

Compressed Super-Resolution of Positive Sources

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Abstract—Atomic norm minimization is a convex optimization framework to recover point sources from a subset of their low-pass observations, or equivalently the underlying frequencies of a spectrally-sparse signal. When the amplitudes of the sources are positive, a positive atomic norm can be formulated, and exact recovery can be ensured without imposing a separation between the sources, as long as the number of observations is greater than the number of sources. However, the classic formulation of the atomic norm requires to solve a semidefinite program involving a linear matrix inequality of a size on the order of the signal dimension, which can be prohibitive. In this letter, we introduce a novel “compressed” semidefinite program, which involves a linear matrix inequality of a reduced dimension on the order of the number of sources. We guarantee the tightness of this program under certain conditions on the operator involved in the dimensionality reduction. Finally, we apply the proposed method to direction finding over sparse arrays based on second-order statistics and achieve significant computational savings.

Index Terms—Atomic norm minimization, positive sources, sparse arrays, dimensionality reduction.

I. INTRODUCTION

SUPER-RESOLUTION [1]–[3] is a signal processing problem aiming at recovering point sources from their low-pass observations. It finds broad applications in applied science from the estimation of the direction of arrivals of far-fields signals in classical array processing, to reverting the distortions introduced by the imperfection of the measurement device in modern imaging modalities.

Algorithms based on convex optimization [3]–[5] have been recently proposed to solve the super-resolution problem without discretizing the grid [6]–[8]. Among those, this letter focuses on atomic norm minimization (ANM, a.k.a. total variation minimization) [5], which proposes to localize the point sources from the output of a semidefinite program (SDP) [9]–[26]; see [2] for a recent overview. ANM inherits well-established advantages of convex estimators, such as amenability to performance analysis and robustness to the presence of noise. It is also a versatile framework that can easily be adapted to fit new measurement models not directly handled by classical approaches [27]–[30]. Additionally, ANM is agnostic to the model order. However, the computational complexity of ANM, essentially driven by the size of the SDP, limits its scalability and remains the most

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prohibitive drawback for practical implantation of this method to real-time systems.

In many imaging applications such as fluorescence microscopy [31], the point sources are positive, which is a prior that can be leveraged to improve performance [32]–[37]. In particular, no separation between positive sources is necessary to guarantee the success of atomic norm minimization as long as the number of observations is greater than the number of sources. In this letter, we propose to solve the super-resolution problem of positive sources using a “compressed” ANM algorithm, which only involves a linear matrix inequality (LMI) of dimension on the order of the number of point sources, instead of the signal length. We guarantee exact reconstruction using the proposed algorithm under certain conditions on the operator involved in the dimensionality reduction, which may lead to significant computational savings. As an illustration, we apply the proposed algorithm in the context of direction finding over sparse arrays from the second-order statistics [38]–[42]. Finally, numerical experiments are provided to demonstrate the effectiveness of the proposed algorithm. Our work is related to the compressed ANM proposed in [43], [44], but focuses on the positive case where we provide guarantees without imposing any separation condition on the sources.

II. PROBLEM FORMULATION AND BACKGROUNDS

Let $\mathbf{a}(\tau) \in \mathbb{C}^N$ be the discrete complex exponential vector $\mathbf{a}(\tau) = [1, e^{i2\pi\tau}, \dots, e^{i2\pi(N-1)\tau}]^\top$ for $\tau \in [0, 1]$. Consider a discrete signal $\mathbf{x}^* = [x_0^*, \dots, x_{N-1}^*] \in \mathbb{C}^N$ modeled as a sparse positive combination of elements of the form $\mathbf{a}(\tau)$, i.e.,

$$\mathbf{x}^* = \sum_{k=1}^p c_k^* \mathbf{a}(\tau_k^*), \quad (1)$$

for some source locations $\{\tau_k^*\}_{k=1}^p \subset [0, 1]$ and some positive amplitudes $\{c_k^*\}_{k=1}^p \in \mathbb{R}^+$. The goal of super-resolution is to recover the parameters $\{\tau_k^*\}$, $\{c_k^*\}$ from possibly a subset of entries of \mathbf{x}^* given by $\mathbf{x}_\Omega^* = \mathbf{P}_\Omega \mathbf{x}^*$, where $\mathbf{P}_\Omega \in \mathbb{C}^{|\Omega| \times N}$ is the matrix that only retains the entries indexed in the subset $\Omega \subseteq \{0, \dots, N-1\}$. The problem is in general ill-posed, in the sense that there could be infinitely many possible configurations of parameters $\{\tau_k\}$, $\{c_k\}$ that are consistent with the observations. Therefore, it is natural to seek for the *sparse decomposition* (1) that contains the smallest number of point sources.

The atomic norm [45] is a general framework to promote sparse solutions to linear inverse problems. Given a generic atomic set $\mathcal{A} \subset \mathbb{C}^N$, the atomic norm $\|\mathbf{x}\|_{\mathcal{A}} \triangleq \inf_{t>0} \{\mathbf{x} \in t\mathcal{A}\}$ of a vector $\mathbf{x} \in \mathbb{C}^N$ is defined by the Minkowski functional of the set \mathcal{A} evaluated at \mathbf{x} .¹ Let $\mathcal{T}(\mathbf{x})$ be the Hermitian Toeplitz matrix whose first column is \mathbf{x} . Specializing the atomic set to

¹Here, the atomic norm should be interpreted broadly as a pseudo-norm since it may not be a norm when \mathcal{A} is not centrally symmetric.

the set of *unphased* complex exponential vector, i.e., $\mathcal{A}_+ = \{\mathbf{a}(\tau) \mid \tau \in [0, 1]\}$, the atomic norm simplifies to [2]

$$\begin{aligned} \|\mathbf{x}\|_{\mathcal{A}_+} &= \inf \left\{ \sum_k c_k \mid \mathbf{x} = \sum_k c_k \mathbf{a}(\tau_k), c_k > 0 \right\} \\ &= \begin{cases} \operatorname{Re}(x_0) & \text{if } \mathcal{T}(\mathbf{x}) \succeq \mathbf{0} \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \quad (2)$$

The decomposition $\mathbf{x} = \sum_k c_k \mathbf{a}(\tau_k)$ that attains the infimum in the first equality of (2) is called the *atomic decomposition*. Of particular interest, it is known that for any vector \mathbf{x}_* of the form (1), as long as $p < N$, its atomic decomposition perfectly recovers the sparse decomposition (1) and, therefore, provides a means to recover $\{c_k^*\}, \{\tau_k^*\}$ [46]. The tightness of the atomic decomposition holds without imposing any separation between the positive point sources, which leads to better resolution than the case of signed amplitudes, where a separation is necessary [47], [48].

Given partial observation \mathbf{x}_Ω^* , one can recover the ground truth signal \mathbf{x}^* by solving the ANM problem as

$$\begin{aligned} \hat{\mathbf{x}}_{\text{ANM}} &:= \operatorname{argmin}_{\mathbf{x} \in \mathbb{C}^N} \|\mathbf{x}\|_{\mathcal{A}_+} \text{ s.t. } \mathbf{x}_\Omega = \mathbf{x}_\Omega^* \\ &= \operatorname{argmin}_{\mathbf{x} \in \mathbb{C}^N} \operatorname{Re}(x_0) \text{ s.t. } \mathbf{x}_\Omega = \mathbf{x}_\Omega^* \text{ and } \mathcal{T}(\mathbf{x}) \succeq \mathbf{0}. \end{aligned} \quad (\text{ANM})$$

This approach is guaranteed to yield a perfect reconstruction of \mathbf{x}^* as long as $p < |\Omega|$, without any need for randomness of the observation set Ω [37]. However, from the computational perspective, (ANM) involves an LMI of dimension N , which in practice may be prohibitive to solve.

III. MAIN RESULTS

Inspired by [43], [44], we propose a novel approach to reduce the computational complexity of (ANM) by projecting the positivity constraint $\mathcal{T}(\mathbf{x}) \succeq \mathbf{0}$ to a lower dimension. Consider a matrix $\mathbf{M} \in \mathbb{C}^{M \times N}$ with $M \leq N$ and full rank, i.e., $\operatorname{rank}(\mathbf{M}) = M$. The compressed positive ANM program is given by

$$\begin{aligned} \hat{\mathbf{x}}_{\text{C-ANM}} &:= \operatorname{argmin}_{\mathbf{x} \in \mathbb{C}^N} \operatorname{Re}(x_0) \\ \text{s.t. } &\mathbf{x}_\Omega = \mathbf{x}_\Omega^* \text{ and } \mathbf{M}\mathcal{T}(\mathbf{x})\mathbf{M}^H \succeq \mathbf{0}. \end{aligned} \quad (\text{C-ANM})$$

Note that the compressed SDP (C-ANM) now contains an LMI of dimension $M \leq N$. An immediate question arising from the definition of (C-ANM) concerns its tightness, i.e., the conditions under which its solution uniquely recovers \mathbf{x}^* and its sparse decomposition. Similar to many linear inverse problems, the tightness of (C-ANM) is related to the existence of a so-called *dual certificate*: an element lying in the dual feasible set and attaining the optimum of the cost function. Lemma 1 characterizes the dual certificate conditions that certify the tightness of (C-ANM), whose proof is given in Appendix A.

Lemma 1 (Dual certificate): Suppose there exists a trigonometric polynomial $Q(\tau) := \sum_{n=0}^{N-1} q_n e^{-i2\pi n\tau}$ with coefficient vector $\mathbf{q} = [q_0, \dots, q_{N-1}]^\top \in \mathbb{C}^N$ such that

- 1) $Q(\tau) \neq 1$.
- 2) The equality $1 - \operatorname{Re}(Q(\tau)) = 0$ holds for $\tau \in \{\tau_k^*\}_{k=1}^p$.
- 3) The coefficient vector \mathbf{q} verifies $\mathbf{q}_{\Omega^c} = \mathbf{0}$.
- 4) There exists a countable collection of trigonometric polynomials $\{P_i(t)\}_i$ with coefficients $\mathbf{p}_i \in \operatorname{span}(\mathbf{M}^H)$ such that $1 - \operatorname{Re}(Q(\tau)) = \sum_i |P_i(\tau)|^2 \geq 0$ for all $\tau \in [0, 1]$.

Then, $\hat{\mathbf{x}}_{\text{C-ANM}} = \mathbf{x}^*$ is the unique solution of (C-ANM).

From Lemma 1, it suffices to construct a trigonometric polynomial $Q(\tau)$ verifying specific conditions to conclude on the tightness of the compressed SDP (C-ANM). The essential difference between the conditions for (C-ANM) stated in Lemma 1, and those for (ANM) [5] on the dual certificate is in the fourth assumption. Herein, the polynomial $1 - \operatorname{Re}(Q(\tau))$ must have a sum-of-squares (SOS) structure over a low-dimensional subspace of trigonometric polynomials whose coefficients lie in the span of \mathbf{M}^H , also called the *sparse-SOS* condition. Note that, in the absence of compression (i.e. $\mathbf{M} = \mathbf{I}$), the sparse-SOS assumption always holds as a consequence of the Fejér-Riesz theorem [49].

A natural question is then: how to design the matrix \mathbf{M} that verifies Lemma 1? In the sequel, we focus on a specific design and discuss its tightness. Let \mathcal{I} be a subset of $\{0, \dots, N-1\}$ with $\{0\} \in \mathcal{I}$ and cardinality $M = |\mathcal{I}|$, and we denote by $\partial\mathcal{I}$ the set of the positive pairwise differences of elements in \mathcal{I} , given by

$$\partial\mathcal{I} = \{j \mid j = i_1 - i_2 \geq 0, (i_1, i_2) \in \mathcal{I} \times \mathcal{I}\}. \quad (3)$$

Let $\mathbf{M} = \mathbf{P}_{\mathcal{I}} \in \mathbb{C}^{M \times N}$ be the subsampling matrix selecting the elements whose indices belong to \mathcal{I} . Theorem 2 states that with this choice of \mathbf{M} , (C-ANM) is tight if $\partial\mathcal{I} \subseteq \Omega$ and the number of sources $p < M$. The proof is given in Appendix B.

Theorem 2 (Exact reconstruction): Let $\mathcal{I} \subseteq \{0, \dots, N-1\}$ be a subset of cardinality $M = |\mathcal{I}|$ verifying $0 \in \mathcal{I}$ and $\partial\mathcal{I} \subseteq \Omega$. Moreover, suppose that $p < M$. Then for the choice $\mathbf{M} = \mathbf{P}_{\mathcal{I}}$, $\hat{\mathbf{x}}_{\text{C-ANM}} = \mathbf{x}^*$ is the unique solution of (C-ANM).

Theorem 2 suggests that the computational complexity can be significantly reduced, where the LMI has a dimension on the order of the number of sources p . For example, consider the case will full observation, i.e., $\Omega = \{0, \dots, N-1\}$. Then, Theorem 2 guarantees the compressed ANM (C-ANM) is exact for any \mathcal{I} as long as $|\mathcal{I}| > p$ and $0 \in \mathcal{I}$. As another example, when $\Omega = \{0, \dots, p\}$, choosing $\mathcal{I} = \Omega$ also reduces the complexity significantly to the order of p .

IV. APPLICATION: DIRECTION FINDING IN SPARSE ARRAYS

In this section, we illustrate the applicability of Theorem 2 for direction finding in sparse arrays from second-order statistics [38], [42], which is a problem of great interest in the array processing literature, as this approach allows the recovery of more sources than the number of antennas, and offers better resolution than its first-order counterpart. We show in particular that (C-ANM) can be applied to reduce the computational complexity of ANM-based recovery, where (C-ANM) can recover the sources by involving an LMI of the size equal to the number of sources, instead of the size of the aperture.

A. Exact Recovery With Infinite Snapshots

We start by introducing some notation. Let \mathcal{J} be the set of integer indices corresponding to the location of the antennas in a linear sparse array. The aperture of \mathcal{J} is assumed to be N , so that \mathcal{J} can be embedded in a uniform array of N elements, i.e., $\mathcal{J} \subseteq \{0, \dots, N-1\}$ and $\{0, N-1\} \in \mathcal{J}$. We denote by $\Omega = \partial\mathcal{J}$ the set of the positive indices of the difference co-array, where $\partial\mathcal{J}$ is given as in (3).

At the time instance $\ell = 1, \dots, L$, the noiseless received signal $\mathbf{u}_\ell^* \in \mathbb{C}^M$ is modeled as $\mathbf{u}_\ell^* = \mathbf{P}_{\mathcal{J}} \sum_{k=1}^p c_{k,\ell}^* \mathbf{a}(\tau_k^*) + \mathbf{w}_\ell$,

TABLE I

RUNTIMES OF THE (ANM) AND (C-ANM) ALGORITHMS OVER CANTOR ARRAYS OF DIFFERENT ORDERS. BOTH ALGORITHMS ARE IMPLEMENTED VIA MATLAB USING THE MOSEK SOLVER IN CVX [51]. WE FIX $p = 8$ IN THE EXPERIMENTS, THE RESULTS ARE AVERAGED OVER 50 TRIALS

Cantor #	Aperture	# elements	(ANM) (s)	(C-ANM) (s)
3	10	$2^3 = 8$	0.30	0.26
4	28	$2^4 = 16$	0.34	0.26
5	82	$2^5 = 32$	0.66	0.29
6	244	$2^6 = 64$	5.83	0.87
7	730	$2^7 = 128$	135.62	8.38

for some zero-mean $c_{k,\ell}^* \in \mathbb{C}$, and $\mathbf{w}_\ell \in \mathbb{C}^{|\mathcal{J}|}$ is a white additive noise with zero-mean and variance σ^2 , which is assumed known. If the sources are incoherent, i.e., obey the second-order statistical property:

$$\mathbb{E} \left[\overline{c_{k,\ell}^*} c_{k',\ell'}^* \right] = \begin{cases} \eta_k^2, & \text{if } k = k', \ell = \ell' \\ 0, & \text{otherwise} \end{cases}, \quad (4)$$

then the covariance matrix $\Sigma_{\mathcal{J}}^* = \mathbb{E}[\mathbf{u}_\ell^* (\mathbf{u}_\ell^*)^\mathsf{H}]$ writes

$$\Sigma_{\mathcal{J}}^* = \mathbf{P}_{\mathcal{J}} \Sigma^* \mathbf{P}_{\mathcal{J}}^\mathsf{H} + \sigma^2 \mathbf{I}_{|\mathcal{J}|}, \quad (5)$$

where $\Sigma^* = \sum_{k=1}^p \eta_k^2 \mathbf{a}(\tau_k) \mathbf{a}(\tau_k)^\mathsf{H}$ is a positive semidefinite Hermitian Toeplitz matrix corresponding the covariance of the observations gathered on the full uniform array $\{0, \dots, N-1\}$, and $\mathbf{I}_{|\mathcal{J}|}$ is an identity matrix of size $|\mathcal{J}|$. Denote by $\mathbf{x}^* \in \mathbb{C}^N$ the first column of Σ^* , where $\mathbf{x}^* = \sum_{k=1}^p \eta_k^2 \mathbf{a}(\tau_k^*)$ is a sparse positive linear combination of p discrete complex exponentials, with frequencies $\{\tau_k^*\}$ encoding the location of the sources. From (5), identifying and rearranging the entries of $\Sigma_{\mathcal{J}}^* - \sigma^2 \mathbf{I}_{|\mathcal{J}|}$ in the coordinates $(i_1, i_2) \in \partial \mathcal{J}$ then equivalently give the observation model $\mathbf{x}_\Omega^* = \mathbf{P}_\Omega \mathbf{x}^*$. Theorem 2 then applies, and the sources can be exactly recovered by applying (C-ANM) on the vector \mathbf{x}_Ω^* . This yields the following corollary.

Corollary 3 (Exact recovery over sparse arrays): If $\mathcal{J} \subseteq \{0, \dots, N-1\}$ is a sparse array with $M = |\mathcal{J}|$ elements and $\{0, N-1\} \in \mathcal{J}$. If $p < M$, then $\{\tau_1^*, \dots, \tau_p^*\}$ can be exactly recovered using (C-ANM) with $\mathbf{M} = \mathbf{P}_{\mathcal{J}}$.

Corollary 3 ensures that (C-ANM) returns the sources by solving an SDP with LMI of size $M = |\mathcal{J}|$, while proceeding to the full-dimensional ANM (ANM) would require to solve an SDP with LMI of size N , equal to the length of the full array. Therefore, the proposed compressed approach can bring an order-of-magnitude reduction in the computational complexity of the problem for appropriate choices of the array and the compression operator. As an example, the benefits are well highlighted when considering the Cantor arrays, which are complete sparse arrays constructed through a fractal process, see [50] for an introduction.

Example 4 (Cantor arrays): If $\mathcal{J} \subseteq \{0, \dots, N-1\}$ is a Cantor set, and if $p < |\mathcal{J}| = M$, then we can recover the spikes by solving a semidefinite program involving a linear matrix inequality of dimension $M = N^{\log(2)/\log(3)} \simeq N^{0.62}$. Experimental runtimes of algorithms (ANM) and (C-ANM) are compared in Table I for different size of Cantor sets.

In addition, it is worth to pay attention to a particular category of sparse arrays, which are called *complete*, that the difference co-array has no holes, i.e., $\partial \mathcal{J} = \{0, \dots, N-1\}$. In that case, Theorem 2 guarantee that running (C-ANM) with any compression matrix $\mathbf{M} = \mathbf{P}_{\mathcal{J}}$ such that $\mathcal{J} \subseteq \mathcal{I}$ would guarantee an exact recovery of at most $|\mathcal{J}| - 1$ sources. Hence, there is a trade-off

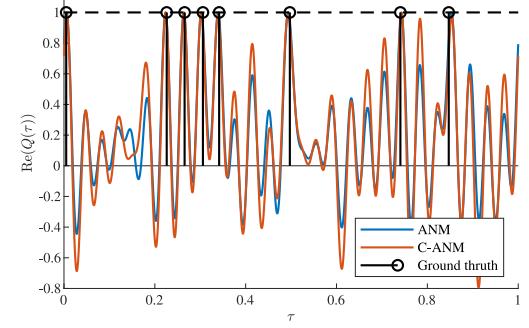


Fig. 1. Empirical localization of the sources using a Cantor array from the second order statistics. The picks of the dual polynomials returned by (C-ANM-Noisy) when $\mathbf{M} = \mathbf{I}$ (ANM), and $\mathbf{M} = \mathbf{P}_{\mathcal{J}}$ (C-ANM) provide an estimate of the ground truth sources. In those settings, $N = 28$, $|\mathcal{J}| = 16$, $p = 8$. We take $L = 100$ snapshots, and SNR is -5 dB.

between the compression ratio of the LMI in (C-ANM) and the number of sources that can effectively be recovered.

B. Recovery Under a Finite Number of Snapshots

In practice, the exact covariance $\Sigma_{\mathcal{J}}^*$ in (5) is imperfectly known, as the number of snapshots L is finite. The empirical covariance of the received signals $\Sigma_{\mathcal{J}} = \frac{1}{L} \sum_{\ell=1}^L \mathbf{u}_\ell^* (\mathbf{u}_\ell^*)^\mathsf{H}$ provides a more accurate estimate of $\Sigma_{\mathcal{J}}^*$ as L increases. Denote by $\mathbf{y}_\Omega \in \mathbb{C}^{|\mathcal{J}|}$ the noisy estimate of \mathbf{x}_Ω^* obtained from $\Sigma_{\mathcal{J}} - \sigma^2 \mathbf{I}_{|\mathcal{J}|}$ in a similar manner as earlier. To adapt to this uncertainty, we formulate the atomic norm denoiser [10] by adding a data fidelity term to the cost function of (C-ANM),

$$\begin{aligned} \hat{\mathbf{x}}_\lambda := \operatorname{argmin}_{\mathbf{x} \in \mathbb{C}^N} \frac{1}{2} \|\mathbf{x}_\Omega - \mathbf{y}_\Omega\|_2^2 + \lambda \operatorname{Re}(x_0) \\ \text{s.t. } \mathbf{M} \mathcal{T}(\mathbf{x}) \mathbf{M}^\mathsf{H} \succeq 0. \end{aligned} \quad (\text{C-ANM-Noisy})$$

where $\lambda > 0$ is a regularization parameter.

In Fig. 1, we compare the localization of the sources using a sparse array from $L = 100$ snapshots using (C-ANM-Noisy) in the absence of compression ($\mathbf{M} = \mathbf{I}$) which corresponds to the original ANM method, and using a compression matrix $\mathbf{M} = \mathbf{P}_{\mathcal{J}}$. We pick a Cantor array with aperture $N = 28$ and $M = 16$ elements. The ground truth signal \mathbf{x}^* impinging on the array is formed by $p = 8$ incoherent sources with equal unit power $\eta_k^2 = 1$. The signal-to-noise ratio (SNR) is defined as $\text{SNR} = \sum_{k=1}^p \eta_k^2 / \sigma^2$ and is set to -5 dB. The sources are identified from the dual solution of (C-ANM-Noisy), namely, by identifying the peaks of the dual polynomial $\operatorname{Re}(Q(\tau)) = \operatorname{Re}(\mathbf{a}(\tau)^\mathsf{H} \mathbf{q})$, where $\mathbf{q} \in \mathbb{C}^N$ is the solution to the dual program of (C-ANM-Noisy) [2]. It can be seen that the compressed ANM approach is able to recover the direction-of-arrivals at a much lower computational complexity.

V. CONCLUSION

In this letter, we showed that super-resolution of point sources can be solved using a compressed ANM algorithm with much lower computational complexity, provided an application of this results to direction finding over sparse arrays. For future work, we aim to study the performance of the compressed ANM in the noisy setting, where it is expected that the compression leads to interesting statistical-computational trade-offs [52].

APPENDIX

A. Proof of Lemma 1

The Lagrange dual program of (C-ANM) reads

$$\begin{aligned} & \underset{\mathbf{q} \in \mathbb{C}^N, \mathbf{S} \in \mathbb{C}^{M \times M}}{\operatorname{argmax}} \operatorname{Re} \langle \mathbf{x}_\Omega, \mathbf{q}_\Omega \rangle \\ \text{s.t. } & \mathbf{S} \succeq 0, \quad \mathbf{q}_{\Omega^c} = \mathbf{0}, \quad \mathcal{T}^*(\mathbf{M}^H \mathbf{S} \mathbf{M}) + \mathbf{K} \mathbf{q} = \mathbf{e}_0, \end{aligned} \quad (6)$$

where $\mathcal{T}^* : \mathbb{C}^{N \times N} \rightarrow \mathbb{C}^N$ is the adjoint to the Toeplitz Hermitian operator \mathcal{T} and is given for any \mathbf{H} by $\mathcal{T}^*(\mathbf{H})_j = \sum_{\ell=1}^{N-j} H_{j+\ell, \ell}$, and $\mathbf{K} \in \mathbb{C}^{N \times N}$ is the diagonal matrix $\mathbf{K} = \operatorname{diag}(1, \frac{1}{2}, \dots, \frac{1}{2})$. Suppose there exists \mathbf{q} that verifies the hypotheses of Lemma 1. Since $\mathbf{p}_i \in \operatorname{span}(\mathbf{M}^H)$, there exists $\mathbf{u}_i \in \mathbb{C}^M$ such that $\mathbf{p}_i = \mathbf{M}^H \mathbf{u}_i$ for all i .

a) *Tightness*: We start by showing that \mathbf{x}^* is a solution to (C-ANM). First, as $Q(\tau) \neq 1$, there exists some i such that $P_i(\tau) \neq 0$, and consequently $\mathbf{p}_i \neq \mathbf{0}$. Since \mathbf{M} is full rank by assumption, this implies that $\mathbf{u}_i \neq \mathbf{0}$ for some i .

Next, the equality $1 - \operatorname{Re}(Q(\tau)) = \sum_i |P_i(\tau)|^2$ holds for any $\tau \in [0, 1]$ if and only if

$$\begin{aligned} \mathbf{e}_0 - \mathbf{K} \mathbf{q} &= \mathcal{T}^* \left(\sum_i \mathbf{p}_i \mathbf{p}_i^H \right) = \mathcal{T}^* \left(\sum_i \mathbf{M}^H \mathbf{u}_i \mathbf{u}_i^H \mathbf{M} \right) \\ &= \mathcal{T}^* \left(\mathbf{M}^H \left(\sum_i \mathbf{u}_i \mathbf{u}_i^H \right) \mathbf{M} \right) = \mathcal{T}^* \left(\mathbf{M}^H \mathbf{S} \mathbf{M} \right), \end{aligned} \quad (7)$$

where we let $\mathbf{S} = \sum_i \mathbf{u}_i \mathbf{u}_i^H \succeq 0$ in the last equality. As $\mathbf{q}_{\Omega^c} = \mathbf{0}$ by assumption, it is evident that the pair (\mathbf{q}, \mathbf{S}) is in the feasible set of the dual problem (6). Evaluating the dual cost function at (\mathbf{q}, \mathbf{S}) yields

$$\begin{aligned} \operatorname{Re} \langle \mathbf{x}_\Omega, \mathbf{q}_\Omega \rangle &= \operatorname{Re} \langle \mathbf{x}_\Omega^*, \mathbf{q}_\Omega \rangle = \operatorname{Re} \langle \mathbf{x}^*, \mathbf{q} \rangle \\ &= \sum_{k=1}^p c_k^* \operatorname{Re} (\mathbf{a}(\tau_k^*)^H \mathbf{q}) = \sum_{k=1}^p c_k^* \operatorname{Re} (Q(\tau_k^*)) \\ &= \sum_{k=1}^p c_k^* = \|\mathbf{x}^*\|_{\mathcal{A}^+}. \end{aligned} \quad (8)$$

By strong duality, \mathbf{x}^* is a solution of (C-ANM).

b) *Uniqueness*: We now prove that \mathbf{x}^* is the unique solution to (C-ANM). Suppose that $\tilde{\mathbf{x}} \in \mathbb{C}^N$ is a solution to (C-ANM), and let $\tilde{\mathbf{x}} = \sum_{k=1}^{\tilde{p}} \tilde{c}_k \mathbf{a}(\tilde{\tau}_k)$ an atomic decomposition with $\|\tilde{\mathbf{x}}\|_{\mathcal{A}^+} = \sum_{k=1}^{\tilde{p}} \tilde{c}_k$. Since $\tilde{\mathbf{x}}$ is a solution, we also have that $\|\tilde{\mathbf{x}}\|_{\mathcal{A}^+} = \|\mathbf{x}^*\|_{\mathcal{A}^+}$ and $\tilde{\mathbf{x}}_\Omega = \mathbf{x}_\Omega^*$.

Denote by $R \subset [0, 1]$ the set of the roots to the equation $1 - \operatorname{Re}(Q(\tau)) = 0$. As $Q(\tau)$ is not the constant polynomial equal to one, the set R is finite and we have $|R| \leq N - 1$. By strong duality, we can further write

$$\begin{aligned} \sum_{k=1}^{\tilde{p}} \tilde{c}_k &= \|\tilde{\mathbf{x}}\|_{\mathcal{A}^+} = \operatorname{Re} \langle \mathbf{x}_\Omega, \mathbf{q}_\Omega \rangle = \operatorname{Re} \langle \tilde{\mathbf{x}}_\Omega, \mathbf{q}_\Omega \rangle = \operatorname{Re} \langle \tilde{\mathbf{x}}, \mathbf{q} \rangle \\ &= \sum_{k=1}^{\tilde{p}} \tilde{c}_k \operatorname{Re} (\mathbf{a}(\tilde{\tau}_k)^H \mathbf{q}) = \sum_{k=1}^{\tilde{p}} \tilde{c}_k \operatorname{Re} (Q(\tilde{\tau}_k)). \end{aligned} \quad (9)$$

We conclude using the positivity of the \tilde{c}_k s that $\operatorname{Re}(Q(\tilde{\tau}_k)) = 1$ for $k = 1, \dots, \tilde{p}$, and therefore that $\{\tilde{\tau}_k\}_{k=1}^{\tilde{p}} \subseteq R$. Let $\mathbf{V}_\Omega \in \mathbb{C}^{|\Omega| \times |R|}$ be the matrix whose column are elements of the form

$\mathbf{a}(\tau)$ with $\tau \in R$. We have that $\mathbf{x}_\Omega^* = \mathbf{V}_\Omega \mathbf{c}^*$ and $\tilde{\mathbf{x}} = \mathbf{V}_\Omega \tilde{\mathbf{c}}$ for some $\mathbf{c}^*, \tilde{\mathbf{c}} \in \mathbb{C}^{|R|}$ with $\|\mathbf{c}^*\|_1 = \|\tilde{\mathbf{c}}\|_1 = \|\mathbf{x}^*\|_{\mathcal{A}^+}$. We conclude using the uniqueness of the solution to the positive linear program [32]

$$\tilde{\mathbf{c}} := \underset{\mathbf{c} \in \mathbb{C}^{|R|}}{\operatorname{argmin}} \|\mathbf{c}\|_1 \text{ such that } \mathbf{c} \geq 0 \text{ and } \mathbf{V}_\Omega \mathbf{c} = \mathbf{x}_\Omega^*, \quad (10)$$

that $\tilde{\mathbf{c}} = \mathbf{c}^* = \tilde{\mathbf{c}}$, and consequently that $\tilde{\mathbf{x}} = \mathbf{x}^*$. We conclude that \mathbf{x}^* is the unique solution to (C-ANM). \square

B. Proof of Theorem 2

In view of Lemma 1, it suffices to show the existence of a trigonometric polynomial $Q(\tau)$ verifying the conditions of Lemma 1 for the compression matrix $\mathbf{M} = \mathbf{P}_\mathcal{I}$.

Denote the Vandermonde matrix $\mathbf{A} = [\mathbf{a}(\tau_1), \dots, \mathbf{a}(\tau_p)] \in \mathbb{C}^{N \times p}$. As long as $p < M$, the matrix $\mathbf{V} = \mathbf{A}^H \mathbf{P}_\mathcal{I}^H \in \mathbb{C}^{p \times M}$ has a non-trivial nullspace. Denote by $\mathbf{u} \in \ker(\mathbf{V})$ a non-zero element of this nullspace. We have that

$$\mathbf{V} \mathbf{u} = \mathbf{A}^H \mathbf{P}_\mathcal{I}^H \mathbf{u} = \mathbf{0}.$$

Let $\mathbf{p} = \mathbf{P}_\mathcal{I}^H \mathbf{u}$, and $P(\tau)$ be the trigonometric polynomial $P(\tau) = \sum_{n=0}^{N-1} p_n e^{-i2\pi n\tau}$. Moreover, let $Q(\tau) = \sum_{n=0}^{N-1} q_n e^{i2\pi n\tau}$ be such that

$$1 - \operatorname{Re}(Q(\tau)) = |P(\tau)|^2, \quad \forall \tau \in [0, 1], \quad (11)$$

which holds if and only if the vector $\mathbf{q} \in \mathbb{C}^N$ satisfies

$$\mathbf{e}_0 - \mathbf{K} \mathbf{q} = \mathcal{T}^*(\mathbf{p} \mathbf{p}^H). \quad (12)$$

We now verify that $Q(\tau)$ meets the conditions of Lemma 1.

- 1) Since $\mathbf{V} \mathbf{e}_0 = \mathbf{A}^H \mathbf{P}_\mathcal{I}^H \mathbf{e}_0 = \mathbf{A}^H \mathbf{e}_0 \neq \mathbf{0}$, which follows by $0 \in \partial \mathcal{I}$, the vector \mathbf{u} is not collinear to \mathbf{e}_0 . Thus \mathbf{p} is not collinear to \mathbf{e}_0 and as \mathbf{K} is a diagonal matrix, from (12) we have

$$\mathbf{q} = \mathbf{K}^{-1} (\mathbf{e}_0 - \mathcal{T}^*(\mathbf{p} \mathbf{p}^H)), \quad (13)$$

is also not collinear to \mathbf{e}_0 . It follows that $Q(\tau) \neq 1$.

- 2) By the assumption $\mathbf{u} \in \ker(\mathbf{V})$, we have that for all $k = 1, \dots, p$,

$$\begin{aligned} 1 - \operatorname{Re}(Q(\tau_k^*)) &= |P(\tau_k^*)|^2 \\ &= \mathbf{a}(\tau_k^*)^H \mathbf{p} \mathbf{p}^H \mathbf{a}(\tau_k^*) \\ &= \mathbf{a}(\tau_k^*)^H \mathbf{P}_\mathcal{I}^H \mathbf{u} \mathbf{u}^H \mathbf{P}_\mathcal{I} \mathbf{a}(\tau_k^*) \\ &= \mathbf{e}_k^H \mathbf{V} \mathbf{u} \mathbf{u}^H \mathbf{V}^H \mathbf{e}_k = 0. \end{aligned}$$

- 3) As $\mathbf{p} \in \operatorname{span}(\mathbf{P}_\mathcal{I}^H)$ is supported over \mathcal{I} , the vector $\mathcal{T}^*(\mathbf{p} \mathbf{p}^H)$ is supported over $\partial \mathcal{I}$. Since $0 \in \partial \mathcal{I}$, the vector \mathbf{e}_0 , and the difference $\mathcal{T}^*(\mathbf{p} \mathbf{p}^H) - \mathbf{e}_0$ are also supported over $\partial \mathcal{I}$. Since \mathbf{K}^{-1} is a diagonal matrix, it leaves the support of the subvectors invariant by multiplication. By (13), \mathbf{q} is supported over $\partial \mathcal{I}$. By the assumption of Theorem 2, we have $\partial \mathcal{I} \subseteq \Omega$, hence $\Omega^c \subseteq \partial \mathcal{I}^c$, and we conclude that $\mathbf{q}_{\Omega^c} = \mathbf{0}$.
- 4) Finally, (11) holds by construction with $\mathbf{p} \in \operatorname{span}(\mathbf{P}_\mathcal{I}^H)$, thus $Q(\tau)$ is sparse-SOS over $\operatorname{span}(\mathbf{P}_\mathcal{I}^H)$.

Invoking Lemma 1 concludes the proof of Theorem 2. \square

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