REPRESENTATIONS AND COHOMOLOGY OF A FAMILY OF FINITE SUPERGROUP SCHEMES

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In memory of Kai Magaard, with admiration

Abstract. We examine the cohomology and representation theory of a family of finite supergroup schemes of the form $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s)$. In particular, we show that a certain relation holds in the cohomology ring, and deduce that for finite supergroup schemes having this as a quotient, both cohomology mod nilpotents and projectivity of modules is detected on proper sub-supergroup schemes. This special case feeds into the proof of a more general detection theorem for unipotent finite supergroup schemes, in a separate work of the authors joint with Iyengar and Krause.

We also completely determine the cohomology ring in the smallest cases, namely $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$ and $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{Z}/p$. The computation uses the local cohomology spectral sequence for group cohomology, which we describe in the context of finite supergroup schemes.

1. Introduction

The calculations in this paper are motivated by the problem of detecting nilpotents in cohomology theories which has a long history. In algebraic topology, the celebrated nilpotence theorem in the stable homotopy category is due to Devinatz–Hopkins–Smith. For mod-p finite group cohomology, Quillen showed that nilpotence is detected upon restriction to elementary abelian subgroups. Suslin proved an analogue of Quillen's detection theorem for cohomology of finite group schemes where the detection family consisted of abelian finite groups schemes isomorphic to $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ (preceded by the work of Suslin-Friedlander-Bendel [3] on infinitesimal finite group schemes and Bendel [2] on unipotent finite group schemes).

In joint work with Iyengar and Krause [7], we study the question of detecting nilpotents in the cohomology of a finite supergroup scheme, or equivalently, a finite dimensional $\mathbb{Z}/2$-graded cocommutative Hopf superalgebra. We establish a detecting family in the case of a unipotent finite supergroup scheme which turns out to have a surprisingly more complicated structure than what one sees in the ungraded case in the detection theorems of Quillen and Suslin. A particularly difficult case arising in the course of the proof of the detection theorem in [7] is that of the degree two cohomology class determined by the central extension of $\mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ by $\mathbb{G}_a^-$, where $\mathbb{G}_a^-$ is a supergroup scheme corresponding to the exterior algebra of a one dimensional super vector space concentrated in odd degree. The outcome of this paper, which feeds into the proof of the general result

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in [7], is that a certain product vanishes in cohomology but this relation does not follow in the usual way from the action of the Steenrod operations.

In the course of producing the desired relation, we study the representation theory and cohomology of finite supergroup schemes of the form $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s)$, where the complement $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ is acting faithfully on the normal sub-supergroup scheme $\mathbb{G}_a^- \times \mathbb{G}_a^-$. We also obtain a great deal of information about the smallest case, computing the cohomology ring of $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{Z}/p$, which is our first result, proved in Section 5. Note that for supergroup schemes, the cohomology is doubly graded: we write $H^{i,j}(G,k)$, where the index $i \in \mathbb{Z}$ is cohomological, and the index $j \in \mathbb{Z}/2$ comes from the internal grading.

Theorem 1.1 (Theorem 5.1 and Remark 5.7)**.** *Let G be either $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{Z}/p$ or $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$, each one being a semidirect product with non-trivial action. Then the cohomology ring $H^{*,*}(G,k)$ is Gorenstein with the Poincaré series given by*

$$\sum_n t^n \dim_k H^{n,*}(G,k) = 1/(1-t)^2.$$

The algebra structure is given as follows. The generators are

$$\zeta \in H^{1,1}(G,k),\ x \in H^{2,0}(G,k),\ \kappa \in H^{p,1}(G,k),\ \lambda_i \in H^{i,1+i}(G,k)\ (1 \leqslant i \leqslant p-1).$$

The relations are

$$\lambda_i\zeta = 0\ (1 \leqslant i \leqslant p-1), \quad x\zeta^{p-1} = 0, \quad \lambda_i\lambda_j = \begin{cases} x\zeta^{p-2} & i+j=p \\ 0 & \textit{otherwise.} \end{cases}$$

One of the techniques we employ for this calculation is the local cohomology spectral sequence which Greenlees [13] developed in the context of cohomology of finite groups. This turns out to be essential to determine that the product $\lambda_i\lambda_{p-i}$ is non-zero. For finite supergroup schemes, this spectral sequence takes the form below, incorporating the modular function δ_G (see Section 4).

Theorem 1.2 (Corollary 4.2 and Corollary 4.5)**.** *Let G be a finite supergroup scheme. Then there is a local cohomology spectral sequence*

$$E_2^{s,t,j} = H^{s,t,j}_{\mathfrak{m}} H^{*,*}(G,M) \Rightarrow H_{-s-t,j+\varepsilon_G}(G, M \otimes \delta_G).$$

Here, the third index $j \in \mathbb{Z}/2$ is given by the internal grading, and δ_G is the modular function *of internal degree $\varepsilon_G \in \mathbb{Z}/2$.*

Suppose that δ_G is trivial, which happens for example in the case where G is unipotent. In this case, if $H^{*,*}(G,k)$ is Cohen–Macaulay then it is Gorenstein, with shift $(0,\varepsilon_G)$.

We also, along the way, make some computations of the structure of the symmetric powers of a faithful two dimensional representation V of $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$. We state it in terms of the dual $V^\#$, because we are interested in cohomology. In the case of $(\mathbb{Z}/p)^s$ this is well known by restricting from $\mathrm{SL}(2,p^s)$, whereas in the case of the Frobenius kernel, the results follow by restricting from $\mathrm{SL}_{2(r)}$ (see, for example, [16, II.2.16]). The following is a tabulation of the results proved in Section 7.

Theorem 1.3. *Let V be a faithful two dimensional representation of $H = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$, $V^\#$ be the dual vector space, and $S^n(V^\#)$ be the module of degree n polynomial functions on V,*

(i) Periodicity: *For $n \geqslant p^{r+s}$ we have $S^n(V^\#) \cong kH \oplus S^{n-p^{r+s}}(V^\#)$, where kH is the group algebra of the finite group scheme H.*

(ii) Projectivity: *$S^n(V^\#)$ is a projective module if and only if n is congruent to -1 modulo p^{r+s}.*
(iii) Uniserial: *For $1 \leqslant i \leqslant p-1$, the module $S^i(V^\#)$ is a uniserial module of dimension $i+1$.*
(iv) Steinberg tensor product: *For $1 \leqslant i \leqslant r+s$ the module $S^{p^i-1}(V^\#)$ is isomorphic to the tensor product of Frobenius twists $S^{p-1}(V^\#) \otimes S^{p-1}(V^\#)^{(1)} \otimes \cdots \otimes S^{p-1}(V^\#)^{(i-1)}$.*
(v) Rank variety: *The rank variety of $S^{p^i-1}(V^\#)$ is an explicitly described linear subspace of affine space $\mathbb{A}^{r+s}$ of codimension i.*

Using Theorem 1.3 to make some spectral sequence computations, the following theorem is proved in Section 8.

Theorem 1.4 (Theorem 8.1)**.** *Let k be a field of odd prime characteristic, and let G be the finite supergroup scheme $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s)$. Then there is a non-zero element $\zeta \in H^{1,1}(G,k)$ such that for all $u \in H^{1,0}(G,k)$ we have $\beta\mathscr{P}^0(u).\zeta^{p^{r+s-1}(p-1)} = 0$.*

The following consequence will be used in our joint work with Iyengar and Krause [7].

Corollary 1.5 (Corollary 8.2)**.** *Let G be a finite unipotent supergroup scheme, with a normal sub-supergroup scheme N such that $G/N \cong \mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$. If the inflation map $H^{1,*}(G/N,k) \to H^{1,*}(G,k)$ is an isomorphism and $H^{2,1}(G/N,k) \to H^{2,1}(G,k)$ is not injective then there exists a non-zero element $\zeta \in H^{1,1}(G,k)$ such that for all $u \in H^{1,0}(G,k)$ we have $\beta\mathscr{P}^0(u).\zeta^{p^{r+s-1}(p-1)} = 0$.*

Throughout this paper, k is a field of odd characteristic. Background on finite supergroup schemes can be found in the “sister paper” [7]. We use [16] as our standard reference for affine group schemes and their representations.

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2. Semidirect products

We begin by recalling, for example from Theorem 2.13 of Molnar [19], the Hopf structure on the smash product of cocommutative Hopf algebras. The same conventions work just as well in the graded cocommutative case, as follows.

Let B be a graded cocommutative Hopf algebra, and A be a Hopf algebra which is a B-module bialgebra. Then the tensor product coalgebra structure on the smash product $A\#B$ makes it a Hopf algebra. In more detail, let $\tau\colon B \otimes A \to A$ be the map giving the action. Then the multiplication on $A\#B$ is

$$(a \otimes h)(b \otimes g) = \sum (-1)^{|h_{(2)}||b|} a\tau(h_{(1)}, b) \otimes h_{(2)}g, \tag{2.1}$$

the comultiplication is

$$\Delta(a \otimes h) = \sum (-1)^{|h_{(1)}||a_{(2)}|}(a_{(1)} \otimes h_{(1)}) \otimes (a_{(2)} \otimes h_{(2)})$$

and the antipode is

$$\boldsymbol{s}(a \otimes h) = \sum (-1)^{(|a|+|h_{(1)}|)|h_{(2)}|}\tau(\boldsymbol{s}(h_{(2)}), \boldsymbol{s}(a)) \otimes \boldsymbol{s}(h_{(1)}).$$

If A is also graded cocommutative, we shall write $A \rtimes B$ for this construction, and call it the *semidirect product* of A and B with action τ. There are obvious maps of Hopf algebras

$$A \longrightarrow A \rtimes B \rightleftarrows B$$

forming a split exact sequence. Theorem 4.1 of the same paper implies that any split exact sequence of graded cocommutative Hopf algebras is isomorphic to a semidirect product.

Recall that if G is a finite supergroup scheme, then its group algebra kG is defined as a linear dual to the coordinate algebra $k[G]$. Hence, it is a finite dimensional graded cocommutative Hopf algebra (see, for example, [7] for more extensive background). We denote by $\mathbb{G}_a^-$ the supergroup scheme with the (self-dual) coordinate algebra $k[v]/(v^2)$ with v an odd primitive element. We denote by $\mathbb{G}_{a(r)}$ the rth Frobenius kernel of the additive group $\mathbb{G}_a$, a finite connected group scheme with coordinate algebra $k[T]/(T^{p^r})$ with T primitive, and group algebra $k\mathbb{G}_{a(r)}$ Recall from [16, 8.7, 7.8] that $k\mathbb{G}_{a(r)}$ has the divided power basis $\{\gamma_0, \ldots, \gamma_{p^r-1}\}$ where $\gamma_i(T^j) = \delta_{ij}$. Writing s_i for $\gamma_{p^{i-1}}$, we have the following explicit formula

$$\gamma_i = \frac{s_1^{\alpha_{i1}} \ldots s_r^{\alpha_{ir}}}{\alpha_{i1}! \ldots \alpha_{ir}!} \tag{2.2}$$

where $i = \sum_{\ell=1}^{r} \alpha_{i\ell} p^{\ell-1}$ is the p-adic decomposition of i. Hence, $s_1, \ldots, s_r$ are algebraic generators of $k\mathbb{G}_{a(r)}$. We identify

$$k\mathbb{G}_{a(r)} = k[s_1, \ldots, s_r]/(s_1^p, \ldots, s_r^p).$$

In terms of the divided power basis the comultiplication in $k\mathbb{G}_{a(r)}$ is given by

$$\Delta(\gamma_i) = \sum_{j=0}^{j=i} \gamma_j \otimes \gamma_{i-j}. \tag{2.3}$$

This formula can be easily checked by applying it to the basis $T^\ell \otimes T^{\ell'}$ of $(k\mathbb{G}_{a(r)} \otimes k\mathbb{G}_{a(r)})^\# \cong k[\mathbb{G}_{a(r)}] \otimes k[\mathbb{G}_{a(r)}]$. In the context of supergroup schemes, we think of $k\mathbb{G}_{a(r)}$ as concentrated in even degree.

Getting back to the discussion of the semidirect product, we are interested in the specific case where A is the group algebra of $\mathbb{G}_a^- \times \mathbb{G}_a^-$, the exterior algebra on two primitive generators v_1 and v_2, and $B = kH$ is the group algebra of the finite group scheme $H = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$. Here, either r or s, but not both, may be equal to zero. We assume that H acts faithfully on $\mathbb{G}_a^- \times \mathbb{G}_a^-$, namely that no proper subgroup scheme of H acts trivially on $\mathbb{G}_a^- \times \mathbb{G}_a^-$, and we write G for $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes (\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s)$. We let $(\mathbb{Z}/p)^s = \langle g_1, \ldots, g_s \rangle$, and write $t_i = g_i - 1 \in k(\mathbb{Z}/p)^s$, so that $\Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i + t_i \otimes t_i$ $(1 \leqslant i \leqslant s)$.

Lemma 2.1. *Given a faithful action of* $H \cong \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ *on* $\mathbb{G}_a^- \times \mathbb{G}_a^-$*, we may choose notation so that* $A = k(\mathbb{G}_a^- \times \mathbb{G}_a^-) \cong \Lambda(v_1, v_2)$ *with* v_2 *invariant, and the map* $\tau\colon kH \otimes A \to A$ *describing the action is given by*

$$\begin{aligned}
&\tau(s_1 \otimes v_1) = v_2, && \\
&\tau(s_i \otimes v_1) = 0 && (2 \leqslant i \leqslant r),\\
&\tau(s_i \otimes v_2) = 0 && (1 \leqslant i \leqslant r),\\
&\tau(t_i \otimes v_1) = \mu_i v_2 && (1 \leqslant i \leqslant s),\\
&\tau(t_i \otimes v_2) = 0 && (1 \leqslant i \leqslant s).
\end{aligned}$$

with $\mu_1, \dots, \mu_s \in k$ *constants, linearly independent over* $\mathbb{F}_p$.

Proof. Since H is unipotent, its action on the vector space spanned by v_1 and v_2 can be upper triangularized. This amounts to choosing an invariant element, and we choose it to be v_2. Then the action of $(\mathbb{Z}/p)^s$ is as shown. We must check that there are enough automorphisms of $\mathbb{G}_{a(r)}$ so that its action may also be written as shown. The general form of an action fixing v_2 is given by $\tau(s_i \otimes v_1) = \alpha_i v_2$ for constants $\alpha_1, \dots, \alpha_r \in k$. Since the action is faithful we have $\alpha_1 \neq 0$, and then by replacing v_2 by $\alpha_1 v_2$ we may assume that $\alpha_1 = 1$. Finally, there is a Hopf algebra automorphism of $\mathbb{G}_{a(r)}$ which replaces s_i by $s_i - \alpha_i s_1$ for $i = 2, \dots, r$, and after applying this automorphism, the action has the given form. □

By abuse of notation, we write v_1 for $v_1 \otimes 1$, v_2 for $v_2 \otimes 1$, s_i for $1 \otimes s_i$ and t_i for $1 \otimes t_i$ in $A \rtimes kH$. These elements satisfy the following relations:

$$\begin{aligned}
&v_1^2 = v_2^2 = v_1 v_2 + v_2 v_1 = 0, && \\
&s_1 v_1 = v_1 s_1 + v_2, && \\
&s_i v_1 = v_1 s_i + (-1)^{i-1} v_2 s_1^{p-1} \dots s_{i-1}^{p-1} && (2 \leqslant i \leqslant r),\\
&s_i v_2 = v_2 s_i && (1 \leqslant i \leqslant r),\\
&t_i v_2 = v_2 t_i && (1 \leqslant i \leqslant s),\\
&t_i v_1 = v_1 t_i + \mu_i v_2 (1 + t_i) && (1 \leqslant i \leqslant s).
\end{aligned}$$

To justify the third relation we observe that the action prescribed by Lemma 2.1 implies that

$$\tau(\gamma_j \otimes v_1) = 0 \qquad (2 \leqslant j \leqslant p^r - 1). \tag{2.4}$$

We compute for $i > 1$:

$$\begin{aligned}
s_i v_1 &= (1 \otimes s_i)(v_1 \otimes 1) = (1 \otimes \gamma_{p^{i-1}})(v_1 \otimes 1)\\
&= \sum_{j=0}^{p^{i-1}} (-1)^{|\gamma_{p^{i-1}-j}||v_1|} \tau(\gamma_j \otimes v_1) \otimes \gamma_{p^{i-1}-j}\\
&= \tau(1 \otimes v_1) \otimes \gamma_{p^{i-1}} + \tau(\gamma_1 \otimes v_1) \otimes \gamma_{p^{i-1}-1}\\
&= v_1 s_i + (-1)^{i-1} v_2 s_1^{p-1} \dots s_{i-1}^{p-1}.
\end{aligned}$$

The second line follows by applying formula (2.1) and comultiplication rule (2.3). All the terms in the sum except for the first two disappear by (2.4) which implies the equality in the third line. Finally, the last equality follows from (2.2) and the p-adic decomposition of $p^{i-1} - 1$.

The other relations on the list follow immediately from Lemma 2.1.

3. Steenrod operations

We shall need to use Steenrod operations in the cohomology of finite supergroup schemes. The discussion of these in the literature is almost, but not completely adequate for our purposes, and so we give a brief discussion here.

If A is a $\mathbb{Z}$-graded cocommutative Hopf algebra over $\mathbb{F}_p$, the discussion in Section 11 of May [18] does the job. Specifically, for p odd, [18, Theorem 11.8] produces natural operations

$$\mathcal{P}^i\colon H^{s,t}(A,k) \to H^{s+(2i-t)(p-1),pt}(A,k)$$
$$\beta\mathcal{P}^i\colon H^{s,t}(A,k) \to H^{s+1+(2i-t)(p-1),pt}(A,k)$$

satisfying, among others, the following properties:

(i) $\mathcal{P}^i = 0$ if either $2i < t$ or $2i > s+t$
$\beta\mathcal{P}^i = 0$ if either $2i < t$ or $2i \geqslant s+t$
(ii) $\mathcal{P}^i(x) = x^p$ if $2i = s+t$
(iii) (Cartan Formula)
$\mathcal{P}^j(xy) = \sum_i \mathcal{P}^i(x)\mathcal{P}^{j-i}(y)$
$\beta\mathcal{P}^j(xy) = \sum_i(\beta\mathcal{P}^i(x)\mathcal{P}^{j-i}(y) + (-1)^{|x|}\mathcal{P}^i(x)\beta\mathcal{P}^{j-i}(y))$
(iv) The $\mathcal{P}^i$ and $\beta\mathcal{P}^i$ satisfy the Adem relations.
(v) $\mathcal{P}^i$ is $\mathbb{F}_p$-linear; that is, $\mathcal{P}^i(x+y) = \mathcal{P}^i(x)+\mathcal{P}^i(y)$, and $\mathcal{P}^i(\lambda x) = \lambda\mathcal{P}^i(x)$ for $\lambda \in \mathbb{F}_p$ and $x, y \in H^{*,*}(A,k)$.

For us, there are two problems with this. The first is that we want to work over a more general field k of characteristic p, not just $\mathbb{F}_p$. As remarked by Wilkerson [21] (bottom of page 140), the only difference is that the operations are no longer k-linear. Rather, they are semilinear, so that (v) should be replaced by

(v) $\mathcal{P}^i$ is k-semilinear; that is, $\mathcal{P}^i(x+y) = \mathcal{P}^i(x) + \mathcal{P}^i(y)$, and $\mathcal{P}^i(\lambda x) = \lambda^p\mathcal{P}^i(x)$ for $\lambda \in k$ and $x, y \in H^{*,*}(A,k)$.

The other problem is that if we wish to apply this to a $\mathbb{Z}/2$-graded object, then the way the indices work involves subtracting an element of $\mathbb{Z}/2$ from an element of $\mathbb{Z}$ and expecting an answer in $\mathbb{Z}$. This clearly doesn't work, so we need to do some re-indexing to take care of this problem. The origin of the problem is that May has chosen to base the indexing of the operations on total degree rather than internal degree. The rationale for doing this is that it avoids the introduction of half-integer indexed operations, but the disadvantage is that it only works for $\mathbb{Z}$-graded objects, and not for example for $\mathbb{Z}/2$-graded objects.

In order to reindex using internal degree rather than total degree, we rename May's $\mathcal{P}^i$ as our $\mathscr{P}^{i-t/2}$. Then we have

$$\mathscr{P}^i\colon H^{s,t}(A,k) \to H^{s+2i(p-1),pt}(A,k)$$
$$\beta\mathscr{P}^i\colon H^{s,t}(A,k) \to H^{s+1+2i(p-1),pt}(A,k).$$

Here, $i \in \mathbb{Z}$ if t is even and $i \in \mathbb{Z}+\frac{1}{2}$ if t is odd. Note that since p is odd, pt is equivalent to t mod 2, so the operations preserve internal degree as elements of $\mathbb{Z}/2$.

These operations are called P^i in Theorem A1.5.2 of Appendix 1 in Ravenel [20]. They are called $\tilde{\mathcal{P}}^i$ in the discussion following Theorem 11.8 of May [18], but he ignores the operations indexed by $\mathbb{Z}+\frac{1}{2}$.

To accommodate the change from $\mathbb{Z}$-grading to $\mathbb{Z}/2$-grading of the Hopf algebra, the following observations pertain to §11 of May [18]. In Definition 11.1, if $C = (E, A, F) \in \mathcal{C}$ with E, A and F $\mathbb{Z}/2$-graded, then the bar resolution $B(C)$ is bigraded, with homological degree $s \in \mathbb{Z}$ and internal degree $t \in \mathbb{Z}/2$. The total degree $s+t$ makes sense as an element of $\mathbb{Z}/2$, and so the sign $(-1)^{s+t}$ makes sense as an integer. In particular, in (3) the definition $\bar{x} = (-1)^{1+\deg x} x$ makes sense with $\deg x$ the total degree, and so $d\colon B(C)_{s,*} \to B(C)_{s-1,*}$ is well defined. In Lemma 11.3, $W \otimes B(C)$ is bigraded with $(W \otimes B(C))_{s,t} = \sum_{i+j=s} W_i \otimes B_{j,t}(C)$ with $s \in \mathbb{Z}$ and $t \in \mathbb{Z}/2$, and in (iii), $(-1)^{\deg w \deg a}$ makes sense as an integer. Finally, of the two conventions for numbering the Steenrod operations, the one used for Theorem 11.8 mixes the homological and internal degrees, and therefore cannot be used. But the alternative numbering used in the remark after the proof of the theorem does not mix the two degrees, and therefore makes sense, and encompasses all the operations once we allow half integer degrees. In particular, the commutation conventions need to avoid mixing degrees. So if x has degree (q, ε) (i.e., homological degree $q \in \mathbb{Z}$ and internal degree $\varepsilon \in \mathbb{Z}/2$) and y has degree (q', ε') then the symmetric braiding sends $x \otimes y$ to $(-1)^{qq'+\varepsilon\varepsilon'} y \otimes x$, and not $(-1)^{(q+\varepsilon)(q'+\varepsilon')} y \otimes x$. So for example $(x \otimes y)^{\otimes p}$ gets shuffled to $(-1)^{m(qq'+\varepsilon\varepsilon')} x^{\otimes p} \otimes y^{\otimes p}$, where $m = (p-1)/2$.

The constants in the definition of $\mathscr{P}^i$ and $\beta\mathscr{P}^i$ also need adjustment, because $(-1)^i$ no longer makes sense when $i \in \mathbb{Z} + \frac{1}{2}$. Instead, we need fourth roots of unity. So we choose an element $\gamma \in k$ with $\gamma^2 = -1$. If $p \equiv 1 \pmod 4$ then γ can be taken to be $m!$ (we continue to write m for $(p-1)/2$), but if $p \equiv 3 \pmod 4$ then γ is an element of $\mathbb{F}_{p^2}$. This may need an extension of scalars, so whenever we need to, we shall assume that k contains a square root of -1. For our theorems given in the introduction, extension of scalars does not affect the validity. The point of introducing the fourth root of unity is that our operations agree with May's.

The definition of the operations $\mathscr{P}^i$ and $\beta\mathscr{P}^i$ is then as follows. If x has degree (q, ε) then

$$\begin{aligned} \mathscr{P}^i(x) &= \gamma^{2i+mq(q+1)}(m!)^{-q} D^{(q-2i)(p-1)}(x), \\ \beta\mathscr{P}^i(x) &= \gamma^{2i+mq(q+1)}(m!)^{-q} D^{(q-2i)(p-1)+1}(x). \end{aligned}$$

These definitions agree with May's whenever $i \in \mathbb{Z}$ and x is even, but also make sense when $i \in \mathbb{Z} + \frac{1}{2}$ and x is odd.

The upshot of this reindexing is that at the expense of introducing half-integer indices for the Steenrod operations, we have made the notation work for $\mathbb{Z}/2$-graded objects. Properties (i) and (ii) have been reindexed, so that (i)–(v) are now as follows:

(i) $\mathscr{P}^i = 0$ if either $i < 0$ or $i > s/2$,
$\beta\mathscr{P}^i = 0$ if either $i < 0$ or $i \geqslant s/2$.
(ii) $\mathscr{P}^i(x) = x^p$ if $i = s/2$.
(iii) (Cartan Formula) If x has degree (q, ε) and y has degree (q', ε') then
$\mathscr{P}^j(xy) = (-1)^{m\varepsilon\varepsilon'} \sum_i \mathscr{P}^i(x)\mathscr{P}^{j-i}(y)$,
$\beta\mathscr{P}^j(xy) = (-1)^{m\varepsilon\varepsilon'} \sum_i (\beta\mathscr{P}^i(x)\mathscr{P}^{j-i}(y) + (-1)^q \mathscr{P}^i(x)\beta\mathscr{P}^{j-i}(y))$.
(iv) The $\mathscr{P}^i$ and $\beta\mathscr{P}^i$ satisfy the Adem relations.
(v) $\mathscr{P}^i$ is k-semilinear; that is, $\mathscr{P}^i(x+y) = \mathscr{P}^i(x) + \mathscr{P}^i(y)$, and $\mathscr{P}^i(\lambda x) = \lambda^p \mathscr{P}^i(x)$ for $\lambda \in k$ and $x, y \in H^{*,*}(A, k)$.

Note that the Adem relations on elements of odd internal degree also have some extra signs. We have not written these out explicitly, as we do not need them.

Proposition 3.1. *The ring $H^{*,*}(\mathbb{G}_a^-, k)$ is a polynomial ring $k[\zeta]$ on a single generator ζ in degree $(1,1)$. The action of the Steenrod operations on $H^{*,*}(\mathbb{G}_a^-, k)$ is given by $\mathscr{P}^{\frac{1}{2}}(\zeta) = \zeta^p$, $\beta\mathscr{P}^{\frac{1}{2}}(\zeta) = 0$.*

Proof. We prove this by reducing the grading modulo two on a $\mathbb{Z}$-graded cocommutative Hopf algebra. The cohomology of a $\mathbb{Z}$-graded Hopf algebra on a primitive exterior generator in degree one is $k[\zeta]$ with ζ in degree $(1,1)$. If we compute the action of the Steenrod operations on this, the action of $\mathscr{P}^{\frac{1}{2}} = \mathcal{P}^1$ and $\beta\mathscr{P}^{\frac{1}{2}} = \beta\mathcal{P}^1$ follows from Theorem 11.8 (ii) of [18], and is given as in the Proposition. Now reduce the grading modulo two. □

We have

$$H^{*,*}(\mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s, k) = \\ k[\zeta] \otimes k[x_1, \dots, x_r] \otimes \Lambda(\lambda_1, \dots, \lambda_r) \otimes k[z_1, \dots, z_s] \otimes \Lambda(y_1, \dots, y_s).$$

The degrees and action of the Steenrod operations are as follows.

	degree	$\mathscr{P}^0$	$\beta\mathscr{P}^0$	$\mathscr{P}^{\frac{1}{2}}$	$\mathscr{P}^1$
ζ	$(1,1)$			ζ^p	
λ_i	$(1,0)$	λ_{i+1}	$-x_i$		0
y_i	$(1,0)$	y_i	z_i		0
x_i	$(2,0)$	x_{i+1}	0		x_i^p
z_i	$(2,0)$	z_i	0		z_i^p

Here, λ_{i+1} and x_{i+1} are taken to be zero if $i = r$.

4. The local cohomology spectral sequence

In this section we sketch the construction of a generalization of the local cohomology spectral sequence to finite supergroup schemes. The spectral sequence was constructed by Benson and Carlson [6] for finite groups; Greenlees gave a more robust construction in [13]. The supergroup version comes with a twist which we now describe.

Recall from Section I.8 of Jantzen [15] that there is a one dimensional representation δ_G of a finite group scheme G, called the *modular function*, and that by Proposition I.8.13 of [15], if Q is a projective kG-module then $\mathsf{Soc}(Q) \cong Q/\mathsf{Rad}(Q) \otimes \delta_G$. This generalizes to finite supergroup schemes, without change in the argument, the only extra feature being that δ_G comes with a parity $\varepsilon_G \in \mathbb{Z}/2$. So for example $\delta_{\mathbb{G}_a^-}$ is the trivial module, but in odd internal degree, so we have $\varepsilon_{\mathbb{G}_a^-} = 1 \in \mathbb{Z}/2$.

The role of the modular function is that it appears in Tate duality which we deduce from the general statement of Auslander-Reiten duality. The latter gives an isomorphism

$$\mathrm{Hom}_k(\underline{\mathrm{Hom}}_G(M,N),k) \cong \underline{\mathrm{Hom}}_G(N, \Omega\nu M) \tag{4.1}$$

(see [1, Proposition I.3.4], also [17, Corollary p. 269]). Here,

$$\nu : \mathsf{StMod}\, G \to \mathsf{StMod}\, G$$

is the Nakayama functor. For a finite supergroup scheme it is given by the formula

$$\nu(-) = - \otimes_k \delta_G$$

(see [8, Section 4]).

Applying (4.1) to $N, \Omega^{n+1}M$, we get Tate duality for finite supergroup schemes:

$$\begin{aligned}\widehat{\mathrm{Ext}}^{-n-1,*}(N,M) &\cong \underline{\mathrm{Hom}}(N,\Omega^{n+1}M)\\ &\cong \mathrm{Hom}_k(\underline{\mathrm{Hom}}(\Omega^{n+1}M,\Omega\nu N),k)\\ &\cong \mathrm{Hom}_k(\underline{\mathrm{Hom}}(\Omega^{n}M,N\otimes\delta_G),k)\\ &\cong \mathrm{Hom}_k(\widehat{\mathrm{Ext}}^{n,*+\epsilon_G}(M,N\otimes\delta_G),k).\end{aligned}$$

In particular, for $M = N = k$, $n \geqslant 0$, this becomes

$$H_{n,j+\varepsilon_G}(G,\delta_G) \cong \widehat{H}^{-n,j+\varepsilon_G}(G,\delta_G) \cong \mathrm{Hom}_k(\widehat{H}^{n-1,*+\epsilon_G}(G,\delta_G),k). \tag{4.2}$$

The local cohomology spectral sequence is triply graded. The gradings are firstly local cohomological, secondly group cohomological, and thirdly internal parity. Repeating the constructions in [13] or [5] (explicitly, Section 3 in [5]) verbatim up to the point where local duality is used to identify negative Tate cohomology and homology gives the following.

Theorem 4.1. *Let G be a finite supergroup scheme. Then there is a spectral sequence*

$$E_2^{s,t,*} = H^{s,t}_{\mathfrak{m}}H^{*,*}(G,k) \Rightarrow \widehat{H}^{s+t-1<0,*}(G,k)$$

converging to the negative part of Tate cohomology.

Now applying the duality isomorphism (4.2), we produce the local cohomology spectral sequence converging to homology.

Corollary 4.2. *Let G be a finite supergroup scheme. Then there is a local cohomology spectral sequence*

$$E_2^{s,t,j} = H^{s,t,j}_{\mathfrak{m}}H^{*,*}(G,k) \Rightarrow H_{-s-t,j+\varepsilon_G}(G,\delta_G).$$

Definition 4.3. A finite supergroup scheme is called *unimodular* if the modular function δ_G is the trivial module in degree ε_G.

Remark 4.4. Finite unipotent supergroup schemes are unimodular since the only one-dimensional representations are given by the trivial module k in either even or odd degree.

Let r be the Krull dimension of $H^{*,*}(G,k)$. Then $H^{i,*}_{\mathfrak{m}}H^{*,*}(G,k) = 0$ for $i > r$. Hence, there is an edge homomorphism of the local cohomology spectral sequence:

$$H^{r,t,j}_{\mathfrak{m}}H^{*,*}(G,k) \to H_{-r-t,j+\varepsilon_G}(G,\delta_G). \tag{4.3}$$

We wish to use the following consequences of Corollary 4.2. The statement of the first Corollary 4.5 is a direct analogue of [13, Corollary 2.3] (see also [6]).

Corollary 4.5. *Let G be a unimodular finite supergroup scheme. If $H^{*,*}(G,k)$ is Cohen–Macaulay, then it is Gorenstein, with shift $(0,\varepsilon_G)$.*

Proof. Since G is unimodular, δ_G is a trivial module. The Cohen–Macaulay assumption on $H^{*,*}(G,k)$ implies that the edge map of (4.3) is an isomorphism which therefore identifies the top local cohomology $H^r_{\mathfrak{m}}(H^{*,*}(G,k))$ with homology $H_{*,*+\epsilon_G}(G,k)$. This is linear dual to cohomology $H^{*,*+\epsilon_G}(G,k)$, and, hence, the top local cohomology module is the injective hull of the trivial $H^{*,*}(G,k)$-module k in the internal degree ε_G. Hence, $H^{*,*}(G,k)$ is Gorenstein (see [14, Theorem 11.26] or [12, Theorem 1.3.4] where the graded case is made explicit). □

Recall that $\{\zeta_1,\dots,\zeta_r\}$ is a *system of parameters* of a (graded) commutative k-algebra A if $k[\zeta_1,\dots,\zeta_r] \subset A$ is a Noether normalization of A, that is, A is a finite module over $k[\zeta_1,\dots,\zeta_r]$. The last corollary is a general property of graded Gorenstein k-algebras.

Corollary 4.6. *Let G be a unimodular finite supergroup scheme. Assume $H^{*,*}(G,k)$ is Cohen–Macaulay, and let $\zeta_1,\dots,\zeta_r$ be a regular homogeneous sequence of parameters in $H^{*,*}(G,k)$. Then the quotient $H^{*,*}(G,k)/(\zeta_1,\dots,\zeta_r)$ is a finite Poincaré duality algebra with dualizing degree $(-r,\varepsilon_G)+\sum_{i=1}^r|\zeta_i|$.*

5. The case $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$

Let $G=(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$. This is a finite supergroup scheme of height one and, hence, kG is isomorphic to the restricted universal enveloping algebra of the three dimensional Lie superalgebra $\mathfrak{g}$ (see, for example, [11, Lemma 4.4.2]). The Lie superalgebra $\mathfrak{g}$ has a basis consisting of odd elements v_1 and v_2, and an even element t. Specializing calculations in Section 2 to this case, we get that the Lie algebra generators satisfy the following relations

$$[v_1,v_2]=0, \quad [t,v_2]=0, \quad [t,v_1]=v_2$$

where $[\ ,\ ]$ is the supercommutator in $\mathfrak{g}$. Thus kG has the following presentation:

$$kG=\frac{k[v_1,v_2,t]}{(v_1^2,v_2^2,v_1v_2+v_2v_1,t^p,tv_2-v_2t,tv_1-v_1t-v_2)}. \tag{5.1}$$

Theorem 5.1. *Let $G=(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$, with $\mathbb{G}_{a(1)}$ acting non-trivially. Then $H^{*,*}(G,k)$ is generated by*

$$\zeta\in H^{1,1}(G,k),\ x\in H^{2,0}(G,k),\ \kappa\in H^{p,1}(G,k),\ \lambda_i\in H^{i,1+i}(G,k)\ (1\leqslant i\leqslant p-1)$$

with the relations

$$\begin{aligned} \lambda_i\zeta &= 0 && (1\leqslant i\leqslant p-1),\\ x\zeta^{p-1} &= 0, && \\ \lambda_i\lambda_j &= 0 && \text{for } i+j\neq p,\\ \lambda_i\lambda_{p-i} &= \alpha_i x\zeta^{p-2} && \text{where } \alpha_i\neq 0. \end{aligned}$$

The Poincaré series is given by

$$\sum_n t^n \dim_k H^{n,*}(G,k)=1/(1-t)^2.$$

Proof. We examine two spectral sequences, the first one given by the semidirect product:

$$H^i(\mathbb{G}_{a(1)},H^{j,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k))\Rightarrow H^{i+j,*}(G,k).$$

Let $V=kv_1\oplus kv_2$ be the two dimensional supervector space generated by v_1,v_2. We have

$$H^{*,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k)\cong S^{*,*}(V^{\#})=k[\zeta,\eta]$$

with ζ and η in degree $(1,1)$, dual to the generators v_1,v_2. For $0\leqslant j\leqslant p-1$,

$$H^{j,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k)\cong S^j(V^{\#})$$

is an indecomposable $k\mathbb{G}_{a(1)}$-module of length $j+1$. It is projective for $j=p-1$, and not otherwise. Hence, we have the following restrictions on dimensions of the E_2 term of the spectral sequence:

$$\dim H^i(\mathbb{G}_{a(1)},H^{j,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k))=1 \text{ for } 0\leqslant j\leqslant p-1, \tag{5.2}$$

$$\dim H^1(\mathbb{G}_{a(1)},H^{p-1,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k))=0, \tag{5.3}$$

$$\dim H^0(\mathbb{G}_{a(1)},H^{p,*}(\mathbb{G}_a^-\times\mathbb{G}_a^-,k))=2. \tag{5.4}$$

To justify the last equality, we do a calculation:

$$H^0(\mathbb{G}_{a(1)}, H^p(\mathbb{G}_a^- \times \mathbb{G}_a^-, k)) = H^0(\mathbb{G}_{a(1)}, S^p(V^\#)) = k\zeta^p \oplus k\eta^p$$

where the last equality is a special case of Lemma 6.1.

We conclude that

$$\dim H^n(G,k) \leqslant \sum_{i+j=n} \dim H^i(\mathbb{G}_{a(1)}, H^j(\mathbb{G}_a^- \times \mathbb{G}_a^-, k)) = n+1 \tag{5.5}$$

for $0 \leqslant n \leqslant p$.

We now examine the spectral sequence

$$H^{*,*}(\mathbb{G}_a^- \times \mathbb{G}_{a(1)}, H^{*,*}(\mathbb{G}_a^-, k)) \Rightarrow H^{*,*}(G,k)$$

corresponding to the central extension

$$1 \to \mathbb{G}_a^- \to G \to \mathbb{G}_a^- \times \mathbb{G}_{a(1)} \to 1$$

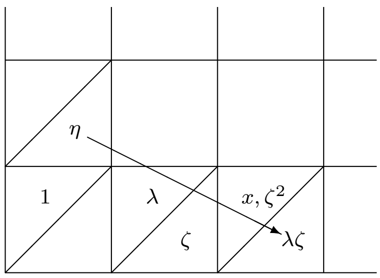
We write

$$\begin{aligned} H^{*,*}(\mathbb{G}_a^- \times \mathbb{G}_{a(1)}, H^{*,*}(\mathbb{G}_a^-, k)) &= H^{*,*}(\mathbb{G}_a^-, k) \otimes H^{*,0}(\mathbb{G}_{a(1)}, k) \otimes H^{*,*}(\mathbb{G}_a^-, k) \\ &= k[\zeta, x] \otimes \Lambda(\lambda) \otimes k[\eta] \end{aligned}$$

with ζ the generator of the first $H^{*,*}(\mathbb{G}_a^-, k)$, x, λ the generators of $H^{*,0}(\mathbb{G}_{a(1)}, k)$, and η the generator of the second $H^{*,*}(\mathbb{G}_a^-, k)$. The degrees of the generators in the spectral sequence are as follows:

$$|\zeta| = (1,0,1), \quad |\lambda| = (1,0,0), \quad |x| = (2,0,0), \quad |\eta| = (0,1,1).$$

Here, the first two indices are the horizontal and vertical directions in the spectral sequence, and the third is the $\mathbb{Z}/2$-grading.



Let $\mathbb{G}_m$ be the multiplicative group scheme. Then $\mathbb{G}_m \times \mathbb{G}_m$ acts on kG (given by the presentation in (5.1)) in such a way that the first copy is acting on v_1 and the second is acting on t. Both copies act on the commutator v_2. Each monomial in the E_2 page of this spectral sequence is then an eigenvector of $\mathbb{G}_m \times \mathbb{G}_m$. The weights are elements of $\mathbb{Z} \times \mathbb{Z}$, and are given by

$$\|\zeta\| = (1,0), \qquad \|\lambda\| = (0,1), \qquad \|x\| = (0,p), \qquad \|\eta\| = (1,1).$$

The differentials in the spectral sequence have to preserve both the weight and the $\mathbb{Z}/2$-grading. The latter implies that x, ζ^2 cannot be hit by $d_2(\eta)$ and, hence, survive to E_∞. Since $\dim H^{1,*}(G,k) \leqslant 2$ by (5.2), we conclude that η must die in E_∞, and hence $d_2(\eta)$ is a non-zero multiple of $\lambda\zeta$. By the Newton–Leibniz rule, we get that a monomial $\lambda^\epsilon \eta^{i_1} x^{i_2} \zeta^{i_3}$ does not survive in E_3 if

$$\{\epsilon = 0 \text{ and } 1 \leqslant i_1 \leqslant p-1\}, \tag{5.6}$$

in which case it is not in the kernel of d_2 or if

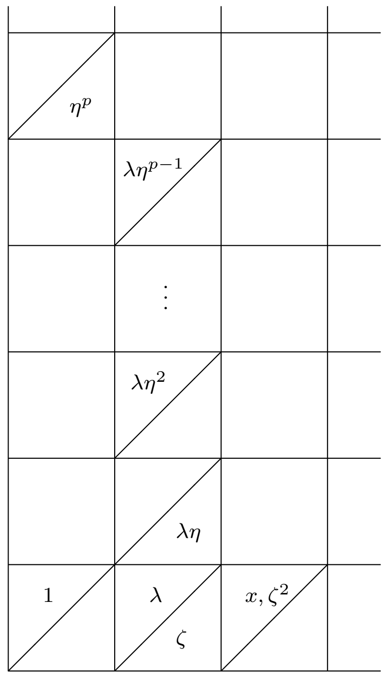
$$\{\epsilon = 1, 0 \leqslant i_1 \leqslant p-2 \text{ and } i_3 \geqslant 1\} \tag{5.7}$$

in which case it is in the image of d_2. On the other hand, $d_2(\eta^p) = 0$.

We conclude that the E_3 page is generated by the permanent cycles λ, ζ and x on the base, the element η^p on the fibre, and $\lambda\eta, \lambda\eta^2, \ldots, \lambda\eta^{p-1}$ in the first column. Moreover, E_3 has the relations

$$(\lambda\eta^i)\zeta = 0, \quad (\lambda\eta^i)(\lambda\eta^j) = 0 \tag{5.8}$$

for $1 \leqslant i, j \leqslant p-2$. Since $\mathscr{P}^{\frac{1}{2}}(\eta) = \eta^p$ and $\mathscr{P}^{\frac{1}{2}}(\lambda\zeta) = 0$, Kudo's transgression theorem ([18, Theorem 3.4]) implies that η^p survives to the E_∞ page of the spectral sequence.



E_3 page

There remains the question of the values of the differentials $d_3, \ldots, d_p$ on the elements $\lambda\eta, \ldots, \lambda\eta^{p-1}$.

Claim 5.2. *The differentials $d_3, \ldots, d_{p-1}$ vanish on the elements $\lambda\eta, \ldots, \lambda\eta^{p-1}$, and $d_p(\lambda\eta^{p-1})$ is a multiple of $x\zeta^{p-1}$.*

Proof of Claim. Suppose some differential d_ℓ is non trivial on $\lambda\eta^i$ and let $\lambda^\varepsilon\eta^{i_1}x^{i_2}\zeta^{i_3}$ be in the target of that differential. If $i_1 \neq 0$, then (5.6), (5.7) imply that $i_3 = 0$ and $\varepsilon = 1$. Hence $\lambda\eta^i$ hits a monomial of the form $\lambda\eta^{i_1}x^{i_2}$. The weights are $\|\lambda\eta^i\| = (i, i+1)$ and $\|\lambda\eta^{i_1}x^{i_2}\| = (i_1, 1 + i_1 + i_2 p)$. Since the weights are preserved, we conclude $i = i_1$, which contradicts the fact that d_ℓ must lower the exponent of η by $\ell - 1$.

Therefore, $i_1 = 0$, and the differential d_ℓ on $\lambda\eta^i$ hits something on the base, a monomial of the form $\lambda^\varepsilon x^{i_2}\zeta^{i_3}$. The weights are $\|\lambda\eta^i\| = (i, i+1)$ and $\|\lambda^\varepsilon x^{i_2}\zeta^{i_3}\| = (i_3, \varepsilon + i_2 p)$. Hence, $i_3 = i > 0$. By (5.6) and (5.7), we have $\varepsilon = 0$. The condition on the total degree gives $i + 1 = 2i_2 + i - 1$, and so $i_2 = 1$. The condition on the second weight gives $i + 1 = i_2 p$, and so $i = p - 1$. Thus the only possible non trivial differential is $d_p(\lambda\eta^{p-1})$, which is some multiple of $x\zeta^{p-1}$. This proves the claim. $\square$

Claim 5.2 immediately implies that $\lambda\eta, \ldots, \lambda\eta^{p-2}$ are (non-trivial) permanent cycles. We also conclude that all differentials up to d_{p-1} vanish on all generators of E_3. Hence, $E_3 = E_p$. It remains to determine the differential d_p on $\lambda\eta^{p-1}$.

Claim 5.3. *$d_p(\lambda\eta^{p-1})$ is a non-zero multiple of $x\zeta^{p-1}$.*

Proof of Claim. We have $\dim H^{p,*}(G,k) \leqslant p+1$ by (5.2). On the other hand, we established at least $p+1$ linearly independent cycles of total degree p in E_∞:

$$\{\eta^p, \lambda x^{\frac{p-1}{2}}, \lambda\eta^2 x^{\frac{p-3}{2}}, \ldots, \lambda\eta^{p-3}x, \zeta^p, x\zeta^{p-2}, \ldots, x^{\frac{p-1}{2}}\zeta\}.$$

Hence, $\lambda\eta^{p-1}$ is not a permanent cycle, since otherwise we would have $\dim H^{p,*}(G,k) \geqslant p+2$. We have already computed that $d_p(\lambda\eta^{p-1})$ is a multiple of $x\zeta^{p-1}$. This proves the claim. □

This completes the determination of the E_∞ page of the spectral sequence of the central extension. We also conclude that $x\zeta^{p-1}$ is zero in $H^{p+1,0}(G,k)$.

To describe the cohomology ring $H^{*,*}(G,k)$, we start by giving elements of $E_\infty^{*,0}$ the same names in $H^{*,*}(G,k)$. We choose a representative $\kappa \in H^{p,1}(G,k)$ of $\eta^p \in E_\infty^{0,p}$; this is a non zero-divisor. Next, choose $\lambda_2, \ldots, \lambda_{p-1}$ to be representatives in $H^{*,*}(G,k)$ of the elements $\lambda\eta, \ldots, \lambda\eta^{p-2}$ in E_∞, as follows. Arguing as before, we see that there is only one dimension in each of these degrees with the correct weight for the action of $\mathbb{G}_m \times \mathbb{G}_m$, so this gives a well defined representative. We also write λ_1 for λ. Thus we have $\|\lambda_i\| = (i-1, i)$.

Using weights, we see that the product $\lambda_i\zeta$ is equal to zero. Similarly, $\lambda_i\lambda_j$ is either zero or a multiple of $x\zeta^{p-2}$, and the latter can only happen when $i+j=p$. In the case where $i+j=p$, we claim that $\lambda_i\lambda_j$ is a non-zero multiple of $x\zeta^{p-2}$. The proof of this claim uses the local cohomology spectral sequence in the form of Corollary 4.6, and this will complete the computation of $H^{*,*}(G,k)$.

Claim 5.4. *$H^{*,*}(G,k)$ is Cohen–Macaulay. A regular homogeneous sequence of parameters is given by $\kappa \in H^{p,1}(G,k)$ and $x+\zeta^2 \in H^{2,0}(G,k)$.*

Proof of Claim. It suffices to show that η^p and $x+\zeta^2$ form a regular sequence in E_∞. Since η^p is a non zero-divisor, this amounts to showing that $x+\zeta^2$ is a non zero-divisor on $E_\infty/(\eta^p)$. The non-zero monomials in $E_\infty/(\eta^p)$ come in two types. The first are the $\zeta^i x^j$ with $j=0$ if $i \geqslant p-1$. Multiplying by $x+\zeta^2$, these go to $\zeta^{i+2}x^j + \zeta^i x^{j+1}$, where the second term is zero if $i \geqslant p-1$ and $j=0$. Ordering lexicographically in (i,j), we see that these are linearly independent, because their leading terms are linearly independent. The second kind of monomials are the $\lambda_i x^j$ with $1 \leqslant i \leqslant p-1$. These go to $\lambda_i x^{j+1}$, which are again linearly independent. □

We are now in a position to complete the proof of Theorem 5.1. Since $H^{*,*}(G,k)$ is Cohen–Macaulay, Corollary 4.6 implies that $H^{*,*}(G,k)/(\kappa, x+\zeta^2)$ has Poincare duality with dualizing element in degree p. The ring $H^{*,*}(G,k)/(\kappa, x+\zeta^2)$ has a basis consisting of $\zeta^i \in H^{i,i}(G,k)$ with $0 \leqslant i \leqslant p$ and $\lambda_i \in H^{i,i-1}(G,k)$ with $1 \leqslant i \leqslant p-1$ (where the second degree is taken mod 2). The top degree dualizing element is ζ^p, which is equivalent modulo $x+\zeta^2$ to $x\zeta^{p-2}$. For each element in $H^{*,*}(G,k)/(\kappa, x+\zeta^2)$ there has to be an element whose product with it is equal to the dualizing element. Applying this to λ_i, we see that $\lambda_i\lambda_{p-i}$ has to be non-zero, and is therefore a non-zero multiple of $x\zeta^{p-2}$. Replacing some of the λ_i by non-zero multiples, we have $\lambda_i\lambda_{p-i} = x\zeta^{p-2}$. We now have all the generators and relations for the ring structure on $H^{*,*}(G,k)$, completing the proof of Theorem 5.1. □

Corollary 5.5. *With G as in Theorem 5.1, we have*

$$\sum_{n\geqslant 0} t^n \dim_k H^{n,*}(G,k) = 1/(1-t)^2.$$

Proof. This is an easy dimension count using the theorem. □

To analyze the case of a more general semidirect product as we do in Section 8, we don't need the force of Theorem 5.1 but only a particular calculation which was obtained as part of the proof.

Corollary 5.6 (of the proof)**.** *In the notation of the proof of Theorem 5.1, we have that $d_p(\lambda\eta^{p-1})$ is a non-zero multiple of $x\zeta^{p-1}$.*

Remark 5.7. The group algebra of the semidirect product $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{Z}/p$ is generated by elements v_1, v_2 and g satisfying $v_1^2 = 0$, $v_2^2 = 0$, $g^p = 1$, $v_1v_2 + v_2v_1 = 0$, $gv_1 = (v_1+v_2)g$, $gv_2 = v_2g$. Writing t for $g-1$, this becomes

$$v_1^2 = v_2^2 = v_1v_2 + v_2v_1 = t^p = 0, \quad tv_2 = v_2t, \quad tv_1 = v_1t + v_2 + v_2t.$$

Substituting $v_2' = v_2 + v_2t$ then gives the presentation of the group algebra studied in this section. Since the cohomology only depends on the algebra structure, not on the comultiplication, we get the same answer as in the case of $(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes \mathbb{G}_{a(1)}$ computed in this section.

6. An invariant theory computation

Let $H = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$, acting on $\mathbb{G}_a^- \times \mathbb{G}_a^-$ as in Section 2, and let G be the semidirect product. In preparation for the computation of $H^{*,*}(G,k)$, we begin with an invariant theory computation.

We have $H^{*,*}(\mathbb{G}_a^- \times \mathbb{G}_a^-, k) \cong k[X,Y]$ where X and Y are in degree $(1,1)$. We choose the notation so that Y is fixed by this action, and X is sent to X plus multiples of Y. In this section, we compute the invariants of such an action. To this end, we consider $k[X,Y]$ to be the ring of polynomial functions on the vector space V with basis v_1 and v_2, so that Y and X form the dual basis of the linear functions on V.

We begin with the case $s = 0$, namely $H = \mathbb{G}_{a(r)}$. In general, an action of a group scheme G on a scheme Z over a scheme S, is given by a map $G \times_S Z \to Z$ satisfying the usual associative law defining an action. Corresponding to this is a map of coordinate rings $k[Z] \to k[G] \otimes_{k[S]} k[Z]$ giving the coaction of $k[G]$ on $k[Z]$. Then the fixed points $k[Z]^G$ is the subring of $k[Z]$ consisting of those f whose image in $k[G] \otimes_{k[S]} k[Z]$ under the comodule maps is equal to $1 \otimes f$.

In our case, we have $k[\mathbb{G}_{a(r)}] = k[t]/(t^{p^r})$ with t a primitive element in the Hopf structure. The action $\mathbb{G}_{a(r)}$ on V corresponds to a map $\mathbb{G}_{a(r)} \times_{\mathrm{Spec}\,k} V \to V$, and then to a map of coordinate rings $k[X,Y] \to k[t]/(t^{p^r}) \otimes k[X,Y]$. The fact that Y is fixed by the action implies that Y maps to $1 \otimes Y$. The fact that X is sent to X plus multiples of Y, together with the identities describing a coaction, imply that X maps to an element of the form $f(t) \otimes Y + 1 \otimes X$ where f is a linear combination of the t^{p^i} with $0 \leqslant i < r$. Faithfulness of the action then implies that the term with $i = 0$ is non-zero. Thus $f(t)$ is primitive, and there is an automorphism of $\mathbb{G}_{a(r)}$ sending $f(t)$ to t. So without loss of generality, the action is given by $X \mapsto t \otimes Y + 1 \otimes X$.

Lemma 6.1. *The invariants of the action of $\mathbb{G}_{a(r)}$ on $k[X,Y]$ are given by*

$$k[X,Y]^{\mathbb{G}_{a(r)}} = k[X^{p^r}, Y].$$

Proof. This is an easy computation. □

Next we describe the case $r = 0$, namely $H = \langle g_1, \dots, g_s\rangle \cong (\mathbb{Z}/p)^s$ with the g_i commuting elements of order p. In this case, the action again fixes Y, and we have $g_i(X) = X - \mu_i Y$ $(1 \leqslant i \leqslant s)$. The fact that the action is faithful is equivalent to the statement that the field elements μ_i are linearly independent over the ground field $\mathbb{F}_p$. Then the orbit product

$$\phi(X,Y) = \prod_{g\in(\mathbb{Z}/p)^s} g(X) = \prod_{(a_1,\dots,a_s)\in(\mathbb{F}_p)^s} X + (a_1\mu_1 + \dots + a_s\mu_s)Y$$

is clearly an invariant.

Lemma 6.2. *The invariants of* $(\mathbb{Z}/p)^s$ *on* $k[X,Y]$ *are given by*

$$k[X,Y]^{(\mathbb{Z}/p)^s} = k[\phi(X,Y),Y],$$

where $\phi(X,Y)$ *is given above.*

Proof. See for example Proposition 2.2 of Campbell, Shank and Wehlau [9]. □

Putting these together, we have the following theorem.

Theorem 6.3. *The invariants of* $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ *on* $k[X,Y]$ *are given by*

$$k[X,Y]^{\mathbb{G}_{a(r)}\times(\mathbb{Z}/p)^s} = k[\phi(X,Y)^{p^r},Y].$$

Proof. This follows by applying first Lemma 6.2 and then Lemma 6.1. □

7. Structure of symmetric powers

We can use the computation of the last section to help us understand the structure of the polynomial functions on the two dimensional space V, as a module for $H = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$. Note that the space of polynomials of degree n is $S^n(V^\#)$, and has a basis consisting of the monomials X^iY^{n-i} for $0 \leqslant i \leqslant n$. In particular, the dimension of $S^n(V^\#)$ is $n+1$.

Lemma 7.1. *Let M be a kH-module whose fixed points M^H are one dimensional. Then M is indecomposable and* $\dim_k(M) \leqslant p^{r+s}$*, with equality if and only if M is projective.*

Proof. Since H is unipotent, kH is a local self-injective algebra. So if M^H is one dimensional, then the injective hull of M is kH. Since kH has dimension p^{r+s}, the lemma follows. □

Theorem 7.2. *For $n < p^{r+s} - 1$, the symmetric nth power $S^n(V^\#)$ is a non-projective indecomposable kH-module. The module $S^{p^{r+s}-1}(V^\#)$ is a free kH-module of rank one.*

Proof. It follows from Theorem 6.3 that $S^n(V^\#)^H$ is one dimensional for $n \leqslant p^{r+s} - 1$. The theorem therefore follows from Lemma 7.1. □

Definition 7.3. Let $f(X,Y) = \sum_{i=0}^n a_iX^iY^{n-i}$ be a degree n homogeneous polynomial in X and Y. Then the *leading term* of f is the term $a_iX^iY^{n-i}$ for the largest value of i with $a_i \neq 0$.

Theorem 7.4. *For $n \geqslant p^{r+s}$, we have* $S^n(V^\#) \cong kH \oplus S^{n-p^{r+s}}(V^\#)$.

Proof. Consider the map $S^{p^{r+s}-1}(V^\#) \to S^n(V^\#)$ given by multiplication by $Y^{n+1-p^{r+s}}$, and the map $S^{n-p^{r+s}}(V^\#) \to S^n(V^\#)$ given by multiplication by $\phi(X,Y)$. Examining the leading terms of the images of monomials under these maps, we see that these maps are injective, the images span and intersect in zero. Therefore $S^n(V^\#)$ is an internal direct sum of $Y^{n+1-p^{r+s}}.S^{p^{r+s}-1}(V^\#)$ and $\phi(X,Y).S^{n-p^{r+s}}(V^\#)$. By Theorem 7.2, the first summand is isomorphic to kH. □

Corollary 7.5. *The kH-module $S^n(V^\#)$ is projective if and only if n is congruent to -1 modulo p^{r+s}.* $\square$

Next, we examine the modules $S^{p^i-1}(V^\#)$ with $1 \leqslant i < r+s$. We have seen that these modules are not projective, but we shall show that the complexity is exactly $r+s-i$, and we shall identify the annihilator of cohomology. The method we use is a variation of the Steinberg tensor product theorem. We also use Carlson's theory of rank varieties, see for example Carlson [10] or §5.8 of [4].

Lemma 7.6. *The kH-module $S^{p-1}(V^\#)$ is a uniserial module whose rank variety is the hyperplane consisting of the points $(\gamma_1, \dots, \gamma_r, \alpha_1, \dots, \alpha_s) \in \mathbb{A}^{r+s}(k)$ such that*

$$-\gamma_1 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s = 0.$$

Proof. We have

$$\begin{aligned} s_1(X^i) &= iX^{i-1}Y \\ s_j(X^i) &= 0 && 2 \leqslant j \leqslant r \\ (g_j - 1)(X^i) &= (X - \mu_j Y)^i - X^i = -i\mu_j X^{i-1}Y + \cdots && 1 \leqslant j \leqslant s \end{aligned}$$

and so if $(\gamma_1, \dots, \gamma_r, \alpha_1, \dots, \alpha_s) \in \mathbb{A}^{r+s}(k) \setminus \{0\}$ then

$$\begin{aligned} &(\gamma_1 s_1 + \cdots + \gamma_r s_r + \alpha_1(g_1 - 1) + \cdots + \alpha_s(g_s - 1))(X^{p-1}) \\ &\qquad = (-\gamma_1 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s)X^{p-2}Y + \cdots \end{aligned}$$

Continuing this way, we have

$$\begin{aligned} &(\gamma_1 s_1 + \cdots + \gamma_r s_r + \alpha_1(g_1 - 1) + \cdots + \alpha_s(g_s - 1))^i(X^{p-1}) \\ &\qquad = i!(-\gamma_1 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s)^i X^{p-1-i}Y^i + \cdots \end{aligned}$$

and finally

$$\begin{aligned} &(\gamma_1 s_1 + \cdots + \gamma_r s_r + \alpha_1(g_1 - 1) + \cdots + \alpha_s(g_s - 1))^{p-1}(X^{p-1}) \\ &\qquad = -(-\gamma_1 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s)^{p-1}Y^{p-1}. \end{aligned}$$

So the restriction to the shifted subgroup defined by $(\gamma_1, \dots, \gamma_r, \alpha_1, \dots, \alpha_s)$ is projective if and only if $-\gamma_1 + \alpha_1\mu_1 + \cdots + \alpha_s\mu_s \neq 0$.

Since there is a non-trivial shifted subgroup such that the restriction is projective, it follows that the module is uniserial. $\square$

Lemma 7.7. *For $1 \leqslant i \leqslant r+s$ the kH-module $S^{p^i-1}(V^\#)$ is isomorphic to the tensor product of Frobenius twists*

$$S^{p-1}(V^\#) \otimes S^{p-1}(V^\#)^{(1)} \otimes \cdots \otimes S^{p-1}(V^\#)^{(i-1)}.$$

Proof. We regard $S^{p-1}(V^\#)^{(j)}$ as the linear span of the p^jth powers of the elements of $S^{p-1}(V^\#)$. Examining monomials, it is apparent that multiplication provides the required isomorphism from the tensor product to $S^{p^i-1}(V^\#)$. $\square$

Theorem 7.8. *For $1 \leqslant i \leqslant r+s$ the rank variety of the module $S^{p^i-1}(V^\#)$ is the linear subspace of $\mathbb{A}^{r+s}$ defined by the first i rows of the $(r+s)\times(r+s)$ matrix*

$$\left(\begin{array}{cccc|ccc}
-1 & 0 & \cdots & 0 & \mu_1 & \cdots & \mu_s \\
0 & -1 & & 0 & \mu_1^p & & \mu_s^p \\
0 & 0 & & 0 & \mu_1^{p^2} & & \mu_s^{p^2} \\
\vdots & & & \vdots & \vdots & & \vdots \\
0 & 0 & & -1 & \mu_1^{p^{r-1}} & \cdots & \mu_s^{p^{r-1}} \\
\hline
0 & 0 & & 0 & \mu_1^{p^r} & & \mu_s^{p^r} \\
\vdots & & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \mu_1^{p^{r+s-1}} & & \mu_s^{p^{r+s-1}}
\end{array}\right)$$

The rows of this matrix are linearly independent, so the complexity of $S^{p^i-1}(V)$ is $r+s-i$.

Proof. It follows from Lemma 7.6 that the rank variety of $S^{p-1}(V)^{(i)}$ is the hyperplane given by the vanishing of the ith row of the above matrix. Now apply Lemma 7.7.

Now by the usual Vandermonde argument, given elements $a_1, \dots, a_s \in k$, the determinant of the matrix

$$\begin{pmatrix} a_1 & \cdots & a_s \\ a_1^p & & a_s^p \\ \vdots & & \vdots \\ a_1^{p^{s-1}} & & a_s^{p^{s-1}} \end{pmatrix}$$

is, up to non-zero scalar, the product of the non-trivial $\mathbb{F}_p$-linear combinations of $a_1, \dots, a_s$, one from each one dimensional subspace (this kind of matrix is called a Moore matrix). It therefore vanishes if and only if they are linearly dependent over $\mathbb{F}_p$.

Applying this to the lower right corner of the matrix in the theorem, the linear independence of the rows of this matrix follows using the fact that the μ_i are linearly independent over $\mathbb{F}_p$. Alternatively, this can be deduced from Theorem 7.2. □

Proposition 7.9. *Let M be a p-dimensional uniserial kH-module. Then there is a subalgebra A of kH of dimension p^{r+s-1} with the following properties:*

(i) *kH is flat as an A-module,*
(ii) *the restriction of M to A is a direct sum of p copies of k with trivial action, and*
(iii) *M is isomorphic to $kH \otimes_A k$ as a kH-module.*

Proof. Let $I \subseteq kH$ be the annihilator of M. Then I is an ideal of codimension p, and M is isomorphic to kH/I. Furthermore, for $n \geqslant 0$ we have $\mathsf{Rad}^n(M) = J^n(kH).M$, and so $M/\mathsf{Rad}^n(M) \cong kH/(I + J^n(kH))$. Since $M/\mathsf{Rad}^2(M)$ has dimension two, so does $kH/(I+J^2(kH))$, and therefore $(I+J^2(kH))/J^2(kH)$ has dimension $r+s-1$. As a vector space, this is isomorphic to $I/(I \cap J^2(kH))$. Choose elements $u_1, \dots, u_{r+s-1} \in I$ which are linearly independent modulo $J^2(kH)$, and let $A = k[u_1, \dots, u_{r+s-1}] \subseteq kH$. Then kH is flat as an A-module, and A acts trivially on M. So we have $\dim_k \operatorname{Hom}_A(k, M) = p$, and therefore

$$\dim_k \operatorname{Hom}_{kH}(kH \otimes_A k, M) = p.$$

Similarly, we have

$$\dim_k \operatorname{Hom}_{kH}(kH \otimes_A k, \mathsf{Rad}(M)) = p - 1.$$

There is therefore a homomorphism from $kH \otimes_A k$ to M whose image does not lie in $\mathsf{Rad}(M)$. Both modules are uniserial of length p, so such a homomorphism is necessarily an isomorphism. □

Theorem 7.10. (i) *There exists a flat embedding $A \to kH$ of a subalgebra A of dimension p^{r+s-1} and an isomorphism $S^{p-1}(V^\#) \cong kH \otimes_A k$.*

(ii) *The cohomology $H^*(kH, S^{p-1}(V^\#))$ is annihilated by*

$$-x_1 + \mu_1^p z_1 + \cdots + \mu_r^p z_r.$$

(iii) *More generally, for $1 \leqslant i \leqslant r+s$, there exists a flat embedding $A_i \to kH$ of a subalgebra A_i of dimension p^{r+s-i} and an isomorphism $S^{p^i-1}(V^\#) \cong kH \otimes_{A_i} k$. The cohomology $H^*(kH, S^{p^i-1}(V^\#))$ is annihilated by the first i elements of the regular sequence*

$$\begin{array}{llll}
-x_1 & & & + \mu_1^p z_1 + \cdots + \mu_r^p z_r \\
& -x_2 & & + \mu_1^{p^2} z_1 + \cdots + \mu_r^{p^2} z_r \\
& & \ddots & \quad \cdots \\
& & & -x_r + \mu_1^{p^r} z_1 + \cdots + \mu_r^{p^r} z_r \\
& & & \quad \mu_1^{p^{r+1}} z_1 + \cdots + \mu_r^{p^{r+1}} z_r \\
& & & \quad \cdots \\
& & & \quad \mu_1^{p^{r+s}} z_1 + \cdots + \mu_r^{p^{r+s}} z_r.
\end{array}$$

Proof. (i) This follows from Lemma 7.6 and Proposition 7.9.

(ii) The annihilator of cohomology consists of the elements of cohomology of H whose restriction to A is zero, and is therefore generated by a degree one element and its image under $\beta\mathscr{P}^0$. Taking into account the Frobenius twist in the relationship between rank variety and cohomology variety for an elementary abelian p-group, the statement follows from Lemma 7.6.

(iii) This follows in the same way, using Lemma 7.7 and Theorem 7.8. □

8. Proof of the main theorem

In this section, we prove Theorem 8.1, using the results of the previous sections.

Theorem 8.1. *Let G be the semidirect product*

$$(\mathbb{G}_a^- \times \mathbb{G}_a^-) \rtimes H$$

where $H = \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ acts faithfully. Then there is a non-zero element $\zeta \in H^{1,1}(G,k)$ such that for all $u \in H^{1,0}(G,k)$ we have $\beta\mathscr{P}^0(u).\zeta^{p^{r+s-1}(p-1)} = 0$.

Proof. In contrast with the case $H = \mathbb{G}_{a(1)}$ studied in Section 5, for more general H we only have one copy of $\mathbb{G}_m$ acting as automorphisms. This acts by scalar multiplication on the generators v_1 and v_2 of $k(\mathbb{G}_a^- \times \mathbb{G}_a^-)$ and centralizes H. So it also acts by scalar multiplication on the generators ζ and η in $H^{1,1}(\mathbb{G}_a^- \times \mathbb{G}_a^-, k) = k[\zeta, \eta]$. As in Section 5 we use weights in $\mathbb{Z}$ for this action. So $\|\zeta\| = \|\eta\| = 1$, and everything in $H^{*,*}(H,k)$ has weight zero.

We compare two spectral sequences. The first is the spectral sequence

$$H^{*,*}(\mathbb{G}_a^- \times H, H^{*,*}(\mathbb{G}_a^-, k)) \Rightarrow H^{*,*}(G,k), \tag{8.1}$$

associated with the central extension

$$1 \to \mathbb{G}_a^- \to G \to \mathbb{G}_a^- \times H \to 1$$

The second is the spectral sequence of the semidirect product

$$H^{*,*}(H, H^{*,*}(\mathbb{G}_a^- \times \mathbb{G}_a^-, k)) \Rightarrow H^{*,*}(G,k). \tag{8.2}$$

As in Section 5, the differentials in these spectral sequences have to preserve weights for the action of $\mathbb{G}_m$.

In the first spectral sequence (8.1), we have

$$d_2(\eta) = (\lambda_1 + \mu_1 y_1 + \cdots + \mu_s y_s)\zeta.$$

Applying the Kudo transgression theorem, we get

$$d_{p+1}(\eta^p) = \mathscr{P}^{\frac{1}{2}} d_2(\eta) = (\lambda_2 + \mu_1^p y_1 + \cdots + \mu_s^p y_s)\zeta^p.$$

Continuing this way,

$$\begin{array}{rl} d_2(\eta) = & (\lambda_1 + \mu_1 y_1 + \dots + \mu_s y_s)\zeta. \\ d_{p+1}(\eta^p) = & (\lambda_2 + \mu_1^p y_1 + \dots + \mu_s^p y_s)\zeta^p \\ \dots & \ddots \quad \dots \\ d_{p^{r-1}+1}(\eta^{p^{r-1}}) = & (\lambda_r + \mu_1^{p^{r-1}} y_1 + \cdots + \mu_s^{p^{r-1}} y_s)\zeta^{p^{r-1}} \\ d_{p^r+1}(\eta^{p^r}) = & (\mu_1^{p^r} y_1 + \dots + \mu_s^{p^r} y_s)\zeta^{p^r} \\ \dots & \dots \\ d_{p^{r+s-1}+1}(\eta^{p^{r+s-1}}) = & (\mu_1^{p^{r+s-1}} y_1 + \cdots + \mu_s^{p^{r+s-1}} y_s)\zeta^{p^{r+s-1}} \end{array}$$

and finally $d_{p^{r+s}}(\eta^{p^{r+s}})$ is in the ideal generated by the previous ones, so $\eta^{p^{r+s}}$ is a universal cycle.

Applying Corollary 5.6 to the restriction of the first spectral sequence (8.1) to the semidirect product of $\mathbb{G}_a^- \times \mathbb{G}_a^-$ with a minimal subgroup of H we conclude that

$$d_p((\lambda_1 + \mu_1 y_1 + \cdots + \mu_s y_s)\eta^{p-1}) \tag{8.3}$$

is non-zero. It has to be something of weight $p-1$, and is therefore something times ζ^{p-1}.

Now, in the second spectral sequence (8.2), Theorem 7.10 shows that the element

$$-x_1 + \mu_1^p z_1 + \cdots + \mu_s^p z_s$$

on the base annihilates ζ^{p-1} on the fibre in the E_2 page. This means that in $H^{*,*}(G,k)$, this product is zero modulo smaller powers of ζ.

Putting these two pieces of information together, we see that (8.3) has to be a non-zero multiple of $(-x_1 + \mu_1^p z_1 + \cdots + \mu_s^p z_s)\zeta^{p-1}$. Therefore, in $H^{*,*}(G,k)$ we have the relation

$$(-x_1 + \mu_1^p z_1 + \cdots + \mu_s^p z_s)\zeta^{p-1} = 0.$$

We now apply Steenrod operations to this relation to obtain further relations. Applying $\mathscr{P}^{\frac{p-1}{2}}$, we obtain

$$(-x_2 + \mu_1^{p^2} z_1 + \cdots + \mu_s^{p^2} z_s)\zeta^{p^2-p} = 0.$$

Continuing this way, applying $\mathscr{P}^{\frac{p(p-1)}{2}}, \mathscr{P}^{\frac{p^2(p-1)}{2}}, \dots$ we have

$$\begin{aligned}
(-x_1 + \qquad\qquad \mu_1^p z_1 + \cdots + \mu_s^p z_s)\zeta^{p-1} &= 0\\
(-x_2 + \qquad \mu_1^{p^2} z_1 + \cdots + \mu_s^{p^2} z_s)\zeta^{p(p-1)} &= 0\\
\cdots \qquad\qquad \cdots&\\
(-x_r + \mu_1^{p^r} z_1 + \cdots + \mu_s^{p^r} z_s)\zeta^{p^{r-1}(p-1)} &= 0\\
(\mu_1^{p^{r+1}} z_1 + \cdots + \mu_s^{p^{r+1}} z_s)\zeta^{p^r(p-1)} &= 0\\
\cdots&\\
(\mu_1^{p^{r+s}} z_1 + \cdots + \mu_s^{p^{r+s}} z_s)\zeta^{p^{r+s-1}(p-1)} &= 0.
\end{aligned}$$

Every linear combination of $x_1, \dots, x_r, z_1, \dots, z_s$ is spanned by the coefficients of the powers of ζ. In particular, this shows that every $x_i\zeta^{p^{r+s-1}(p-1)}$ and every $z_i\zeta^{p^{r+s-1}(p-1)}$ is zero in $H^{*,*}(G,k)$. This completes the proof. □

Finally, we deduce a corollary to be used in [7].

Corollary 8.2. *Let G be a finite unipotent supergroup scheme, with a normal sub-supergroup scheme N such that $G/N \cong \mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$. If the inflation map $H^{1,*}(G/N,k) \to H^{1,*}(G,k)$ is an isomorphism and $H^{2,1}(G/N,k) \to H^{2,1}(G,k)$ is not injective then there exists a non-zero element $\zeta \in H^{1,1}(G,k)$ such that for all $u \in H^{1,0}(G,k)$ we have $\beta\mathscr{P}^0(u).\zeta^{p^{r+s-1}(p-1)} = 0$.*

Remark 8.3. The condition that the inflation map is an isomorphism on $H^{1,*}$ effectively decodes the fact that G/N is the maximal quotient of prescribed form. See [7] for more details on how it arises.

Proof. Recall that

$$H^{*,*}(G/N,k) \cong k[\zeta] \otimes k[x_1,\dots,x_r] \otimes \Lambda(\lambda_1,\dots,\lambda_r) \otimes k[z_1,\dots,z_s] \otimes \Lambda(y_1,\dots,y_s)$$

with ζ in degree $(1,1)$ and the rest of the generators in even internal degree. If the inflation map $H^{2,1}(G/N,k) \to H^{2,1}(G,k)$ is not an isomorphism then the kernel contains an element of the form $u\zeta$ with $u \in H^{1,0}(G/N,k)$, $\zeta \in H^{1,1}(G/N,k)$. The five term sequence corresponding to the extension $1 \to N \to G \to G/N \to 1$,

$$H^{1,1}(G/N,k) \longrightarrow H^{1,1}(G,k) \longrightarrow H^{1,1}(N,k)^G \xrightarrow{d_2} H^{2,1}(G/N,k) \longrightarrow H^{2,1}(G,k)$$

gives an element $0 \neq \eta \in H^{1,1}(N,k)^G$ such that $d_2(\eta) = u\zeta$. Now $H^{1,1}(N,k) \cong \mathrm{Hom}(N,\mathbb{G}_a^-)$ (see [7, Lemma 4.1]), so corresponding to η there is a G-invariant surjective homomorphism $N \to \mathbb{G}_a^-$. Letting $N_1 \leqslant N$ be the kernel of this homomorphism, it follows that N_1 is normal in G. Looking at the map of five term sequences given by factoring out N_1, we see that we might as well replace G by G/N_1 and N by N/N_1, since the hypotheses of the corollary are preserved, and the conclusion for G/N_1 inflates to the same conclusion for G.

We are left in a situation where we have a short exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 1\\
 & & \downarrow\cong & & \downarrow = & & \downarrow\cong & & \\
1 & \longrightarrow & \mathbb{G}_a^- & \longrightarrow & G & \longrightarrow & \mathbb{G}_a^- \times \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s & \longrightarrow & 1.
\end{array}$$

The fact that $d_2(\eta) = u\zeta$ means that the restrictions of $d_2(\eta)$ to the two factors $\mathbb{G}_a^-$ and $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ of the quotient are both zero. So the restriction of the extension to these two factors gives abelian subgroups. It is then easy to see that the restricted extensions split, and so G has subgroups $\mathbb{G}_a^- \times \mathbb{G}_a^-$ and $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^s$ satisfying the conditions for a semidirect product. This puts us in the situation of Theorem 8.1, and the Corollary is proved. □

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