar v, h).
$$

Note that from Lemma 3.2, we have

$$
|Y_g(\bar v)| \le C_2\frac{\varepsilon^2}{\lambda}. \tag{3.46}
$$

We will use the notations $\mathcal{G}_1, \mathcal{G}_2, \mathcal{D}$ to denote three good terms of $\mathcal{G}$, that is $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{D}$ where

$$\begin{aligned}
&\mathcal{G}_1(U) := \frac{\sigma_\varepsilon}{2}\int_{\mathbb{R}} a'\Big(h - \tilde{h}_\varepsilon - \frac{p(v)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big)^2 d\xi,\\
&\mathcal{G}_2(U) = \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi,\\
&\mathcal{D}(U) = \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\tilde{v}_\varepsilon))|^2 d\xi.
\end{aligned} \tag{3.47}$$

We first notice that since $p(\bar{v}) - p(\tilde{v}_\varepsilon)$ is constant for v satisfying either $p(v) - p(\tilde{v}_\varepsilon) < -\delta_3$ or $p(v) - p(\tilde{v}_\varepsilon) > \delta_3$, we have

$$\mathcal{D}(\bar{U}) = \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\tilde{v}_\varepsilon))|^2 \mathbf{1}_{\{|p(v)-p(\tilde{v}_\varepsilon)|\le \delta_3\}} d\xi.$$

We also note that

$$\begin{aligned}
|p(v)-p(\bar{v})| &= |(p(v)-p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(\bar{v}))|\\
&= |(\psi_{\delta_3} - I)(p(v)-p(\tilde{v}_\varepsilon))|\\
&= (|p(v)-p(\tilde{v}_\varepsilon)| - \delta_3)_+.
\end{aligned} \tag{3.48}$$

Therefore,

$$\begin{aligned}
\mathcal{D}(U) &= \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\tilde{v}_\varepsilon))|^2 d\xi\\
&= \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\tilde{v}_\varepsilon))|^2 (\mathbf{1}_{\{|p(v)-p(\tilde{v}_\varepsilon)|\le \delta_3\}} + \mathbf{1}_{\{|p(v)-p(\tilde{v}_\varepsilon)|> \delta_3\}}) d\xi\\
&= \mathcal{D}(\bar{U}) + \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\bar{v}))|^2 d\xi\\
&\ge \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\bar{v}))|^2 d\xi,
\end{aligned} \tag{3.49}$$

which also yields

$$\mathcal{D}(U) - \mathcal{D}(\bar{U}) = \int_{\mathbb{R}} a|\partial_\xi (p(v)-p(\bar{v}))|^2 d\xi \ge 0. \tag{3.50}$$

On the other hand, since $Q(v|\tilde{v}_\varepsilon) \ge Q(\bar{v}|\tilde{v}_\varepsilon)$, we have

$$|\sigma_\varepsilon| \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon)\, d\xi \ge \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) = |\sigma_\varepsilon| \int_{\mathbb{R}} |a'| \left(Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)\right) d\xi \ge 0. \tag{3.51}$$

We will first show the following lemma.

Lemma 3.3. *There exist* $C, \varepsilon_0, \delta_0 > 0$*, such that for any* $\varepsilon < \varepsilon_0$*,* $\varepsilon/\lambda < \delta_0$*, and* $\lambda < 1/2$*, the following is true whenever* $|Y(U)| \le \varepsilon^2$*:*

$$0 \le \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \le \mathcal{G}_2(U) \le C\frac{\varepsilon^2}{\lambda}, \tag{3.52}$$

$$\int_{\mathbb{R}} |a'||p(v) - p(\bar{v})|^2\, d\xi + \int_{\mathbb{R}} |a'||p(v) - p(\bar{v})|\, d\xi \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U), \tag{3.53}$$

$$\int_{\mathbb{R}} |a'|\big||p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2\big|\, d\xi \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U), \tag{3.54}$$

$$\begin{aligned}\int_{\mathbb{R}} |a'|\,|p(v|\tilde{v}_\varepsilon) - p(\bar{v}|\tilde{v}_\varepsilon)|\, d\xi + \int_{\mathbb{R}} |a'|\,|Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)|\, d\xi + \int_{\mathbb{R}} |a'||v - \bar{v}|\, d\xi \\ \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})).\end{aligned} \tag{3.55}$$

Proof. We split the proof into several steps.
Step 1: The estimate (3.41) with (3.51) gives (3.52).

Step 2: Note first that since $(y - \delta_3/2)_+ \ge \delta_3/2$ whenever $(y - \delta_3)_+ > 0$, we have

$$(y - \delta_3)_+ \le (y - \delta_3/2)_+ \mathbf{1}_{\{y - \delta_3 > 0\}} \le (y - \delta_3/2)_+ \left(\frac{(y - \delta_3/2)_+}{\delta_3/2}\right) \le \frac{2}{\delta_3}(y - \delta_3/2)_+^2. \tag{3.56}$$

Hence, to show (3.53), it is enough to show it only for the quadratic part, with $\bar{v}$ defined with $\delta_3/2$ instead of δ_3. We will keep the notation $\bar{v}$ for this case below.

Step 3: Since $|a'| = (\lambda/\varepsilon)|\tilde{v}_\varepsilon'|$, thanks to (2.2) and (3.41), we get

$$\begin{aligned} 2\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} Q(v|\tilde{v}_\varepsilon)\, d\xi &\le \frac{2\varepsilon}{\inf_{[-1/\varepsilon, 1/\varepsilon]} |a'|} \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon)\, d\xi \\ &\le C\frac{\varepsilon}{\lambda\varepsilon}\frac{\varepsilon^2}{\lambda} = C\left(\frac{\varepsilon}{\lambda}\right)^2. \end{aligned}$$

Hence, there exists $\xi_0 \in [-1/\varepsilon, 1/\varepsilon]$ such that $Q(v(\xi_0), \tilde{v}_\varepsilon(\xi_0)) \le C(\varepsilon/\lambda)^2$. For δ_0 small enough, and using (2.42), we have

$$|(p(v) - p(\tilde{v}_\varepsilon))(\xi_0)| \le C\frac{\varepsilon}{\lambda}.$$

Thus, if δ_0 is small enough such that $C\varepsilon/\lambda \le \delta_3/2$, then we have from (3.48) that

$$(p(v) - p(\bar{v}))(\xi_0) = 0.$$

Therefore using (3.49), we obtain that for any $\xi \in \mathbb{R}$,

$$\begin{aligned} |(p(v) - p(\bar{v}))(\xi)| &= \left|\int_{\xi_0}^{\xi} \partial_\zeta (p(v) - p(\bar{v}))\, d\zeta\right| \\ &\le \sqrt{|\xi| + |\xi_0|}\sqrt{\int_{\mathbb{R}} \left|\partial_\zeta (p(v) - p(\bar{v}))\right|^2 d\zeta} \\ &\le C\sqrt{|\xi| + \frac{1}{\varepsilon}}\sqrt{\mathcal{D}(U)}. \end{aligned} \tag{3.57}$$

For any ξ such that $|(p(v)-p(\bar v))(\xi)|>0$, we have from (3.48) that $|(p(v)-p(\tilde v_\varepsilon))(\xi)|>\delta_3$. Thus using (2.36) and (2.32), we have $Q(v(\xi)|\tilde v_\varepsilon(\xi))\geq\alpha$, for some constant $\alpha>0$ depending only on δ_3. Hence

$$\mathbf{1}_{\{|p(v)-p(\bar v)|>0\}}\leq\frac{Q(v|\tilde v_\varepsilon)}{\alpha}. \tag{3.58}$$

In the next computation, we split the integral in two parts, and use (3.57)-(3.58) to have

$$\begin{aligned}
\int_{\mathbb{R}}|a'||p(v)-p(\bar v)|^2\,d\xi &\leq \int_{-\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}^{\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}|a'||p(v)-p(\bar v)|^2\,d\xi+\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}|a'||p(v)-p(\bar v)|^2\,d\xi\\
&\leq\left(\sup_{\left[-\sqrt{\frac{\lambda}{\varepsilon^3}},\sqrt{\frac{\lambda}{\varepsilon^3}}\right]}|p(v)-p(\bar v)|^2\right)\int_{-\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}^{\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}|a'|\mathbf{1}_{\{|p(v)-p(\bar v)|>0\}}\,d\xi\\
&\qquad+C\mathcal{D}(U)\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}|a'|\left(|\xi|+\frac{1}{\varepsilon}\right)d\xi\\
&\leq C\mathcal{D}(U)\left(\sqrt{\frac{\lambda}{\varepsilon^3}}\int_{\mathbb{R}}|a'|\frac{Q(v|\tilde v_\varepsilon)}{\alpha}\,d\xi+2\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}|a'||\xi|\,d\xi\right).
\end{aligned}$$

Therefore we have

$$\int_{\mathbb{R}}|a'||p(v)-p(\bar v)|^2\,d\xi\leq C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U).$$

Indeed, using (3.41) and (2.1) (recalling $|a'|=(\lambda/\varepsilon)|\tilde v_\varepsilon'|$), we have

$$\int_{\mathbb{R}}|a'||p(v)-p(\bar v)|^2\,d\xi\leq C\mathcal{D}(U)\left(\sqrt{\frac{\varepsilon}{\lambda}}+\lambda\varepsilon\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}e^{-c\varepsilon|\xi|}|\xi|\,d\xi\right),$$

and for the last term, we take δ_0 small enough such that for any $\varepsilon/\lambda\leq\delta_0$,

$$\lambda\varepsilon\int_{|\xi|\geq\frac{1}{\varepsilon}\sqrt{\frac{\lambda}{\varepsilon}}}e^{-c\varepsilon|\xi|}|\xi|\,d\xi=\frac{\lambda}{\varepsilon}\int_{|\xi|\geq\sqrt{\frac{\lambda}{\varepsilon}}}e^{-c|\xi|}|\xi|d\xi\leq\frac{\lambda}{\varepsilon}\int_{|\xi|\geq\sqrt{\frac{\lambda}{\varepsilon}}}e^{-\frac{c}{2}|\xi|}d\xi=\frac{2\lambda}{c\varepsilon}e^{-\frac{c}{2}\sqrt{\frac{\lambda}{\varepsilon}}}\leq\sqrt{\frac{\varepsilon}{\lambda}}.$$

As mentioned in Step 2, recall that $\bar v=\bar v_{\delta_3/2}$ in the above estimate. Then using (3.48), we have

$$\begin{aligned}
\int_{\mathbb{R}}|a'||p(v)-p(\bar v_{\delta_3})|^2\,d\xi&=\int_{\mathbb{R}}|a'|(|p(v)-p(\tilde v_\varepsilon)|-\delta_3)_+^2\,d\xi\\
&\leq\int_{\mathbb{R}}|a'|(|p(v)-p(\tilde v_\varepsilon)|-\delta_3/2)_+^2\,d\xi\\
&=\int_{\mathbb{R}}|a'||p(v)-p(\bar v_{\delta_3/2})|^2\,d\xi\leq C\mathcal{D}(U)\sqrt{\frac{\varepsilon}{\lambda}}.
\end{aligned}$$

Likewise, using (3.48) and (3.56) with $y:=|p(v)-p(\tilde v_\varepsilon)|$, we have

$$\int_{\mathbb{R}}|a'||p(v)-p(\bar v_{\delta_3})|\,d\xi\leq\frac{2}{\delta_3}\int_{\mathbb{R}}|a'||p(v)-p(\bar v_{\delta_3/2})|^2\,d\xi\leq C\mathcal{D}(U)\sqrt{\frac{\varepsilon}{\lambda}}.$$

Hence, we obtain (3.53).

Step 4: We use $|p(\bar v)-p(\tilde v_\varepsilon)|\le \delta_3$ and (3.53) to show

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|\,\big||p(v)-p(\tilde v_\varepsilon)|^2-|p(\bar v)-p(\tilde v_\varepsilon)|^2\big|\,d\xi\\
&\quad\le \int_{\mathbb{R}} |a'||p(v)-p(\bar v)||p(v)+p(\bar v)-2p(\tilde v_\varepsilon)|\,d\xi\\
&\quad\le \int_{\mathbb{R}} |a'||p(v)-p(\bar v)|\left(|p(v)-p(\bar v)|+2|p(\bar v)-p(\tilde v_\varepsilon)|\right)d\xi\\
&\quad\le \int_{\mathbb{R}} |a'|\left(|p(v)-p(\bar v)|^2+2\delta_3|p(v)-p(\bar v)|\right)d\xi\\
&\quad\le C\mathcal{D}(U)\sqrt{\frac{\varepsilon}{\lambda}},
\end{aligned}$$

which gives (3.54).

Step 5: First, since $\tilde v_\varepsilon\in[v_-/2,v_-]$ for ε_0 small enough, it follows from the definition of the relative pressure (2.19) that

$$|p(v|\tilde v_\varepsilon)-p(\bar v|\tilde v_\varepsilon)|=|(p(v)-p(\bar v))-p'(\tilde v_\varepsilon)(v-\bar v)|\le |p(v)-p(\bar v)|+C|v-\bar v|.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|\,|p(v|\tilde v_\varepsilon)-p(\bar v|\tilde v_\varepsilon)|\,d\xi+\int_{\mathbb{R}}|a'||v-\bar v|\,d\xi\\
&\quad\le C\int_{\mathbb{R}}|a'||p(v)-p(\bar v)|\,d\xi+C\int_{\mathbb{R}}|a'||v-\bar v|\,d\xi.
\end{aligned}$$

To control the last term above, we use (2.34) as follows: If $|v-\bar v|>0$, we have from the definition of $\bar v$ that $|p(\bar v)-p(\tilde v_\varepsilon)|=\delta_3$. Then using (2.36), we find

$$|\bar v-\tilde v_\varepsilon|\ge \min(c_3^{-1}\delta_3, v_-/2-\varepsilon_0).$$

Taking δ_* in 2) of Lemma 2.4 such that $\varepsilon_0\le\delta_*/2$ and $\min(c_3^{-1}\delta_3,v_-/2-\varepsilon_0)\ge\delta_*$, we use (2.34) with $w=\tilde v_\varepsilon$, $u=\bar v$ and $v=v$ to find that there exists a constant $C>0$ such that

$$C|v-\bar v|\le Q(v|\tilde v_\varepsilon)-Q(\bar v|\tilde v_\varepsilon).$$

Therefore, using (3.53) and (3.51) , we find

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|\,|p(v|\tilde v_\varepsilon)-p(\bar v|\tilde v_\varepsilon)|\,d\xi+\int_{\mathbb{R}}|a'||v-\bar v|\,d\xi\\
&\quad\le C\int_{\mathbb{R}}|a'||p(v)-p(\bar v)|\,d\xi+C\int_{\mathbb{R}}|a'|\big(Q(v|\tilde v_\varepsilon)-Q(\bar v|\tilde v_\varepsilon)\big)\,d\xi\\
&\quad\le C\mathcal{D}(U)\sqrt{\frac{\varepsilon}{\lambda}}+C[\mathcal{G}_2(U)-\mathcal{G}_2(\bar U)].
\end{aligned}$$

This together with (3.51) completes the proof of (3.55). □

We first recall Y in (2.21) as

$$Y=-\int_{\mathbb{R}} a'\frac{|h-\tilde h_\varepsilon|^2}{2}d\xi-\int_{\mathbb{R}}a'Q(v|\tilde v_\varepsilon)d\xi+\int_{\mathbb{R}}a\Big(-\partial_\xi p(\tilde v_\varepsilon)(v-\tilde v_\varepsilon)+\partial_\xi\tilde h_\varepsilon(h-\tilde h_\varepsilon)\Big)d\xi.$$

We split Y into three parts Y_g, Y_b and Y_l as below: Y_g consists of the terms related to $v-\tilde{v}_\varepsilon$, while Y_b and Y_l consist of terms related to $h-\tilde{h}_\varepsilon$. While Y_b is quadratic in U, the term Y_l is linear in $h-\tilde{h}_\varepsilon$. More precisely, Y can be decomposed as

$$Y = Y_g + Y_b + Y_l,$$

where

$$\begin{aligned} Y_g &:= -\frac{1}{2\sigma_\varepsilon^2}\int_{\mathbb{R}} a'|p(v)-p(\tilde{v}_\varepsilon)|^2 d\xi - \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\mathbb{R}} a\partial_\xi p(\tilde{v}_\varepsilon)(v-\tilde{v}_\varepsilon) d\xi \\ &\quad + \frac{1}{\sigma_\varepsilon}\int_{\mathbb{R}} a\partial_\xi \tilde{h}_\varepsilon \big(p(v)-p(\tilde{v}_\varepsilon)\big) d\xi, \end{aligned}$$

$$Y_b := -\frac{1}{2}\int_{\mathbb{R}} a'\Big(h-\tilde{h}_\varepsilon - \frac{p(v)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big)^2 d\xi - \frac{1}{\sigma_\varepsilon}\int_{\mathbb{R}} a'\big(p(v)-p(\tilde{v}_\varepsilon)\big)\Big(h-\tilde{h}_\varepsilon - \frac{p(v)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big) d\xi.$$

$$Y_l = \int_{\mathbb{R}} a\partial_\xi \tilde{h}_\varepsilon \Big(h-\tilde{h}_\varepsilon - \frac{p(v)-p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon}\Big) d\xi.$$

Notice that the first part Y_g is independent of h, and $Y_g(\bar{U})$ was used to absorb the bad term $\mathcal{B}$ in Proposition 3.4, while Y_b and Y_l are useless because $\mathcal{B}$ does not depend on $h-\tilde{h}_\varepsilon$. Therefore we need show that $Y_g(U)-Y_g(\bar{U})$, $Y_b(U)$ and $Y_l(U)$ are negligible by other terms. We now prove the following lemma.

Lemma 3.4. *There exist constants $\delta_0, \varepsilon_0, C, C^* > 0$ such that for any $\varepsilon < \varepsilon_0$, and any $\lambda < 1/2$ with $\varepsilon/\lambda < \delta_0$, the following statements hold true.*

1 *For any U such that $|Y(U)| \le \varepsilon^2$,*

$$|\mathcal{B}(U) - \mathcal{B}(\bar{U})| \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C\frac{\varepsilon}{\lambda}[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})], \tag{3.59}$$

$$|\mathcal{B}(U)| \le C^*\frac{\varepsilon^2}{\lambda} + C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U). \tag{3.60}$$

2 *For any U such that $|Y(U)| \le \varepsilon^2$ and $\mathcal{D}(U) \le \frac{C^*}{4}\frac{\varepsilon^2}{\lambda}$,*

$$|Y_g(U) - Y_g(\bar{U})| + |Y_b(U)| \le C\frac{\varepsilon^2}{\lambda}, \tag{3.61}$$

$$\begin{aligned} |Y_g(U) - Y_g(\bar{U})| + |Y_b(U)| &\le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})] \\ &\quad + \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathcal{G}_1(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\mathcal{G}_2(\bar{U}), \end{aligned} \tag{3.62}$$

$$|Y_l(U)|^2 \le \frac{\varepsilon^2}{\lambda}\mathcal{G}_1(U). \tag{3.63}$$

Proof. We split the proof in several steps.

Step 1: Recall the bad term $\mathcal{B}$ in (2.21). Using (2.39), (2.42) and $|\tilde{v}_\varepsilon'| + |a''| \le C|a'|$, and then (3.41), we have

$$|\mathcal{B}(\bar{U})| \le C\int_{\mathbb{R}} |a'|Q(\bar{v}|\tilde{v}_\varepsilon)\, d\xi \le C\int_{\mathbb{R}} |a'|Q(v|\tilde{v}_\varepsilon)\, d\xi \le C^*\frac{\varepsilon^2}{\lambda}. \tag{3.64}$$

Moreover, using (3.54) and (3.55) together with $|\tilde{v}_\varepsilon'| \le C\frac{\varepsilon}{\lambda}|a'|$, we have

$$|\mathcal{B}(U) - \mathcal{B}(\bar{U})| \le C\mathcal{D}(U)\sqrt{\frac{\varepsilon}{\lambda}} + C\frac{\varepsilon}{\lambda}[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})].$$

Combining the above two estimates together with (3.52), we have (3.60).

Step 2: We show (3.61) as follows: Using (3.53)-(3.55), we have

(3.65)

$$\begin{aligned}|Y_g(U) - Y_g(\bar{U})| &\le C\int_{\mathbb{R}} |a'|\Big(\big||p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2\big| + \big|Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)\big| \\ &\qquad + |v - \bar{v}| + |p(v) - p(\bar{v})|\Big)\, d\xi \\ &\le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})).\end{aligned}$$

Then using (3.52) and $\mathcal{D}(U) - \mathcal{D}(\bar{U}) \le \mathcal{D}(U) \le C\varepsilon^2/\lambda$, we have $|Y_g(U) - Y_g(\bar{U})| \le C\varepsilon^2/\lambda$. Next, recalling $\mathcal{G}_1$ in (3.47), we have

$$|Y_b(U)| \le C\mathcal{G}_1(U) + C\int_{\mathbb{R}} |a'||p(v) - p(\tilde{v}_\varepsilon)|^2\, d\xi \le C(\mathcal{G}_1(U) + |\mathcal{B}(U)|).$$

Since

$$\mathcal{G}_1(U) \le \int_{\mathbb{R}} |a'|\left(|h - \tilde{h}_\varepsilon|^2 + |p(v) - p(\tilde{v}_\varepsilon)|^2\right)\, d\xi,$$

using (3.41) and (3.60), we have $|Y_b(U)| \le C\varepsilon^2/\lambda$.

Step 3: We first estimate the term $\int a'(p(v) - p(\tilde{v}_\varepsilon))(h - \tilde{h}_\varepsilon + (p(v) - p(\tilde{v}_\varepsilon))/\sigma) d\xi$ in Y_b using Young's inequality with ε/λ as follows:

$$\begin{aligned}&\Big|\int a'(p(v) - p(\tilde{v}_\varepsilon))\left(h - \tilde{h}_\varepsilon + (p(v) - p(\tilde{v}_\varepsilon))/\sigma\right) d\xi\Big| \\ &\qquad \le \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathcal{G}_1(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\int_{\mathbb{R}} |a'||p(v) - p(\tilde{v}_\varepsilon)|^2\, d\xi.\end{aligned}$$

Since (3.59) and the first inequality in (3.64) yield

$$\begin{aligned}\int_{\mathbb{R}} |a'||p(v) - p(\tilde{v}_\varepsilon)|^2\, d\xi &\le |\mathcal{B}(U) - \mathcal{B}(\bar{U})| + |\mathcal{B}(\bar{U})| \\ &\le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C\frac{\varepsilon}{\lambda}[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})] + C\mathcal{G}_2(\bar{U}),\end{aligned}$$

we have

$$\begin{aligned}&\Big|\int a'(p(v) - p(\tilde{v}_\varepsilon))\left(h - \tilde{h}_\varepsilon + (p(v) - p(\tilde{v}_\varepsilon))/\sigma\right) d\xi\Big| \\ &\quad \le \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathcal{G}_1(U) + C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C\frac{\varepsilon}{\lambda}[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})] + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\mathcal{G}_2(\bar{U}).\end{aligned}$$

Therefore, this estimate together with $\lambda/\varepsilon > \delta_0^{-1} \gg 1$ and (3.65) implies

$$\begin{aligned}&|Y_g(U) - Y_g(\bar{U})| + |Y_b(U)| \\ &\quad \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathcal{D}(U) + C[\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})] + \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathcal{G}_1(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\mathcal{G}_2(\bar{U}),\end{aligned}$$

which proves (3.62).

Step 4: Using Cauchy-Schwartz inequality together with $|\tilde{h}'_\varepsilon| \le C\frac{\varepsilon}{\lambda}|a'|$, we find

$$|Y_l(U)|^2 \le \left(\frac{\varepsilon}{\lambda}\right)^2 \left[\int_{\mathbb{R}} |a'|\,d\xi\right] \int_{\mathbb{R}} |a'||h - \tilde{h}_\varepsilon + (p(v) - p(\tilde{v}_\varepsilon))/\sigma|^2\,d\xi \le C\frac{\varepsilon^2}{\lambda}\mathcal{G}_1(U),$$

which gives (3.63). □

3.7. **Proof of Proposition 3.1.** We now prove the main Proposition of the paper. We split the proof into two steps, depending on the strength of the dissipation term $\mathcal{D}(U)$.

Step 1: We first consider the case where $\mathcal{D}(U) \ge 4C^*\frac{\varepsilon^2}{\lambda}$, where the constant C^* is defined as in Lemma 3.4. Then using (3.60), we find that for δ_0 small enough,

$$\begin{aligned}\mathcal{R}(U) := -\frac{|Y(U)|^2}{\varepsilon^4} + \left(1 + \delta_0\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U)| - \mathcal{G}(U) &\le 2|\mathcal{B}(U)| - \mathcal{D}(U)\\ &\le 2C^*\frac{\varepsilon^2}{\lambda} + \left(2C\sqrt{\frac{\varepsilon}{\lambda}} - 1\right)\mathcal{D}(U)\\ &\le 2C^*\frac{\varepsilon^2}{\lambda} - \frac{1}{2}\mathcal{D}(U) \le 0,\end{aligned}$$

which gives the desired result.

Step 2: We now assume the other alternative, i.e., $\mathcal{D}(U) \le 4C^*\frac{\varepsilon^2}{\lambda}$.
We will use Proposition 3.4 to get the desired result. First of all, we have (3.46), and for the small constant δ_3 of Proposition 3.4 associated to the constant C_2 of (3.46), we have $|p(\bar{v}) - p(\tilde{v}_\varepsilon)| \le \delta_3$.
Let us take δ_0 small enough such that $\delta_0 \le \delta_3^8$. Using

$$Y_g(\bar{U}) = Y(U) - (Y_g(U) - Y_g(\bar{U})) - Y_b(U) - Y_l(U),$$

we have

$$|Y_g(\bar{U})|^2 \le 4|Y(U)|^2 + 4|Y_g(U) - Y_g(\bar{U})|^2 + 4|Y_b(U)|^2 + 4|Y_l(U)|^2,$$

which can be written as

$$-4|Y(U)|^2 \le -|Y_g(\bar{U})|^2 + 4|Y_g(U) - Y_g(\bar{U})|^2 + 4|Y_b(U)|^2 + 4|Y_l(U)|^2.$$

Thus we find that for any $\varepsilon < \varepsilon_0(\le \delta_3)$ and $\varepsilon/\lambda < \delta_0$,

$$\begin{aligned}\mathcal{R}(U) &\le -\frac{4|Y(U)|^2}{\varepsilon\delta_3} + \left(1 + \delta_0\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U)| - \mathcal{G}(U)\\ &\le -\frac{|Y_g(\bar{U})|^2}{\varepsilon\delta_3} + \left(1 + \delta_0\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(\bar{U})| - \mathcal{G}_2(\bar{U}) - (1-\delta_3)\mathcal{D}(U)\\ &\qquad + \frac{4}{\varepsilon\delta_3}|Y_g(U) - Y_g(\bar{U})|^2 + \frac{4}{\varepsilon\delta_3}|Y_b(U)|^2 + \frac{4}{\varepsilon\delta_3}|Y_l(U)|^2\\ &\qquad + \left(1 + \delta_0\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(U) - \mathcal{B}(\bar{U})| - (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) - \mathcal{G}_1(U) - \delta_3\mathcal{D}(U).\end{aligned}$$

To control the square of $|Y_g(U) - Y_g(\bar{U})| + |Y_b(U)|$, we multiply the bound of (3.61) and the bound of (3.62) to find

$$
\begin{aligned}
&\frac{1}{\varepsilon\delta_3}|Y_g(U) - Y_g(\bar{U})|^2 + \frac{1}{\varepsilon\delta_3}|Y_b(U))|^2 \\
&\le \frac{C}{\delta_3}\left[\left(\frac{\varepsilon}{\lambda}\right)^{3/2}\mathcal{D}(U) + \frac{\varepsilon}{\lambda}(\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left(\frac{\varepsilon}{\lambda}\right)^{3/4}\mathcal{G}_1(U) + \left(\frac{\varepsilon}{\lambda}\right)^{1/4}\frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U})\right] \\
&\le C\delta_0^{1/8}\left[\mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1(U) + \frac{\varepsilon}{\lambda}\mathcal{G}_2(\bar{U})\right].
\end{aligned}
$$

Using also (3.63) and (3.59) together with (3.50), therefore we find that for δ_0 small enough with $\delta_0 \le \delta_3^8$,

$$
\mathcal{R}(U) \le -\frac{|Y_g(\bar{U})|^2}{\varepsilon\delta_3} + \left(1 + \delta_3\frac{\varepsilon}{\lambda}\right)|\mathcal{B}(\bar{U})| - \left(1 - \delta_3\frac{\varepsilon}{\lambda}\right)\mathcal{G}_2(\bar{U}) - (1-\delta_3)\mathcal{D}(\bar{U}). \tag{3.66}
$$

Since the above quantities $Y_g(\bar{U})$, $\mathcal{B}(\bar{U}) = \mathcal{B}_1(\bar{U}) + \mathcal{B}_2(\bar{U})$, $\mathcal{G}_2(\bar{U})$ and $\mathcal{D}(\bar{U})$ depends only on $\bar{v}$ through $\bar{U}$, it follows from Proposition 3.4 that $\mathcal{R}(U) \le 0$. Hence we complete the proof of Proposition 3.1.

Appendix A. Proof of Lemma 2.7

We show the following lemma which contains Lemma 2.7.

Lemma A.1. *Let*

$$
g(x) := 2x - 2x^2 - \frac{4}{3}x^3 + \frac{4\theta}{3}\Big(-x^2 - 2x\Big)^{3/2},
$$

where $\theta = \sqrt{5 - \frac{\pi^2}{3}} \approx 1.308$. *The following statements are true.*

1 *For any* $x \in [-2, -\frac{1+\sqrt{3}}{2}]$, $g''(x) > 0$.
2 *For any* $x \in (-\frac{1+\sqrt{3}}{2}, -1]$, $g'(x) > 0$.
3 *The function* g' *has exactly two roots* x_1 *and* x_2 *on* $[-1, 0]$. *The smaller one* x_1 *belongs to* $(-1 + \sqrt{2}/2, -1 + \sqrt{3}/2)$, *and is the only local maximum of* g *on* $(-1, 0)$.
4 *The function* g *is negative on* $(-2, 0)$.

The point 4 is the result of Lemma 2.7.

Proof. **Step 1.** Note that

$$
-x^2 - 2x = 1 - (1+x)^2.
$$

This function is increasing on $(-2, -1)$. So, for $-2 \le x \le -\frac{1+\sqrt{3}}{2}$ we have

$$
1 - (1+x)^2 \le 1 - \frac{1}{4}(1 - \sqrt{3})^2 = 1 - \frac{1}{4}(1 + 3 - 2\sqrt{3}) = \frac{\sqrt{3}}{2}. \tag{A.1}
$$

Then we have

$$
\begin{aligned}
\frac{g'(x)}{2} &= 1 - 2x - 2x^2 - 2\theta(1+x)\sqrt{1 - (1+x)^2}, \\
\frac{g''(x)}{2} &= -2 - 4x - 4\theta\sqrt{1 - (x+1)^2} + \frac{2\theta}{\sqrt{1 - (x+1)^2}}.
\end{aligned}
$$

So, thanks to (A.1), if $-2 \le x \le -\frac{1+\sqrt{3}}{2}$:

$$\frac{g''(x)}{2} \ge -2 - 4x - 4\theta\sqrt{\frac{\sqrt{3}}{2}} + 2\theta\sqrt{\frac{2}{\sqrt{3}}}.$$

But we have

$$-4\theta\sqrt{\frac{\sqrt{3}}{2}} + 2\theta\sqrt{\frac{2}{\sqrt{3}}} \approx -2.06 > -2.1, \qquad \text{and} \quad -\frac{1+\sqrt{3}}{2} < -\frac{4.1}{4}.$$

Therefore:

$$\frac{g''(x)}{2} > -4.1 - 4x > 0, \quad \text{whenever} \quad -2 \le x \le -\frac{1+\sqrt{3}}{2}.$$

This proves the point 1 of the lemma.

Step 2. We have

$$g'(x) = \underbrace{2 - 4x - 4x^2}_{=:h_1(x)} - \underbrace{4\theta(x+1)\sqrt{1-(1+x)^2}}_{=:h_2(x)}. \tag{A.2}$$

Note that $-\frac{1+\sqrt{3}}{2}$ and $-\frac{1-\sqrt{3}}{2}(> -1)$ are the two roots of h_1. Therefore $h_1 > 0$ on $(-\frac{1+\sqrt{3}}{2}, -1]$. The function $x+1$ is non-positive on this interval, so we have also $h_2 \le 0$ on the same interval. Therfore $g' > 0$ on that interval. This proves the point 2.

Step 3. For any root x of g',

$$P(x) := (h_1(x))^2 - (h_2(x))^2 = 0.$$

Note that P is a polynomial of order 4, so it has at most 4 roots. Using special roots of h_1 and h_2, we find that

$$P(-2) = (h_1(-2))^2 > 0, \qquad P\left(\frac{-1+\sqrt{3}}{2}\right) = -(h_2)^2 < 0, \qquad P(-1) = (h_1(-1))^2 > 0.$$

Hence P has at least two roots on $(-2,-1)$. Therefore P (and g') cannot have more than 2 roots on $[-1,0]$. However:

$$g'(-1+\sqrt{2}/2) = 2(\sqrt{2}-\theta) > 0, \qquad g'(-1+\sqrt{3}/2) = (2-3\theta)\sqrt{3}-1 < 0, \qquad g'(0) = 2 > 0.$$

So g' has exactly two roots in $[-1,0]$. One root x_1 is in $(-1+\sqrt{2}/2, -1+\sqrt{3}/2)$ and the other root x_2 is in $(-1+\sqrt{3}/2, 0)$. Moreover, g is increasing on $(-1, x_1)$ and on $(x_2, 0)$, and decreasing on (x_1, x_2). Hence, g has a local maximum at x_1 and a a local minimum at x_2.

Step 4. The function g is continuous on $[-2,0]$, so it attains its maximum on this interval. Assume that this maximum is reached at $x_* \in (-2,0)$. At this point it verifies both $g'(x_*) = 0$ and $g''(x_*) \le 0$. From Steps 1 and 2, we have $x_* \in (-1,0)$. But from Step 3, we have $x_* = x_1 \in (-1+\sqrt{2}/2, -1+\sqrt{3}/2)$.

Let us consider

$$h_1'(x) = 4 - 8(1+x),$$
$$\sqrt{1-(1+x)^2}\,h_2'(x) = 4\theta(1-2(1+x)^2).$$

We see that these functions are decreasing on $(-1+\sqrt{2}/2, -1+\sqrt{3}/2)$, and non-positive at $-1+\sqrt{2}/2$, that is,

$$\text{for } x \in (-1+\sqrt{2}/2, -1+\sqrt{3}/2), \quad h_i'(x) \le 0 \quad \text{and} \quad h_i''(x) \le 0, \quad \text{for } i = 1, 2.$$

Since $g(0) = 0$, and $g(x_1)$ is supposed to be a global maximum, we have $g(x_1) \ge 0$ and

$$I = \int_{-1+\frac{\sqrt{2}}{2}}^{x_1} g'(y)\,dy = g(x_1) - g(-1+\frac{\sqrt{2}}{2}) \ge -g(-1+\frac{\sqrt{2}}{2}) > 0.107.$$

But using the monotonicity of h_1' and h_2', and $h'(x_1) = h_2'(x_1)$ (since $g'(x_1) = 0$), we have

$$\begin{aligned} I &= \int_{-1+\frac{\sqrt{2}}{2}}^{x_1} (h_1'(y) - h_2'(y))\,dy \le (x_1 - (-1+\sqrt{2}/2))(h_1'(-1+\sqrt{2}/2) - h_2'(x_1)) \\ &= (x_1 - (-1+\sqrt{2}/2))(h_1'(-1+\sqrt{2}/2) - h_1'(x_1)) \\ &\le \frac{\sqrt{3}-\sqrt{2}}{2}(h_1'(-1+\sqrt{2}/2) - h_1'(-1+\sqrt{3}/2)). \end{aligned}$$

Since

$$\frac{\sqrt{3}-\sqrt{2}}{2} < 0.2, \quad h_1(-1+\frac{\sqrt{2}}{2}) - h_1(-2+2\frac{\sqrt{3}}{2}) = 2\sqrt{2} - 2(\sqrt{3} - \frac{1}{2}) < 0.4,$$

we have

$$I \le 0.08,$$

which contradicts with $I > 0.107$. Hence g reaches his maximum only at 0 or -2. Since $g(-2) = -4/3$, and $g(0) = 0$, therefore

$$g(x) < 0 \qquad \text{for every } x \in [-2, 0).$$

□

Appendix B. Proof of Lemma 2.9

Let $\{P_n : [-1,1] \to \mathbb{R}\}_{n \ge 0}$ be an orthonormal basis of the Legendre polynomials, that are solutions to Legendre's differential equations:

$$\text{(B.3)} \qquad \frac{d}{dx}\Big((1-x^2)\frac{d}{dx}P_n(x)\Big) = -n(n+1)P_n(x),$$

and satisfy the orthonormality in $L^2[-1,1]$, i.e., $\int_{-1}^1 P_i P_j = \delta_{ij}$ and $\int_{-1}^1 P_i^2 = 1$.
Then, for any $w \in L^2[-1,1]$, we have $w = \sum_{i=0}^\infty c_i P_i$, $c_i = \int_{-1}^1 w(x)P_i(x)dx$.
In particular, we see that $P_0(x) = \frac{1}{\sqrt{2}}$, thus $c_0 P_0 = \frac{1}{2}\int_{-1}^1 w dx =: \bar{w}$, which is an average of

w over $[-1,1]$. Then, since $w-\bar{w}=\sum_{i=1}^{\infty} c_i P_i$, using (B.3), we have

$$\begin{aligned}\int_{-1}^{1}(1-x^2)|w'|^2dx &= -\int_{-1}^{1}\Big((1-x^2)w'\Big)'wdx = -\int_{-1}^{1}\Big((1-x^2)w'\Big)'(w-\bar{w})dx\\ &= -\sum_{i\geq 1}\sum_{j\geq 1}\int_{-1}^{1} c_i\Big((1-x^2)P_i'\Big)'c_jP_jdx\\ &= \sum_{i\geq 1}\sum_{j\geq 1}\int_{-1}^{1} c_ic_j i(i+1)P_iP_jdx\\ &= \sum_{i\geq 1}\int_{-1}^{1} i(i+1)c_i^2P_i^2dx \geq 2\sum_{i\geq 1}\int_{-1}^{1}c_i^2P_i^2dx = 2\int_{-1}^{1}(w-\bar{w})^2dx.\end{aligned}$$

Therefore, we have

$$\int_{-1}^{1}(w-\bar{w})^2dx \leq \frac{1}{2}\int_{-1}^{1}(1-x^2)|w'|^2dx.$$

By a change of variable as $W(x):=w(2x-1)$, we have

$$\int_{0}^{1}(W-\bar{W})^2dx \leq \frac{1}{2}\int_{0}^{1}x(1-x)|W'|^2dx,$$

where $\bar{W}=\int_0^1 Wdx$.

References

[1] Adimurthi, Shyam Sundar Ghoshal, and G. D. Veerappa Gowda. L^p stability for entropy solutions of scalar conservation laws with strict convex flux. *J. Differential Equations*, 256(10):3395–3416, 2014.

[2] B. Barker, J. Humpherys, O. Laffite, K. Rudd, and Zumbrun. K. Stability of isentropic navier-stokes shocks,. *Appl. Math. Lett.*, 21:742–747, 2008.

[3] D. Bresch and B. Desjardins. Existence of global weak solutions for 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238:211–223, 2003.

[4] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2d viscous shallow water model and for compressible navier-stokes models,. *J. Math. Pures Appl.*, 86(9):362–368, 2006.

[5] D. Bresch, B. Desjardins, and C.K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems,. *Comm. Partial Differential Equations,*, 28:843–868, 2003.

[6] S. Chapman and T.G. Cowling. The mathematical theory of non-uniform gases. *Cambridge University Press, London*, 3rd ed., 1970.

[7] K. Choi and A. Vasseur. Short-time stability of scalar viscous shocks in the inviscid limit by the relative entropy method. *SIAM J. Math. Anal.*, 47:1405–1418, 2015.

[8] P. Constantin, T. D. Drivas, H. Q. Nguyen, and F. Pasqualotto. Compressible fluids and active potentials,. *https://arxiv.org/pdf/1803.04492.pdf*, 2018.

[9] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179, 1979.

[10] R. J. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, 28(1):137–188, 1979.

[11] B. Haspot. New formulation of the compressible navier-stokes equations and parabolicity of the density,. *arXiv:1411.5501.*

[12] B. Haspot. Porous media, fast diffusion equations and the existence of global weak solution for the quasi-solution of compressible Navier-Stokes equations,. *Hyperbolic Problems: Theory, Numerics, Applications-Methods and Applications of Analysis,*, 20(2):141–164, 2013.

[13] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D,. *posted on arXiv,*, 2014.

[14] B. Haspot. Weak-strong uniqueness for compressible Navier-Stokes system with degenerate viscosity coefficient and vacuum in one dimension. *Commun. Math. Sci.*, 15(3):587–591, 2017.

[15] J. Humpherys, O. Laffite, and Zumbrun. K. Stability of isentropic viscous shock profiles in the high-mach number limit,. *Comm. Math. Phys.*, 293:1–36, 2010.

[16] M.-J. Kang. Non-contraction of intermediate admissible discontinuities for 3-D planar isentropic magnetohydrodynamics. *Kinet. Relat. Models*, 11(1):107–118, 2018.

[17] M.-J. Kang. L^2-type contraction for shocks of scalar viscous conservation laws with strictly convex flux. *posted on arXiv*, 2019.

[18] M.-J. Kang and A. Vasseur. Uniqueness of entropy shocks to the isentropic euler system in a class of inviscid limits from a large family of navier-stokes systems. *In Preparation.*

[19] M.-J. Kang and A. Vasseur. Criteria on contractions for entropic discontinuities of systems of conservation laws. *Arch. Ration. Mech. Anal.*, 222(1):343–391, 2016.

[20] M.-J. Kang and A. Vasseur. L^2-contraction for shock waves of scalar viscous conservation laws. *Annales de l'Institut Henri Poincaré (C) : Analyse non linéaire*, 34(1):139156, 2017.

[21] M.-J. Kang, A. Vasseur, and Y. Wang. L^2-contraction for planar shock waves of multi-dimensional scalar viscous conservation laws. *http://arxiv.org/pdf/1609.01825.pdf.*

[22] N. Leger. L^2 stability estimates for shock solutions of scalar conservation laws using the relative entropy method. *Arch. Ration. Mech. Anal.*, 199(3):761–778, 2011.

[23] N. Leger and A. Vasseur. Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations. *Arch. Ration. Mech. Anal.*, 201(1):271–302, 2011.

[24] T.-P. Liu and Y. Zeng. Time-asymptotic behavior of wave propagation around a viscous shock profile,. *Comm. Math. Phys.*, 290:23–82, 2009.

[25] C. Mascia and K. Zumbrun. Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems,. *Comm. Pure Appl. Math.*, 57:841–876, 2004.

[26] A. Matsumura and K. Nishihara. On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas,. *Japan J. Appl. Math.*, 2:17–25, 1985.

[27] A. Matsumura and Y. Wang. Asymptotic stability of viscous shock wave for a one-dimensional isentropic model of viscous gas with density dependent viscosity,. *Methods Appl. Anal.*, 17:279–290, 2010.

[28] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible navier-stokes equations,. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[29] D. Serre and A. Vasseur. L^2-type contraction for systems of conservation laws. *J. Éc. polytech. Math.*, 1:1–28, 2014.

[30] D. Serre and A. Vasseur. About the relative entropy method for hyperbolic systems of conservation laws. *Contemp. Math. AMS*, 658:237–248, 2016.

[31] D. Serre and A. Vasseur. The relative entropy method for the stability of intermediate shock waves; the rich case. *Discrete Contin. Dyn. Syst.*, 36(8):4569–4577, 2016.

[32] V.V. Shelukhin. On the structure of generalized solutions of the one-dimensional equations of a polytropic viscous gas,. *J. Appl. Math. Mech., 48(1984), 665–672; translated from Prikl. Mat. Mekh. 48(1984), no. 6, 912–920.*

[33] A. Vasseur. Recent results on hydrodynamic limits. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 323–376. Elsevier/North-Holland, Amsterdam, 2008.

[34] A. Vasseur. Relative entropy and contraction for extremal shocks of conservation laws up to a shift. In *Recent advances in partial differential equations and applications*, volume 666 of *Contemp. Math.*, pages 385–404. Amer. Math. Soc., Providence, RI, 2016.

[35] A. Vasseur and Y. Wang. The inviscid limit to a contact discontinuity for the compressible navier-stokes-fourier system using the relative entropy method. *SIAM J. Math. Anal.*, 47(6):4350–4359, 2015.

[36] A. Vasseur and L. Yao. Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic navier-stokes equations with density dependent viscous coefficient,. *Commun. Math. Sci.*, 14(8):2215–2228, 2016.

(Moon-Jin Kang)
Department of Mathematic & Research Institute of Natural Sciences,
Sookmyung Women's University, Seoul 140-742, Korea
E-mail address: moonjinkang@sookmyung.ac.kr

(Alexis F. Vasseur)
Department of Mathematics,
The University of Texas at Austin, Austin, TX 78712, USA
E-mail address: vasseur@math.utexas.edu