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Chvátal Rank in Binary Polynomial Optimization

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Abstract. Recently, several classes of cutting planes have been introduced for binary polynomial optimization. In this paper, we present the first results connecting the combinatorial structure of these inequalities with their Chvátal rank. We determine the Chvátal rank of all known cutting planes and show that almost all of them have Chvátal rank 1. We observe that these inequalities have an associated hypergraph that is β -acyclic. Our second goal is to derive deeper cutting planes; to do so, we consider hypergraphs that admit β -cycles. We introduce a novel class of valid inequalities arising from odd β -cycles, that generally have Chvátal rank 2. These inequalities allow us to obtain the first characterization of the multilinear polytope for hypergraphs that contain β -cycles. Namely, we show that the multilinear polytope for cycle hypergraphs is given by the standard linearization inequalities, flower inequalities, and odd β -cycle inequalities. We also prove that odd β -cycle inequalities can be separated in linear time when the hypergraph is a cycle hypergraph. This shows that instances represented by cycle hypergraphs can be solved in polynomial time. Last, to test the strength of odd β -cycle inequalities, we perform numerical experiments that imply that they close a significant percentage of the integrality gap.

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Keywords: binary polynomial optimization • Chvátal-Gomory cuts • multilinear polytope

1. Introduction

In recent work, Del Pia and Khajavirad (2017) introduced the multilinear polytope. In order to define it, let V be a ground set, let E be a set of subsets of cardinality at least two of V , and denote by G the hypergraph (V, E) . The *multilinear polytope* of G , denoted by MP_G , consists of the convex hull of the binary points $z \in \mathbb{Z}^{V \cup E}$ that satisfy $z_e = \prod_{v \in e} z_v$ for every $e \in E$. The combinatorial structure of the multilinear polytope is highlighted by the fact that its face defined by $z_e = 0$, $\forall e \in E$, is an affine transformation of the set covering polytope.

The multilinear polytope plays a fundamental role in integer programming. In fact, the problem of minimizing a multivariate polynomial function over all binary points can be reformulated as a linear program over the multilinear polytope. In this reformulation, each $v \in V$ corresponds to a variable of the original polynomial problem, and each $e \in E$ corresponds to a nonlinear monomial in the original objective function.

The strong NP-hardness of binary polynomial programming (Garey and Johnson 1979) indicates the high complexity of the multilinear polytope. A well-known polyhedral relaxation of the multilinear polytope is the so-called *standard linearization*, denoted by MP_G^{LP} , and defined by

$$\begin{aligned} z_v &\leq 1 & \forall v \in V. & (\alpha_v) \\ -z_e &\leq 0 & \forall e \in E. & (\nu_e) \\ \sum_{v \in e} z_v - z_e &\leq |e| - 1 & \forall e \in E, & (\varepsilon_e) \\ -z_v + z_e &\leq 0 & \forall v \in e, \forall e \in E. & (\delta_{v,e}) \end{aligned}$$

This linearization follows from Fortet (Fortet 1960, Glover and Woolsey 1974). Because the binary points in MP_G^{LP} coincide with the vertices of MP_G , our binary polynomial program can be now reformulated as an integer linear program over the standard linearization.

In order to obtain tighter polyhedral relaxations of the multilinear polytope, several classes of valid inequalities for MP_G have been defined (Del Pia and Khajavirad 2017, 2018a, b, 2021; Crama and Rodríguez-Heck 2017). These cutting planes have been shown to drastically improve the performance of global solvers

(Del Pia et al. 2020). All these inequalities have been derived by directly exploiting the combinatorial nature of the multilinear polytope. In this paper, we take a different approach. Namely, we leverage both the integer programming and combinatorial optimization aspects of the multilinear polytope in order to provide the first links between valid inequalities for MP_G and Chvátal-Gomory (CG) cuts (Schrijver 1986). CG cuts, which are defined in Section 1.1, provide a fundamental class of valid inequalities for general integer programming problems and have been the subject of extensive research (see Schrijver 1986 and Conforti et al. 2014 and references therein). These inequalities are used by all high-performance algorithms for solving integer programs and have been one of the reasons for the great leap in the success of solvers to handle real-world problems in the past 20 years (Lodi 2010). If we restrict our attention to the case in which the objective function is a quadratic polynomial, the study of the Chvátal rank of known valid inequalities for the Boolean quadric polytope (Padberg 1989) has been developed in Boros et al. (1992) and Bonami et al. (2018). The first paper focuses on the specific case in which the graph representing the instance is complete. The authors show that, under this assumption, the Chvátal closure is given by adding to the formulation the triangle inequalities. The second paper generalizes this result to any instance by proving that the Chvátal closure of the Boolean quadric polytope is obtained by adding the odd-cycle inequalities.

In Section 2, we show that running intersection inequalities (Del Pia and Khajavirad 2021) are CG cuts for MP_G^{LP} . Because running intersection inequalities subsume two-link inequalities (Crama and Rodríguez-Heck 2017) and flower inequalities (Del Pia and Khajavirad 2018a), this result shows that almost all cutting planes defined thus far in the literature are CG cuts for MP_G^{LP} . In addition, our result implies that MP_G^{LP} has Chvátal rank 1 when G is kite-free β -acyclic, a class introduced in Del Pia and Khajavirad (2021) that includes γ -acyclic hypergraphs. To obtain this result, we heavily exploit the running intersection property, which has been extensively used in the database and machine learning communities (Beeri et al. 1983, Lauritzen 1996). We refer the reader to Section 1.1 for an overview of the various types of cycles in a hypergraph.

All running intersection inequalities correspond to β -acyclic hypergraphs. In order to derive cutting planes with higher Chvátal rank, in Section 3 we consider hypergraphs that contain β -cycles. We observe that several applications of binary polynomial optimization are represented by hypergraphs that contain multiple β -cycles, including problems arising from the area of computer vision and from the Bernasconi model in theoretical physics (Mertens and Bessenrodt 1998, POLIP 2014, Crama and Rodríguez-Heck 2017, Elloumi et al. 2019, Del Pia et al. 2020). In particular, we introduce the *odd β -cycle inequalities*, a novel class of valid inequalities for MP_G , arising from odd β -cycles, that generally have Chvátal rank 2. These inequalities generalize odd-cycle inequalities for the Boolean quadric polytope (Padberg 1989) and their lifting by node addition obtained in Del Pia and Khajavirad (2017). This also completes one task of the paper, because we have now determined the Chvátal rank of all the already known cutting planes for the multilinear polytope.

The remaining sections of this paper are devoted to assessing whether odd β -cycle inequalities can provide some benefit in solving instances of binary polynomial optimization. In Section 4, we start by studying the separation problem for odd β -cycle inequalities. Although, in general, it is not clear whether this problem can be solved efficiently, we consider the particular setting where G is a cycle hypergraph. In this case, we show that the separation problem can be solved in linear time. The study of this special case is further developed in Section 6.

We then present an indication of the theoretical power of the odd β -cycle inequalities. In fact, the introduction of these inequalities allows us to provide the first characterization of MP_G in a setting that allows β -cycles. Sections 5 and 6 are devoted to this convex hull characterization. In particular, in Section 5 we present a procedure that will be essential in the proofs of Section 6. Namely, we introduce a technique that allows us to exploit a description of any multilinear polytope MP_G to obtain a description of $MP_{G'}$, where G' is a new hypergraph obtained from G by replacing any node with a new edge. We remark that this method holds for any hypergraph G . Moreover, this technique provides a general way to iteratively extend convex hull characterizations and decomposability results and, in particular, allows us to extend all known decomposability results for the multilinear polytope (Del Pia and Khajavirad 2018a, b, 2021). In Section 6, we return to studying cycle hypergraphs, and we present a perfect formulation of MP_G in this setting. Indeed, we show that MP_G is fully characterized by the standard linearization, flower inequalities, and odd β -cycle inequalities, when G is a cycle hypergraph. We remark that all previous perfect formulations results for the multilinear polytope were under the assumption, among others, that the hypergraph could not have any β -cycle (Crama and Rodríguez-Heck 2017; Buchheim et al. 2018; Del Pia and Khajavirad 2018a, 2021). Furthermore, this result also attests the theoretical power of odd β -cycle inequalities. In particular, together with the positive result of Section 4, this implies that these instances can be solved in polynomial time. To the best of our knowledge, instances represented by cycle hypergraphs represent a new class of instances of binary polynomial

optimization for which we can find an optimal solution in polynomial time. Last, we observe that our explicit description of MP_G implies that MP_G^{LP} has Chvátal rank at most 2, provided that G is a cycle hypergraph.

On the other hand, the aim of Section 7 is to understand if the odd β -cycle inequalities can be useful also from a practical point of view. In particular, our goal is to compute the reduction in the integrality gap obtained by using a subset of the odd β -cycle inequalities. Namely, we only use inequalities corresponding to β -cycles of length 3 or 4. We tested them on instances coming from the two applications mentioned earlier: the one arising in computer vision (Crama and Rodríguez-Heck 2017) and the one in theoretical physics (Bernasconi 1987, Mertens and Bessenrodt 1998). Our numerical results indicate that the odd β -cycle inequalities can be a very useful tool. As a matter of fact, just using a subset of them leads to an average reduction in the integrality gap by 44% for computer vision application and 60% for the theoretical physics one. These results, together with the well-known computational impact of odd-cycle inequalities in the quadratic setting (Bonami et al. 2018), motivate our belief that the odd β -cycle inequalities could give rise to an improvement in state-of-the-art solvers.

We close the introduction by observing that studying extended formulations and representing the objective function with a hypergraph is not the only way to approach binary polynomial optimization. There is a line of work that focuses on using particular types of graphs to represent the problem and finding properties of such graphs that lead to classes of instances that can be solved in polynomial time (Bienstock and Muñoz 2018, Hojny et al. 2019). In particular, the first paper shows that binary polynomial optimization problems can be solved in polynomial time if the corresponding intersection graphs has bounded treewidth. On the other hand, the second paper introduces additional monomials and uses digraphs to represent the problem. They show that if the digraph related to this new extended formulation is acyclic in the undirected sense, then there is an algorithm able to solve the corresponding instance in polynomial time. Graphs have also been used in the context of Pseudo-boolean optimization. In fact, Crama et al. (1990) provides a direct method to find the optimal solution of binary polynomial optimization problems whose corresponding co-occurrence graphs have bounded treewidth in polynomial time. The difference with the papers mentioned previously is that here the algorithm does not use a polyhedral approach. In the field of pseudo-Boolean optimization, the goal is to optimize set functions with a closed algebraic expression. Any such function can be equivalently written as a polynomial function whose variables are allowed to only have binary values. Some seminal papers on this topic are Hammer et al. (1963) and Hammer and Rudeanu (1968), and we refer to Boros and Hammer (2002) for a thorough survey. We remark that instances that can be represented by cycle hypergraphs do not fall in general into any of the previous categories that can be solved by a polynomial time algorithm. Last but not least, we mention that a different popular way to solve binary polynomial optimization problems exploits quadratization of the objective function. Here the idea is to increase the number of variables and constraints of the problem, in order to write the original problem as a binary quadratic problem in a higher dimensional space. In this way, it is then possible to use the literature that has been developed for the quadratic case. Some papers in this field are as follows: Rosenberg (1975), Freedman and Drineas (2005), Buchheim and Rinaldi (2007), Boros and Gruber (2012), Ishikawa (2009, 2011), Elloumi et al. (2019), and Boros et al. (2020).

1.1. Definitions

In this section we present some definitions that will be heavily used in this paper (Fagin 1983; Schrijver 1986; Del Pia and Khajavirad 2017, 2021). We start by describing the definitions connected to CG cuts. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron, with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Let $\pi x \leq \pi_0$ be a valid inequality for P , with $\pi \in \mathbb{Z}^n$. Then, $\pi x \leq \lfloor \pi_0 \rfloor$ is a *Chvátal-Gomory cut*, or more compactly a *CG cut*. Equivalently, $\pi x \leq \lfloor \pi_0 \rfloor$ is CG cut if and only if there exists a vector $u \in \mathbb{R}^m$ such that $u \geq 0$, $uA = \pi$, $ub = \pi_0$. The set of points in P that satisfy all CG cuts is called the *Chvátal closure* of P , which we denote by $C(P)$. One can then iteratively define the t -th *Chvátal closure* of P as the Chvátal closure of the $(t-1)$ th Chvátal closure of P , that is, $C^t(P) := C(C^{t-1}(P))$. The smallest t for which $C^t(P) = \text{conv}(P \cap \mathbb{Z}^n)$ is called the *Chvátal rank* of P . Similarly, the *Chvátal rank of an inequality* $cx \leq d$ is the number t for which $cx \leq d$ is valid for $C^t(P)$ but not for $C^{t-1}(P)$.

As mentioned previously, there are several definitions of cycles in a hypergraph. In this paper, we will only use concepts related to Berge-cycles, γ -cycles, and β -cycles, which we provide next. We refer the reader to Fagin (1983) for further notions of cycles in hypergraphs.

Let $G = (V, E)$ be a hypergraph. A *Berge-cycle* of length m , for some $m \geq 2$, is a sequence $v_1, e_1, v_2, e_2, \dots, v_m, e_m, v_1$ such that $v_1, v_2, \dots, v_m$ are distinct nodes, $e_1, e_2, \dots, e_m$ are distinct edges, $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, m-1$, $v_m, v_1 \in e_m$. A hypergraph is *Berge-acyclic* if it contains no Berge-cycles. We say that a Berge-cycle is a γ -cycle if $m \geq 3$ and the node v_i belongs to e_{i-1}, e_i and no other e_j for all $i = 2, \dots, m$. A hypergraph is said to be γ -acyclic if it does not contain any γ -cycles. Next, we define a β -cycle as a γ -cycle such that the node

v_1 belongs to e_1, e_m and cannot belong to any other e_j . Then, a hypergraph is β -acyclic if there are no β -cycles in it.

We close this section with some definitions regarding hypergraphs that will be useful in the rest of the paper. Consider a hypergraph $G = (V, E)$. Given two edges $e, f \in E$, we say that e is *adjacent* to f if $e \cap f \neq \emptyset$. A hypergraph G is *connected* if for any two distinct nodes $v_i, v_j \in G$, there is a sequence $v_i, e_i, v_q, e_q, \dots, e_r, v_j$ such that $v_i, v_q, \dots, v_j$ are distinct nodes of G , $e_i, e_q, \dots, e_r$ are distinct edges of G , and every node belongs to the edge that come before it and after it in the sequence. Let V' be a subset of V . A hypergraph (V', E') is a *partial hypergraph* of G if $E' \subseteq E$. The *section hypergraph* of G induced by V' is the hypergraph (V', E') , where $E' = \{e \in E : e \subseteq V'\}$. The *connected components* of G are the maximal connected section hypergraphs of G . The *support hypergraph* of a valid inequality $az \leq b$ for MP_G , is the hypergraph $G(a) = (V(a), E(a))$, where $V(a) := \{v \in V : a_v \neq 0\} \cup \{v \in V : \exists e \in E \text{ s.t. } v \in e, a_e \neq 0\}$, and $E(a) := \{e \in E : a_e \neq 0\}$. By a_t we denote the entry of the vector a corresponding to the variable z_t for $t \in V \cup E$.

2. Running Intersection Inequalities Are CG Cuts

In this section, we analyze the Chvátal rank of running intersection inequalities, a class of valid inequalities for the multilinear polytope introduced in Del Pia and Khajavirad (2021). First, we give the formal definition of running intersection inequalities.

A family F of subsets of a finite set V has the *running intersection property* if there exists an ordering $s_1, s_2, \dots, s_m$ of the sets in F such that, for each $i = 2, \dots, m$, there exists $j < i$ such that $s_i \cap (\cup_{k < i} s_k) \subseteq s_j$. An ordering $s_1, s_2, \dots, s_m$ satisfying the above condition is called a *running intersection ordering* of F . Each running intersection ordering $s_1, s_2, \dots, s_m$ of F induces a collection of sets

$$N(s_1) := \emptyset, \quad N(s_i) := s_i \cap \left(\bigcup_{k < i} s_k \right) \quad \text{for } i = 2, \dots, m.$$

In the remainder of the paper, for a nonnegative integer m , we denote by $[m]$ the set $\{1, \dots, m\}$.

Definition 1. Consider a hypergraph $G = (V, E)$. Let $e_0 \in E$ and let $e_i, i \in [m]$, be a collection of edges in E , adjacent to e_0 , such that the family $\tilde{E} := \{e_0 \cap e_i : i \in [m]\}$ has the running intersection property. Consider a running intersection ordering of $\tilde{E}$ with the corresponding sets $N(e_0 \cap e_i)$, for $i \in [m]$. For each $i \in [m]$ with $N(e_0 \cap e_i) \neq \emptyset$, let u_i be a node in $N(e_0 \cap e_i)$. We define a running intersection inequality as

$$- \sum_{\substack{i \in [m] \\ N(e_0 \cap e_i) \neq \emptyset}} z_{u_i} + \sum_{v \in e_0 \setminus \bigcup_{i \in [m]} e_i} z_v + \sum_{i \in [m]} z_{e_i} - z_{e_0} \leq n_0 + |\{i \in [m] : N(e_0 \cap e_i) = \emptyset\}| - 1, \quad (1)$$

where n_0 is the number of nodes in e_0 not contained in any edge $e_i, i \in [m]$. We refer to e_0 as the center and to $e_i, i \in [m]$, as the neighbors.

The reader can find an illustration of an example of the support hypergraph of such inequalities in figure 2 in Del Pia and Khajavirad (2021).

Theorem 1. Running intersection inequalities are CG cuts for MP_G^{LP} .

Proof. In order to make this proof simpler to describe, we will be using the following class of redundant valid inequalities for MP_G^{LP} :

$$z_e \leq 1 \quad \forall e \in E. \quad (\eta_e)$$

Consider a running intersection Inequality (1); let us denote it by $az \leq b$. If $m = 0$, then the inequality is in the standard linearization, hence we assume $m \geq 1$. To show that it is a CG cut for MP_G^{LP} , we provide a non-negative combination of the inequalities $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}$, and η_e , which we denote by $\pi z \leq \pi_0$. We indicate by $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}$, and η_e the multipliers associated with the inequalities $(\alpha_v), (v_e), (\varepsilon_e), (\delta_{v,e})$, and (η_e) , respectively.

Let e_0 and $\tilde{E}$ be as in Definition 1, and let $e_0 \cap e_1, e_0 \cap e_2, \dots, e_0 \cap e_m$ be a running intersection ordering of the family $\tilde{E}$. We partition the nodes of $e_0 \cap \bigcup_{i \in [m]} e_i$ in two sets U, W , where $U := \{u_i : i \in [m], N(e_0 \cap e_i) \neq \emptyset\}$ contains the nodes whose variables belong to the first sum of (1), and $W := (e_0 \cap \bigcup_{i \in [m]} e_i) \setminus U$. Define $\gamma := |e_0 \cap \bigcup_{i \in [m]} e_i| \geq 1$. Then, the multipliers are defined as follows. For ease of exposition, all the multipliers not explicitly defined are set to zero.

1. Set $\varepsilon_{e_0} := 1/\gamma, v_{e_0} := 1 - 1/\gamma, \alpha_v := 1 - 1/\gamma$, for every $v \in e_0 \setminus \bigcup_{i \in [m]} e_i$.

Next, we define the multipliers $\delta_{v,e}, \eta_e$ of the edges $e_1, \dots, e_m$ recursively.

2. Going backward, consider e_i with $i = m, \dots, 1$.

2.1. Set $\delta_{w,e_i} := 1/\gamma$ for every $w \in W \cap e_i \setminus \bigcup_{j>i} e_j$.

2.2. For each index $j > i$ such that $N(e_0 \cap e_j) \neq \emptyset$, $u_j \in e_i$, and $u_j \notin e_\ell$ for $\ell = i+1, \dots, j-1$, set $\delta_{u_j,e_i} := 1 - (\delta_{u_j,e_j} - 1/\gamma)$.

2.3. If $N(e_0 \cap e_i) \neq \emptyset$, then set $\delta_{u_i,e_i} := 1 - \sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i}$.

Otherwise, set $\eta_{e_i} := 1 - \sum_{v \in e_i} \delta_{v,e_i}$.

We observe that, in order to prove this theorem, it suffices to show that the inequality $\pi z \leq \lfloor \pi_0 \rfloor$ is a CG cut for MP_G^{LP} and is equal to $az \leq b$. The proof of this fact is rather technical and requires some involved steps that rely on the combinatorial structure of the support hypergraph of (1). For ease of exposition, we simply state here the statements of the three claims that complete the proof. The interested reader can find the corresponding proofs in Section 8.1.

Claim 1. The multipliers $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}, \eta_e$ are nonnegative.

Claim 2. The left-hand side of $az \leq b$ is equal to πz .

Claim 3. The right-hand side of $az \leq b$ coincides with $\lfloor \pi_0 \rfloor$. $\square$

We remark that Theorem 1 yields an alternative proof of the validity of the running intersection inequalities. We recall that running intersection inequalities are a generalization of both two-link and flower inequalities; therefore, Theorem 1 immediately shows that also such cutting planes are CG cuts for MP_G^{LP} . In Del Pia and Khajavirad (2021), the authors introduce kite-free β -acyclic hypergraphs, a class that in particular contains all γ -acyclic hypergraphs. The authors further show that, for hypergraphs in this class, the multilinear polytope coincides with the running intersection relaxation. As a consequence, Theorem 1 implies that MP_G^{LP} has Chvátal rank 1 when G is kite-free β -acyclic.

3. Odd β -Cycle Inequalities

In this section, our aim is to introduce valid inequalities for MP_G that are deeper than the running intersection inequalities, in the sense that their Chvátal rank can be larger than 1. Because all the inequalities considered in Section 2 correspond to β -acyclic hypergraphs, we decide to consider here hypergraphs that contain β -cycles. We start by defining our *odd β -cycle inequalities*. In the remainder of the paper, given a β -cycle $C = v_1, e_1, v_2, \dots, v_m, e_m, v_1$ in a hypergraph $G = (V, E)$, we denote by $V(C) := \{v_1, \dots, v_m\} \subseteq V$, and by $E(C) := \{e_1, \dots, e_m\} \subseteq E$.

Definition 2. Consider a hypergraph $G = (V, E)$, let $C = v_1, e_1, v_2, \dots, v_m, e_m, v_1$ be a β -cycle in G , and let E^-, E^+ be a partition of $E(C)$ such that $k := |E^-|$ is odd and $e_1 \in E^-$. Let $D := \{e_{p+1}, e_{p+2}, \dots, e_m\}$, where e_p is the last edge in C that belongs to E^- . We denote by $f_1, \dots, f_k$ the subsequence of $e_1, \dots, e_m$ of the edges in E^- . Let $S_1 := (\cup_{e \in E^-} e) \setminus \cup_{e \in E^+} e$ and $S_2 := V(C) \setminus \cup_{e \in E^-} e$. With this notation in place, we make the following assumptions:

(a) Every node $v \in \bigcup_{i=1}^m e_i$ is contained in at most two edges among $e_1, \dots, e_m$.

(b) For every edge $e_i \in E^+ \setminus D$, every edge in E^- adjacent to e_i (if any) is either e_{i-1} or e_{i+1} .

(c) No edge in D is adjacent to an edge f_i with i even.

(d) At least one of the following two conditions holds:

(d1) For every $v \in S_1$, either v is contained in just one edge $e \in E^-$, or it is contained in two edges f_i, f_j with i odd and j even.

(d2) For every $e' \in E^-$ and $e'' \in D$ such that $e' \cap e'' \neq \emptyset$, then either $e' = e_1$, $e'' = e_m$ or $e' = e_p$, $e'' = e_{p+1}$.

We then define the odd β -cycle inequality corresponding to C and E^- as

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \quad (2)$$

We observe that assumption (a) is equivalent to

(a') All γ -cycles of G that can be formed using edges of C are β -cycles.

To see that (a) implies (a'), let C' be a γ -cycle of G with $E(C') \subseteq \{e_1, \dots, e_m\}$. We have $V(C') \subseteq \bigcup_{e \in E(C')} e \subseteq \bigcup_{i=1}^m e_i$, and (a) implies that each node in $V(C')$ is contained in at most two edges in $E(C')$; thus, C' is also a β -cycle. Conversely, to show that (a') implies (a), we assume by contradiction that there exists $v \in \bigcup_{i=1}^m e_i$ that is contained in three distinct edges $e_{i_1}, e_{i_2}, e_{i_3} \in \{e_1, \dots, e_m\}$. Without loss of generality, let us assume that $i_1 < i_2 < i_3$. Let P be the substring of C from e_{i_1} to e_{i_3} , and define $C' := v, P, v$. By substring we mean a contiguous sequence within a string. By definition of β -cycle, the node v is not in $V(C)$, thus v does not appear in the

sequence P . Because v is contained in three different edges in $E(C')$, we have shown that C' is a γ -cycle that is not a β -cycle.

It is simple to see that odd β -cycle inequalities are never valid for MP_G^{LP} . This can be checked by considering the vector $\bar{z}$ in MP_G^{LP} defined as follows, given a β -cycle C and sets E^-, E^+ satisfying assumptions (a)–(d). For every edge $e_i \in E^-$, we set $\bar{z}_{v_i} := 1/2$, $\bar{z}_{v_{i+1}} := 1/2$, $\bar{z}_v := 1$ for every $v \in e_i \setminus \{v_i, v_{i+1}\}$, and $\bar{z}_{e_i} := 0$, where v_i and v_{i+1} are the nodes in C coming immediately before and after e_i . Next, we consider the edges in E^+ . Let $e_i \in E^+$. For every $v \in e_i$ such that its corresponding variable has not been defined yet, we set $\bar{z}_v := 1/2$. Moreover, we define $\bar{z}_{e_i} := 1/2$. All the remaining entries of $\bar{z}$ can be set to 0.

Later in this section we will show the following result.

Theorem 2. *Odd β -cycle Inequalities (2) are inequalities of Chvátal rank at most 2 for MP_G^{LP} . In particular, they are valid for MP_G .*

Although the definition of odd β -cycle inequalities is not straightforward, it yields a large class of inequalities that contains odd-cycle inequalities for the Boolean quadric polytope (Padberg 1989), as well as their lifting by node addition obtained in corollary 10 in Del Pia and Khajavirad (2017). In particular, the inequalities given in Del Pia and Khajavirad (2017) have the same form (2) of our odd β -cycle inequalities, but they are defined only in the special case where the hypergraph $G = (V, E)$, with edges $e_1, \dots, e_m$, satisfies $m \geq 3$, and every edge e_i has nonempty intersection only with e_{i-1} and e_{i+1} for every $i \in \{1, \dots, m\}$, where, for convenience, we define $e_{m+1} := e_1$ and $e_0 := e_m$. If $m = 3$, it is also required that $e_1 \cap e_2 \cap e_3 = \emptyset$. In this paper, we refer to a hypergraph of this type as a *cycle hypergraph*. If G is a cycle hypergraph, we now explain why assumptions (a)–(d) are trivially satisfied. (a) Every node in G is contained in at most two edges. (b) Every edge e_i in $E(C)$ intersects only e_{i-1} and e_{i+1} . (c) Each $e \in D$ is not adjacent to edges $f_2, f_3, \dots, f_{k-1}$. (d1) Every $v \in S_1$ is either contained in just one edge $e \in E^-$, or in exactly two consecutive edges f_i, f_{i+1} . Moreover, assumption (d2) is also satisfied.

Next, we provide an example of odd β -cycle inequalities.

Example 1. Let $G = (V, E)$ be the hypergraph depicted in Figure 1 and defined by

$$V = \{v_1, \dots, v_9\}, E = \{e_{1278}, e_{23}, e_{349}, e_{4578}, e_{569}, e_{16}\},$$

where the edge e_l contains the nodes with indices in l .

A β -cycle in G is given by

$$C = v_1, e_{1278}, v_2, e_{23}, v_3, e_{349}, v_4, e_{4578}, v_5, e_{569}, v_6, e_{16}, v_1.$$

If we define $E^- := \{e_{1278}, e_{23}, e_{349}\}$, it is simple to check that C and E^- satisfy assumptions (a)–(d) in Definition 2. In particular, assumption (d1) is satisfied and (d2) is not. The corresponding odd β -cycle Inequality (2) is

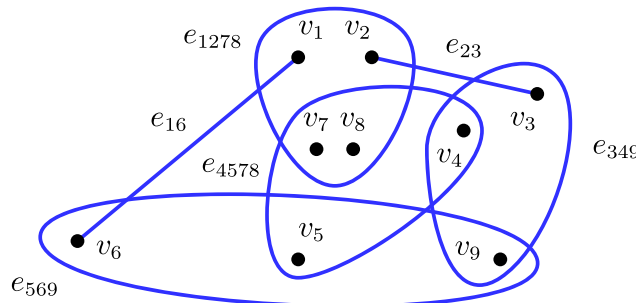
$$z_{v_2} + z_{v_3} - z_{e_{1278}} - z_{e_{23}} - z_{e_{349}} - z_{v_5} - z_{v_6} + z_{e_{4578}} + z_{e_{569}} + z_{e_{16}} \leq 1. \quad (3)$$

Using the same cycle C , we could define instead $E^- := \{e_{1278}, e_{23}, e_{4578}\}$. Also, in this case, C and E^- satisfy assumptions (a)–(d) in Definition 2. However, in this case, assumption (d2) is satisfied and (d1) is not. The corresponding odd β -cycle Inequality (2) is

$$z_{v_2} + z_{v_7} + z_{v_8} - z_{e_{1278}} - z_{e_{23}} - z_{e_{4578}} - z_{v_6} + z_{e_{349}} + z_{e_{569}} + z_{e_{16}} \leq 3. \quad (4)$$

It can be checked that (3) and (4) are both facet defining for MP_G , for example, by using PORTA (PORTA 2015). In this example, MP_G has a total of 156 facet-defining inequalities; 35 of them are in the standard linearization, 2 are flower inequalities, and 70 are odd β -cycle inequalities.

Figure 1. (Color online) Hypergraph Containing β -Cycles



The hypergraph G given in this example satisfies assumption (a) for any choice of C . In this case, all the remaining assumptions are necessary for the validity of (2). Indeed, for any of the three assumptions (b), (c), and (d), there exists an inequality of Form (2) that does not satisfy that specific assumption but satisfies the remaining two and is not valid for MP_G . We now provide three examples that show this. We use the same cycle of before, that is, $C = v_1, e_{1278}, v_2, e_{23}, v_3, e_{349}, v_4, e_{4578}, v_5, e_{569}, v_6, e_{16}$, and v_1 , for all the three examples. What changes throughout the examples is the partition E^-, E^+ of E . Consider first the case in which $|E^-| = 3$ with $f_1 = e_{1278}, f_2 = e_{4578}, f_3 = e_{569}$. It follows that $D = \{e_{16}\}$ and $E^+ \setminus D = \{e_{23}, e_{349}\}$. Here, only assumption (b) does not hold. The corresponding inequality of Form (2) is $z_{v_5} + z_{v_7} + z_{v_8} - z_{e_{1278}} - z_{e_{4578}} - z_{e_{569}} - z_{v_3} + z_{e_{16}} + z_{e_{23}} + z_{e_{349}} \leq 3$, which is not valid for MP_G . In fact, it can be checked that both $z_{v_4} + z_{v_5} + z_{v_7} + z_{v_8} - z_{e_{1278}} - z_{e_{4578}} - z_{e_{569}} - z_{v_3} + z_{e_{16}} + z_{e_{23}} + z_{e_{349}} \leq 4$ and $z_{v_5} + z_{v_7} + z_{v_8} + z_{v_9} - z_{e_{1278}} - z_{e_{4578}} - z_{e_{569}} - z_{v_3} + z_{e_{16}} + z_{e_{23}} + z_{e_{349}} \leq 4$ are facet-defining inequalities for MP_G , which would not be possible if the previous inequality was valid. Next, let $f_1 = e_{23}, f_2 = e_{349}$, and $f_3 = e_{4578}$. It follows that $E^+ = D$ in this case. In particular, $D = \{e_{569}, e_{16}, e_{1278}\}$. It is easy to see that (c) is the only violated assumption. Similarly to the previous case, we observe that the corresponding inequality, $z_{v_3} + z_{v_4} - z_{e_{23}} - z_{e_{349}} - z_{e_{4578}} - z_{v_1} - z_{v_6} + z_{e_{16}} + z_{e_{1278}} + z_{e_{569}} \leq 1$, is not valid, as $z_{v_3} + z_{v_4} + z_{v_9} - z_{e_{23}} - z_{e_{349}} - z_{e_{4578}} - z_{v_1} - z_{v_6} + z_{e_{16}} + z_{e_{1278}} + z_{e_{569}} \leq 2$ induces a facet of MP_G . The last example considers $|E^-| = 5$ with $f_1 = e_{23}, f_2 = e_{349}, f_3 = e_{4578}, f_4 = e_{569}$, and $f_5 = e_{16}$, while $E^+ = D = \{e_{1278}\}$. It follows that assumptions (b) and (c) hold, whereas assumption (d) does not, because both (d1) and (d2) are not satisfied in this case. The resulting inequality derived from (2) is $z_{v_3} + z_{v_4} + z_{v_5} + z_{v_6} + z_{v_9} - z_{e_{23}} - z_{e_{349}} - z_{e_{4578}} - z_{e_{569}} - z_{e_{16}} + z_{e_{1278}} \leq 3$. The fact that the inequality $z_{v_3} + z_{v_4} + z_{v_5} + z_{v_6} + z_{v_7} + z_{v_8} + z_{v_9} - z_{e_{23}} - z_{e_{349}} - z_{e_{4578}} - z_{e_{569}} - z_{e_{16}} + z_{e_{1278}} \leq 5$ is facet defining for MP_G implies that the previous inequality is not valid for MP_G . $\diamond$

Theorem 2 states that odd β -cycle inequalities have Chvátal rank at most 2 for MP_G^{LP} . This leaves open the possibility that the Chvátal rank of these inequalities could be always equal to 1. We show next that this is not the case, by providing an example of an odd β -cycle inequality that is not valid for the Chvátal closure of MP_G^{LP} .

Example 2. Let $G = (V, E)$ be the cycle hypergraph depicted in Figure 2 and defined by

$$V = \{v_1, \dots, v_9\}, E = \{e_{123}, e_{345}, e_{4567}, e_{678}, e_{89}, e_{129}\}.$$

By defining $E^- = \{e_{345}, e_{678}, e_{129}\}$, we obtain the odd β -cycle inequality

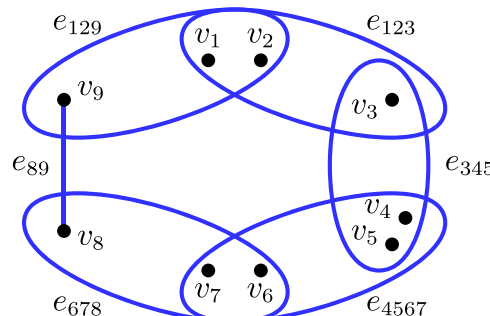
$$-z_{e_{345}} - z_{e_{678}} - z_{e_{129}} + z_{e_{123}} + z_{e_{4567}} + z_{e_{89}} \leq 1. \quad (5)$$

By corollary 3 in Hartmann et al. (1999), it follows immediately that (5) has Chvátal rank 2. Indeed, corollary 3 holds because MP_G is full-dimensional, (5) is facet defining for MP_G (both these facts are proved in Del Pia and Khajavirad 2017), and finally all the components of (5) are relatively prime integers.

Usually proving directly that an inequality has Chvátal rank greater than 1 is hard, and not many proofs of this kind are available in the literature. However, in this case we are able to provide a direct and simple proof, which we give next.

In order for (5) to have Chvátal rank 2, we verify that (5) is not valid for the Chvátal closure $\mathcal{C}(\text{MP}_G^{\text{LP}})$ of MP_G^{LP} . Assume, for a contradiction, that (5) is valid for $\mathcal{C}(\text{MP}_G^{\text{LP}})$. Because G is a cycle hypergraph, it follows from corollary 10 in Del Pia and Khajavirad (2017) that (5) is facet defining for MP_G . As $\text{MP}_G \subseteq \mathcal{C}(\text{MP}_G^{\text{LP}})$, all the vectors in MP_G that satisfy at Equality (5) belong to $\mathcal{C}(\text{MP}_G^{\text{LP}})$ too. Therefore, (5) is facet defining for $\mathcal{C}(\text{MP}_G^{\text{LP}})$ as well. It then follows that (5) is a CG cut for MP_G^{LP} ; therefore, if we maximize the left-hand side of (5) over MP_G^{LP} , we should get an objective value π_0 such that $\lfloor \pi_0 \rfloor = 1$. However, it can be checked that the vector

Figure 2. (Color online) Cycle Hypergraph



defined by $z_{v_1} = z_{v_2} = z_{v_3} = z_{v_4} = z_{v_5} = z_{v_6} = z_{v_7} = 1/2$, $z_{v_8} = z_{v_9} = 1$, $z_{e_{123}} = z_{e_{4567}} = 1/2$, $z_{e_{345}} = z_{e_{678}} = z_{e_{129}} = 0$, and $z_{e_{89}} = 1$ is feasible to MP_G^{LP} and yields $\pi_0 = 2$. We obtained a contradiction; thus, we conclude that (5) is not valid for $\mathcal{C}(\text{MP}_G^{\text{LP}})$. $\diamond$

In order to prove Theorem 2, we show that each odd β -cycle inequality can be obtained as a CG cut for the flower relaxation of MP_G , that is, the polyhedron obtained from MP_G^{LP} by adding all flower inequalities. Let us recall here the definition of flower inequalities.

Definition 3. Consider a hypergraph $G = (V, E)$. Let $f \in E$ and let $T \subseteq E \setminus \{f\}$ be a subset of edges adjacent to f such that $f \cap e \cap e' = \emptyset$ for all $e, e' \in T$ with $e \neq e'$. Then the flower inequality centered at f with neighborhood T , is given by

$$\sum_{v \in f \setminus \bigcup_{e \in T} e} z_v + \sum_{e \in T} z_e - z_f \leq \left| f \setminus \bigcup_{e \in T} e \right| + |T| - 1. \quad (\theta_f)$$

Flower inequalities were introduced in Del Pia and Khajavirad (2018a). However, our definition is more general than the original one for three reasons. (i) In Definition 3, the set T could be empty, whereas in the previous definition it must be nonempty. (ii) The condition $f \cap e \cap e' = \emptyset$ in Definition 3 replaces the previous stronger assumption $e \cap e' = \emptyset$. (iii) In Definition 3, we require that each edge in T is adjacent to f , whereas originally it was assumed that $|f \cap e| \geq 2$ for every $e \in T$. Flower inequalities in Definition 3 are still a special case of running intersection inequalities. This follows immediately by observing that the set $\{f \cap e : e \in T\}$ has the running intersection property, because $f \cap e \cap e' = \emptyset$ for every $e, e' \in T$, $e \neq e'$ by Definition 3. Therefore, Theorem 1 implies that they are CG cuts for MP_G^{LP} .

Proposition 1. Odd β -cycle inequalities are CG cuts for the flower relaxation of MP_G .

Proof. The overall scheme of the proof is similar to the one of Theorem 1. Consider an odd β -cycle Inequality (2), we denote it by $az \leq b$. We provide a nonnegative combination $\pi z \leq \pi_0$ of the inequalities $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}$, and θ_f such that π coincides with a , and $\lfloor \pi_0 \rfloor$ is equal to b . We denote by $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}$, and θ_f the multipliers associated with the inequalities (α_v) , (v_e) , (ε_e) , $(\delta_{v,e})$, and (θ_f) , respectively.

For every $f_i \in E^-$, we define the set $T_i := \{e \in E^+ : e \cap f_i \neq \emptyset\}$. We denote by (θ_{f_i}) the flower inequality with center f_i and neighbors T_i . Because of (a), we have that $e \cap e' \cap e'' = \emptyset$ for every three edges in $E(C)$. In particular, this is true whenever $e \in E^-$ and $e', e'' \in E^+$. Hence, the flower inequalities (θ_{f_i}) are well defined. If $\sum_{i=1}^k |T_i \cap D| \geq 1$, we set $T := \sum_{i=1}^k |T_i \cap D|$. In this case, because we are dealing with a cycle, we have that $\sum_{i=1}^k |T_i \cap D| \neq 1$; thus, $T \geq 2$. Otherwise, if $\sum_{i=1}^k |T_i \cap D| = 0$, we set $T := 2$. For ease of exposition, all multipliers not explicitly defined are set to zero. The nonzero multipliers are defined in the following rules.

1. For i odd, set $\theta_{f_i} := 1/T$, and $v_{f_i} := (T-1)/T$. For i even, set $\theta_{f_i} := (T-1)/T$, and $v_{f_i} := 1/T$.
2. For every v contained in only one edge $f_i \in E^-$ and in no edge of E^+ , set $\alpha_v := (T-1)/T$ if i is odd, and set $\alpha_v := 1/T$ if i is even.
3. Consider the multipliers regarding inequalities involving edges in $E^+ \setminus D$. Note that $E^+ \setminus D$ can be partitioned into maximal length substrings of $e_1, \dots, e_p$. For each such substring $e_i, \dots, e_{i+h}$:
 - 3.1. Note that $e_{i-1} = f_j$ for some j . Then set $\delta_{v_{i+1}, e_i} := 1 - \theta_{f_j}$.
 - 3.2. For $l = i+1, \dots, i+h-1$, set $\delta_{v_l, e_l} := \theta_{f_j}$, and then $\delta_{v_{i+1}, e_l} := 1 - \theta_{f_j}$.
 - 3.3. Set $\delta_{v_{i+h}, e_{i+h}} := \theta_{f_j}$.
4. Let us focus here on the edges in D and their related inequalities. For every edge $e_i \in D$, define the number $\Delta_i := |\{f \in E^- : e_i \cap f \neq \emptyset\}|$.
 - 4.1. Set $\delta_{v_{p+2}, e_{p+1}} := 1 - \Delta_{p+1}/T$.
 - 4.2. For $i = p+2, \dots, m-1$, set $\delta_{v_i, e_i} := \sum_{j=p+1}^{i-1} \Delta_j/T$, and set $\delta_{v_{i+1}, e_i} := 1 - \sum_{j=p+1}^i \Delta_j/T$.
 - 4.3. Set $\delta_{v_m, e_m} := \sum_{j=p+1}^{m-1} \Delta_j/T$.

We remark that, when $\sum_{i=1}^k |T_i \cap D| = 0$, then $D = \emptyset$; thus, we do not need to consider rule 4.

The thesis is obtained by showing that the inequality $\pi z \leq \lfloor \pi_0 \rfloor$ is a CG cut for the flower relaxation of MP_G . Moreover, $\pi z \leq \lfloor \pi_0 \rfloor$ is equal to $az \leq b$. Next, we state the three claims that yield the proof of this statement. The proofs of these claims can be found in Section 8.2.

Claim 4. The multipliers $\alpha_v, v_e, \varepsilon_e, \delta_{v,e}, \theta_f$ are nonnegative.

Claim 5. The left-hand side of $az \leq b$ coincides with πz .

Claim 6. The right-hand side of $az \leq b$ is equal to $\lfloor \pi_0 \rfloor$. $\square$

We are now ready to give the straightforward proof of Theorem 2.

Proof of Theorem 2. We observe that the thesis of Theorem 2 follows from combining Proposition 1 with the fact that flower inequalities are CG cuts for MP_G^{LP} , as observed after Definition 3. $\square$

4. Separation of the Odd β -Cycle Inequalities on Cycle Hypergraphs

Understanding whether the separation problem for the odd β -cycle inequalities can be solved in polynomial time is an open question in the general setting. However, we are able to provide a positive result for the special case in which the whole hypergraph that represents the instance is a cycle hypergraph. In fact, we show that under this assumption the separation problem can be solved very efficiently. More formally, given a cycle hypergraph G and $\bar{z} \in MP_G^{LP}$, we can either understand if $\bar{z}$ satisfies all odd β -cycle inequalities, or we can find an odd β -cycle inequality violated by $\bar{z}$ in linear time. We recall that the definition of cycle hypergraph can be found in Section 3 just after the statement of Theorem 2.

Theorem 3. *Given a cycle hypergraph G and $\bar{z} \in MP_G^{LP}$, it is possible to check in linear time whether $\bar{z}$ satisfies all the odd β -cycle inequalities valid for MP_G , or find one odd β -cycle inequality valid for MP_G that is violated by $\bar{z}$.*

Proof. We start by expressing any odd β -cycle Inequality (2) in a more convenient way. We observe that $\lfloor k/2 \rfloor = (|E^-| - 1)/2$ in (2), because $k = |E^-|$ is odd. This means that an equivalent form of (2) is given by

$$\sum_{v \in S_1} (1 - z_v) + \sum_{e \in E^-} \left(\frac{1}{2} + z_e \right) - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \sum_{v \in S_2} z_v - \sum_{e \in E^+} z_e \geq \frac{1}{2}. \quad (6)$$

Next, $m = |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + |S_2| + 2\sigma$, where $\sigma = |\{i : e_i \in E^+, e_{i+1} \in E^-\}|$ is the same σ that appears in the proof of Claim 6. Second, observe that $\sigma = |E^+| - |S_2|$. Then, $|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| = m + |S_2| - 2|E^+|$. Therefore, (6) is equivalent to

$$\sum_{v \in S_1} (1 - z_v) + \sum_{e \in E^-} \left(z_e - \frac{1}{2} \right) + \sum_{v \in S_2} (z_v - 1) + \sum_{e \in E^+} (1 - z_e) \geq \frac{1}{2}. \quad (7)$$

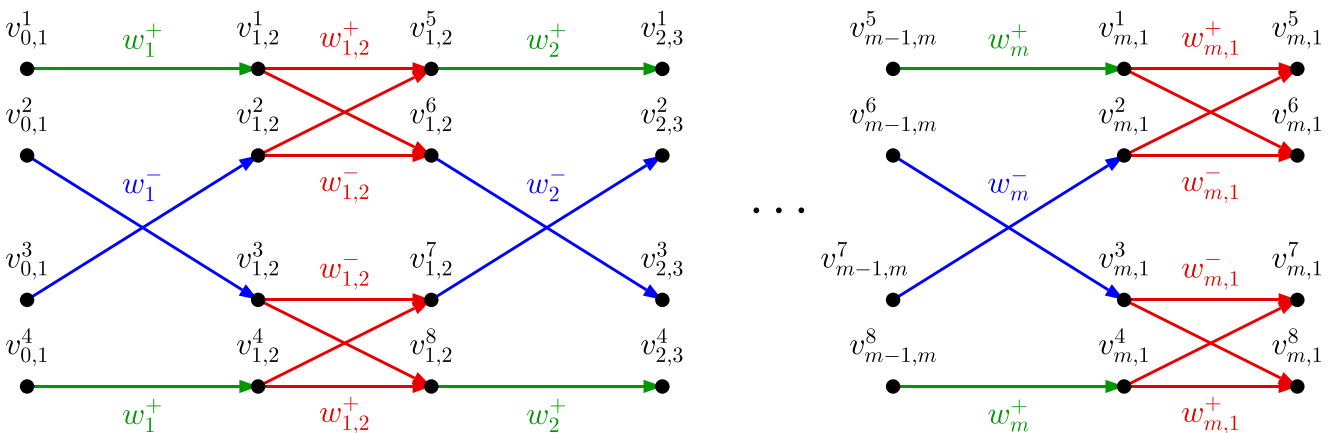
We next show that, in order to solve the separation problem, it suffices to find sets S_1 , E^- , S_2 , and E^+ that satisfy Definition 2 and minimize the left-hand side of (7), and checking whether the resulting value is greater than or equal to the right-hand side of (7), given a vector $\bar{z} \in MP_G^{LP}$. Actually, we remark that, given a partition E^-, E^+ of E , the corresponding desired sets S_1 and S_2 can be found automatically. Indeed, these sets can be computed as

$$S_1 = \bigcup_{e_i \in E^-} (e_i \setminus (e_{i-1} \cup e_{i+1})) \cup \bigcup_{\substack{i=1, \dots, m \\ e_i, e_{i+1} \in E^-}} e_i \cap e_{i+1}, \quad S_2 = \bigcup_{\substack{i=1, \dots, m \\ e_i, e_{i+1} \in E^+}} \hat{u}_i, \quad (8)$$

where $e_{m+1} = e_1$ and $\hat{u}_i = \arg \min_{v \in e_i \cap e_{i+1}} \bar{z}_v$. The definition of $\hat{u}_i$ is because our goal is to minimize the left-hand side of (7).

Similar to the result in Barahona and Mahjoub (1986), we reduce this problem to solving a shortest path problem on a digraph. Let $G = (V, E)$ be the cycle hypergraph, and let $\bar{z} \in MP_G^{LP}$ the vector of the

Figure 3. (Color online) Digraph G'



separation problem. We next construct a digraph $G' = (V', A')$, where $|V'| = 8|E| + 4$ and $|A'| = 12|E|$. A picture of G' can be found in Figure 3.

For the remainder of the section, we will use the following convention regarding the indices: that is, $m + 1 = 1$ and $0 = m$. Each intersection $e_i \cap e_{i+1}$ in G , for $i = 1, \dots, m - 1$, is represented by eight nodes in G' , namely $v_{i,i+1}^j$ where $j = 1, \dots, 8$. Note that $e_1 \cap e_m$ is represented instead by 12 nodes in G' : 8 of them are at the right of the digraph, $v_{m,1}^1, \dots, v_{m,1}^8$, and the remaining 4 are at the left of the digraph: $v_{0,1}^1, v_{0,1}^2, v_{0,1}^3$, and $v_{0,1}^4$.

Every edge in G is replaced with 12 arcs in G' . Let us consider e_1 first. The two arcs that correspond to $e_1 \in E^+$ are $(v_{0,1}^1, v_{1,2}^1)$ and $(v_{0,1}^4, v_{1,2}^4)$, whereas the arcs that represent $e_1 \in E^-$ are $(v_{0,1}^2, v_{1,2}^3)$ and $(v_{0,1}^3, v_{1,2}^2)$. The other eight arcs address the issues arising whether e_1 and e_2 are both in E^- or E^+ , or if one of them is in E^- and the other belongs to E^+ . The arcs $(v_{1,2}^1, v_{1,2}^5)$ and $(v_{1,2}^4, v_{1,2}^8)$ deal with the case $e_1, e_2 \in E^+$. Similarly, $(v_{1,2}^2, v_{1,2}^6)$ and $(v_{1,2}^3, v_{1,2}^7)$ handle the case $e_1, e_2 \in E^-$. The remaining arcs correspond to the cases where e_1 and e_2 are not in the same set. Indeed, $(v_{1,2}^1, v_{1,2}^6)$ and $(v_{1,2}^4, v_{1,2}^7)$ represent the case in which $e_1 \in E^+$ and $e_2 \in E^-$. On the other hand, $(v_{1,2}^2, v_{1,2}^5)$ and $(v_{1,2}^3, v_{1,2}^8)$ correspond to the case where $e_1 \in E^-$ and $e_2 \in E^+$. The construction of the other arcs is analogous, where $v_{0,1}^j, v_{1,2}^j$ are replaced by, respectively, $v_{i-1,i}^{j+4}$ and $v_{i,i+1}^j$, for $j = 1, \dots, 4$ and $i = 2, \dots, m$. We observe that G' is acyclic.

We now describe the weight vector w whose entries are the weights on the arcs of G' . For each $i = 2, \dots, m$, the arcs $(v_{i-1,i}^5, v_{i,i+1}^1)$, $(v_{i-1,i}^8, v_{i,i+1}^4)$ have equal weight, which we denote by w_i^+ . Likewise, the arcs $(v_{0,1}^1, v_{1,2}^1)$ and $(v_{0,1}^4, v_{1,2}^4)$ are both assigned the weight w_1^+ . These weights are defined as

$$w_i^+ := -\bar{z}_{e_i}, \text{ for } i = 1, \dots, m.$$

Next, we deal with the weights of the arcs corresponding to $e_i \in E^-$, which are $(v_{0,1}^2, v_{1,2}^3)$, $(v_{0,1}^3, v_{1,2}^2)$, and $(v_{i-1,i}^6, v_{i,i+1}^3)$, $(v_{i-1,i}^7, v_{i,i+1}^2)$, for $i = 2, \dots, m$. The arcs $(v_{i-1,i}^6, v_{i,i+1}^3)$, $(v_{i-1,i}^7, v_{i,i+1}^2)$ are given weight w_i^- , for $i = 2, \dots, m$, whereas the arcs $(v_{1,2}^2, v_{1,2}^5)$, $(v_{1,2}^3, v_{1,2}^8)$ have both weight w_1^- . Such weights are

$$w_i^- := \sum_{v \in e_i \setminus \{e_{i-1}, e_{i+1}\}} (1 - \bar{z}_v) + \left(\frac{1}{2} + \bar{z}_{e_i}\right), \text{ for } i = 1, \dots, m.$$

It remains to define the weights $w_{i,i+1}^+$ and $w_{i,i+1}^-$. Both arcs $(v_{i,i+1}^1, v_{i+1,i+2}^5)$, $(v_{i,i+1}^4, v_{i+1,i+2}^8)$ have weight $w_{i,i+1}^+$, defined as

$$w_{i,i+1}^+ := \bar{z}_{e_i}, \text{ for } i = 1, \dots, m.$$

Finally, the arcs $(v_{i,i+1}^2, v_{i+1,i+2}^6)$, $(v_{i,i+1}^3, v_{i+1,i+2}^7)$ have equal weight $w_{i,i+1}^-$, which is

$$w_{i,i+1}^- := \sum_{v \in e_i \cap e_{i+1}} (1 - \bar{z}_v) - 1, \text{ for } i = 1, \dots, m.$$

The remaining weights are set equal to 0. Some weights can be negative. However, this does not constitute an issue, because G' is acyclic and therefore there cannot be any negative-cost cycles in G' .

It is easy to see that the weight of any path in G' from $v_{0,1}^1$ to $v_{m,1}^8$ or from $v_{0,1}^2$ to $v_{m,1}^7$ corresponds to the left-hand side of an odd β -cycle inequality for MP_G evaluated at $\bar{z}$. In fact, given any path in G' from $v_{0,1}^1$ to $v_{m,1}^8$ or from $v_{0,1}^2$ to $v_{m,1}^7$, we can construct a partition E^-, E^+ of E as follows. If we are considering a path from $v_{0,1}^1$ to $v_{m,1}^8$, then $e_1 \in E^+$. Otherwise, $e_1 \in E^-$. For $i = 2, \dots, m$, the edge e_i belongs to E^+ if either the arc $(v_{i-1,i}^5, v_{i,i+1}^1)$ or the arc $(v_{i-1,i}^8, v_{i,i+1}^4)$ is in the chosen path. Similarly, for $i = 2, \dots, m$, the edge e_i is in E^- if either $(v_{i-1,i}^6, v_{i,i+1}^3)$ or $(v_{i-1,i}^7, v_{i,i+1}^2)$ is part of the selected path. Observe that E^-, E^+ indeed form a partition of E , and also that $|E^-|$ is odd. We then compute the sets S_1 and S_2 by using (8). We finally recall that, because G is a cycle hypergraph, assumptions (a)–(d) are valid for any partition E^-, E^+ , and therefore also for the partition that was just defined. Hence, the inequality of Form (2) found by using the sets E^-, E^+, S_1 , and S_2 defined previously is a valid odd β -cycle inequality. By definition of the weight vector w , it follows that the weight of the chosen path is exactly equal to the value of the left-hand side of this odd β -cycle inequality when we consider $z = \bar{z}$.

On the other hand, we next show that for any odd β -cycle inequality of MP_G we can find a path in G' with weight lower than or equal to the value of the left-hand side of the chosen odd β -cycle inequality when evaluated in $z = \bar{z}$. Indeed, we can use the same construction of before backward in order to define the path. We observe, however, that in this case the weight of the constructed path can be lower than the value of the left-hand side of the selected odd β -cycle inequality when $z = \bar{z}$. This happens whenever there exists a node in S_2 in the odd β -cycle inequality that is not a minimizer of $\min_{v \in e_i \cap e_{i+1} : e_i, e_{i+1} \in E^+} \bar{z}_v$.

At this point, we compute two shortest paths in G' : the one from $v_{0,1}^1$ to $v_{m,1}^8$ and the one from $v_{0,1}^2$ to $v_{m,1}^7$. We choose the one yielding lower weight between the two, and check whether its weight is less than $1/2$, which is

the right-hand side of (7). If so, we found a violated odd β -cycle inequality. Namely, the one that can be constructed from the path with minimum weight, by following the procedure explained previously. Otherwise, we know that $\bar{z}$ satisfies all odd β -cycle inequalities for MP_G .

Lastly, we observe that the running time of this procedure is linear. In fact, the number of nodes and arcs of G' is linear in the number of nodes and edges of G . Then, both computing the weight vector w and solving the shortest path can be done in linear time. In particular, solving the shortest path problem in G' takes $O(|E'|)$ operations, since G' is an acyclic digraph. $\square$

5. Combining Perfect Formulations

In this section, we present a result of a different flavor that is of independent interest and that is used in the characterization of MP_G given in the next section. It deals with extending the multilinear polytope of a hypergraph G to the multilinear polytope of a new hypergraph G' obtained from G by replacing any node with a new edge containing arbitrarily many new nodes. To the best of our knowledge, this is the first result of this type for the multilinear polytope. We first show a very general lemma that allows us to combine two perfect formulations that overlap only in one variable.

Lemma 1. *Let $P = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R} : A(x, y) \leq b\}$, $Q = \{(y, z) \in \mathbb{R} \times \mathbb{R}^{n_2} : C(y, z) \leq d\}$ be polytopes with binary vertices. Then $R = \{(x, y, z) \in \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} : A(x, y) \leq b, C(y, z) \leq d\}$ is a polytope with binary vertices.*

Proof. Because $P \subseteq [0, 1]^{n_1} \times [0, 1]$ and $Q \subseteq [0, 1] \times [0, 1]^{n_2}$, the polyhedron R is contained in $[0, 1]^{n_1} \times [0, 1] \times [0, 1]^{n_2}$. Let $(\bar{x}, \bar{y}, \bar{z})$ be a vertex of R . We want to show that $(\bar{x}, \bar{y}, \bar{z}) \in \{0, 1\}^{n_1} \times \{0, 1\} \times \{0, 1\}^{n_2}$. Because it is a vertex, there exist $n_1 + 1 + n_2$ linearly independent constraints among $A(x, y) \leq b$, $C(y, z) \leq d$ that are active at $(\bar{x}, \bar{y}, \bar{z})$. In the inequalities from $A(x, y) \leq b$, only $n_1 + 1$ variables can appear with nonzero coefficient, so at most $n_1 + 1$ of these constraints can be linearly independent. Similarly there can be at most $n_2 + 1$ linearly independent constraints among the inequalities $C(y, z) \leq d$. Therefore, there are only two ways the $n_1 + 1 + n_2$ inequalities defining $(\bar{x}, \bar{y}, \bar{z})$ can be distributed among the systems $A(x, y) \leq b$ and $C(y, z) \leq d$: either $n_1 + 1$ are in $A(x, y) \leq b$ and n_2 are in $C(y, z) \leq d$, or n_1 are in $A(x, y) \leq b$ and $n_2 + 1$ are in $C(y, z) \leq d$. By symmetry, we can assume, without loss of generality, that we are in the first case.

The vector $(\bar{x}, \bar{y})$ is in P because it satisfies $A(x, y) \leq b$. Moreover, $n_1 + 1$ linearly independent constraints among $A(x, y) \leq b$ are active at $(\bar{x}, \bar{y})$. This implies that $(\bar{x}, \bar{y})$ is a vertex of P . Since P has binary vertices, we obtain $(\bar{x}, \bar{y}) \in \{0, 1\}^{n_1} \times \{0, 1\}$. In particular, these $n_1 + 1$ linearly independent constraints imply the constraint $y = \bar{y}$.

Consider now the vector $(\bar{y}, \bar{z})$. This vector is in Q , and it satisfies at equality n_2 linearly independent constraints among $C(y, z) \leq d$. Moreover, it also satisfies the equation $y = \bar{y}$. This equation must be linearly independent from the latter n_2 constraints, because it was obtained from the first $n_1 + 1$ constraints defining $(\bar{x}, \bar{y}, \bar{z})$. Hence, $(\bar{y}, \bar{z})$ must be a vertex of the polytope $F := \{(y, z) \in Q : y = \bar{y}\}$. Because $\bar{y} \in \{0, 1\}$ and Q is contained in $[0, 1] \times [0, 1]^{n_2}$, we have that F is a face of Q . Because Q has binary vertices, so does F , and we obtain $(\bar{y}, \bar{z}) \in \{0, 1\} \times \{0, 1\}^{n_2}$. We have shown that $(\bar{x}, \bar{y}, \bar{z}) \in \{0, 1\}^{n_1} \times \{0, 1\} \times \{0, 1\}^{n_2}$; thus, R has binary vertices. $\square$

Theorem 4. *Let $G = (V, E)$ be a hypergraph, let $w \in V$, and let f be a nonempty set of nodes disjoint from V . We define the hypergraph $G' = (V', E')$ as follows:*

$$\begin{aligned} V' &:= (V \setminus \{w\}) \cup f, \\ E' &:= \{f\} \cup \{e \in E : w \notin e\} \cup \{e \cup f : e \in E, w \in e\}. \end{aligned}$$

Then, a perfect formulation of the multilinear polytope $MP_{G'}$ is obtained from an external description of MP_G by

1. *Replacing z_w with z_f in every inequality of MP_G , and*
2. *Adding the standard linearization of the edge f .*

Proof. Let R be the polyhedron obtained by performing the operations in the statement. We can then apply Lemma 1, where $P = MP_G$, Q is the polytope defined by the standard linearization of the edge f , and y is the variable z_f . We obtain that R is a polytope with binary vertices. Note that R lives in a space of dimension $n = |V'| + |E'| = |V| + |E| + |f|$. Therefore, to conclude the proof, we only need to show that $R \cap \{0, 1\}^n = MP_{G'} \cap \{0, 1\}^n$. For ease of notation, in this proof, we denote by S_G and $S_{G'}$ the set of binary point in MP_G and $MP_{G'}$, respectively.

First, we prove the inclusion $R \cap \{0, 1\}^n \supseteq S_{G'}$. Let $\bar{z}'$ be a vector in $S_{G'}$. As $\bar{z}'$ is binary, we only need to show that $\bar{z}'$ is in R . Clearly $\bar{z}'$ satisfies all inequalities of the standard linearization of f . Let $\bar{z}$ be the vector in the space of G obtained from $\bar{z}'$. Formally, $\bar{z}$ is constructed by setting $\bar{z}_v := \bar{z}'_v$ for every $v \in V \setminus \{w\}$, $\bar{z}_w := \bar{z}'_f$, $\bar{z}_e := \bar{z}'_e$ for every $e \in E$ with $w \notin e$, and $\bar{z}_e := \bar{z}'_{e \cup f}$ for every $e \in E$ with $w \in e$. Clearly, $\bar{z} \in S_G$, thus it satisfies the constraints in the linear description of MP_G . Because $\bar{z}_w = \bar{z}'_f$, we obtain that $\bar{z}'$ satisfies the constraints of R obtained by applying the operation 1 in the statement. Therefore, we have shown $\bar{z}' \in R$.

Next, we show the reverse inclusion $R \cap \{0, 1\}^n \subseteq S_{G'}$. Let $\bar{z}'$ be a binary vector not in $S_{G'}$. If $\bar{z}'_f \neq \prod_{v \in f} \bar{z}'_v$, then $\bar{z}'$ does not satisfy some inequality in the standard linearization of f and so $\bar{z}'$ is not in R . Assume now $\bar{z}'_f = \prod_{v \in f} \bar{z}'_v$. Because $\bar{z}'$ is not in $S_{G'}$, there exists an edge $g' \in E'$ with $g' \neq f$ such that $\bar{z}'_{g'} \neq \prod_{v \in g'} \bar{z}'_v$. Let g be the edge of G corresponding to g' . As above, we define the vector $\bar{z}$ in the space of G obtained from $\bar{z}'$. Then $\bar{z}_g \neq \prod_{v \in g} \bar{z}_v$; thus, $\bar{z}$ is not in S_G . This means that there exists an inequality in the linear description of MP_G that is not satisfied by $\bar{z}$. The inequality obtained from it by applying the operation 1 is not satisfied by $\bar{z}'$, because $\bar{z}_w = \bar{z}'_f$, which implies that $\bar{z}'$ is not in R . $\square$

Theorem 4 can be used to extend all known decomposition results for the multilinear polytope (Del Pia and Khajavirad (2018a, b, c, 2021)). Let G be a hypergraph, and let G_1, G_2 be section hypergraphs of G such that MP_G is decomposable into MP_{G_1} and MP_{G_2} . Let $w \in V(G)$, let G' be obtained from G as described in Theorem 4, and let G'_1, G'_2 be the section hypergraphs of G' corresponding to G_1 and G_2 . Then, Theorem 4 implies that $MP_{G'}$ is decomposable into $MP_{G'_1}$ and $MP_{G'_2}$. We remind the reader that the definition of section hypergraph can be found in Section 1.1.

We remark that Theorem 4 can be also used to characterize the multilinear polytope of laminar hypergraphs, providing a simple proof of theorem 10 in Del Pia and Khajavirad (2018a). In fact, we can exploit Theorem 4 to iteratively construct MP_G , where G is a laminar hypergraph. The new proof is elementary, as it does not rely on the result by Conforti and Cornuéjols regarding the connection between integral polyhedra and balanced matrices (theorem 6.13 in Cornuéjols 2001). Next, we present the idea of this simpler proof.

Let $G = (V, E)$ be laminar. Because G is laminar, we can assume without loss of generality that there exists an edge $\bar{e} \in E$ that contains V , that is, $\bar{e} = V$. Observe that we can partition the edges of G in sets L_i , for $i = 1, \dots, r$ for some $r \geq 1$, such that an $e \in E$ belongs to L_i if and only if there exist precisely $i - 1$ distinct edges that properly contain e . We begin our construction starting from the hypergraph G' that has only one node and no edges. We apply Theorem 4 to the only node v in G' . We replace v with an edge f , which at the end of the construction will be equal to $\bar{e}$. The cardinality of f is equal to the sum of the number of nodes in $\bar{e}$ that do not belong to any edge of L_2 , plus the cardinality of L_2 . Note that all edges of L_2 are subsets of $\bar{e}$, by definition of $\bar{e}$ and the fact that G is laminar. Next, we expand the nodes of f corresponding to edges of L_2 one by one, provided that $L_2 \neq \emptyset$, by using again Theorem 4. Each of these nodes is replaced by an edge whose cardinality is equal to the sum of the number of nodes that are in the corresponding edge of L_2 and do not belong to any edge of L_3 , plus the number of edges of L_3 that are subsets of this specific edge of L_2 (if any). When all the nodes corresponding to edges in L_2 have been replaced, we start replacing the nodes corresponding to edges of L_3 by applying Theorem 4, and so on. After replacing all the nodes corresponding to edges of L_r , we will have obtained the laminar hypergraph G .

6. The Multilinear Polytope of Cycle Hypergraphs

In this section, we provide an indication of the theoretical power of odd β -cycle inequalities. These results provide the first characterizations of MP_G for a nontrivial family of hypergraphs G that contain β -cycles. First, we show that the multilinear polytope of cycle hypergraphs is fully described by the standard linearization and odd β -cycle inequalities, if each pair of edges intersect in exactly one node.

Proposition 2. *Let G be a cycle hypergraph with edges $e_1, \dots, e_m$ such that $|e_i \cap e_{i+1}| = 1$ for every $i = 1, \dots, m$, where, for convenience, we define $e_{m+1} := e_1$. Then, MP_G is given by the system comprised of the standard linearization and the odd β -cycle inequalities.*

Proof. We show this result by induction on the number of edges. We first show the base case, which is when G has three edges. We assume $G = (V, E)$, with $E = \{e_1, e_2, e_3\}$ such that $|e_1 \cap e_2| = |e_2 \cap e_3| = |e_1 \cap e_3| = 1$. Let $\bar{v}$ be the only node in $e_1 \cap e_2$. We define $F^1 := \{z \in MP_G : z_{\bar{v}} = 0\}$, $F^2 := \{z \in MP_G : z_{\bar{v}} = 1\}$. By definition, MP_G is a binary polytope; hence, MP_G is the convex hull of the union of F^1 and F^2 . Observe that we know a perfect formulation of F^1 and of F^2 . In fact, the underlying hypergraph of F^1 is $(V \setminus \{\bar{v}\}, \{e_3\})$, whereas the underlying hypergraph of F^2 is $(V \setminus \{\bar{v}\}, \{e_1 \setminus \{\bar{v}\}, e_2 \setminus \{\bar{v}\}, e_3\})$. Both these hypergraphs are Berge-acyclic, and by theorem 7 in Del Pia and

Khajavirad (2018a), we know that their perfect formulations coincide with the corresponding standard linearizations. Therefore, we apply Balas' formulation (see theorem 2.1 in Balas 1998) for the union of polytopes and obtain a perfect formulation for MP_G in the extended space. Let P be the polyhedron described by Balas' formulation for the convex hull of the union of F^1 and F^2 . We denote by $\text{proj}(P)$ the projection of P on the space of the variables $\{z_t : t \in V \cup E\}$. It follows immediately that P is an integral polyhedron, and therefore also $\text{proj}(P)$ is integral. Hence, we have that $MP_G = \text{proj}(P)$. In order to achieve the thesis, it suffices to show the correctness of the following claim.

Claim 7. *$\text{proj}(P)$ is described by the inequalities from the standard linearization and the odd β -cycle inequalities.*

In the proof of Claim 7, we use Fourier-Motzkin elimination on the additional variables that arise in the extended formulation obtained from Balas' theorem. Therefore, this proof requires many steps, and is rather long. In order to streamline the presentation, we decided not to include the Fourier-Motzkin computations. However, they can be found in Section 8.3.1

Let us move on to the inductive step. Consider G a cycle hypergraph with m edges, $e_1, \dots, e_m$, such that $|e_i \cap e_{i+1}| = 1$ for every $i = 1, \dots, m$, where $e_{m+1} = e_1$. Let $v' := e_1 \cap e_2$, and $v'' := e_{m-1} \cap e_m$. We set $f := \{v', v''\}$, and define $G' = (V, E')$, where $E' := E \cup \{f\}$. We also consider two hypergraphs G_1, G_2 that are the section hypergraphs of G' induced by $e_1 \cup e_m$, and $\bigcup_{i=2}^{m-1} e_i$, respectively. Note that G_1 and G_2 satisfy the assumptions of Proposition 2 and both have less than m edges. Hence, we apply the inductive hypothesis in order to obtain a perfect formulation of MP_{G_1} and MP_{G_2} . Observe also that G_1 and G_2 satisfy the hypothesis of the decomposition result given by theorem 1 in Del Pia and Khajavirad (2018b), because $G_1 \cup G_2 = G'$, and $G_1 \cap G_2 = (\{v', v''\}, \{f\})$ is a complete hypergraph. This implies that a perfect formulation for $MP_{G'}$ is obtained by combining the perfect formulations of MP_{G_1} and MP_{G_2} .

In order to achieve a description of MP_G , we need to project the variable z_f out of the system defining $MP_{G'}$. We do so by applying the Fourier-Motzkin elimination procedure. Once z_f is no longer present in the system, we obtain a perfect formulation for MP_G . Therefore, $MP_G = \text{proj}(MP_{G'})$, where $\text{proj}(MP_{G'})$ is the projection of $MP_{G'}$ on the space of the variables $\{z_t : t \in V \cup E\}$.

Claim 8. *$\text{proj}(MP_{G'})$ consists only of inequalities from the standard linearization, and odd β -cycle inequalities.*

Similar to what we did with the previous claim, the calculations that show the correctness of Claim 8 are in Section 8.3.2. Because of the fact that $MP_G = \text{proj}(MP_{G'})$ and of Claim 8, we can conclude the proof. $\square$

Next, we present the main result of this section. Specifically, we prove that the multilinear polytope of general cycle hypergraphs is characterized by the standard linearization and all the flower and odd β -cycle inequalities.

Theorem 5. *Let $G = (V, E)$ be a cycle hypergraph. Then, MP_G is given by the system comprised of the standard linearization, the flower inequalities, and the odd β -cycle inequalities. Moreover, the Chvátal rank of MP_G^{LP} is at most 2.*

Proof. We prove this theorem using Proposition 2 and Theorem 4 and subsequently projecting out the additional variables introduced by the application of Theorem 4.

Let $G = (V, E)$ be a cycle hypergraph with edges $e_1, \dots, e_m$. First, we create a new hypergraph $G' = (V', E')$ obtained by contracting every intersection $e_i \cap e_{i+1}$ to a new node w_i , for every $i \in [m]$, where we define $e_{m+1} := e_1$. Observe that G' satisfies the hypothesis of Proposition 2, because it is still a cycle hypergraph and now every nonempty intersection of two edges has cardinality equal to one. Hence, $MP_{G'}$ is described by the system consisting of the standard linearization and the odd β -cycle inequalities. Then, starting from G' , we construct a second hypergraph $G'' = (V, E'')$ by applying Theorem 4 to every node w_i , for $i \in [m]$, in order to obtain the same node set V of the original hypergraph G . In this way, we obtain $MP_{G''}$. By the application of Theorem 4, it follows that $E'' \neq E$. Indeed, the application of Theorem 4 yields a new edge for every node w_i , and therefore $E'' = E \cup \{f_1, \dots, f_m\}$, where $f_i := e_i \cap e_{i+1}$.

Thus, in order to achieve a description of MP_G , it remains to eliminate the variables corresponding to the new edges $f_1, \dots, f_m$. This means that $MP_G = \text{proj}(MP_{G''})$, where $\text{proj}(MP_{G''})$ is the projection of $MP_{G''}$ on the space of the variables $\{z_t : t \in V \cup E\}$. We project out the variables $z_{f_1}, \dots, z_{f_m}$ using the Fourier-Motzkin elimination procedure.

Claim 9. *The system describing $\text{proj}(MP_{G''})$ is given by inequalities from the standard linearization, flowers inequalities, and odd β -cycle inequalities.*

Once again, we decided not to present the Fourier-Motzkin calculations, because they are pretty long. They can be found in Section 8.4.

At this point, we recall that it suffices to use Theorems 1 and 2 in order to prove the last part of Theorem 5. In fact, from Theorems 1 and 2, it follows immediately that MP_G^{LP} has Chvátal rank at most 2. This concludes the proof of the theorem. $\square$

We close this section by observing that all the instances of binary polynomial optimization represented by cycle hypergraphs can be solved in polynomial time, as the corresponding linear program in the extended space can be solved in polynomial time. This can be easily obtained by combining Theorems 3 and 5.

Corollary 1. *Let G be a cycle hypergraph. Then, optimizing a linear function over MP_G can be done in polynomial time.*

Proof. By the equivalence of separation and optimization (Conforti et al. 2014), it suffices to show that all the inequalities defining MP_G can be separated efficiently. From Theorem 5, this implies that we are interested in the separation of the odd β -cycle inequalities, the flower inequalities, and the standard linearization inequalities. The first result is proved in Theorem 3. Second, the number of flower and the standard linearization inequalities in the system describing MP_G is polynomial in $|V|$ and $|E|$. In particular, there are only $3|E|$ flower inequalities in this system. This follows by the definition of cycle hypergraph. Lastly, the number of standard linearization inequalities is bounded by $|V| + 2|E| + |V||E|$. Therefore, it is possible to separate over both the flower inequalities and the standard linearization inequalities simply by checking them one by one. $\square$

7. Numerical Results

Given the indication of the theoretical power of the odd β -cycle inequalities presented in Theorem 5, we wanted to gain some further insight about the practical effectiveness of these inequalities. In particular, we focused on cases when the hypergraph representing the instance is not a cycle hypergraph, because we now know that such instances can be solved in polynomial time. In order to do so, we chose two problems. The first problem is the image restoration problem in computer vision, as described in Crama and Rodríguez-Heck (2017), whereas the second is the low autocorrelation binary sequence problem that emerged from theoretical physics (Bernasconi 1987, Mertens and Bessenrodt 1998, POLIP 2014, MINLPLib 2020). These problems have been commonly used to test procedures for binary polynomial optimization (Crama and Rodríguez-Heck 2017, Elloumi et al. 2019, Del Pia et al. 2020). Our goal is to understand the percentage of the integrality gap that is closed by applying only the odd β -cycle inequalities. We remark that these problems are indeed not represented by cycle hypergraphs, and a polynomial-time separation for the odd β -cycle inequalities in these settings is not known. Therefore, because of the exponential number of these inequalities, we only consider a subset of them in our computational experiments. Namely, we only use the ones arising from β -cycles of length 3 or 4. Hence, we define two relaxations of the original problem that are obtained by adding to the standard linearization all the odd β -cycle inequalities coming from β -cycles of length 3 and all the inequalities arising from β -cycles of lengths 3 and 4, respectively. We denote the corresponding polytopes by MP_G^3 and MP_G^4 . The percentage of the integrality gap closed is computed as in Fischetti and Lodi (2007). The precise expression is $100 - 100(\text{opt}(MP_G) - \text{opt}(MP_G^3)) / (\text{opt}(MP_G) - \text{opt}(MP_G^{LP}))$ and $100 - 100(\text{opt}(MP_G) - \text{opt}(MP_G^4)) / (\text{opt}(MP_G) - \text{opt}(MP_G^{LP}))$ for the two relaxations respectively, where $\text{opt}(\cdot)$ denotes the optimal value of the objective function when optimized over that specific polytope.

We first present the results for the image restoration problem and then the ones for the low autocorrelation binary sequence problem. We observe that the running intersection inequalities have been proved to be very effective in solving the image restoration problem. On the other hand, they are not a useful tool for the low autocorrelation binary sequence problem, as mentioned in Del Pia et al. (2020). As the reader will see in the next sections, the odd β -cycle inequalities are able to close a significant amount of the integrality gap for such problems. Therefore, this might indicate that using inequalities with a higher Chvátal rank can lead to better performances. We close this paragraph by noticing that the instances from the low auto-correlation binary sequence problem are much harder to solve than the ones of the image restoration problem. This is probably because the hypergraphs representing the instances of the first problem are considerably more dense than the ones coming from the image restoration problem, as it is explained in Elloumi et al. (2019).

7.1. Image Restoration Problem

Next, we briefly describe the test set for this problem, while we refer to Crama and Rodríguez-Heck (2017) for a thorough explanation of these instances' construction. The test set involves 45 instances. There are three

image types (top left, center, and cross) and three image sizes (10×10 , 10×15 , and 15×15). Each combination of image type and size leads to five instances depending on the level of perturbation that is applied to them. In particular, one instance corresponds to no perturbation, two instances have a low level of perturbation, whereas the last two correspond to a high level.

We recall that in our experiments we compute the percentage of the integrality gap closed by two relaxations of the problem, whose polytopes are MP_G^3 and MP_G^4 . The results can be found in Table 1.

It is immediate to see that using these cuts yields a considerable reduction in the integrality gap. In fact, one can see that the integrality gap is reduced on average by 24% just when using the odd β -cycle inequalities corresponding to cycles of length 3. This percentage increases to 44% when we also consider inequalities corresponding to cycles of length 4.

7.2. Low Autocorrelation Binary Sequence Problem

The test set for this problem can be found in both POLIP (2014) and MINLPLib (2020). We start this section by observing that for many instances of the test set an optimal solution is not known. Precisely, this is true for 27 instances over 45. For these instances we cannot compute the reduction in the integrality by using

Table 1. Results of Computer Vision Instances

Instance	Gap reduced MP_G^3	Gap reduced MP_G^4
Top left, 10×10 , none	27%	48%
Top left, 10×15 , none	26%	47%
Top left, 15×15 , none	26%	47%
Top left, 10×10 , low 1	25%	47%
Top left, 10×10 , low 2	25%	47%
Top left, 10×15 , low 1	26%	47%
Top left, 10×15 , low 2	25%	47%
Top left, 15×15 , low 1	26%	46%
Top left, 15×15 , low 2	26%	46%
Top left, 10×10 , high 1	24%	41%
Top left, 10×10 , high 2	24%	41%
Top left, 10×15 , high 1	23%	41%
Top left, 10×15 , high 2	24%	42%
Top left, 15×15 , high 1	23%	41%
Top left, 15×15 , high 2	23%	40%
Center, 10×10 , none	25%	46%
Center, 10×15 , none	25%	46%
Center, 15×15 , none	25%	47%
Center, 10×10 , low 1	24%	45%
Center, 10×10 , low 2	24%	45%
Center, 10×15 , low 1	24%	46%
Center, 10×15 , low 2	24%	45%
Center, 15×15 , low 1	25%	46%
Center, 15×15 , low 2	25%	46%
Center, 10×10 , high 1	24%	41%
Center, 10×10 , high 2	23%	41%
Center, 10×15 , high 1	23%	41%
Center, 10×15 , high 2	24%	41%
Center, 15×15 , high 1	23%	40%
Center, 15×15 , high 2	23%	40%
Cross, 10×10 , none	25%	45%
Cross, 10×15 , none	25%	45%
Cross, 15×15 , none	25%	46%
Cross, 10×10 , low 1	24%	45%
Cross, 10×10 , low 2	24%	44%
Cross, 10×15 , low 1	24%	44%
Cross, 10×15 , low 2	24%	44%
Cross, 15×15 , low 1	24%	45%
Cross, 15×15 , low 2	25%	45%
Cross, 10×10 , high 1	23%	40%
Cross, 10×10 , high 2	23%	41%
Cross, 10×15 , high 1	23%	41%
Cross, 10×15 , high 2	23%	41%
Cross, 15×15 , high 1	23%	40%
Cross, 15×15 , high 2	23%	41%

the expression $100-100(\text{opt}(\text{MP}_G)-\text{opt}(\text{MP}_G^3))/(\text{opt}(\text{MP}_G)-\text{opt}(\text{MP}_G^{\text{LP}}))$, because the value $\text{opt}(\text{MP}_G)$ is missing. Therefore, we replaced $\text{opt}(\text{MP}_G)$ in the previous formula with the best primal bound, that is, the value of the best feasible solution. Such bounds are regularly updated in MINLPLib (2020). It can be easily checked that the reduction of the integrality gap that we provide in these cases is a lower bound for the true reduction of the integrality gap.

The results of the computational experiments are displayed in Table 2. Two instances, bern 20.3 and bern 25.3, were not considered in our study, as their optimal solution can be found by simply solving the standard linearization. This is because the hypergraphs representing the two instances are actually Berge-acyclic, and therefore, the standard linearization is a perfect formulation for these problems (Del Pia and Khajavirad 2018a). Moreover, in this case, we decided to use only odd β -cycle inequalities corresponding to β -cycles of length 3. This is because these instances are significantly harder to solve than the ones in the previous section. We were able to compute the reduction in the integrality gap for 18 of the 43 considered instances in the test set. The reason behind it is that the linear programs corresponding to the remaining instances resulted in an error in GUROBI (2019) caused by the huge size of the matrix, even if we only used a subset of inequalities.

We observe that the odd β -cycle inequalities produce a remarkable reduction in the integrality gap when applied to these hard problems. In particular, these percentages are considerably higher than the corresponding ones for the instances of the image restoration problem. In fact, on average the integrality gap is reduced by 60% just by considering all the inequalities coming from β -cycles of length 3.

Given the results of these experiments, we believe that an interesting future work would be to perform a more thorough computational study and implementing the odd β -cycle inequalities in a branch and bound routine.

8. Technical Proofs

8.1. Claims in the Proof of Theorem 1

8.1.1. Proof of Claim 1. Let $\tilde{G}$ be the hypergraph $(e_0, \tilde{E})$, and note that $\tilde{G}$ can contain loops and parallel edges. There is a one-to-one correspondence between edges in E and in $\tilde{E}$. Given an edge $e \in E$, we denote by $\tilde{e}$ the corresponding edge in $\tilde{E}$ defined by $\tilde{e} := e_0 \cap e$. Vice versa, given an edge $\tilde{e} \in \tilde{E}$, we denote by e the corresponding edge in E .

For every edge e_i , $i \in [m]$, let J_i be the set of indices $j \in \{i+1, \dots, m\}$ for which the condition in rule 2.2 holds. Subsequently, we define recursively the sets

$$C_i := J_i \cup \left(\bigcup_{j \in J_i} C_j \right) \quad \text{for every } i \in [m],$$

Table 2. Results of Low Autocorrelation Binary Sequence Instances

Instance	Gap reduced MP_G^3
bern 20.5	58%
bern 20.10	64%
bern 20.15	63%
bern 25.6	64%
bern 25.13	64%
bern 30.4	39%
bern 30.8	64%
bern 30.15	63%
bern 35.4	39%
bern 35.9	64%
bern 40.5	59%
bern 40.10	64%
bern 45.5	59%
bern 45.11	62%
bern 50.6	64%
bern 50.13	64%
bern 55.6	64%
bern 60.8	64%

starting from $i = m$ and moving to $i = 1$. Moreover, for every $i \in [m]$, let

$$W^i := \{w \in W : \delta_{w,e_h} \neq 0, \text{ for some } h \in \{i\} \cup C_i\},$$

$$U^i := \{u_j \in U : \delta_{u_j,e_h} \neq 0, \text{ for some } h \in \{i\} \cup C_i, h \neq j\}.$$

The next claim provides a fundamental tool for the arguments in this proof.

Subclaim 1. For every $i \in [m]$, we have

$$\sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} = \frac{|W^i|}{\gamma} + \frac{|U^i|}{\gamma}. \quad (9)$$

If $N(\tilde{e}_i) = \emptyset$, then u_i is not defined, and the sum ranges over all $v \in e_i$.

Proof of Subclaim 1. We fix $i \in [m]$ and prove (9). In order to do this, it will be useful to write the left-hand side of (9) explicitly, in terms of the multipliers defined in rules 2.1 and 2.2. We obtain

$$\sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} = \sum_{\substack{v \in W \cap e_i \\ v \notin \bigcup_{j>i} e_j}} \frac{1}{\gamma} + \sum_{j \in J_i} \left(1 - \left(\delta_{u_j,e_j} - \frac{1}{\gamma}\right)\right). \quad (10)$$

If $j \in J_i$, then $u_j \in \tilde{e}_i \cap \tilde{e}_j$; thus, $\tilde{e}_i$ and $\tilde{e}_j$ belong to the same connected component of $\tilde{G}$ that we denote by C . Hence, the right-hand side of (10) only depends on the values of the multipliers $\delta_{v,e}$ for edges $\tilde{e}$ of C .

Let $p := |E(C)| \geq 1$. We can assume, without loss of generality, that the edges of C correspond to the first p edges in the running intersection ordering of $\tilde{E}$. That is, the edges of C are $\tilde{e}_1, \dots, \tilde{e}_p$. We observe that this can be done without loss of generality, since C is a connected component of $\tilde{G}$, and the considered multipliers depend exclusively on multipliers corresponding to the other edges of C . Recall that we only need to prove (9) for the fixed index $i \in [p]$. Instead, we prove (9) for all indices $i \in [p]$. We do so by induction on i , starting from $i = p$ and going backward to 1. Thus, we no longer think of i as a fixed index.

Consider the base case $i = p$. Then $J_p = C_p = \emptyset$. Therefore, the second sum of (10) reduces to $|W \cap e_p|/\gamma = |W^p|/\gamma$, and the third sum is zero. This completes the proof of the base case, because $U^p = \emptyset$.

Before showing the inductive step, we also provide the argument for the case $i = p - 1$ because the idea is the same, but the calculations are significantly simpler. Let $i = p - 1$. If $J_{p-1} = \emptyset$, then also $C_{p-1} = \emptyset$ and therefore the third sum of (10) is zero, and the second sum is equal to $|W \cap e_{p-1} \setminus e_p|/\gamma = |W^{p-1}|/\gamma$. This is true because, in this case, $W^{p-1} = \{w \in W : \delta_{w,e_{p-1}} \neq 0\}$. Note also that U^{p-1} is the empty set, because $J_{p-1} = \emptyset$ implies that there is no node $u \in U \setminus \{u_{p-1}\}$ for which $\delta_{u,e_{p-1}} \neq 0$.

Thus, we now assume that $J_{p-1} = \{p\}$, which means that $N(\tilde{e}_p) \neq \emptyset$ and $u_p \in \tilde{e}_{p-1}$. Then, $C_{p-1} = \{p\} \cup C_p = \{p\}$. In this case, the second sum of (10) reduces to $|W \cap e_{p-1} \setminus e_p|/\gamma$, and the third sum is

$$1 - \left(\delta_{u_p,e_p} - \frac{1}{\gamma}\right) = 1 - \left(1 - \sum_{v \in e_p \setminus \{u_p\}} \delta_{v,e_p} - \frac{1}{\gamma}\right) = \sum_{v \in e_p \setminus \{u_p\}} \delta_{v,e_p} + \frac{1}{\gamma} = \frac{|W^p|}{\gamma} + \frac{1}{\gamma},$$

where we have used the definition of δ_{u_p,e_p} in rule 2.3, and the fact that $\sum_{v \in e_p \setminus \{u_p\}} \delta_{v,e_p} = |W^p|/\gamma$. We obtain

$$\sum_{v \in e_{p-1} \setminus \{u_{p-1}\}} \delta_{v,e_{p-1}} = \frac{|W \cap e_{p-1} \setminus e_p|}{\gamma} + \frac{|W^p|}{\gamma} + \frac{1}{\gamma} = \frac{|W^{p-1}|}{\gamma} + \frac{|U^{p-1}|}{\gamma}.$$

This holds because, by definition, $W^{p-1} = \{w \in W : \delta_{w,e_h} \neq 0, \text{ for some } h \in \{p-1, p\}\}$, and every node w in W has exactly one edge e for which $\delta_{w,e} \neq 0$. This second fact follows directly from rule 2.1. Hence, W^{p-1} is the disjoint union of $W \cap e_{p-1} \setminus e_p$ and $W^p = W \cap e_p$. Similarly, we show that $U^{p-1} = \{u_p\}$. By definition, $U^{p-1} = \{u_j \in U : \delta_{u_j,e_h} \neq 0, \text{ for some } h \in \{p-1, p\}, h \neq j\}$. When $h = p$, then the only nonzero multiplier δ_{u_j,e_p} is for $j = p$, hence $h = j$, and when $h = p - 1$, the multipliers $\delta_{u_j,e_{p-1}}$ are nonzero only if $h \neq j$. However, h needs to be different from j , hence the only multiplier that satisfies this condition is $\delta_{u_p,e_{p-1}}$. This concludes the proof that $U^{p-1} = \{u_p\}$.

We now prove the inductive step. Let $i \in \{2, \dots, p\}$, we suppose that (9) holds for all $j \geq i$, and we show that (9) still holds for $i - 1$. The second sum of (10) is then $|W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j|/\gamma$, and the third sum is

$$\begin{aligned} \sum_{j \in J_{i-1}} \left(1 - \left(\delta_{u_j, e_j} - \frac{1}{\gamma}\right)\right) &= \sum_{j \in J_{i-1}} \left(1 - \left(1 - \sum_{v \in e_j \setminus \{u_j\}} \delta_{v, e_j} - \frac{1}{\gamma}\right)\right) \\ &= \sum_{j \in J_{i-1}} \left(\sum_{v \in e_j \setminus \{u_j\}} \delta_{v, e_j} + \frac{1}{\gamma}\right) = \sum_{j \in J_{i-1}} \left(\frac{|W^j|}{\gamma} + \frac{|U^j|}{\gamma}\right) + \frac{|J_{i-1}|}{\gamma}, \end{aligned}$$

where we have used the definition of δ_{u_j, e_j} in rule 2.3, and the induction hypothesis. We obtain

$$\begin{aligned} \sum_{v \in e_{i-1} \setminus \{u_{i-1}\}} \delta_{v, e_{i-1}} &= \frac{|W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j|}{\gamma} + \sum_{j \in J_{i-1}} \left(\frac{|W^j|}{\gamma} + \frac{|U^j|}{\gamma}\right) + \frac{|J_{i-1}|}{\gamma} \\ &= \frac{|W^{i-1}|}{\gamma} + \frac{|U^{i-1}|}{\gamma}. \end{aligned}$$

The last equality follows from two facts that we show next. The first is that W^{i-1} is the disjoint union of $W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$ and $\bigcup_{j \in J_{i-1}} W^j$, where also all these unions are disjoint.

We prove first that $W^{i-1} = (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j) \cup \bigcup_{j \in J_{i-1}} W^j$, by showing the two set inclusions. Recall that $W^{i-1} = \{w \in W : \delta_{w, e_h} \neq 0, \text{ for some } h \in \{i-1\} \cup C_{i-1}\}$. We show first the inclusion $\subseteq$. Let $w \in W^{i-1}$. Then either $\delta_{w, e_{i-1}} \neq 0$, or $\delta_{w, e_h} \neq 0$ with $h \in C_{i-1}$. If the first case holds, then $w \in (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j)$ by rule 2.1. Assume then that there exists $h \in C_{i-1}$ such that $\delta_{w, e_h} \neq 0$. It implies that $h \in J_{i-1} \cup (\bigcup_{j \in J_{i-1}} C_j)$. Then either $h \in J_{i-1}$ or $h \in \bigcup_{j \in J_{i-1}} C_j$. If $h \in J_{i-1}$, then $\delta_{w, e_h} \neq 0$ and $w \in W^h$. On the other hand, if $h \in \bigcup_{j \in J_{i-1}} C_j$, it means that there exists $j' \in J_{i-1}$ such that $h \in C_{j'}$. Then $w \in W^{j'}$, by definition of $W^{j'}$. We prove now the reverse set inclusion $\supseteq$. Let $w \in (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j) \cup \bigcup_{j \in J_{i-1}} W^j$. Assume first that $w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$. By rule 2.1, we have $\delta_{w, e_{i-1}} \neq 0$, thus $w \in W^{i-1}$. Consider now the other case $w \in \bigcup_{j \in J_{i-1}} W^j$. This means that there exists $\bar{j} \in J_{i-1}$ such that $w \in W^{\bar{j}}$. It follows that there exists $h \in \{\bar{j}\} \cup C_{\bar{j}}$ such that $\delta_{w, e_h} \neq 0$. Note that $\{\bar{j}\} \cup C_{\bar{j}} \subseteq C_{i-1}$; hence, $h \in C_{i-1}$. Therefore, $w \in W^{i-1}$.

Next we show that W^{i-1} is the disjoint union of the set $W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$, and of the sets W^j , with $j \in J_{i-1}$. Before moving forward, we recall that for every $w \in W$, there exists a unique edge e such that $\delta_{w, e} \neq 0$. Assume for a contradiction that these unions are not disjoint; that is, there exists $w \in W^{i-1}$ such that either $w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$ and there exists $\bar{j} \in J_{i-1}$ such that $w \in W^{\bar{j}}$, or there exist two distinct indices $j', j'' \in J_{i-1}$ such that $w \in W^{j'}$, $w \in W^{j''}$.

Consider the first case. By assumption $w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$, which implies that $w \notin e_j$ for all $j \geq i$. Observe that, by rule 2.2, all indices $j \in J_{i-1}$ are greater than or equal to i . Therefore, by definition of W^j , this set contains only nodes w' for which $\delta_{w', e_h} \neq 0$ with $h \geq i$. Because $\delta_{w, e_{i-1}} \neq 0$ by rule 2.1, and because there exists only one edge e such that $\delta_{w, e} \neq 0$, we can conclude that $w \notin W^j$, which is a contradiction.

Assume that the second case holds. Without loss of generality we suppose that $j' < j''$. As remarked before there exists a unique index $h \geq i$ such that $\delta_{w, e_h} \neq 0$. By definition of $W^{j'}, W^{j''}$, it follows that $h \in (\{j'\} \cup C_{j'}) \cap (\{j''\} \cup C_{j''})$. We show $j' \notin C_{j''}$. As $j' < j''$, and $C_{j''}$ only contains indices strictly larger than $j'' > j'$, then $j' \notin C_{j''}$. Next, we show $j'' \notin C_{j'}$. By expanding the definition of $C_{j'}$, observe that $C_{j'}$ is a collection of some sets J_q with $q \geq j'$. Assume by contradiction $j'' \in C_{j'}$. Then, there exists an index $k \geq j'$ such that $j'' \in J_k$, which means that, by rule 2.2, $\delta_{u_{j''}, e_k} \neq 0$. By assumption, $j'' \in J_{i-1}$; hence, e_{i-1} is the only edge e , different from $e_{j''}$ such that $\delta_{u_{j''}, e} \neq 0$. This is a contradiction because $k \geq j' > i-1$. Therefore, we have $j'' \notin C_{j'}$.

As $j' \neq j''$, $j' \notin C_{j''}$, and $j'' \notin C_{j'}$, we have $h \in C_{j'} \cap C_{j''}$. Because $h \in C_{j'}$, there exists an index $k' \geq j'$, $k' \neq h$ such that $h \in J_{k'}$; therefore, $\delta_{u_{h'}, e_{k'}} \neq 0$. Similarly, because $h \in C_{j''}$, there exists $k'' \geq j''$, $k'' \neq h$ such that $\delta_{u_{h'}, e_{k''}} \neq 0$. However, $\delta_{u_{h'}, e} \neq 0$ with $e \neq e_h$ is satisfied only by one edge e ; thus, $e_{k'} = e_{k''}$. Let us denote this edge by e_k . Because of the recursive nature of the set $C_{j'}$, there exists a sequence of edges $e_{1'}, e_{2'}, \dots, e_{r'}$ such that $e_{1'} = e_{j'}$, $e_{r'} = e_k$, and $p' \in J_{(p-1)'}$, for every $p \in \{2, \dots, r\}$. Similarly, let $e_{1''}, e_{2''}, \dots, e_{t''}$ be a sequence such that $e_{1''} = e_{j''}$, $e_{t''} = e_k$, and $p'' \in J_{(p-1)''}$, for every $p \in \{2, \dots, t\}$. Because $k \in J_{(r-1)'} \cap J_{(t-1)''}$ and there exists a unique edge $e \neq e_k$ for which $\delta_{u_{k'}, e} \neq 0$, it follows that $e_{(r-1)'} = e_{(t-1)''}$. Recursively, we apply the same uniqueness property until we arrive to the first edge of the shortest sequence. Either $r \neq t$ or $r = t$. Consider the first case. If $r < t$, it means that $e_{1'} = e_{q''}$ for some $q > 1$. Because $e_{1'} = e_{j'}$, this implies in turn that $j' \in C_{j''}$, which is a contradiction as showed earlier. Then suppose $r > t$. Similarly, we can conclude that $j'' \in C_{j'}$, which again is a contradiction as proved previously. Hence assume that $r = t$. In this case, $e_{1'} = e_{1''}$, which means that $e_{j'} = e_{j''}$. This is a

contradiction because e_j' and e_j'' are different edges by assumption. Therefore, $h \notin C_j' \cap C_j''$, and we have showed that W^{i-1} is the disjoint union of the set $W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j$ and the set W_j^i , with $j \in J_{i-1}$.

The second fact is that $U^{i-1} = J_{i-1} \cup \bigcup_{j \in J_{i-1}} U_j^i$, and in addition U^{i-1} is the disjoint union of J_{i-1} , U_j^i , for $j \in J_{i-1}$. The proof of this is very similar to the one regarding W^{i-1} . This concludes the proof that (9) holds and of the subclaim. $\diamond$

Because $\gamma \geq 1$, we only need to prove that the multipliers δ_{u_j, e_i} , δ_{u_i, e_i} , and η_{e_i} are nonnegative for every edge e_i , $i \in [m]$. We first consider multipliers δ_{u_i, e_i} and η_{e_i} , which are defined in rule 2.3. If we are defining η_{e_i} instead of δ_{u_i, e_i} , it means that $N(e_0 \cap e_i) = \emptyset$, which implies that node u_i does not exist. Therefore, to show that multipliers δ_{u_i, e_i} and η_{e_i} are nonnegative, we can equivalently show that $\sum_{v \in e_i \setminus \{u_i\}} \delta_{v, e_i} \leq 1$, for every $i \in [m]$.

By Subclaim 1, we obtain

$$\sum_{v \in e_i \setminus \{u_i\}} \delta_{v, e_i} = \frac{|W^i|}{\gamma} + \frac{|U^i|}{\gamma} \leq \frac{|W|}{\gamma} + \frac{|U|}{\gamma} = \frac{|e_0 \cap \bigcup_{i \in [m]} e_i|}{\gamma} = 1,$$

where the second equality holds because W, U form a partition of $e_0 \cap \bigcup_{i \in [m]} e_i$, and the last equality is true by definition of γ .

Finally, the multipliers δ_{u_j, e_i} are nonnegative because $\delta_{u_j, e_i} = 1 + 1/\gamma - \delta_{u_j, e_j}$ and we just showed that $\delta_{u_j, e_j} \leq 1$. $\square$

8.1.2. Proof of Claim 2. We check that every entry of π coincides with the corresponding component of a , the left-hand side of the inequality of type (1) that was fixed at the beginning of the proof of Theorem 1.

Each variable corresponding to an edge different from e_0 or its neighbors $e_1, \dots, e_m$ does not appear in $az \leq b$, and it does not appear in the inequality $\pi z \leq \lfloor \pi_0 \rfloor$ either, because their corresponding multipliers are not explicitly defined and therefore are set to zero. An analogous argument holds for the nodes v that do not belong to e_0 .

Consider the variable z_{e_0} . The only constraints chosen with nonzero multipliers in which it appears are (ε_{e_0}) and (ν_{e_0}) . The first constraint is selected with multiplier equal to $1/\gamma$, whereas the second with multiplier $1 - 1/\gamma$ by rule 1. By summing these two inequalities we obtain that the entry of π related to z_{e_0} is equal to -1 , as is the coefficient of z_{e_0} in az .

Similarly, consider variables z_v , for $v \in e_0 \setminus \bigcup_{i \in [m]} e_i$. Each of them is involved in just two constraints among the ones picked with nonzero multiplier. These two inequalities are (ε_{e_0}) and (α_v) . Once we sum these two constraints chosen with the multipliers described in rule 1, we obtain that the resulting coefficients of these variables in πz are all equal to 1.

Let $w \in W \cap \bigcup_{i \in [m]} e_i$. The corresponding variable z_w is present again in only two constraints among the selected ones: (ε_{e_0}) and (δ_{w, e_i}) , where i is the largest index in the running intersection ordering of $\tilde{E}$ such that $\tilde{e}_i$ contains the node w . By rules 1 and 2.1, the multipliers corresponding to these two inequalities are $1/\gamma$. Then the component of π corresponding to z_w is equal to 0, because in one inequality it has coefficient $+1$ and in the other it has coefficient -1 .

Now consider a node $u \in U$. Let $\tilde{e}', \tilde{e}''$ be the two edges in $\tilde{E}$ that contain u , with, respectively, largest and second largest index in the running intersection ordering of $\tilde{E}$. This time the variable z_u is present in three different constraints: (ε_{e_0}) , $(\delta_{u, e'})$, $(\delta_{u, e''})$. The corresponding multipliers are $\varepsilon_{e_0} = 1/\gamma$, $\delta_{u, e'}$, and $\delta_{u, e''} = 1 - (\delta_{u, e'} - 1/\gamma)$. Therefore, the coefficient of z_u in $\pi z \leq \lfloor \pi_0 \rfloor$ is equal to

$$\frac{1}{\gamma} - \left(\delta_{u, e'} + 1 - \delta_{u, e'} + \frac{1}{\gamma} \right) = \frac{1}{\gamma} - 1 - \frac{1}{\gamma} = -1,$$

as it is in the left-hand side of (1), and therefore in a .

We only need to check the coefficients of the variables z_{e_i} corresponding to the edges e_i with $i \in [m]$. The variable z_{e_i} appears in several δ_{v, e_i} inequalities. Because of rules 2.1, 2.2, and 2.3, the entry in π corresponding to z_{e_i} is given either by $\delta_{u_i, e_i} + \sum_{v \in e_i \setminus \{u_i\}} \delta_{v, e_i}$ if $N(e_0 \cap e_i) \neq \emptyset$, or by $\eta_{e_i} + \sum_{v \in e_i} \delta_{v, e_i}$. In both cases, by the definition of δ_{u_i, e_i} and η_{e_i} , respectively, we can conclude that the coefficient corresponding to z_{e_i} in πz is equal to 1, for $i \in [m]$. $\square$

8.1.3. Proof of Claim 3. Denote by K_i , $i = 1, \dots, l$, the connected components of $\tilde{G}$. For every $i = 1, \dots, l$, let $p_i := |E(K_i)|$. Moreover, let us order the edges of $E(K_i)$ such that they follow the same order in which they appear in the running intersection ordering of $\tilde{E}$. Then let $\tilde{e}_j$, with $j = 1, \dots, p_i$, be the order of the edges of $E(K_i)$, for every $i \in \{1, \dots, l\}$.

Notice that for every connected component K_i , $i = 1, \dots, l$, only the first edge $\tilde{e}_{i_1}$ satisfies $N(\tilde{e}_{i_1}) = \emptyset$; thus, it is the only edge that may contribute to $\lfloor \pi_0 \rfloor$, because it is the only edge for which η_e is possibly nonzero. This contribution is equal to

$$\eta_{e_{i_1}} = 1 - \sum_{v \in e_{i_1}} \delta_{v, e_{i_1}} = 1 - \frac{|W^{i_1}|}{\gamma} - \frac{|U^{i_1}|}{\gamma},$$

where the second equality comes from Subclaim 1.

By rules 1, 2.1, 2.2, and 2.3 we obtain that $\lfloor \pi_0 \rfloor$ is equal to

$$\left\lfloor \frac{|e_0| - 1}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^l \eta_{e_{i_1}} \right\rfloor = \left\lfloor \frac{|e_0| - 1}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^l \left(1 - \frac{|W^{i_1}|}{\gamma} - \frac{|U^{i_1}|}{\gamma}\right) \right\rfloor. \quad (11)$$

By Definition 1, we know that we need (11) to be equal to $|e_0 \setminus \bigcup_{i \in [m]} e_i| + |\{i \in [m] : N(e_0 \cap e_i) = \emptyset\}| - 1$. Observe that (11) is equal to

$$\left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + |\{i \in [m] : N(e_0 \cap e_i) = \emptyset\}| + \left\lfloor \frac{|e_0| - 1}{\gamma} - \frac{1}{\gamma} \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| - \sum_{i=1}^l \frac{|W^{i_1}| + |U^{i_1}|}{\gamma} \right\rfloor.$$

Then we simply need to show that

$$\left\lfloor \frac{|e_0|}{\gamma} - \frac{1}{\gamma} \left(1 + \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^l (|W^{i_1}| + |U^{i_1}|) \right) \right\rfloor = -1. \quad (12)$$

Note that W^{i_1} and U^{i_1} contain exactly all the nodes in $V(K_i)$. Moreover, W^{i_1} and U^{i_1} are disjoint and therefore $|W^{i_1}| + |U^{i_1}| = |V(K_i)|$, for every connected component K_i . Then, the $2l$ sets W^{i_1}, U^{i_1} for $i = 1, \dots, l$, form a partition of $e_0 \cap \bigcup_{i \in [m]} e_i$. Thus, $\sum_{i=1}^l (|W^{i_1}| + |U^{i_1}|) = |e_0 \cap \bigcup_{i \in [m]} e_i|$. It follows that the left-hand side of (12) can be written as

$$\left\lfloor \frac{|e_0|}{\gamma} - \frac{1}{\gamma} \left(1 + \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \left| e_0 \cap \bigcup_{i \in [m]} e_i \right| \right) \right\rfloor = \left\lfloor \frac{|e_0|}{\gamma} - \frac{1}{\gamma} (1 + |e_0|) \right\rfloor = \left\lfloor -\frac{1}{\gamma} \right\rfloor = -1.$$

We can therefore conclude that the claim holds. $\square$

8.2. Claims in the Proof of Proposition 1

8.2.1. Proof of Claim 4. We recall from the definition of T that $T \geq 2$. Then, all multipliers defined in rule 1 and rule 2 are nonnegative. Observe that the multipliers θ_{f_i} are defined only in rule 1. Therefore, all multipliers defined in rule 3 are either $1/T$ or $(T-1)/T$ and thus nonnegative. Notice that $\sum_{i=p+1}^m \Delta_i = T$. Therefore, all the sums of Δ_j considered in rule 4 are between zero and T . Hence, also the multipliers defined in rule 4 are nonnegative. $\square$

8.2.2. Proof of Claim 5. Each variable corresponding to an edge not in the cycle does not appear in any inequality of the type (2), and it does not appear in $\pi z \leq \lfloor \pi_0 \rfloor$ either, because its corresponding multipliers are not explicitly defined in rules 1–4 and thus are set to zero. A similar argument is true for the nodes v that do not belong to any edge of the cycle.

Observe also that the variables representing nodes that are contained in edges of E^+ , but do not belong to S_2 , are not present in any of the inequalities chosen with nonzero multiplier. This does not represent a problem, because these variables do not appear in az .

It is easy to check that the entries of π corresponding to $f \in E^-$ and to $v \in S_1 \cup S_2$ satisfy the thesis. In fact, each of the corresponding variables appears in only two inequalities chosen with nonzero multiplier and these multipliers sum to 1. Indeed, variables related to $f_i \in E^-$ are present only in the inequalities corresponding to the multipliers θ_{f_i} and v_{f_i} , which sum to 1 by rule 1. Similarly, each z_v for $v \in S_1$ is present only in one or two

different θ_f , because of assumption (a). If it is present only in one flower inequality, then the multiplier α_v is chosen accordingly such that the two multipliers sum to 1, by rule 2. If, on the other hand, z_v is contained in two flower inequalities, we must analyze two different cases: whether assumption (d1) or assumption (d2) holds. Suppose that assumption (d1) is satisfied. This implies that the multipliers of the two flower inequalities sum to 1, because of rule 1. Then, assume that assumption (d2) holds. We remark that T is exactly equal to 2 here. If D is nonempty, it follows by assumption (a), whereas if D is empty, it follows from the definition of T . Therefore, the two flower inequalities both have multiplier $1/2$, and the coefficient of z_v in πz is equal to 1.

The rest of the proof works both for assumption (d1) and assumption (d2). Consider $v_i \in S_2$. Then the only nonzero multipliers that affect its component in π are $\delta_{v_i, e_{i-1}}$ and δ_{v_i, e_i} . By rules 3.2 and 4.2, the sum of these two values is equal to 1.

Next, we focus on variables z_e , with $e \in E^+$. We start analyzing the coefficients of the variables corresponding to $e \in E^+ \setminus D$. We restrict our attention on one maximal length substring of $E^+ \setminus D$. Let $e_i, \dots, e_{i+h}$ be the substring, and note that $e_{i-1} = f_j$ for some j . Recall that, by assumption (b), only e_i and e_{i+h} are adjacent to edges that belong to E^- . Observe that, if $e_i = e_{i+h}$, then z_{e_i} appears with nonzero coefficient only in the two flower inequalities (θ_{f_j}) and $(\theta_{f_{j+1}})$. By rule 1, the corresponding multipliers sum to one. Assume now $e_i \neq e_{i+h}$. The variable z_{e_i} is present in just two inequalities: the flower inequality (θ_{f_j}) , and (δ_{v_{i+1}, e_i}) . By rule 3.1, the sum of the related multipliers is equal to 1. The coefficients of $z_{e_{i+1}}, \dots, z_{e_{i+h-1}}$ are correct in $\pi z \leq \lfloor \pi_0 \rfloor$, because these variables are present in only two inequalities with nonzero multipliers and, by rule 3.2, they sum to 1. Consider the entry of π corresponding to the variable $z_{e_{i+h}}$. This variable appears in two inequalities: one chosen with multiplier equal to $\delta_{v_{i+h}, e_{i+h}}$, which is equal to θ_{f_j} , and in the flower inequality corresponding to $e_{i+h+1} = f_{j+1}$. By rule 1, the multiplier of this flower inequality is equal to $1 - \theta_{f_j}$, because there are no edges of E^- between e_i and e_{i+h} . Hence, the coefficient of $z_{e_{i+h}}$ in $\pi z \leq \lfloor \pi_0 \rfloor$ is correctly equal to 1, as it is in a too.

We check the correctness of the coefficients of z_{e_i} , with $e_i \in D$; thus, we assume that D is nonempty. Consider the first edge of D , that is e_{p+1} . Because of assumption (c), the variable corresponding to it appears in Δ_{p+1} flower inequalities, whose centers are edges f_j with j odd, and in $(\delta_{v_{p+2}, e_{p+1}})$ if $|D| \geq 2$. By rules 1 and 4.1, all these multipliers sum to 1. Then consider edges $e_{p+i} \in D \setminus \{e_{p+1}, e_m\}$. Because of assumption (c), each of these variables is present in Δ_{p+i} flower inequalities, with centers in the edges $f_j \in E^-$ with j odd, and $(\delta_{v_{p+i}, e_{p+i}})$. Then this sum is equal to

$$\begin{aligned} \frac{\Delta_{p+i}}{T} + \delta_{v_{p+i}, e_{p+i}} + \delta_{v_{p+i+1}, e_{p+i}} &= \frac{\Delta_{p+i}}{T} + \frac{\sum_{j=p+1}^{p+i-1} \Delta_j}{T} + 1 - \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} \\ &= \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} + 1 - \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} = 1, \end{aligned}$$

where the first equality comes from the definition of the multipliers in rule 4.2. We only need to verify what happens for z_{e_m} . This variable is present in Δ_m flower inequalities all with multiplier $1/T$, and in (δ_{v_m, e_m}) . Then, by rules 1 and 4.3, its coefficient is equal to

$$\frac{\Delta_m}{T} + \frac{\sum_{j=p+1}^{m-1} \Delta_j}{T} = \frac{\sum_{j=p+1}^m \Delta_j}{T} = 1,$$

where the last equality follows from $\sum_{i=p+1}^m \Delta_i = T$. $\square$

8.2.3. Proof of Claim 6. Assume first that assumption (d1) holds. This implies that $D \neq \emptyset$. In fact, if this is not the case, then we would have that $e_m = f_k$ and $\emptyset \neq f_1 \cap f_k \subseteq S_1$, by assumption (a). However, both 1 and k are odd indices, which contradict (d1). We obtain the following formula for $\lfloor \pi_0 \rfloor$, where the first two sums come from inequalities (θ_{f_i}) , depending on i being odd or even, and the last two from α_v :

$$\begin{aligned} &\left| \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{1}{T} \left(\left| f_i \setminus \bigcup_{e \in T_i} e \right| + |T_i| - 1 \right) + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{T-1}{T} \left(\left| f_i \setminus \bigcup_{e \in T_i} e \right| + |T_i| - 1 \right) + \right. \\ &\quad \left. + \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{T-1}{T} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{1}{T} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| \right|. \end{aligned}$$

Recall that S_1 is the set of nodes contained only in edges of E^- . Then observe that $|S_1|$ is equal to

$$\begin{aligned} & \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{T-1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \\ & + \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{T-1}{T} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{1}{T} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right|. \end{aligned} \quad (13)$$

In fact, each node v of S_1 is either contained in two edges of E^- , one odd and one even, or only in one edge of E^- , by assumptions (a) and (d1). In the first case, v appears only in the first two sums and is counted one single time in each sum, once with multiplier $1/T$ and once with $(T-1)/T$. Summing these two quantities, we obtain that every node $v \in S_1$ of this type contributes by 1 in (13). Consider the second case, that is, v is in only one edge of E^- . If v belongs to f_i for some i odd, it contributes by $1/T$ in the first sum and by $(T-1)/T$ in the third sum. Otherwise, it contributes by $(T-1)/T$ in the second sum and by $1/T$ in the last sum. Hence, also when v is contained in just one edge, by summing these two terms we see that v contributes by 1 in (13). Then $|S_1| \leq (13)$. To prove that $(13) \leq |S_1|$, it suffices to see that only nodes in S_1 are considered in (13) and that no node is considered twice with the same multiplier.

Therefore, we get the following expression for $\lfloor \pi_0 \rfloor$:

$$|S_1| + \left\lfloor \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{1}{T} (|T_i \cap D| + |T_i \setminus D| - 1) + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{T-1}{T} (|T_i \setminus D| - 1) \right\rfloor, \quad (14)$$

$$= |S_1| + \left\lfloor 1 + \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{|T_i \setminus D|}{T} - \frac{1}{T} \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{T-1}{T} |T_i \setminus D| - \frac{T-1}{T} \left\lfloor \frac{k}{2} \right\rfloor \right\rfloor, \quad (15)$$

$$= |S_1| + \left\lfloor 1 + \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{|T_i \setminus D|}{T} + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{T-1}{T} |T_i \setminus D| - \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{T} \right\rfloor,$$

where the first line comes also from the fact that $|T_i \cap D| = 0$ for all i even, by assumption (c), and the first equality follows from the definition of T .

Because of assumption (b), notice that $\sum_{\substack{i \in [k] \\ i \text{ odd}}} |T_i \setminus D| = \sum_{\substack{i \in [k] \\ i \text{ even}}} |T_i \setminus D|$, because both sums count how many times the sign of the variables z_e changes from negative to positive in the part of the β -cycle corresponding to the edges of $E(C) \setminus D$. Let us denote this quantity by σ . Then $\lfloor \pi_0 \rfloor$ is equal to

$$\begin{aligned} & |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\sigma}{T} + \frac{T-1}{T} \sigma - \frac{1}{T} \right\rfloor = |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \sigma - \frac{1}{T} \right\rfloor \\ & = |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \sigma + \left\lfloor -\frac{1}{T} \right\rfloor = |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \sigma - 1 = |S_1| - \left\lfloor \frac{k}{2} \right\rfloor + \sigma. \end{aligned} \quad (16)$$

In order for $\lfloor \pi_0 \rfloor$, we need to check that $-\lfloor k/2 \rfloor + \sigma = -|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \lfloor k/2 \rfloor$. This is equivalent to $\sigma + |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| = 2\lfloor k/2 \rfloor = k - 1$. The latter equality is true because every edge in E^- , except the last one, is either accounted for in $|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}|$ or in σ , if its succeeding edge belongs to $E^+ \setminus D$.

Now assume that (d2) holds. Observe that, when assumption (d2) holds, $T = 2$. In fact, when $D \neq \emptyset$, assumption (a) implies that T is equal to 2. On the other, when $D = \emptyset$, it follows that $T = 2$ from its definition. We start by the case in which D , and therefore also E^+ , is nonempty. In this case $\lfloor \pi_0 \rfloor$ is given by

$$\left\lfloor \sum_{i \in [k]} \frac{1}{2} \left(\left| f_i \setminus \bigcup_{e \in T_i} e \right| + |T_i| - 1 \right) + \sum_{i \in [k]} \frac{1}{2} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| \right\rfloor.$$

By similar arguments to the previous case, S_1 is equal to

$$\sum_{i \in [k]} \frac{1}{2} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \sum_{i \in [k]} \frac{1}{2} \left| f_i \setminus \left(\bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right|.$$

Thus, at this point, we obtain that $\lfloor \pi_0 \rfloor$ is equal to

$$|S_1| + \left\lfloor \sum_{\substack{i \in [k] \\ i \text{ odd}}} \frac{1}{2} (|T_i \cap D| + |T_i \setminus D| - 1) + \sum_{\substack{i \in [k] \\ i \text{ even}}} \frac{1}{2} (|T_i \setminus D| - 1) \right\rfloor = |S_1| + \left\lfloor 1 + \sum_{i \in [k]} \frac{1}{2} (|T_i \setminus D| - 1) \right\rfloor. \quad (17)$$

The equality comes from the fact that in this case the only indices i for which $|T_i \cap D| \neq 0$ are $i = p + 1, m$. More precisely they are exactly equal to 1. Because of this, the calculations are analogous to the ones of the first part of the proof.

Next, we deal with the case in which $D = \emptyset$ and $E^+ \neq \emptyset$. The only difference with the previous proof is that $|T_i \cap D| = 0$ for every $i \in [k]$ in (14), and therefore, there is no +1 at the beginning of the floor in (17). By doing the same computations as in (16), we see that in this case

$$\lfloor \pi_0 \rfloor = |S_1| - \left\lfloor \frac{k}{2} \right\rfloor + \sigma - 1.$$

Therefore, we now want to check that $-\lfloor k/2 \rfloor + \sigma - 1 = -|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \lfloor k/2 \rfloor$. This is true, because we have that $\sigma + |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| = k$. Indeed now for all the edges e_i of E^- either e_{i+1} is in E^- too, or it belongs to $E^+ \setminus D$, which is equal to E^+ .

It remains to study the case in which $E^+ = \emptyset$. By the fact that $\lfloor \pi_0 \rfloor$ is equal to (17), $T_i = \emptyset$ for all $i \in [k]$, we get that

$$\lfloor \pi_0 \rfloor = |S_1| + \left\lfloor -\frac{k}{2} \right\rfloor.$$

In order to achieve the thesis, it suffices to check that $-\lfloor k/2 \rfloor = -|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| + \lfloor k/2 \rfloor$. It is easy to see that this is true, once we observe that $|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-\}| = k$ in this case.

We can conclude that b coincides with $\lfloor \pi_0 \rfloor$ in every case. $\square$

8.3. Claims in the Proof of Proposition 2

8.3.1. Proof of Claim 7. We recall that the perfect formulations of F^1, F^2 coincide with their corresponding standard linearizations, which we now write explicitly. Hence, the perfect formulation of F^1 is

$$\begin{aligned} z_{e_1} &= z_{e_2} = z_{\bar{v}} = 0 \\ 0 &\leq z_v \leq 1 & \forall v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3) \\ -z_{e_3} &\leq 0 \\ \sum_{v \in e_3} z_v - z_{e_3} &\leq |e_3| - 1 \\ z_v &\leq 1 & \forall v \in e_3 \\ -z_v + z_{e_3} &\leq 0 & \forall v \in e_3. \end{aligned}$$

Similarly, F^2 is described by the standard linearization:

$$\begin{aligned} z_{\bar{v}} &= 1 \\ -z_e &\leq 0 & \forall e \in \{e_1, e_2, e_3\} \\ \sum_{v \in e_1 \setminus \{\bar{v}\}} z_v - z_{e_1} &\leq |e_1| - 2 \\ \sum_{v \in e_2 \setminus \{\bar{v}\}} z_v - z_{e_2} &\leq |e_2| - 2 \\ \sum_{v \in e_3} z_v - z_{e_3} &\leq |e_3| - 1 \\ z_v &\leq 1 & \forall v \in V \setminus \{\bar{v}\} \\ -z_v + z_{e_1} &\leq 0 & \forall v \in e_1 \setminus \{\bar{v}\} \\ -z_v + z_{e_2} &\leq 0 & \forall v \in e_2 \setminus \{\bar{v}\} \\ -z_v + z_{e_3} &\leq 0 & \forall v \in e_3. \end{aligned}$$

The application of Balas' formulation for the union of polytopes gives us the following perfect formulation of MP_G :

$$z = z^1 + z^2, \quad (18)$$

$$0 \leq \lambda \leq 1$$

$$z_{e_1}^1 = z_{e_2}^1 = z_{\bar{v}}^1 = 0$$

$$0 \leq z_v^1 \leq 1 - \lambda \quad \forall v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3), \quad (19)$$

$$-z_{e_3}^1 \leq 0, \quad (20)$$

$$\sum_{v \in e_3} z_v^1 - z_{e_3}^1 \leq (|e_3| - 1)(1 - \lambda), \quad (21)$$

$$z_v^1 \leq 1 - \lambda \quad \forall v \in e_3, \quad (22)$$

$$-z_v^1 + z_{e_3}^1 \leq 0 \quad \forall v \in e_3, \quad (23)$$

$$z_{\bar{v}}^2 = \lambda$$

$$-z_e^2 \leq 0 \quad \forall e \in \{e_1, e_2, e_3\}$$

$$\sum_{v \in e_1 \setminus \{\bar{v}\}} z_v^2 - z_{e_1}^2 \leq (|e_1| - 2)\lambda$$

$$\sum_{v \in e_2 \setminus \{\bar{v}\}} z_v^2 - z_{e_2}^2 \leq (|e_2| - 2)\lambda$$

$$\sum_{v \in e_3} z_v^2 - z_{e_3}^2 \leq (|e_3| - 1)\lambda$$

$$z_v^2 \leq \lambda \quad \forall v \in V \setminus \{\bar{v}\}$$

$$-z_v^2 + z_{e_1}^2 \leq 0 \quad \forall v \in e_1 \setminus \{\bar{v}\}$$

$$-z_v^2 + z_{e_2}^2 \leq 0 \quad \forall v \in e_2 \setminus \{\bar{v}\}$$

$$-z_v^2 + z_{e_3}^2 \leq 0 \quad \forall v \in e_3,$$

where the variables z^1 and z^2 arise from the systems defining F^1 and F^2 , respectively. Therefore, it remains to use Fourier-Motzkin elimination on the additional variables z^1, z^2, λ .

From (18), we see immediately that $z_{\bar{v}} = z_{\bar{v}}^1 + z_{\bar{v}}^2 = \lambda$. Similarly, $z_{e_1} = z_{e_1}^2$ and $z_{e_2} = z_{e_2}^2$. Hence, projecting out the variables $z_{e_1}^1, z_{e_1}^2, z_{e_2}^1, z_{e_2}^2, z_{\bar{v}}^1, z_{\bar{v}}^2$ is trivial.

Then, we begin projecting the variables coming from F^1 , starting from z_v^1 for every $v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3)$. These variables only appear in (18) and (19). We write these constraints in form of inequality, splitting them in inequalities in which z_v^1 has negative coefficient and inequalities in which its coefficient is positive.

$$\begin{array}{r} -z_v^1 \leq 0 \\ z_v - z_v^1 - z_v^2 \leq 0 \\ \hline z_v^1 \leq 1 - \lambda \\ -z_v + z_v^1 + z_v^2 \leq 0 \end{array}$$

Therefore, from the Fourier-Motzkin elimination of these variables we obtain the following inequalities:

$$\begin{array}{ll} -z_v + z_v^2 \leq 0 & \forall v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3) \\ z_v - z_v^2 \leq 1 - \lambda & \forall v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3). \end{array}$$

Next, we deal with $z_{e_3}^1$. This variable is present in (18), (20), (21), and (23). By splitting the inequalities in two sets, we have

$$\begin{array}{r} -z_{e_3}^1 \leq 0 \\ \sum_{v \in e_3} z_v^1 - z_{e_3}^1 \leq (|e_3| - 1)(1 - \lambda) \\ z_{e_3} - z_{e_3}^1 - z_{e_3}^2 \leq 0 \\ \hline -z_v^1 + z_{e_3}^1 \leq 0 \quad \forall v \in e_3 \\ -z_{e_3} + z_{e_3}^1 + z_{e_3}^2 \leq 0. \end{array}$$

After summing every pair of inequalities in which $z_{e_3}^1$ has different sign, we only keep in the formulation the following inequalities:

$$\sum_{v \in e_3} z_v^1 - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda), \quad (24)$$

$$\begin{aligned} -z_v^1 + z_{e_3} - z_{e_3}^2 &\leq 0 & \forall v \in e_3, \\ -z_{e_3} + z_{e_3}^2 &\leq 0 & \forall v \in e_3, \end{aligned} \quad (25)$$

as $-z_v^1 \leq 0$ for every $v \in e_3$ is implied by the last two inequalities of the previous system, and $\sum_{v \in e_3 \setminus \{w\}} z_v^1 \leq (|e_3| - 1)(1 - \lambda)$ is implied by (22) for every $w \in e_3$.

It remains to project out the variables z_v^1 , with $v \in e_3$, in order to eliminate all the variables arising from F^1 . These variables appear in (18), (22), (24), and (25):

$$-z_v^1 + z_{e_3} - z_{e_3}^2 \leq 0 \quad (26)$$

$$z_v - z_v^1 - z_v^2 \leq 0, \quad (27)$$

$$z_v^1 \leq 1 - \lambda, \quad (28)$$

$$-z_v + z_v^1 + z_v^2 \leq 0, \quad (29)$$

$$\sum_{v \in e_3} z_v^1 - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda). \quad (30)$$

Observe that when we projected out the variables z_v^1 , with $v \in e_1 \cup e_2 \setminus (\{\bar{v}\} \cup e_3)$, we did not impose an elimination order among these variables. This is because no two variables z_v^1, z_w^1 , with $v \neq w$, appeared in the same inequality. However, it is not the case now because of the presence of (30). Then, let us see how the inequalities change after we eliminate one specific variable, let it be $z_{\hat{v}}^1$ such that $\hat{v} \in e_3$. We leave in the formulation the inequalities:

$$-z_{\hat{v}} + z_{e_3}^2 + z_{e_3} - z_{e_3}^2 \leq 0, \quad (31)$$

$$z_{\hat{v}} - z_{\hat{v}}^2 \leq 1 - \lambda, \quad (32)$$

$$z_{\hat{v}} - z_{\hat{v}}^2 + \sum_{v \in e_3 \setminus \{\hat{v}\}} z_v^1 - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda). \quad (33)$$

In fact, the other inequalities are redundant. Indeed, the inequality obtained by (26) + (28) is implied by (31) + (32), and (26) + (30) is implied by (22). Moreover, (27) + (29) provides a trivial inequality, that is, $0 \leq 0$.

Note that (33) contains all the remaining z_v^1 variables, all of them with coefficient +1. Recall that, by Fourier-Motzkin elimination, we can sum (33) with only inequalities in which $-z_v^1$ is present, in order to project out the remaining variables. In the system there are only two inequalities of this type: $-z_v^1 + z_{e_3} - z_{e_3}^2 \leq 0$ and $z_v - z_v^1 - z_v^2 \leq 0$. Similarly to the case in which we projected out $z_{\hat{v}}^1$, we obtain a redundant inequality when we sum (33) with $-z_v^1 + z_{e_3} - z_{e_3}^2 \leq 0$, for the next chosen v in the elimination order. Hence, the only way in which (33) may lead to a nonredundant inequality is by summing it with $z_v - z_v^1 - z_v^2 \leq 0$. This argument holds for any node of e_3 in the elimination order. Then, we can conclude that after we eliminate all the variables z_v^1 , with $v \in e_3$, (24) has become $\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda)$. Moreover, observe that every time we remove a variable z_v^1 , we obtain the corresponding Inequalities (31) and (32).

We are done with the variables resulted from F^1 . At this point, the system has become

$$\begin{aligned} z_{\bar{v}} &= \lambda \\ 0 &\leq \lambda \leq 1 \\ -z_{e_1} &\leq 0 \\ -z_{e_2} &\leq 0 \\ -z_{e_3}^2 &\leq 0 \\ -z_{e_3} + z_{e_3}^2 &\leq 0 \\ z_v^2 &\leq \lambda & \forall v \in V \setminus \{\bar{v}\}, \\ -z_v + z_v^2 &\leq 0 & \forall v \in e_1 \cup e_2 \setminus (\bar{v} \cup e_3) \end{aligned} \quad (34)$$

$$z_v - z_v^2 \leq 1 - \lambda \quad \forall v \in V \setminus \{\bar{v}\}, \quad (35)$$

$$\begin{aligned}
-z_v^2 + z_{e_1} &\leq 0 & \forall v \in e_1 \setminus \{\bar{v}\} \\
-z_v^2 + z_{e_2} &\leq 0 & \forall v \in e_2 \setminus \{\bar{v}\} \\
-z_v^2 + z_{e_3}^2 &\leq 0 & \forall v \in e_3 \\
-z_v + z_v^2 + z_{e_3} - z_{e_3}^2 &\leq 0 & \forall v \in e_3 \\
\sum_{v \in e_1 \setminus \{\bar{v}\}} z_v^2 - z_{e_1} &\leq (|e_1| - 2)\lambda \\
\sum_{v \in e_2 \setminus \{\bar{v}\}} z_v^2 - z_{e_2} &\leq (|e_2| - 2)\lambda \\
\sum_{v \in e_3} z_v^2 - z_{e_3}^2 &\leq (|e_3| - 1)\lambda \\
\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} + z_{e_3}^2 &\leq (|e_3| - 1)(1 - \lambda).
\end{aligned}$$

Next, we deal with variables coming from F^2 , starting with $z_{e_3}^2$. For simplicity, we display the inequalities in which $z_{e_3}^2$ appears. We divide them in two sets:

$$\begin{aligned}
-z_{e_3}^2 &\leq 0 \\
-z_v + z_v^2 + z_{e_3} - z_{e_3}^2 &\leq 0 & \forall v \in e_3,
\end{aligned} \tag{36}$$

$$\sum_{v \in e_3} z_v^2 - z_{e_3}^2 \leq (|e_3| - 1)\lambda, \tag{37}$$

$$\begin{aligned}
-z_{e_3} + z_{e_3}^2 &\leq 0 \\
-z_v^2 + z_{e_3}^2 &\leq 0 & \forall v \in e_3,
\end{aligned} \tag{38}$$

$$\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda). \tag{39}$$

After applying Fourier-Motzkin elimination on this variable, we keep the following inequalities:

$$\begin{aligned}
-z_{e_3} &\leq 0 \\
-z_v^2 &\leq 0 & \forall v \in e_3 \setminus (e_1 \cup e_2) \\
-z_v + z_v^2 &\leq 0 & \forall v \in e_3 \\
-z_v + z_{e_3} &\leq 0 & \forall v \in e_3 \\
-z_v + z_v^2 - z_w^2 + z_{e_3} &\leq 0 & \forall v, w \in e_3, v \neq w,
\end{aligned} \tag{40}$$

$$\begin{aligned}
\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} &\leq (|e_3| - 1)(1 - \lambda) \\
\sum_{v \in e_3} z_v^2 - z_{e_3} &\leq (|e_3| - 1)\lambda \\
\sum_{v \in e_3} z_v - z_{e_3} &\leq |e_3| - 1,
\end{aligned} \tag{41}$$

where (40) comes from (36) + (38) when the nodes v in the two inequalities are distinct. We discarded (36) + (39), as it is implied by (35). Similarly, we did not write (37) + (38), because it is entailed by (34). Let us remark that we decide to keep some redundant inequalities in the system as long as they contain only variables in the original space, like for Example (41), because they will be useful in the next arguments.

Now we continue with the projection of the other variables. We focus first on z_v^2 with $v \in e_1 \setminus (\{\bar{v}\} \cup e_3)$. For $v \in e_2 \setminus (\{\bar{v}\} \cup e_3)$ the calculations are similar, as it suffices to swap the roles of e_1 and e_2 . Therefore, we are not going to repeat the computations. As before, we write here the inequalities involving such z_v^2 :

$$z_v - z_v^2 \leq 1 - \lambda \quad \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3), \tag{42}$$

$$-z_v^2 + z_{e_1} \leq 0 \quad \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3), \tag{43}$$

$$\begin{aligned}
z_v^2 &\leq \lambda & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3) \\
-z_v + z_v^2 &\leq 0 & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3),
\end{aligned} \tag{44}$$

$$\sum_{v \in e_1 \setminus \{\bar{v}\}} z_v^2 - z_{e_1} \leq (|e_1| - 2)\lambda. \tag{45}$$

Observe that here, like when we projected out variables z_v^1 for $v \in e_3$, there is one inequality, (45), that contains all the variables z_v^2 . Then, let us fix a node $\hat{v} \in e_1 \setminus (\{\bar{v}\} \cup e_3)$. After performing Fourier-Motzkin elimination on this variable, we keep the following set of inequalities:

$$\begin{aligned} z_{\hat{v}} &\leq 1 \\ z_{e_1} &\leq \lambda \\ -z_{\hat{v}} + z_{e_1} &\leq 0 \\ z_{\hat{v}} + \sum_{v \in e_1 \setminus \{\bar{v}, \hat{v}\}} z_v^2 - z_{e_1} &\leq 1 + (|e_1| - 3)\lambda. \end{aligned} \quad (46)$$

In fact, (42) + (44) provides the trivial inequality $0 \leq 1 - \lambda$, and (43) + (45) produces $\sum_{v \in e_1 \setminus \{\bar{v}, \hat{v}\}} z_v^2 \leq (|e_1| - 2)\lambda$, which is implied by (34).

Observe that (46) contains all the z_v^2 variables that still need to be projected out, with $v \in e_1 \setminus (\{\bar{v}\} \cup e_3)$. When we sum (46) with (43), chosen for the variable z_v that we are currently eliminating, we obtain a redundant inequality. In fact, it is dominated by the bounds $z_w \leq 1$, for all $w \in e_1 \setminus (\{\bar{v}\} \cup e_3)$ for which z_w^2 has already been projected out, and $z_v^2 \leq \lambda$, for the variables that are still to be eliminated. Hence, (46) can lead to a nonredundant inequality only when it is summed with (42). Thus, if we add (46) to (42) for every $v \in e_1 \setminus (\{\bar{v}\} \cup e_3)$, then we can conclude that (45) leads to

$$\sum_{v \in e_1 \setminus (\{\bar{v}\} \cup e_3)} z_v + \sum_{v \in e_1 \cap e_3} z_v^2 - z_{e_1} \leq |e_1| - 2.$$

Observe that the second sum actually involves only one variable, because $|e_1 \cap e_3| = 1$ by assumption.

At this point, the variables left to project out are z_v^2 , for $v \in e_3$, and λ . We focus on z_v^2 , with $v \in e_3 \setminus (e_1 \cup e_2)$. Consider a specific node $\hat{v}$ in this set. Then $z_{\hat{v}}^2$ is present in the following system of inequalities:

$$-z_{\hat{v}}^2 \leq 0, \quad (47)$$

$$z_{\hat{v}} - z_{\hat{v}}^2 \leq 1 - \lambda, \quad (48)$$

$$-z_w + z_w^2 - z_{\hat{v}}^2 + z_{e_3} \leq 0 \quad \forall w \in e_3, w \neq \hat{v}, \quad (49)$$

$$\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} \leq (|e_3| - 1)(1 - \lambda), \quad (50)$$

$$-z_{\hat{v}} + z_{\hat{v}}^2 \leq 0, \quad (51)$$

$$z_{\hat{v}}^2 \leq \lambda, \quad (52)$$

$$-z_{\hat{v}} + z_{\hat{v}}^2 - z_w^2 + z_{e_3} \leq 0 \quad \forall w \in e_3, w \neq \hat{v}, \quad (53)$$

$$\sum_{v \in e_3} z_v^2 - z_{e_3} \leq (|e_3| - 1)\lambda. \quad (54)$$

We first provide the inequalities that we do not discard after this iteration of Fourier-Motzkin elimination and next we explain why we removed the other inequalities. We keep

$$\begin{aligned} z_{\hat{v}} &\leq 1 \\ z_{\hat{v}} + \sum_{v \in e_3 \setminus \{\hat{v}\}} (z_v - z_v^2) - z_{e_3} &\leq (|e_3| - 1)(1 - \lambda) + \lambda, \end{aligned} \quad (55)$$

$$z_{\hat{v}} + \sum_{v \in e_3 \setminus \{\hat{v}\}} z_v^2 - z_{e_3} \leq 1 + (|e_3| - 2)\lambda. \quad (56)$$

All inequalities coming from (47) are redundant. Inequality (47) + (51) is equivalent to the sum of $-z_{\hat{v}} + z_{e_3} \leq 0$ and $-z_{e_3} \leq 0$; however, these two inequalities are already in the system, and therefore, (47) + (51) is redundant. Moreover, (47) + (52) is equivalent to $0 \leq \lambda$, which is always true, and (47) + (53) is implied by $-z_{\hat{v}} + z_{e_3} \leq 0$ and $-z_w^2 \leq 0$ for all $w \in e_3, w \neq \hat{v}$. Observe that if $w \in e_3 \setminus (e_1 \cup e_2)$, then $-z_w^2 \leq 0$ is in the system; otherwise, $-z_w^2 \leq 0$ is obtained by the sum of $-z_w^2 + z_{e_1} \leq 0$ and $-z_{e_1} \leq 0$, depending on $w = e_3 \cap e_1$ or $w = e_3 \cap e_2$. Furthermore, (47) + (54) is implied by (34) and $-z_{e_3} \leq 0$.

Consider Inequality (48). When summed with (51), it provides the trivial inequality $0 \leq 1 - \lambda$. Additionally, (48) + (53) is redundant as it is implied by $-z_v + z_v^2 - z_w^2 + z_{e_3} \leq 0$ and $z_v - z_v^2 \leq 1 - \lambda$, for $v \neq w, v \in e_3$. Such a v exists, because we have not projected out the nodes in the intersections $e_1 \cap e_3$ and $e_2 \cap e_3$ yet.

Also, all inequalities deriving from (49) are redundant. The calculations regarding (49) + (51), (49) + (52), and (49) + (53) are similar to the previous cases in which we considered (53). Inequality (49) + (54) is implied instead by $-z_w + z_w^2 \leq 0$ and (34) for all $w \neq \hat{v}$.

Then, let us analyze what happens to (50). The sum (50) + (51) can be obtained by summing (35) and $-z_{e_3} \leq 0$. Similarly, (50) + (53) is implied by $-z_w^2 \leq 0$, for all $w \neq \hat{v}$ and (35). Last, observe that (50) + (54) coincides with (41), which is already in the system.

Now, observe that both (55) and (56) contain all the variables z_v^2 corresponding to the other nodes in $e_3 \setminus (e_1 \cup e_2)$. Like for Inequalities (33) and (46), notice that (55) and (56) might become nonredundant only when they are summed with, respectively, (52) and (48), where these inequalities are chosen for the node v whose variable is currently being eliminated. Therefore, after projecting out all variables corresponding to nodes in $e_3 \setminus (e_1 \cup e_2)$, (55) and (56) become

$$\begin{aligned} \sum_{v \in e_3 \setminus (e_1 \cup e_2)} z_v + \sum_{e_3 \cap (e_1 \cup e_2)} (z_v - z_v^2) - z_{e_3} &\leq |e_3 \setminus (e_1 \cup e_2)| + 1 - \lambda \\ \sum_{v \in e_3 \setminus (e_1 \cup e_2)} z_v + \sum_{e_3 \cap (e_1 \cup e_2)} z_v^2 - z_{e_3} &\leq |e_3 \setminus (e_1 \cup e_2)| + \lambda. \end{aligned}$$

For ease of notation, let $\{w_1\} = e_1 \cap e_3$ and $\{w_2\} = e_2 \cap e_3$. There are only three variables left to eliminate: z_{w_1} , z_{w_2} , and λ . Consider z_{w_1} . It is present in

$$z_{w_1} - z_{w_1}^2 \leq 1 - \lambda, \quad (57)$$

$$-z_{w_1}^2 + z_{e_1} \leq 0, \quad (58)$$

$$-z_{w_2} + z_{w_2}^2 - z_{w_1}^2 + z_{e_3} \leq 0, \quad (59)$$

$$\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_1} - z_{w_1}^2 + z_{w_2} - z_{w_2}^2 - z_{e_3} \leq |e_3 \setminus (e_1 \cup e_2)| + 1 - \lambda, \quad (60)$$

$$z_{w_1}^2 \leq \lambda, \quad (61)$$

$$-z_{w_1} + z_{w_1}^2 \leq 0, \quad (62)$$

$$-z_{w_1} + z_{w_1}^2 - z_{w_2}^2 + z_{e_3} \leq 0, \quad (63)$$

$$\sum_{v \in e_1 \setminus \{\hat{v}, w_1\}} z_v + z_{w_1}^2 - z_{e_1} \leq |e_1| - 2, \quad (64)$$

$$\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_1}^2 + z_{w_2}^2 - z_{e_3} \leq |e_3 \setminus (e_1 \cup e_2)| + \lambda. \quad (65)$$

We keep these inequalities in the formulation:

$$\begin{aligned} z_{w_1} &\leq 1 \\ -z_{w_2}^2 + z_{e_3} &\leq 1 - \lambda \\ \sum_{v \in e_1 \setminus \{\hat{v}\}} z_v - z_{e_1} &\leq |e_1| - 1 - \lambda \\ \sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2}^2 - z_{e_3} &\leq |e_3 \setminus e_2| \\ -z_{w_1} + z_{e_1} &\leq 0 \\ -z_{w_1} + z_{e_1} - z_{w_2}^2 + z_{e_3} &\leq 0 \\ \sum_{v \in e_1 \setminus \{\hat{v}, w_1\}} z_v - z_{w_2} + z_{w_2}^2 - z_{e_1} + z_{e_3} &\leq |e_1| - 2 \\ \sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_2}^2 + z_{e_1} - z_{e_3} &\leq |e_3 \setminus (e_1 \cup e_2)| + \lambda \\ \sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_3} &\leq |e_3 \setminus e_2| \\ \sum_{v \in (e_3 \cup e_1) \setminus \{\hat{v}, w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_1} - z_{e_3} &\leq |e_3 \setminus (e_1 \cup e_2)| + |e_1| - 1 - \lambda. \end{aligned} \quad (66)$$

All the other inequalities are redundant. Let us check them one by one. Inequality (57) + (62) is equal to $0 \leq 1 - \lambda$, which is always true because $0 \leq \lambda \leq 1$. This is the only inequality certainly redundant arising from (57). Then, let us move on to (58). The result of (58) + (61) is equal to $z_{e_1} \leq \lambda$, which is already in the system. We can discard also (58) + (64), because $z_v \leq 1$ holds for every variable corresponding to nodes v for which we have already projected out z_v^2 .

Now consider inequalities coming from (59). Inequality (59) + (61) is redundant because it can be obtained equivalently by summing $-z_{w_2} + z_{e_3} \leq 0$ and $z_{w_2}^2 \leq \lambda$. Similarly, (59) + (62) is implied by $-z_{w_1} + z_{e_3} \leq 0$ and $-z_{w_2} + z_{w_2}^2 \leq 0$. Along the same lines, we can obtain (59) + (63) as the sum of $-z_{w_1} + z_{e_3} \leq 0$ and $-z_{w_2} + z_{e_3} \leq 0$. Finally, Inequality (59) + (65) can be discarded, as the combination of $z_v \leq 1$, for $v \in e_3 \setminus \{w_1, w_2\}$, $-z_{w_1} + z_{w_1}^2 \leq 0$, and $z_{w_2}^2 \leq \lambda$ provides the same inequalities.

It remains to check the inequalities originating from (60). The first redundant inequality is (60) + (62), because $z_{w_2} - z_{w_2}^2 \leq 1 - \lambda$, $-z_{e_3} \leq 0$, and $z_v \leq 1$, for all $v \in e_3 \setminus \{w_1, w_2\}$. Similarly, we can remove (60) + (63) from the formulation, because it can be obtained by summing $z_{w_2} - z_{w_2}^2 \leq 1 - \lambda$, $-z_{w_2}^2 \leq 0$, and $z_v \leq 1$, if $v \in e_3 \setminus \{w_1, w_2\}$. Ultimately, (60) + (65) is implied by (41) and the usual bound on the variables z_v . We finished with projecting out the variable $z_{w_1}^2$.

Next, we apply Fourier-Motzkin elimination on $z_{w_2}^2$. This variable is present in the following inequalities:

$$z_{w_2} - z_{w_2}^2 \leq 1 - \lambda, \quad (67)$$

$$-z_{w_2}^2 + z_{e_2} \leq 0, \quad (68)$$

$$-z_{w_2}^2 + z_{e_3} \leq 1 - \lambda, \quad (69)$$

$$-z_{w_1} + z_{e_1} - z_{w_2}^2 + z_{e_3} \leq 0, \quad (70)$$

$$\sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_3} \leq |e_3 \setminus e_2|, \quad (71)$$

$$\sum_{v \in (e_3 \cup e_1) \setminus \{\bar{v}, w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_1} - z_{e_3} \leq |(e_3 \cup e_1) \setminus e_2| - \lambda, \quad (72)$$

$$z_{w_2}^2 \leq \lambda, \quad (73)$$

$$-z_{w_2} + z_{w_2}^2 \leq 0, \quad (74)$$

$$\sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v + z_{w_2}^2 - z_{e_2} \leq |e_2| - 2, \quad (75)$$

$$\sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2}^2 - z_{e_3} \leq |e_3| - 1, \quad (76)$$

$$\sum_{v \in e_1 \setminus \{\bar{v}, w_1\}} z_v - z_{w_2} + z_{w_2}^2 - z_{e_1} + z_{e_3} \leq |e_1| - 2, \quad (77)$$

$$\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_2}^2 + z_{e_1} - z_{e_3} \leq |e_3 \setminus (e_1 \cup e_2)| + \lambda, \quad (78)$$

We keep the following inequalities and discard the rest:

$$\begin{aligned} & z_{w_2} \leq 1 \\ & -z_{w_2} + z_{e_2} \leq 0 \\ & \sum_{v \in e_2 \setminus \{\bar{v}\}} z_v - z_{e_2} \leq |e_2| - 1 - \lambda \\ & \sum_{v \in e_1 \setminus \{\bar{v}, w_1\}} z_v - z_{w_2} - z_{e_1} + z_{e_2} + z_{e_3} \leq |e_1| - 2 \\ & \sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v - z_{w_1} + z_{e_1} - z_{e_2} + z_{e_3} \leq |e_2| - 2 \\ & \sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{e_1} + z_{e_2} - z_{e_3} \leq |e_3| - 2 + \lambda \\ & \sum_{v \in V \setminus \{\bar{v}\}} z_v - z_{e_1} - z_{e_2} - z_{e_3} \leq |V| - 2 - \lambda. \end{aligned}$$

Here we show why we decided to discard all other inequalities. Let us start from analyzing the ones coming from (67). Inequality (67) + (74) is equal to $0 \leq 1 - \lambda$; hence, it is always true because λ is bounded to lie

in $[0, 1]$. Also (67) + (76) is redundant, as it is implied by (41). Observe that (67) + (77) can be obtained as sum of (66) and $-z_{w_1} + z_{e_3} \leq 0$; therefore, we are allowed to ignore it. Similarly, (67) (78) is equal to the sum of (41) and $-z_{w_1} + z_{e_1} \leq 0$.

Next, consider the inequalities coming from (68). Note that (68) + (73) was already found when projecting out nodes in $e_2 \setminus (e_1 \cup e_3)$. Inequalities (68) + (75) and (68) + (76) are implied by the fact that $z_v \leq 1$, for all $v \neq w_2$.

All inequalities resulting from (69) are redundant. In fact, (69) + (73) can be obtained by summing inequalities already in the system, which are $-z_v + z_{e_3} \leq 0$, $z_v \leq 1$ for any $v \in e_3$, $v \neq w_2$. Then, (69) + (74) is equal to (δ_{w_2, e_3}) , which was found earlier. Inequality (69) + (75) is redundant for similar reasons as (67) + (77). Instead, (69) + (76) is implied by $z_v \leq 1$. Additionally, (69) + (77) can be achieved as sum of (66), $-z_{w_1} + z_{e_3} \leq 0$, and $-z_{w_2} + z_{e_3} \leq 0$. Moreover, (69) + (78) can be discarded, as it is implied by $z_v \leq 1$ and $z_{e_1} \leq \lambda$, for all involved nodes v .

We move on and consider (70). Notice that (70) + (73) is implied by $-z_{w_1} + z_{e_3} \leq 0$ and $z_{e_1} \leq \lambda$. Furthermore, (70) + (74) is equal to the sum of $-z_{w_2} + z_{e_3} \leq 0$ and $-z_{w_1} + z_{e_1} \leq 0$. Also, (70) + (76) can be obtained by $z_v \leq 1$, for all $v \in e_3 \setminus \{w_2\}$, and $-z_{w_1} + z_{e_1} \leq 0$. Similarly, inequality (70) + (77) is implied by the sum of $z_v \leq 1$, for suitable nodes v , $-z_{w_1} + z_{e_3} \leq 0$, and $-z_{w_2} + z_{e_3} \leq 0$. Last, (70) + (78) is redundant, because it is the sum of $z_v \leq 1$, $-z_{w_1} + z_{e_1} \leq 0$, and $z_{e_1} \leq \lambda$, all inequalities that are already in the system.

Here we focus on the next to last inequality in which $z_{w_2}^2$ has negative coefficient. (71) + (73) is implied by (41). Inequality (71) + (74) instead is redundant because of $z_v \leq 1$, and $-z_{e_3} \leq 0$. Then, (71) + (75) is implied by (41), $z_v \leq 1$, and $-z_{e_2} \leq 0$. Furthermore, (71) + (76) is similar to (71) + (75). Also, (71) + (77) is redundant because of $z_v \leq 1$ and $-z_{e_1} \leq 0$. Observe that (71) + (78) is obtained by summing (41), $z_v \leq 1$ for the remaining z_v , and $z_{e_1} \leq \lambda$, $z_{e_3} \leq 0$.

We focus now on (72). Similar to before, (72) + (73) is equal to the sum of (41), $z_v \leq 1$ for the variables not included in (41), and $-z_{e_1} \leq 0$. Analogously, (72) + (74) is implied by (66), $z_v \leq 1$, and $-z_{e_3} \leq 0$. Inequality (72) + (76) can be obtained by (66), (41), $z_v \leq 1$, and $-z_{e_3} \leq 0$. Along the same lines, (72) + (77) coincides with summing (66), $z_v \leq 1$, and $-z_{e_1} \leq 0$. Finally, (72) + (78) is redundant because it is implied by (41), $z_v \leq 1$, and $-z_{e_3} \leq 0$.

At this point, the only variables appearing in the formulation are z and λ . The system is

$$\begin{aligned}
& z_{\bar{v}} = \lambda \\
& 0 \leq \lambda \leq 1 \\
& -z_{e_1} \leq 0 \\
& -z_{e_2} \leq 0 \\
& -z_{e_3} \leq 0 \\
& -\lambda + z_{e_1} \leq 0 \\
& -\lambda + z_{e_2} \leq 0 \\
& z_v \leq 1 & \forall v \in V \setminus \{\bar{v}\} \\
& -z_v + z_{e_1} \leq 0 & \forall v \in e_1 \setminus \{\bar{v}\} \\
& -z_v + z_{e_2} \leq 0 & \forall v \in e_2 \setminus \{\bar{v}\} \\
& -z_v + z_{e_3} \leq 0 & \forall v \in e_3 \\
& \sum_{v \in e_1 \setminus \{\bar{v}\}} z_v + \lambda - z_{e_1} \leq |e_1| - 1 \\
& \sum_{v \in e_2 \setminus \{\bar{v}\}} z_v + \lambda - z_{e_2} \leq |e_2| - 1 \\
& \sum_{v \in e_3} z_v - z_{e_3} \leq |e_3| - 1 \\
& \sum_{v \in e_1 \setminus \{\bar{v}, w_1\}} z_v - z_{w_2} - z_{e_1} + z_{e_2} + z_{e_3} \leq |e_1| - 2 \\
& \sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v - z_{w_1} + z_{e_1} - z_{e_2} + z_{e_3} \leq |e_2| - 2 \\
& \sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v - \lambda + z_{e_1} + z_{e_2} - z_{e_3} \leq |e_3| - 2 \\
& \sum_{v \in V \setminus \{\bar{v}\}} z_v + \lambda - z_{e_1} - z_{e_2} - z_{e_3} \leq |V| - 2.
\end{aligned} \tag{79}$$

By (79), it suffices to substitute λ with $z_{\bar{v}}$ in the previous system in order to project out the variable λ . Because the system consists of all the standard linearization inequalities and the four odd β -cycle inequalities that arise in this case, we obtain that the claim holds when the hypergraph G contains only three edges. $\square$

8.3.2. Proof of Claim 8. Here, we present the computations that are required to project the variable z_f out of the system. We also remove redundant inequalities from the obtained formulation. We perform this projection by applying the Fourier-Motzkin elimination procedure. We separate in two sets the inequalities of the formulation for $MP_{G'}$ in which z_f appears with coefficient -1 and $+1$, respectively.

$$-z_f \leq 0, \quad (80)$$

$$z_{v'} + z_{v''} - z_f \leq 1, \quad (81)$$

$$-z_v + z_{e_1} + z_{e_m} - z_f \leq 0 \quad \text{where } v = e_1 \cap e_m, \quad (82)$$

$$\sum_{v \in e_1 \cup e_m} z_v - z_{e_1} - z_{e_m} - z_f \leq |e_1 \cup e_m| - 2, \quad (83)$$

$$\sum_{v \in S_1(G_2)} z_v - \sum_{e \in E^-(G_2)} z_e - \sum_{v \in S_2(G_2)} z_v + \sum_{e \in E^+(G_2)} z_e \leq |S_1(G_2)| - |I(G_2)| + \left\lfloor \frac{|E^-(G_2)|}{2} \right\rfloor, \quad (84)$$

$$-z_{v'} + z_f \leq 0, \quad (85)$$

$$-z_{v''} + z_f \leq 0, \quad (86)$$

$$\sum_{v \in e_1 \setminus (\{v'\} \cup e_m)} z_v - z_{v''} - z_{e_1} + z_{e_m} + z_f \leq |e_1| - 2, \quad (87)$$

$$\sum_{v \in e_m \setminus (\{v''\} \cup e_1)} z_v - z_{v'} + z_{e_1} - z_{e_m} + z_f \leq |e_m| - 2, \quad (88)$$

$$\sum_{v \in S_1(G_2)} z_v - \sum_{e \in E^-(G_2)} z_e - \sum_{v \in S_2(G_2)} z_v + \sum_{e \in E^+(G_2)} z_e \leq |S_1(G_2)| - |I(G_2)| + \left\lfloor \frac{|E^-(G_2)|}{2} \right\rfloor. \quad (89)$$

In this system, Inequality (84) holds for every odd subset $E^-(G_2)$ of $E(G_2)$ containing f , whereas Inequality (89) holds for each odd subset $E^-(G_2)$ of $E(G_2)$ such that $f \notin E^-(G_2)$. In these inequalities, $I(G_2)$ denotes the set of edges in $E^-(G_2)$ such that also the next edge in the β -cycle belongs to $E^-(G_2)$ as well. We refer the reader to the definition of odd β -cycle inequalities for the meaning of the sets $E^+(G_2)$, $S_1(G_2)$, and $S_2(G_2)$. Before moving on, we remark that Inequalities (82), (83), (87), and (88) are the odd β -cycle inequalities corresponding to G_1 . In particular, Inequalities (82) and (83) are the odd β -cycle inequalities in which $f \in E^-$, whereas (87) and (88) are the odd β -cycle inequalities that arise when $f \in E^+$.

The sums (80) + (85), (80) + (86), (80) + (87), (80) + (88), (80) + (89), (81) + (85), (81) + (86), (81) + (87), (81) + (88), (81) + (89), (82) + (85), (82) + (86), (82) + (87), (82) + (88), (83) + (85), (83) + (86), (83) + (87), (83) + (88), (84) + (85), (84) + (86), and (84) + (89) have a support hypergraph that is Berge-acyclic. This is true, by the assumptions on G and because none of these sums contains all edges of E . By Del Pia and Khajavirad (2018a), we know that the only nonredundant inequalities for these hypergraphs are those of the standard linearization. Observe that only Inequalities (81) + (85) and (81) + (86) belong to the standard linearization. However, they are already present in the system because of the inductive hypothesis. All the remaining inequalities cited previously do not belong to the standard linearization and therefore are redundant for the multilinear polytopes deriving from their support hypergraphs. Moreover, by proposition 6 in Del Pia and Khajavirad (2017), it follows that these inequalities are redundant also for MP_G , because the support hypergraphs of any of these sums is a partial hypergraph of G . We recall that the definitions of support hypergraph and of partial hypergraph are given in Section 1.1.

Then, it remains to check (82) + (89), (83) + (89), (84) + (87), and (84) + (88). Consider (84) + (87). We define $E^-(G) = \{e_1\} \cup E^-(G_2) \setminus \{f\}$, and denote by $S_1(G)$ and $S_2(G)$, the corresponding sets in Definition 2. Notice immediately that $|E^-(G)| = 1 + |E^-(G_2) \setminus \{f\}|$; therefore, $|E^-(G)|$ is an odd number. Moreover, $\lfloor |E^-(G)|/2 \rfloor = \lfloor |E^-(G_2)|/2 \rfloor$, and $\lfloor |\{e_1\}|/2 \rfloor = 0$. We need to examine the different cases where each of e_2, e_{m-1} belongs to $E^-(G_2)$ or $E^+(G_2)$.

Then, let us start from the case $e_2 \in E^-(G_2)$. With this assumption, the node v' is in $S_1(G)$, because $e_1, e_2 \in E^-(G)$. Indeed, $v' \in S_1(G_2)$ and the corresponding variable does not appear in (87); therefore, $z_{v'}$ appears with coefficient $+1$ in (84) + (87). Next, we further assume that $e_{m-1} \in E^+(G_2)$; hence, $v'' \in S_2(G)$. The variable $z_{v''}$ does not appear in (84), whereas it appears with coefficient -1 in (87). Observe that this is exactly what we expect from the definition of odd β -cycle. We need to check the correctness of the right-hand side. It follows that $|S_1(G)| = |e_1| - 2 + |S_1(G_2)|$, as $S_1(G)$ is the disjoint union of $e_1 \setminus (\{v'\} \cup e_m)$ and $S_1(G_2)$.

Second, $|I(G_2)| = |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-(G)\}|$, because e_1 replaces f as predecessor of e_2 . Therefore, we obtain the desired right-hand side in this case.

Then, consider $e_2, e_{m-1} \in E^-(G_2)$. As a consequence, we have that $e_{m-1} \in E^-(G), e_m \in E^+(G)$; therefore, $z_{v''}$ must not be present in the resulting inequality. This is true, because there is $-z_{v''}$ in the left-hand side of (87) and $+z_{v''}$ in (84) because $e_{m-1}, f \in E^-(G_2)$. Once we sum the two inequalities, $z_{v''}$ vanishes. However, we need to be more careful in the analysis of the correctness of the right-hand side. In fact, in this case, $|S_1(G)| = |e_1| - 2 + |S_1(G_2) \setminus \{v''\}| = |e_1| - 2 + |S_1(G_2)| - 1$. Moreover, we now have that $|\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-(G)\}|$ is equal to $|I(G_2)| - 1$, because the edge after e_{m-1} does not belong to $E^-(G)$. Then, the right-hand side of the odd β -cycle of G corresponding to $E^-(G)$ in this case is equal to

$$\begin{aligned} |S_1(G)| - |\{i \in \{1, \dots, m\} : e_i, e_{i+1} \in E^-(G)\}| + \left\lfloor \frac{|E^-(G)|}{2} \right\rfloor &= \\ = |e_1| - 2 + |S_1(G_2)| - 1 - |I(G_2)| + 1 + \left\lfloor \frac{|E^-(G_2)|}{2} \right\rfloor &= \\ = |e_1| - 2 + |S_1(G_2)| - |I(G_2)| + \left\lfloor \frac{|E^-(G_2)|}{2} \right\rfloor, \end{aligned}$$

which coincides with the sum of the right-hand sides of (87) and (84). This concludes the case $e_2 \in E^-(G_2)$.

Now assume instead that $e_2 \in E^+(G_2)$. Therefore, $z_{v''}$ does not appear in (87) or (84); hence, this variable is not present in their sum. Let $e_{m-1} \in E^+(G_2)$. This case is similar to the previous case $e_2 \in E^-(G_2), e_{m-1} \in E^+(G_2)$. Moreover, the case $e_{m-1} \in E^-(G_2)$ has similar calculations to the previous case $e_2, e_{m-1} \in E^-(G_2)$.

We can conclude that by summing (84) with (87), we obtain the odd β -cycle inequalities arising from G . Similar arguments hold in the other sums (82) + (89), (83) + (89), and (84) + (88). $\square$

8.4. Proof of Claim 9

We apply the Fourier-Motzkin elimination on variables $z_{f_1}, \dots, z_{f_m}$ one by one. At every step, we remove redundant inequalities from the formulation. We start from projecting out the variable z_{f_1} . We write here the inequalities in which this variable is present, and we divide the inequalities in two sets: one in which z_{f_1} has coefficient -1 , whereas in the second set, its coefficient is equal to $+1$.

$$\sum_{v \in f_1} z_v - z_{f_1} \leq |f_1| - 1 \quad (90)$$

$$-z_{f_1} + z_{e_1} \leq 0 \quad (91)$$

$$-z_{f_1} + z_{e_2} \leq 0 \quad (92)$$

$$\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{f \in S_2} z_f + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor \quad (93)$$

$$-z_v + z_{f_1} \leq 0 \quad \forall v \in f_1 \quad (94)$$

$$\sum_{v \in e_1 \setminus (f_1 \cup f_m)} z_v + z_{f_1} + z_{f_m} - z_{e_1} \leq |e_1 \setminus (f_1 \cup f_m)| + 1 \quad (95)$$

$$\sum_{v \in e_2 \setminus (f_1 \cup f_2)} z_v + z_{f_1} + z_{f_2} - z_{e_2} \leq |e_2 \setminus (f_1 \cup f_2)| + 1 \quad (96)$$

$$\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{f \in S_2} z_f + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor \quad (97)$$

In this system, Inequality (93) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_1, e_2 \notin E^-$, whereas Inequality (97) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_1, e_2 \in E^-$. In these inequalities, $E^+ = \{e_1, \dots, e_m\} \setminus E^-$. Moreover, the set S_1 contains the edges $f_i = e_i \cap e_{i+1}$ such that $e_i, e_{i+1} \in E^-$ and the nodes in V contained only in one edge among $e_1, \dots, e_m$, and this edge belongs to E^- . The set S_2 contains the edges f_i such that $e_i, e_{i+1} \in E^+$. Therefore, in Inequality (93), we have $f_1 \in S_2$, and in Inequality (97), we have $f_1 \in S_1$. We did not write the inequalities $-z_{f_1} \leq 0$ and $z_{f_1} \leq 1$ in this system; the first comes from the addition of the standard linearization of f_1 and the second derives from the replacement of z_{w_1} with z_{f_1} in $z_{w_1} \leq 1$. This is because these two inequalities are redundant, and we discard them. In fact, the first inequality can be obtained by summing $-z_{f_1} + z_{e_1} \leq 0$ and $-z_{e_1} \leq 0$. The second inequality is redundant because it is implied by $-z_v + z_{f_1} \leq 0$ and $z_v \leq 1$ for any $v \in f_1$.

Observe that, without doing any calculations, we know a priori that the inequalities obtained from the sums (91) + (97), (92) + (97), (93) + (95), (93) + (96), and (93) + (97) are redundant for the new system. In fact, the support hypergraph of each of these inequalities is γ -acyclic, because either the edge e_1 or e_2 is missing, and by theorem 14 in Del Pia and Khajavirad (2018a), the only nonredundant inequalities for these hypergraphs are the standard linearization and the flower inequalities. However, Inequalities (91) + (97), (92) + (97), (93) + (95), (93) + (96), and (93) + (97) do not belong to these categories and therefore are redundant for the multilinear polytopes deriving from their support hypergraphs. By proposition 6 in Del Pia and Khajavirad (2017), it follows that Inequalities (91) + (97), (92) + (97), (93) + (95), (93) + (96), and (93) + (97) are redundant also for the new system of inequalities, because the support hypergraphs of any of these sums is a partial hypergraph of $(V, E'' \setminus \{f_1\})$.

Also, the inequalities obtained from (90) + (94), (91) + (95), and (92) + (96) are redundant for the new system, as they can be achieved equivalently by summing, for each variable appearing in the left-hand side, the bound that every variable is less than or equal to 1.

On the other hand, all the remaining inequalities obtained by the Fourier-Motzkin elimination might be facet defining for the resulting polytope and will not be discarded. Inequalities (90) + (95) and (90) + (96) are the flower inequalities with center e_1 , e_2 , and neighbor f_m, f_2 , respectively. Inequality (90) + (97) has the same form of Inequality (97) if we redefine S_1 by replacing f_1 with all its nodes. Notice that the obtained inequality is the first step in achieving the odd β -cycle inequalities in which the nodes in $e_1 \cap e_2$ belong to the resulting set S_1 .

The inequalities obtained from (92) are analogous to (91), as the role of z_{e_1} in (91) is the same of z_{e_2} in (92). Hence, we only consider the inequalities arising from (91). These are achieved by summing (91) with (94) and (96). From the first, we get (δ_{v,e_1}) for all $v \in e_1 \cap e_2$, whereas from the second, we gain the flower inequality with center e_2 and neighbors $\{e_1, f_2\}$.

It remains to check what happens for (93) + (94). In this case, we obtain $|e_1 \cap e_2|$ inequalities. Each of these inequalities has the same form of (93), where the set S_2 is redefined by replacing the edge f_1 with one of the nodes in f_1 . This inequality will lead to the odd β -cycles in which there is one node $v \in e_1 \cap e_2$ in S_2 .

We are done with eliminating the variable z_{f_1} and now we move on to z_{f_2} . Next, we consider the system obtained from the Fourier-Motzkin elimination of variable z_{f_1} . To project out the variable z_{f_2} , we focus on the inequalities with a nonzero coefficient for z_{f_2} .

$$\sum_{v \in f_2} z_v - z_{f_2} \leq |f_2| - 1 \quad (98)$$

$$-z_{f_2} + z_{e_2} \leq 0 \quad (99)$$

$$-z_{f_2} + z_{e_3} \leq 0 \quad (100)$$

$$\frac{\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{p \in S_2} z_p + \sum_{e \in E^+} z_e \leq |S_1| - |\{i: e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor}{-z_v + z_{f_2} \leq 0 \quad \forall v \in f_2} \quad (101)$$

$$-z_v + z_{f_2} \leq 0 \quad \forall v \in f_2 \quad (102)$$

$$\sum_{v \in e_2 \setminus f_2} z_v + z_{f_2} - z_{e_2} \leq |e_2 \setminus f_2| \quad (103)$$

$$\sum_{v \in e_2 \setminus (f_2 \cup f_3)} z_v + z_{f_2} + z_{f_3} - z_{e_3} \leq |e_3 \setminus (f_2 \cup f_3)| + 1 \quad (104)$$

$$\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{p \in S_2} z_p + \sum_{e \in E^+} z_e \leq |S_1| - |\{i: e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor \quad (105)$$

Like in the previous system, Inequality (101) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_2, e_3 \notin E^-$, whereas Inequality (105) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_2, e_3 \in E^-$. In these inequalities, $E^+ = \{e_1, \dots, e_m\} \setminus E^-$. The set S_1 contains the edges f_i , for $i \in \{2, \dots, m\}$ such that $e_i, e_{i+1} \in E^-$, all the nodes in $e_1 \cap e_2$ if $e_1, e_2 \in E^-$, and all the nodes in V contained only in one edge among $e_1, \dots, e_m$, and this edge belongs to E^- . The set S_2 contains the edges f_i , for $i \in \{2, \dots, m\}$, such that $e_i, e_{i+1} \in E^+$, and one node in $e_1 \cap e_2$ if $e_1, e_2 \in E^+$. In particular, in Inequality (101), we have $f_2 \in S_2$, and in Inequality (105), we have $f_2 \in S_1$.

We remark that the structure of Inequalities (98)–(105) is almost identical to that of System (90)–(97). In particular, f_1, e_1, e_2 are replaced by f_2, e_2, e_3 , respectively. The only difference is between Inequality (95) and (103), because the variable z_{f_1} has been already projected out. The same analysis of before holds in this case, except for the inequality obtained by (98) + (103). In fact, in this case, we obtain inequality (ε_{e_2}) of the standard linearization instead of a flower inequality.

Recursively, we project out all the additional variables until we are left with only z_{f_m} . This variable is present in the following inequalities:

$$\sum_{v \in f_m} z_v - z_{f_m} \leq |f_m| - 1, \quad (106)$$

$$-z_{f_m} + z_{e_1} \leq 0, \quad (107)$$

$$-z_{f_m} + z_{e_m} \leq 0, \quad (108)$$

$$\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{p \in S_2} z_p + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor, \quad (109)$$

$$-z_v + z_{f_m} \leq 0 \quad \forall v \in f_n, \quad (110)$$

$$\sum_{v \in e_1 \setminus f_m} z_v + z_{f_m} - z_{e_1} \leq |e_1 \setminus f_m|, \quad (111)$$

$$\sum_{v \in e_m \setminus f_m} z_v + z_{f_m} - z_{e_m} \leq |e_m \setminus f_m|, \quad (112)$$

$$\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor. \quad (113)$$

Inequality (109) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_m, e_1 \notin E^-$, and Inequality (113) holds for each odd subset E^- of $\{e_1, \dots, e_m\}$ with $e_m, e_1 \in E^-$. The set S_1 contains the edge f_m if $e_m, e_1 \in E^-$, all the nodes in $e_i \cap e_{i+1}$, for $i \in \{1, \dots, m-1\}$, if $e_i, e_{i+1} \in E^-$, and all the nodes in V contained only in one edge among $e_1, \dots, e_m$, and this edge belongs to E^- . The set S_2 contains the edge f_m if $e_m, e_1 \in E^+$, and one node in $e_i \cap e_{i+1}$, for $i \in \{1, \dots, m-1\}$, if $e_i, e_{i+1} \in E^+$. In particular, in Inequality (109), we have $f_m \in S_2$, and in Inequality (113), we have $f_m \in S_1$.

The variable z_{f_m} can be projected out by following the same arguments of the previous steps, because Inequalities (106)–(113) have the same structure. However, observe that the inequalities obtained by (106) + (113) and (109) + (110) are exactly the odd β -cycle inequalities for which e_m, e_1 both belong to either E^- or E^+ .

Once we have eliminated all variables z_{f_i} from the description of $MP_{G''}$, the obtained system contains only inequalities of the standard linearization, flower inequalities, and odd β -cycle inequalities. $\square$

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