Optimal Control of Polynomial Hybrid Systems via Convex Relaxations

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***Abstract*—This paper considers the optimal control for hybrid systems whose trajectories transition between distinct subsystems when state-dependent constraints are satisfied. Though this class of systems is useful while modeling a variety of physical systems undergoing contact, the construction of a numerical method for their optimal control has proven challenging due to the combinatorial nature of the state-dependent switching and the potential discontinuities that arise during switches. This paper constructs a convex relaxation-based approach to solve this optimal control problem by formulating the problem in the space of relaxed controls, which gives rise to a linear program whose solution is proven to compute the globally optimal controller. This conceptual program is solved using a sequence of semidefinite programs whose solutions are proven to converge from below to the true solution of the original optimal control problem. Finally, a method to synthesize the optimal controller is developed. Using an array of examples, the performance of the proposed method is validated on problems with known solutions and also compared to a commercial solver.**

I. Introduction

Controlled hybrid dynamical systems can describe the dynamics of a variety of physical systems in which the evolution of the system undergoes sudden changes due to the satisfaction of state-dependent constraints such as in bipeds [1], automotive sub-systems [2], aircraft control [3], and biological systems [4]. Given the practical applications of such systems, the development of algorithms to perform optimal control of hybrid systems has drawn considerable interest among theoreticians and practitioners. The theoretical development of both necessary and sufficient conditions for the optimal control of hybrid controlled systems has been considered using extensions of the Pontryagin Maximum Principle [5]–[7] and Dynamic Programming [8]–[10], respectively. Recent work has even linked these approaches [11]. Typically, these methods have assumed that the sequence of transitions between the systems was known *a priori*. Practitioners, as a result, have fixed the sequence of transitions and used gradient-based methods to locally optimize over the time spent and control applied within each subsystem [12]–[15].

Recent work has focused on the development of numerical optimal control techniques for mechanical systems undergoing contact without specifying the ordering of visited subsystems. One approach to address the optimal control problem has focused on the construction of a novel notion of derivative [16]. Though this method still requires fixing the total number of visited subsystems, assuming *a priori* knowledge of the visited subsystems, and performs optimization only over the initial condition, this gradient-based approach is able to find the locally optimal ordering of subsystems under certain regularity conditions on the nature of the state-dependent switching. Other approaches have relaxed satisfaction of the unilateral constraint directly and instead focused on treating constraint satisfaction as a continuous decision variable that can be optimized using traditional numerical methods to find local minima [17]–[19].

This paper develops a numerical approach to find the *global* optimum to the hybrid optimal control problem when the vector field of each hybrid "mode" is a polynomial function. It relies on treating the optimal control problem in the relaxed sense wherein the original problem is lifted to the space of measures [20], [21]. In the instance of classical dynamical systems, this lifting renders the optimal control problem linear in the space of relaxed controls [22]; however, there are few numerical methods to tackle this relaxed problem directly.

Recent developments in semidefinite programming have made it possible to solve this lifted optimal control problem for classical dynamical systems by relying on moment-based relaxations [23]. By solving the problem over truncated moment sequences, it is possible to transform the optimal control problem into either a finite-dimensional linear or finite-dimensional semidefinite program. Either transformation of the relaxed problem is proven to provide a lower bound on the optimal cost. In fact, this bound converges to the true optimal cost as the moment sequence extends to infinity under the assumption that the incremental cost is convex in control. Recent work has also shown how the optimal control policy can be extracted for systems that are affine in control [24], [25]. Unfortunately this relaxed control formulation for controlled hybrid systems, the subsequent development of a numerically implementable convex relaxation, and optimal control synthesis have remained unaddressed.

Note that the focus of this paper is on the development of an optimal control approach for hybrid systems with state-dependent rather than controlled switching. A variety of numerical methods have been proposed to perform optimal control for systems with controlled switching [26]–[32]. In contrast to the controlled switching case, after state-dependent switching, the state is allowed to change discontinuously.

The contributions of this paper are three-fold: first, Sec-

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tion IV provides a conceptual infinite dimensional linear programming-based approach for the optimal control of hybrid systems with state-dependent switching; second, Section V presents a semidefinite programming-based sequence of relaxations to this infinite dimensional linear program that is proven to generate a sequence of convergent lower bounds to the true optimal cost; finally Section V provides a method to generate a sequence of controllers that converge to the true optimal control. The remainder of this paper is organized as follows: Section II defines the class of systems under consideration and their executions, Section III describes how to lift executions of the hybrid system to the space of measures, and Section VI illustrates the efficacy of the proposed method on a variety of systems.

II. Preliminaries

This section introduces the notation used throughout the paper, defines controlled hybrid systems, and formulates the optimal control problem of interest.

A. Notation

Given an element $y \in \mathbb{R}^n$, let $[y]_i$ denote the i-th component of y. We use the same convention for any multidimensional vector space. Let card denote the cardinality of a set. Let $\mathbb{R}[y]$ denote the ring of real polynomials in the variable y and $\mathbb{R}_k[y]$ denote the space of real multivariate polynomials of total degree less than or equal to k. Let $\{A_i\}_{i\in\mathcal{I}}$ be a family of non-empty sets indexed by i, the *disjoint union* of this family is $\coprod_{i\in\mathcal{I}} A_i = \bigcup_{i\in\mathcal{I}}(A_i \times \{i\})$. Let $\iota_i : A_i \to \coprod_{i\in\mathcal{I}} A_i$ be the canonical injection defined as $\iota_i(a) = (a, i)$ whose inverse is $\pi_i : \coprod_{i\in\mathcal{I}} A_i \to A_i$. Note $\pi_i(\iota_j(a)) = \varnothing$ if $i \neq j$. For convenience, denote $x_i := \pi_i(x)$ for all $x \in \coprod_{i\in\mathcal{I}} A_i$. Similarly define a projection operator onto the indexing set $\lambda : \coprod_{i\in\mathcal{I}} A_i \to \mathcal{I}$ such that $\lambda(\iota_i(a)) = i$. Let conv denote the convex hull of a set. Let a.e. denote "almost everywhere".

Let $\mathbb{1}_S$ be the indicator function on a set S. We say a function is pointwise bounded if its range is a bounded set. Suppose Y is a measurable metric space, then let $C(Y)$ be the space of continuous functions on Y, let $C_b(Y)$ be the space of bounded continuous functions on Y, let $AC(I)$ be the space of absolutely continuous functions on $I \subset \mathbb{R}$, let $L^1(Y)$ be the space of L^1 functions with respect to Lebesgue measure on Y, let $L^1(\mu)$ be the space of L^1 functions with respect to the measure μ, and let $\mathcal{M}(Y)$ be the space of finite signed Radon measures on Y endowed with the total variation norm (denoted by $\|\cdot\|$), whose positive cone $\mathcal{M}_+(K)$ is the space of finite unsigned Radon measures on Y. Any $\mu \in \mathcal{M}(Y)$ is an element of the dual to $C(Y)$ via the *duality pairing*

$$\langle \mu, v \rangle := \int_Y v(z)\, d\mu(z), \quad \forall v \in C(Y). \tag{1}$$

Let the *support* of $\mu \in \mathcal{M}(Y)$ be denoted as spt(μ). A probability measure is a non-negative, unsigned measure whose integral is one. Denote the dual to a vector space V as V'.

Suppose $Y_1 \subset Y$ is a compact set endowed with the subspace topology, then define the *zero extension* of any $f \in L^1(Y_1)$ as

$$\hat{f}(y) = \begin{cases} f(y), & \text{if } y \in Y_1 \\ 0, & \text{if } y \in Y \backslash Y_1 \end{cases} \tag{2}$$

Define the *zero extension* of $\mu \in \mathcal{M}(Y_1)$ as $\hat{\mu}(B) = \mu(B \cap Y_1)$ for all subsets B in the Borel σ-algebra of Y. Let $\mu_{y_1|y_2} \in \mathcal{M}(Y_1)$ denote the conditional probability measure of $\mu \in \mathcal{M}(Y_1 \times Y_2)$ on Y_1 given $y_2 \in Y_2$, and let $\mu_{y_2} \in M(Y_2)$ denote the marginal of μ on Y_2.

Suppose $Y \subset \mathbb{R}^n$, define the *convolution* of $\mu \in \mathcal{M}_+(Y)$ and $\theta \in L^1(\mathbb{R}^n)$, denoted as $\mu * \theta \in \mathcal{M}_+(\mathbb{R}^n)$, as

$$(\mu * \theta)(B) = \int_Y \int_{\mathbb{R}^n} \mathbb{1}_B(x + y)\theta(y)\, dy\, d\mu(x) \tag{3}$$

for all subsets B in the Borel σ-algebra of $\mathbb{R}^n$. If Y_1, Y_2 are measurable spaces, $\mu \in \mathcal{M}(Y_1)$, and $f : Y_1 \to Y_2$ is a Borel function, let $f_\# \mu \in \mathcal{M}(Y_2)$ denote the *pushforward* of μ through f, given by

$$(f_\# \mu)(B) := \mu(f^{-1}(B)) \tag{4}$$

for any B in the Borel σ-algebra of Y_2. Note for every $f_\#\mu$-integrable function $v : Y_2 \to \mathbb{R}$

$$\int_{Y_2} v\, d(f_\#\mu) = \int_{Y_1} v \circ f\, d\mu. \tag{5}$$

B. Controlled Hybrid Systems

Consider the following class of *controlled hybrid systems*:

Definition 1. *A* controlled hybrid system *is a tuple* $\mathcal{H} = (\mathcal{I}, \mathcal{E}, \mathcal{D}, U, \mathcal{F}, \mathcal{S}, \mathcal{R})$, *where:*

- $\mathcal{I}$ *is a finite set indexing the discrete states of* $\mathcal{H}$*;*
- $\mathcal{E} \subset \mathcal{I} \times \mathcal{I}$ *is a set of edges, forming a directed graph structure over* $\mathcal{I}$*;*
- $\mathcal{D} = \coprod_{i\in\mathcal{I}} X_i$ *is a disjoint union of domains, where each* X_i *is a compact subset of* $\mathbb{R}^{n_i}$*, and* $n_i \in \mathbb{N}$*;*
- U *is a compact subset of* $\mathbb{R}^m$ *that describes the range of control inputs, where* $m \in \mathbb{N}$*;*
- $\mathcal{F} = \{F_i\}_{i\in\mathcal{I}}$ *is the set of vector fields, where each* $F_i : \mathbb{R} \times X_i \times U \to \mathbb{R}^{n_i}$ *is a Lipschitz continuous vector field defining the dynamics of the system on* X_i*;*
- $\mathcal{S} = \coprod_{e\in\mathcal{E}} S_e$ *is a disjoint union of guards, where each* $S_{(i,i')} \subset \partial X_i$ *is a compact, co-dimension 1 guard defining a state-dependent transition from* X_i *to* $X_{i'}$*; and,*
- $\mathcal{R} = \{R_e\}_{e\in\mathcal{E}}$ *is a set of continuous reset maps, where each map* $R_{(i,i')} \colon S_{(i,i')} \to X_{i'}$ *defines the transition from guard* $S_{(i,i')}$ *to* $X_{i'}$.

For convenience, we refer to these controlled hybrid systems as just hybrid systems and refer to a vertex within the graph associated with a hybrid system as a *mode*. Though the range space of control inputs are assumed to be the same in each mode, this is not restrictive since we can always concatenate all the control inputs in different modes. The compactness of each X_i ensures the optimal control problem defined below is well-posed. Since the focus of this paper is on the optimal control of deterministic hybrid systems, we avoid any ambiguity during the transition between modes by making the following assumption:

Assumption 2. *Guards do not intersect with themselves or the images of reset maps. The controlled vector fields in each mode has nonzero normal component on the guard for all control inputs in* U*.*

Next, we define a *hybrid trajectory* of a hybrid system up to time $T > 0$ in Fig. 1. Step 1 initializes the hybrid

Require: $t = 0$, $T > 0$, $i \in \mathcal{I}$, $(x_0, i) \in \mathcal{D}$, and $u : \mathbb{R} \to U$ Lebesgue measurable.

1: Set $\gamma(0) = (x_0, i)$.
2: **loop**
3: Let $I \subset [t, T]$ and $\phi \in AC(I; X_i)$ such that:
  (i) $\dot{\phi}(s) = F_i(s, \phi(s), u(s))$ for almost every $s \in I$ with respect to the Lebesgue measure on I with $(\phi(t), i) = \gamma(t)$ and
  (ii) for any other $\hat{\phi}: \hat{I} \to X_i$ satisfying (i), $\hat{I} \subset I$.
4: Let $t' = \sup I$ and $\gamma(s) = (\phi(s), i)$ for each $s \in [t, t')$.
5: **if** $t' = T$, **or** $\nexists (i, i') \in \mathcal{E}$ such that $\phi(t') \in S_{(i,i')}$ **then**
6: Stop.
7: **end if**
8: Let $(i, i') \in \mathcal{E}$ be such that $\phi(t') \in S_{(i,i')}$.
9: Set $\gamma(t') = (R_{(i,i')}(\phi(t')), i')$, $t = t'$, and $i = i'$.
10: **end loop**

Fig. 1: The procedure to define a trajectory of hybrid system $\mathcal{H}$.

trajectory at a given point (x_0, i) at time $t = 0$. Step 3 defines ϕ to be the maximal integral curve of F_i under the control u beginning from the initial point. Step 4 defines the hybrid trajectory on a finite interval as the curve ϕ with associated index i. As described in Steps 5 - 7, the hybrid trajectory terminates when it either reaches the terminal time T or hits $\partial X_i \setminus \bigcup_{(i,i') \in \mathcal{E}} S_{(i,i')}$ where no transition is defined. Steps 8 and 9 define a discrete transition to a new domain using a reset map where evolution continues again as a classical dynamical system by returning to Step 3. Note that this definition is a rephrasing of [33, Fig. 8] and is meant to formalize what is meant by a solution to a hybrid system. This paper applies this definition only to ensure the existence of solutions to hybrid systems. A description of how to implement this definition can be found in [33]. Denote the space of such hybrid trajectories as $\mathcal{X}$. Note that for any t at which a hybrid trajectory γ is defined, $\gamma(t) = (\gamma_{\lambda(\gamma(t))}(t), \lambda(\gamma(t)))$.

Trajectories of hybrid systems can undergo an infinite number of discrete transitions in a finite amount of time. Since the state of the trajectory after these Zeno behaviors occur may not be well defined [34] and because the focus of this paper is on optimal control for deterministic hybrid systems, we make the following assumption:

Assumption 3. *$\mathcal{H}$ has no Zeno trajectories.*

C. Problem Formulation

This paper is interested in finding a (γ, u) satisfying Algorithm 1 from a given initial condition x_0, that reaches a target set while minimizing a cost function. To formulate this problem, define the *target set*, $X_T \subset \mathcal{D}$, as $X_T = \coprod_{i \in \mathcal{I}} X_{T_i}$, where X_{T_i} is a compact subset of X_i for each $i \in \mathcal{I}$. To avoid any ambiguity, we make the following assumption :

Assumption 4. *The target set does not intersect any guards.*

Given a $T > 0$ and an initial point $(x_0, j) \in \mathcal{D}$, a pair of functions (γ, u) satisfying Algorithm 1 is called an *admissible pair* if $\gamma(T) \in X_T$. In this instance, γ is called an *admissible trajectory* and u is called an *admissible control*. The time T at which the admissible trajectory reaches the target set is called the *terminal time*. Denote the space of admissible trajectories and controls by $\mathcal{X}_T$ and $\mathcal{U}_T$, respectively. The space of admissible pairs is denoted as $\mathcal{P}_T \subset \mathcal{X}_T \times \mathcal{U}_T$. Without loss of generality, we make the following assumption:

Assumption 5. *The initial condition is not in any guard.*

For any admissible pair (γ, u), the associated cost is:

$$J(\gamma, u) := \int_0^T h_{\lambda(\gamma(t))}\left(t, \gamma_{\lambda(\gamma(t))}(t), u(t)\right) dt + \\ + H_{\lambda(\gamma(T))}\left(\gamma_{\lambda(\gamma(T))}(T)\right) \tag{6}$$

where $h_i : [0, T] \times X_i \times U \to \mathbb{R}$ and $H_i : X_{T_i} \to \mathbb{R}$ are integrable. Our goal is to find an admissible pair that minimizes (6), which we refer to as *Hybrid Optimal Control Problem (HOCP)*:

$$\inf_{(\gamma,u) \in \mathcal{P}_T} J(\gamma, u) \qquad (HOCP)$$

The optimal cost of $(HOCP)$ is denoted as J^*.

III. The Hybrid Liouville Equation

This section constructs measures whose supports model the evolution of families of trajectories, an equivalent form of J, and an equivalent form of Algorithm 1 in the space of measures. These transformations make a convex formulation of $(HOCP)$ feasible.

Consider the projection γ_i of a hybrid trajectory γ onto mode $i \in \mathcal{I}$. Define the *occupation measure* in mode $i \in \mathcal{I}$ associated with γ, denoted by $\mu^i(\cdot \mid \gamma) \in \mathcal{M}_+([0, T] \times X_i)$, as

$$\mu^i(A \times B \mid \gamma) := \int_0^T \mathbb{1}_{A \times B}(t, \gamma_i(t))\, dt \tag{7}$$

for all subsets $A \times B$ in the Borel σ-algebra of $[0, T] \times X_i$. Note that $\gamma_i(t)$ may not be defined for all $t \in [0, T]$, but we use the same notation and let $\mathbb{1}_{A \times B}(t, \gamma_i(t)) = 0$ whenever $\gamma_i(t)$ is undefined. The quantity $\mu^i(A \times B \mid \gamma)$ is equal to the amount of time the graph of the trajectory, $(t, \gamma_i(t))$, spends in $A \times B$. Define the *initial measure*, $\mu_0^i(\cdot \mid \gamma) \in \mathcal{M}_+(X_i)$, as

$$\mu_0^i(B \mid \gamma) := \mathbb{1}_B(\gamma_i(0)) \tag{8}$$

for all subsets B in the Borel σ-algebra of X_i; define the *terminal measure*, $\mu_T^i(\cdot \mid \gamma) \in \mathcal{M}_+(X_{T_i})$, as

$$\mu_T^i(B \mid \gamma) := \mathbb{1}_B(\gamma_i(T)) \tag{9}$$

for all subsets B in the Borel σ-algebra of X_{T_i}.

One can show that the occupation measure, initial measure, and the terminal measure satisfy a linear equation whose solution can model the evolution of a nonlinear dynamical system [23]. This result enables one to formulate nonlinear optimal control problems as infinite dimensional linear programs [23, Theorem 2.3]. Unfortunately the linear equation over measures is unable to describe the transitions between hybrid modes. However, these transitions can be described using *guard measures*. Define the *guard measure*, $\mu^{S_{(i,i')}}(\cdot \mid \gamma) \in \mathcal{M}_+([0, T] \times S_{(i,i')})$, as

$$\mu^{S_{(i,i')}}(A \times B \mid \gamma) := \text{card}\{t \in A \mid \lim_{\tau \to t^-} \gamma_i(\tau) \in B\} \tag{10}$$

for all subsets $A \times B$ in the Borel σ-algebra of $[0, T] \times S_{(i,i')}$, given any pair $(i, i') \in \mathcal{E}$. The guard measure counts the number of times a given trajectory passes through the guard.

Next, define the occupation measure in $i \in \mathcal{I}$ associated with (γ, u), denoted $\mu^i(\cdot \mid \gamma, u) \in \mathcal{M}_+([0,T] \times X_i \times U)$, as

$$\mu^i(A \times B \times C \mid \gamma, u) := \int_0^T \mathbb{1}_{A\times B\times C}(t, \gamma_i(t), u(t))\, dt \quad (11)$$

for all subsets $A\times B\times C$ in the Borel σ-algebra of $[0,T]\times X_i \times U$. For convenience, it is useful to collect the initial, average, terminal, and guard occupation measures in each mode. That is, define $\mu_0^{\mathcal{I}}(\cdot \mid \gamma) \in \mathcal{M}_+(\mathcal{D})$ as $\mu_0^{\mathcal{I}}(\cdot, i \mid \gamma) := \mu_0^i(\cdot \mid \gamma)$ for each $i \in \mathcal{I}$. For convenience, we refer to $\mu_0^{\mathcal{I}}$ as an initial measure and write μ_0^i when we refer to the i-th slice of $\mu_0^{\mathcal{I}}$. We define and refer to $\mu^{\mathcal{I}}(\cdot \mid \gamma, u) \in \mathcal{M}_+([0,T] \times \mathcal{D} \times U)$, $\mu_T^{\mathcal{I}}(\cdot \mid \gamma, u) \in \mathcal{M}_+(X_T)$, and $\mu^{\mathcal{S}}(\cdot \mid \gamma, u) \in \mathcal{M}_+([0,T] \times \mathcal{S})$ similarly.

Using these definitions, we can rewrite the cost function J:

Lemma 6. *Let $\mu^{\mathcal{I}}(\cdot \mid \gamma, u)$ and $\mu_T^{\mathcal{I}}(\cdot \mid \gamma)$ be the occupation measure and terminal measure associated with the pair (γ, u), respectively. Then the cost function can be expressed as*

$$J(\gamma, u) = \sum_{i\in\mathcal{I}} \langle \mu^i(\cdot \mid \gamma, u), h_i\rangle + \sum_{i\in\mathcal{I}} \langle \mu_T^i(\cdot \mid \gamma), H_i\rangle. \quad (12)$$

Proof: Notice that h_i and H_i are measurable, and the rest follows directly from (6), (9), and (11). ■

Despite the cost function being a nonlinear function of the admissible pair in the space of functions, the analogous cost function over the space of measures is linear. A similar analogue holds true for the dynamics of the system. That is, the occupation measure associated with an admissible pair satisfies a linear equation over measures. To formulate this linear equation, let $\mathcal{L}_i : C^1([0,T] \times X_i) \to C([0,T] \times X_i \times U)$ be a linear operator that acts on a test function v, defined as

$$(\mathcal{L}_i v)(t, x, u) := \frac{\partial v(t,x)}{\partial t} + \sum_{k=1}^{n_i} \frac{\partial v(t,x)}{\partial [x]_k} [F_i(t,x,u)]_k \quad (13)$$

for all $i \in \mathcal{I}$. Using the dual relationship between measures and functions, we define $\mathcal{L}_i' : C([0,T] \times X_i \times U)' \to C^1([0,T] \times X_i)'$ as the adjoint operator of $\mathcal{L}_i$, satisfying $\langle \mathcal{L}_i' \mu, v\rangle = \langle \mu, \mathcal{L}_i v\rangle$ for all $\mu \in \mathcal{M}([0,T] \times X_i \times U)$ and $v \in C^1([0,T] \times X_i)$.

Each of these adjoint operators can describe the evolution of trajectories of the system within each mode [23]. However in the instance of hybrid systems, trajectories may not just begin evolving within a mode at $t = 0$. Instead a trajectory can enter a mode either by starting from inside it at $t = 0$, or by being reset into it. Similarly a trajectory can terminate in a mode either by reaching the terminal time, or by hitting a guard and transitioning. To formalize this, we first modify reset maps to also act on time by defining $\tilde{R}_{(i,i')} : [0,T] \times S_{(i,i')} \to [0,T] \times X_{i'}$ by $\tilde{R}_{(i,i')}(t,x) = (t, R_{(i,i')}(x))$ for all $(i,i') \in \mathcal{E}$ and $(t,x) \in [0,T] \times S_{(i,i')}$. To describe trajectories of a controlled hybrid system using measures, we rely on the following result of [35, (16)]:

Lemma 7. *Given an admissible pair (γ, u), its initial measure, occupation measure, terminal measure, and guard measure satisfy the following linear equation over measures:*

$$\begin{aligned}\delta_0 \otimes \mu_0^i(\cdot \mid \gamma) + \mathcal{L}_i' \mu^i(\cdot \mid \gamma, u) + \sum_{(i',i)\in\mathcal{E}} \tilde{R}_{(i',i)\#}\mu^{S_{(i',i)}}(\cdot \mid \gamma) \\ = \delta_T \otimes \mu_T^i(\cdot \mid \gamma) + \sum_{(i,i')\in\mathcal{E}} \mu^{S_{(i,i')}}(\cdot \mid \gamma), \quad \forall i \in \mathcal{I},\end{aligned} \quad (14)$$

where (14) *holds in the sense that it is true for all test functions in* $C^1([0,T] \times X_i)$.

Now one can ask whether the converse relationship holds: does an arbitrary set of measures, $\mu_0^{\mathcal{I}} \in \mathcal{M}_+(\mathcal{D})$, $\mu^{\mathcal{I}} \in \mathcal{M}_+([0,T] \times \mathcal{D} \times U)$, $\mu_T^{\mathcal{I}} \in \mathcal{M}_+(X_T)$, and $\mu^{\mathcal{S}} \in \mathcal{M}_+([0,T] \times \mathcal{S})$, that satisfy (14) correspond to an initial measure, $\mu_0^{\mathcal{I}}(\cdot \mid \gamma)$, occupation measure, $\mu^{\mathcal{I}}(\cdot \mid \gamma, u)$, terminal measure, $\mu_T^{\mathcal{I}}(\cdot \mid \gamma)$, and guard measure, $\mu^{\mathcal{S}}(\cdot \mid \gamma)$ for some admissible pair (γ, u)? To answer this question, consider a family of hybrid trajectories modeled by a non-negative probability measure $\rho \in \mathcal{M}_+(\mathcal{X})$, and define an *average occupation measure* $\zeta^i \in \mathcal{M}_+([0,T] \times X_i)$ in each mode $i \in \mathcal{I}$ for the family of trajectories as

$$\zeta^i(A \times B) := \int_{\mathcal{X}} \mu^i(A \times B \mid \gamma)\, d\rho(\gamma) \quad (15)$$

for any $i \in \mathcal{I}$ and $A \times B$ in the Borel σ-algebra of $[0,T] \times X_i$; Define the *average initial measure* ζ_0^i, *average terminal measure* ζ_T^i, and *average guard measure* $\zeta^{S_{(i,i')}}$ similarly.

To prove the converse of Lemma 7, we define the *Hybrid Liouville Equation* whose solution can be disintegrated into a set of measures that we eventually prove are related to ρ in Theorem 12.

Lemma 8. *Let $\mu_0^{\mathcal{I}} \in \mathcal{M}_+(\mathcal{D})$, $\mu^{\mathcal{I}} \in \mathcal{M}_+([0,T] \times \mathcal{D} \times U)$, $\mu_T^{\mathcal{I}} \in \mathcal{M}_+(X_T)$, and $\mu^{\mathcal{S}} \in \mathcal{M}_+([0,T] \times \mathcal{S})$ satisfy the* Hybrid Liouville Equation *(HLE), which is defined as*

$$\delta_0 \otimes \mu_0^i + \mathcal{L}_i' \mu^i + \sum_{(i',i)\in\mathcal{E}} \tilde{R}_{(i',i)\#}\mu^{S_{(i',i)}} = \delta_T \otimes \mu_T^i + \sum_{(i,i')\in\mathcal{E}} \mu^{S_{(i,i')}} \quad (16)$$

for each $i \in \mathcal{I}$. Then each measure μ^i can be disintegrated as

$$d\mu^i(t,x,u) = d\nu^i_{u|t,x}(u)\, d\mu^i_{t,x}(t,x) = d\nu^i_{u|t,x}(u)\, d\tilde{\mu}^i_{x|t}(x) dt \quad (17)$$

where $\nu^i_{u|t,x}$ is a stochastic kernel on U given $(t,x) \in [0,T] \times X_i$, $\mu^i_{t,x}$ is the (t,x)-marginal of μ^i, and $\tilde{\mu}^i_{x|t}$ is a conditional measure on X_i given $t \in [0,T]$.

Proof: Since each measure μ^i is finite measure defined on a Euclidean space, using [36, Theorem 5.3.1], they can be disintegrated as $d\mu^i(t,x,u) = d\nu^i_{u|t,x}(u)\, d\mu^i_{t,x}(t,x)$ where $\nu^i_{u|t,x}$ is a stochastic kernel on U given $(t,x) \in [0,T] \times X_i$, and $\mu^i_{t,x}$ is the (t,x)-marginal of μ^i. Using the same argument, we disintegrate $\mu^i_{t,x}$ into $d\mu^i_{t,x}(t,x) = d\mu^i_{x|t}(x)\, d\mu^i_t(t)$ where μ^i_t is the t-marginal of $\mu^i_{t,x}$. To show the measure μ^i_t is absolutely continuous with respect to the Lebesgue measure, notice:

$$\begin{aligned}\langle \mu_t^i, \dot{\psi}\rangle = \langle \delta_T \otimes \mu_T^i + \sum_{(i,i')\in\mathcal{E}} \mu^{S_{(i,i')}} - \delta_0 \otimes \mu_0^i + \\ - \sum_{(i',i)\in\mathcal{E}} \tilde{R}_{(i',i)\#}\mu^{S_{(i',i)}}, \psi\rangle,\end{aligned} \quad (18)$$

for any $\psi \in C^1([0,T])$. The desired result then follows [37, Exercise 5.8.78]. Since μ^i is finite and therefore its t-marginal measure is also finite, using the Radon-Nikodym Theorem, $\exists\ l \in L^1([0,T])$ such that $d\mu^i_t(t) = l(t)\, dt$. Letting $d\tilde{\mu}^i_{x|t} := l(t)\, d\mu^i_{x|t}$ for all $t \in [0,T]$, then $d\mu^i_{x|t}(x)\, d\mu^i_t(t) = l(t)\, d\mu^i_{x|t}(x)\, dt = d\tilde{\mu}^i_{x|t}(x)\, dt$ and (17) follows. ■

For convenience, denote $\tilde{\mu}^i_{x|t}$ by $\mu^i_{x|t}$ and define:

$$\begin{aligned}\sigma^i &:= \delta_0 \otimes \mu_0^i + \sum_{(i',i)\in\mathcal{E}} \tilde{R}_{(i',i)\#}\mu^{S_{(i',i)}}, \\ \eta^i &:= \delta_T \otimes \mu_T^i + \sum_{(i,i')\in\mathcal{E}} \mu^{S_{(i,i')}}\end{aligned} \quad (19)$$

Using (17), HLE can also be written as a non-homogeneous PDE that holds in the sense of distributions:

$$\partial_t \mu^i_{t,x} + D_x \cdot (\bar{F}_i \mu^i_{t,x}) = \sigma^i - \eta^i, \tag{20}$$

where

$$\bar{F}_i(t,x) := \int_U F_i(t,x,u)\, d\nu^i_{u|t,x}(u) \in \text{conv } F_i(t,x,U). \tag{21}$$

Note that even when F_i is Lipschitz continuous, $\bar{F}_i$ may not be Lipschitz continuous. By applying integration by parts, we can write

$$\begin{aligned} \int_0^T\int_{X_i} \big(\partial_t v(t,x) + \nabla_x v(t,x)\cdot \bar{F}_i\big)\, d\mu^i_{x|t}(x)\, dt \\ + \int_{[0,T]\times X_i} v(t,x)\, d\sigma^i(t,x) = \int_{[0,T]\times X_i} v(t,x)\, d\eta^i(t,x) \end{aligned} \tag{22}$$

for any test function $v \in C^1([0,T]\times X_i)$. We later show in Corollary 10 that σ^i and η^i capture the trajectories that enter and leave domain i, respectively.

Next, we prove the converse of Lemma 7 using Theorems 9 and 12. These converse theorems prove that a solution to the Hybrid Liouville Equation can be identified with a solution to the hybrid system under certain regularity conditions on the vector fields in each mode. This result enables us to formulate (HOCP) as an optimization problem over measures, as described in Section IV. We start by showing $\mu^i_{x|t}$ is related to the solution of the ODE $\bar{F}_i$. As shown in the Appendix A, (Theorem 25), when $\bar{F}_i$ satisfies certain regularity conditions (e.g. Lipschitz continuity), the relationship between $\mu^i_{x|t}$ and $\bar{F}_i$ is clear, but to deal with solutions to a non-smooth ODE, we construct the notion of evaluation maps that act on the space of absolutely continuous functions. Let $\Gamma_i := AC([0,T];\mathbb{R}^{n_i})$ be the space of absolutely continuous functions from $[0,T]$ to $\mathbb{R}^{n_i}$ endowed with the norm $\|\cdot\| : \gamma \mapsto |\gamma(0)| + \int_0^T |\dot{\gamma}(t)|\, dt$. Define an *evaluation map* $e_t : [0,t]\times[t,T]\times\Gamma_i \to \mathbb{R}^{n_i}$ as $e_t(s,\tau,\gamma) = \gamma(t)$ on $s \le t \le \tau$ for each $t \in [0,T]$. The evaluation map allows us to establish the following relationship:

Theorem 9. *Let $\mu^i_{x|t}$, σ^i, η^i satisfy the PDE* (22) *for some $i \in \mathcal{I}$, where $\bar{F}_i$ is defined as in* (21). *Assume $\bar{F}_i$ is pointwise bounded. Then there exists a measure $\rho^i \in \mathcal{M}_+([0,T]\times[0,T]\times\Gamma_i)$ such that*

(a) ρ^i is concentrated on the triplets (s,τ,γ), where $s \le \tau$, and $\gamma \in \Gamma_i$ are solutions of the ODE $\dot{\gamma}(t) = \bar{F}_i(t,\gamma(t))$ for a.e. $t \in [s,\tau]$.

(b) $\mu^i_{x|t} = (e_t)_\# \rho^i$ for a.e. $t \in [0,T]$.

Proof: See Appendix B. ∎

Theorem 9 establishes a connection between the measure $\mu^i_{x|t}$ that solves the PDE (22) and trajectories that satisfy the dynamics in mode i. We next show those trajectories start and terminate in the support of σ^i and η^i, respectively.

Corollary 10. *Let $\mu^i_{x|t}$, σ^i, and η^i satisfy the PDE* (22) *for some i and let $\bar{F}_i$ which is defined in* (21) *be pointwise bounded. Let ρ^i be defined as in Theorem 9. Define maps $r^1, r^2 \in [0,T]\times[0,T]\times\Gamma_i \to [0,T]\times\mathbb{R}^{n_i}$ by $r^1 : (s,\tau,\gamma) \mapsto (s,\gamma(s))$ and $r^2 : (s,\tau,\gamma) \mapsto (\tau,\gamma(\tau))$. Then $r^1_\#\rho^i = \sigma^i$ and $r^2_\#\rho^i = \eta^i$.*

Proof: Recall in the proof of Theorem 9 we mollified σ^i and η^i using a family of smooth mollifiers to obtain smooth measures σ^i_ϵ and η^i_ϵ. We also defined a *tight* family of measures $\{\rho^i_\epsilon\}_\epsilon \subset \mathcal{M}_+([0,T]\times[0,T]\times\Gamma_i)$ that converges to ρ^i in the narrow sense. The connection between each ρ^i_ϵ in that family and the mollified measures σ^i_ϵ and η^i_ϵ was established via measures $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$.

For all $\varphi \in C_b([0,T]\times\mathbb{R}^{n_i})$, it follows from (55), (43), and (42) that $\int_{[0,T]\times[0,T]\times\Gamma_i} \varphi(s,\gamma(s))\, d\rho^i_\epsilon(s,\tau,\gamma) = \int_{[0,T]\times\mathbb{R}^{n_i}} \varphi(s,x)\, \sigma^i_\epsilon(s,x)$. Since the families $\{\sigma^i_\epsilon\}_\epsilon$ and $\{\rho^i_\epsilon\}_\epsilon$ are tight as was shown in the proof of Theorem 9, we may let $\epsilon \downarrow 0$ to obtain $\int_{[0,T]\times[0,T]\times\Gamma_i} \varphi(r^1(s,\tau,\gamma))\, d\rho^i(s,\tau,\gamma) = \int_{[0,T]\times\mathbb{R}^{n_i}} \varphi(s,x)\, \sigma^i(s,x)$. This is also true for all measurable functions φ because $C_b(\mathbb{R}^{n_i+1})$ is dense in $L^1(\sigma^i)$ [37, Corollary 4.2.2], as a result $r^1_\#\rho^i = \sigma^i$. The result for η^i can be proved in a similar manner. ∎

Theorem 9 illustrates that measures satisfying HLE in mode $i \in \mathcal{I}$ correspond to trajectories $\gamma \in \Gamma_i$ of the convexified inclusion, $\dot{\gamma}(t) \in \text{conv } F_i(t,\gamma(t),U)$, rather than the original specified dynamics within each mode of the system. To ensure that the there is no gap between the original dynamics and its convexified inclusion, we make the following assumption:

Assumption 11. *The set $F_i(t,x,U)$ is compact and convex for all t, x, and $i \in \mathcal{I}$.*

The above condition is sufficient to ensure that measures satisfying HLE correspond exactly to trajectories described according to Algorithm 1 [38, p. 529]. Assumption 11 is satisfied if, for example, F_i is control affine and U is compact and convex.

As a consequence of Corollary 10 and Assumption 11, any triplet $(s,\tau,\gamma) \in \text{spt}(\rho^i)$ can be viewed as a trajectory γ in mode i that is well defined on $[s,\tau]$ and satisfies $(s,\gamma(s)) \in \text{spt}(\sigma^i)$, $(\tau,\gamma(\tau)) \in \text{spt}(\eta^i)$. Such trajectories in different modes are related by reset maps and can be combined together to be admissible trajectories for the hybrid system. To illustrate this, define an evaluation map that acts on the trajectories of the hybrid system $e^i_t : \mathcal{X} \to X_i$ as $e^i_t(\gamma) = \gamma_i(t)$ if $\lambda(\gamma(t)) = i$ and $e^i_t(\gamma) = \varnothing$ otherwise for each $i \in \mathcal{I}$. We can establish a relationship between admissible trajectories and measures that satisfy (22) for each i:

Theorem 12. *Let $\mu^i_{x|t}$, σ^i, and η^i satisfy the PDE* (22) *for some i and let $\bar{F}_i$ which is defined in* (21) *be pointwise bounded. Then there exists a non-negative measure $\rho \in \mathcal{M}_+(\mathcal{X})$ such that*

(a) For any hybrid trajectory $\gamma \in spt(\rho)$, γ is defined on $[0,T]$ and satisfies $\gamma(0) \in spt(\mu^{\mathcal{I}}_0)$, $\gamma(T) \in spt(\mu^{\mathcal{I}}_T)$.

(b) For a.e. $t \in [0,T]$, $\mu^i_{x|t} = (e^i_t)_\#\rho$.

(c) If $\sum_{i\in\mathcal{I}} \mu^i_0(X_i) = 1$, then ρ is a probability measure.

(d) If $\sum_{i\in\mathcal{I}} \mu^i_0(X_i) = 1$, then $\mu^i_{t,x}$ (resp. μ^i_0, μ^i_T, μ^{S_e}) is the average occupation measure (resp. average initial measure, average terminal measure, average guard measure) generated by the family of admissible trajectories in the support of ρ for each mode $i \in \mathcal{I}$ and $e \in \mathcal{E}$. Moreover, $\sum_{i\in\mathcal{I}} \mu^i_{t,x}([0,T]\times X_i) = T$, $\sum_{i\in\mathcal{I}} \mu^i_T(X_{T_i}) = 1$, and $\sum_{e\in\mathcal{E}} \mu^{S_e}([0,T]\times S_e) \le C$ for some constant $C < +\infty$.

Proof: See Appendix C. ■

Notice in Theorem 12 if we define μ_0^i to be Dirac measure supported at x_0 if $x_0 \in X_i$ or zero otherwise, then spt$(\rho) \subset \mathcal{X}_T$. Finally, we establish a relationship between the solution measures and the underlying control input when the dynamics are control-affine, which enables control synthesis:

Corollary 13. *Let U be convex. For each $i \in \mathcal{I}$, suppose there exists pointwise bounded functions $f_i : \mathbb{R} \times X_i \to \mathbb{R}^{n_i}$ and $g_i : \mathbb{R} \times X_i \to \mathbb{R}^{n_i \times m}$ such that $F_i(t,x,u) = f_i(t,x) + g_i(t,x)u$ for all $t, x, u \in [0,T] \times X_i \times U$. Let $\nu^i_{u|t,x}$ be defined as in (17) and let ρ be defined as in Theorem 12. Then $t \mapsto (\gamma(t), \int_U u \, d\nu^{\lambda(\gamma(t))}_{u|t,\gamma_{\lambda(\gamma(t))}}(u))$ is an admissible pair for all $\gamma \in spt(\rho)$, where*

$$\int_U u \, d\nu^i_{u|t,x}(u) := \begin{bmatrix} \int_U [u]_1 \, d\nu^i_{u|t,x}(u) \\ \vdots \\ \int_U [u]_m \, d\nu^i_{u|t,x}(u) \end{bmatrix} \tag{23}$$

is an $m \times 1$ real vector for each t, x, and $i \in \mathcal{I}$.

Proof: For any $\gamma \in$ spt(ρ), $\dot{\gamma}_i(t) = f_i(t, \gamma_i(t)) + g_i(t, \gamma_i(t)) \cdot \int_U u \, d\nu^i_{u|t,\gamma_i(t)}(u)$ for a.e. $t \in [0,T]$. Since $\nu^i_{u|t,x}$ is a stochastic kernel and U is convex, $\int_U u \, d\nu^i_{u|t,\gamma_i(t)}(u) \in U$ for all $i \in \mathcal{I}$. Thus $t \mapsto (\gamma(t), \int_U u \, d\nu^{\lambda(\gamma(t))}_{u|t,\gamma_{\lambda(\gamma(t))}}(u))$ is an admissible pair. ■

IV. Infinite Dimensional Linear Program

This section formulates $(HOCP)$ as an infinite-dimensional linear program over the space of measures, proves it computes the solution to $(HOCP)$, and illustrates how its solution can be used for control synthesis. To formulate the cost function for these hybrid trajectories in measure-theoretic form and to make control synthesis feasible, we make the following assumption:

Assumption 14. *U is convex and for each $i \in \mathcal{I}$, there exists pointwise bounded functions $f_i : \mathbb{R} \times X_i \to \mathbb{R}^{n_i}$ and $g_i : \mathbb{R} \times X_i \to \mathbb{R}^{n_i \times m}$ such that $F_i(t,x,u) = f_i(t,x) + g_i(t,x)u$ for all $t, x, u \in [0,T] \times X_i \times U$.*

First define μ_0^i to be Dirac measure supported at x_0 if $x_0 \in X_i$ or zero otherwise and the optimization problem (P) as:

$$\begin{aligned} \inf_{\Gamma} \quad & \sum_{i\in\mathcal{I}} \langle \mu^i, h_i \rangle + \sum_{i \in \mathcal{I}} \langle \mu_T^i, H_i \rangle && (P) \\ \text{s.t.} \quad & \delta_0 \otimes \mu_0^i + \mathcal{L}_i' \mu^i + \sum_{(i',i)\in\mathcal{E}} \tilde{R}_{(i',i)\#} \mu^{S_{(i',i)}} = \delta_T \otimes \mu_T^i + \sum_{(i,i')\in\mathcal{E}} \mu^{S_{(i,i')}} && \forall i \in \mathcal{I}, \\ & \mu^i, \mu_T^i \geq 0 && \forall i \in \mathcal{I}, \\ & \mu^{S_e} \geq 0 && \forall e \in \mathcal{E}, \end{aligned}$$

where the infimum is taken over a tuple of measures $\Gamma = (\mu^{\mathcal{I}}, \mu_T^{\mathcal{I}}, \mu^{\mathcal{S}}) \in \mathcal{M}_+([0,T] \times \mathcal{D} \times U) \times \mathcal{M}_+(X_T) \times \mathcal{M}_+([0,T] \times \mathcal{S})$ and for each mode $i \in \mathcal{I}$, where μ_0^i is a Dirac measure if $x_0 \in X_i$ or zero otherwise. The dual to (P) is given as:

$$\begin{aligned} \sup_{v} \quad & \sum_{i \in \mathcal{I}} \langle \mu_0^i(x), v_i(0,x) \rangle && (D) \\ \text{s.t.} \quad & \mathcal{L}_i v_i(t,x) + h_i(t,x,u) \geq 0 && \forall (t,(x,i),u) \in [0,T] \times \mathcal{D} \times U, \\ & v_i(T,x) \leq H_i(x) && \forall (x,i) \in X_T, \\ & v_i(t,x) \leq v_{i'}(t, R_{(i,i')}(x)) && \forall (t,(x,(i,i'))) \in [0,T] \times \mathcal{S}, \end{aligned}$$

where the supremum is taken over the function $v \in C^1([0,T] \times \mathcal{D})$ and for each mode $i \in \mathcal{I}$. For convenience, denote the $i \in \mathcal{I}$ slice of v using subscript i. We have the following result from [39, Theorem 3.10]:

Theorem 15. *If either (P) or (D) is feasible, then there is no duality gap between (P) and (D).*

Next, we illustrate (P) is well-posed:

Lemma 16. *If (P) is feasible, then the minimum to (P), p^*, is attained.*

Proof: Let $(\mu^{\mathcal{I}}, \mu_T^{\mathcal{I}}, \mu^{\mathcal{S}})$ be a feasible solution to (P), and therefore they satisfy HLE (16). Using Theorem 12 we know the tuple of measures $(\frac{1}{T}\mu^{\mathcal{I}}, \mu_T^{\mathcal{I}}, \frac{1}{C}\mu^{\mathcal{S}})$ belongs to the unit ball B_1 of $\mathcal{M}([0,T] \times \mathcal{D} \times U) \times \mathcal{M}(X_T) \times \mathcal{M}([0,T] \times \mathcal{S})$ for some $C < +\infty$. By the Banach-Alaoglu Theorem, B_1 is weak-* sequentially compact. Since the operators $\tilde{R}_{e\#}$ and $\mathcal{L}_i'$ are bounded (because $\mathcal{L}_i$ is bounded) and therefore continuous, the set of $(\frac{1}{T}\mu^{\mathcal{I}}, \mu_T^{\mathcal{I}}, \frac{1}{C}\mu^{\mathcal{S}})$ satisfying HLE is a closed subset of $B_1 \cap \mathcal{M}_+([0,T] \times \mathcal{D} \times U) \times \mathcal{M}_+(X_T) \times \mathcal{M}_+([0,T] \times \mathcal{S})$, and therefore is also weak-* sequentially compact. Since the cost function is continuous, p^* is attained. ■

Now we prove that (P) solves $(HOCP)$:

Theorem 17. *Let (P) be feasible and suppose $h_i(t,x,\cdot)$ is convex for all $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times X_i$. Then $p^* = J^*$.*

Proof: Suppose (γ^*, u^*) is an optimal admissible pair to (OCP). By Lemma 7, its initial measures, occupation measures, terminal measures and guard measures are supported on proper domains and satisfy (16). Furthermore, $\xi_0^i = \mu_0^i$ for any $i \in \mathcal{I}$. Therefore these measures are a feasible solution to (P) with cost J^*, and $p^* \leq J^*$ follows.

We next prove $p^* \geq J^*$. Suppose $(\mu^{\mathcal{I}*}, \mu_T^{\mathcal{I}*}, \mu^{\mathcal{S}*})$ is an optimal solution to (P) which exists according to Lemma 16. The optimal tuple satisfies (16). By Theorem 12, there exists a probability measure $\rho \in \mathcal{M}_+(\mathcal{X}_T)$ such that $\mu^{i*}_{t,x}$ coincides with the occupation measures of a family of admissible trajectories in the support of ρ, when restricted to mode i. We abuse notation in the remainder of this proof and define $[\tilde{u}_i(t,x)]_j := \int_U [u]_j \, d\nu^{i*}_{u|t,x}(u)$ for any $i \in \mathcal{I}$ and $j \in \{1, \cdots, m\}$. Notice

$$\begin{aligned} q^* &= \sum_{i\in\mathcal{I}} \Big(\int_{[0,T]\times X_i \times U} h_i(t,x,u) d\nu^{i*}_{u|t,x}(u) d\mu^{i*}_{t,x}(t,x) + \\ &\qquad + \int_{X_T} H_i(x) d\mu_T^{i*}(x) \Big) &(24)\\ &\geq \sum_{i\in\mathcal{I}} \Big(\int_{[0,T]\times X_i} h_i(t,x,\tilde{u}_i(t,x)) d\mu^{i*}_{t,x}(t,x) + \\ &\qquad + \int_{X_T} H_i(x) \, d\mu_T^{i*}(x) \Big) &(25)\\ &= \int_{\mathcal{X}_T} \sum_{i\in\mathcal{I}} \Big(\int_{[0,T]} h_i\left(t, \gamma_i(t), \tilde{u}_i\left(t, \gamma_i(t)\right)\right) dt + \\ &\qquad + H_i(\gamma_i(T)) \Big) d\rho(\gamma) &(26)\\ &= \int_{\mathcal{X}_T} J\left(\gamma, \tilde{u}_{\lambda(\gamma(\cdot))}(\cdot, \gamma_{\lambda(\gamma(\cdot))}(\cdot))\right) d\rho(\gamma) \geq J^*, &(27) \end{aligned}$$

where in (24) we disintegrate the measure μ^{i*} into $\nu^{i*}_{u|t,x}$ and $\mu^i_{t,x}$ according to Lemma 8; (25) is obtained from the convexity of $h_i(t,x,\cdot)$ and the fact that $\nu^{i*}_{u|t,x}$ is a probability measure; in (26) we apply Theorem 12 and then interchange the order of summation and integration; (27) follows because we let $h_i = 0$ where $\gamma_i(t)$ is undefined and $(\gamma, \tilde{u}_{\lambda_\gamma(\cdot)}(\cdot, \gamma_{\lambda_\gamma(\cdot)}(\cdot)))$ is an admissible pair (according to Corollary 13) and since ρ is a probability measure. ∎

The previous result provides an extension of the weak formulation in [23] to hybrid systems, and ensures (P) can be solved to find a solution to $(HOCP)$ in a convex manner. Next we describe how to perform control synthesis with the solution of (P).

Theorem 18. *Let (P) be feasible and suppose $h_i(t, x, \cdot)$ is convex for all $i \in \mathcal{I}$ and $(t, x) \in [0, T] \times X_i$, and suppose the optimal trajectory γ^* is unique dt-a.e. Let $\Gamma^* = (\mu^{\mathcal{I}*}, \mu_T^{\mathcal{I}*}, \mu^{\mathcal{S}*})$ be a vector of measures that achieves the infimum of (P), then*

(a) One can disintegrate $\mu^{\mathcal{I}*}$ in each mode $i \in \mathcal{I}$ as $d\mu^{i*}(t,x,u) = d\nu^{i*}_{u|t,x}(u) d\mu^{i*}_{t,x}(t,x) = d\nu^{i*}_{u|t,x}(u) d\mu^{i*}_{x|t}(x) dt$. Moreover, $\mu^{i*}_{t,x}(t,x)$ coincides with the occupation measures of γ^* in each mode $i \in \mathcal{I}$ a.e.

(b) For each $i \in \mathcal{I}$ and $j \in \{1, \cdots, m\}$, let $[\tilde{u}_i(t,x)]_j := \int_U [u]_j \, d\nu^{i*}_{u|t,x}(u)$ for all $(t,x) \in spt(\mu^{i*}_{t,x})$, where $d\nu^{i*}_{u|t,x}$ is defined as in (a). If $\tilde{u}(t,x,i) := \tilde{u}_i(t,x)$ for all $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times X_i$, then $J(\gamma^*, \tilde{u}(\cdot, \gamma^*(\cdot))) = J^*$.

Proof: To prove (a) note that the decomposition of $\mu^{\mathcal{I}*}$ exists as a result of Lemma 8. Using the proof of Theorem 17, $J\left(\gamma, \tilde{u}_{\lambda(\gamma(\cdot))}(\cdot, \gamma_{\lambda(\gamma(\cdot))}(\cdot))\right) = J^*$ for any $\gamma \in \text{spt}(\rho)$, and therefore every admissible pair $\left(\gamma(\cdot), \tilde{u}_{\lambda(\gamma(\cdot))}(\cdot, \gamma_{\lambda(\gamma(\cdot))}(\cdot))\right)$ must be optimal. Since the optimal trajectory γ^* is assumed to be unique dt-a.e., $\gamma(t) = \gamma^*(t)$ for a.e. $t \in [0,T]$, $\forall \gamma \in \text{spt}(\rho)$. According to Theorem 12, $\mu^{i*}_{t,x}$ coincides with the occupation measure of γ^* in each mode $i \in \mathcal{I}$ a.e. Part (b) follows by noticing $J(\gamma, \tilde{u}(\cdot, \gamma(\cdot))) = J^*$ and $\gamma(t) = \gamma^*(t)$ a.e. for all $\gamma \in \text{spt}(\rho)$. ∎

Theorem 18 illustrates how one can construct a feedback controller using the conditional measure $\nu^{i*}_{u|t,x}$. Notice $[\tilde{u}_i(t,x)]_j := \int_U [u]_j \, d\nu^{i*}_{u|t,x}(u)$ can be equivalently written as

$$[\tilde{u}_i(t,x)]_j \int_U d\mu^{i*}(t,x,u) = \int_U [u]_j \, d\mu^{i*}(t,x,u). \quad (28)$$

Therefore $\tilde{u}_i$ can also be constructed by computing the Radon-Nikodym derivative using the optimal measures from the solution to (P). In the next section, this result is used to construct a sequence of controllers that converge to the optimal control. Finally notice that in the hypothesis of Theorem 18 we do not assume the uniqueness of the optimal control law, i.e., there may exist different control laws u_1 and u_2, such that $J^* = J(\gamma^*, u_1) = J(\gamma^*, u_2)$. Instead we only assume that the optimal trajectory is unique almost everywhere.

V. Numerical Implementation

This sections describes a solution to the infinite-dimensional problem (P) via a sequence of finite-dimensional approximations formulated as semidefinite programs (SDP)s. These SDPs are generated by representing the measures in (P) using a truncated sequence of moments and restricting the functions in (D) to polynomials of finite degree. The solutions to any of the SDPs in this sequence can be used to synthesize an approximation to the optimal controllers. To formulate this SDP relaxation, we restrict our interest to polynomial hybrid optimal control problems:

Assumption 19. *The functions f_i, g_i, h_i, and H_i are polynomials. that is, $[f_i]_j, [g_i]_{jk} \in \mathbb{R}[t,x]$, $h_i \in \mathbb{R}[t,x,u]$, and $H_i \in \mathbb{R}[x]$ for all $i \in \mathcal{I}$, $j \in \{1, \cdots, n_i\}$, and $k \in \{1, \cdots, m\}$.*

Note that in the notation $\mathbb{R}[t,x,u]$, we refer to x as an indeterminate in X_i with dimension n_i. In addition, for convenience, the dimension n_i of x is omitted when it is clear in context. We also make the following assumption:

Assumption 20. *For each $i \in \mathcal{I}$ and $(i, i') \in \mathcal{E}$, there exists polynomials $h_{X_{i_j}} \in \mathbb{R}[x]$ for all $j \in \{1, \cdots, n_{X_i}\}$, $h_{T_{i_j}} \in \mathbb{R}[x]$ for all $j \in \{1, \cdots, n_{T_i}\}$, $h_{U_j} \in \mathbb{R}[u]$ for all $j \in \{1, \cdots, n_U\}$, and $h_{(i,i')_j} \in \mathbb{R}[x]$ for all $j \in \{1, \cdots, n_{(i,i')}\}$ such that the following holds:*

$$X_i = \left\{x \in \mathbb{R}^{n_i} \mid h_{X_{i_j}}(x) \geq 0, \forall j \in \{1, \cdots, n_{X_i}\}\right\} \quad (29)$$

$$X_{T_i} = \left\{x \in \mathbb{R}^{n_i} \mid h_{T_{i_j}}(x) \geq 0, \forall j \in \{1, \cdots, n_{T_i}\}\right\} \quad (30)$$

$$U = \left\{u \in \mathbb{R}^m \mid h_{U_j}(u) \geq 0, \forall j \in \{1, \cdots, n_U\}\right\} \quad (31)$$

$$S_{(i,i')} = \left\{x \in \partial X_i \mid h_{(i,i')_j}(x) \geq 0, \forall j \in \{1, \cdots, n_{(i,i')}\}\right\}. \quad (32)$$

Since X_i and X_{T_i} are also compact, note that Putinar's condition is satisfied by adding the redundant constraint $M - \|x\|_2^2$ for some large enough M [40, Theorem 2.14].

To derive the SDP relaxation, we begin with a few preliminaries. Any polynomial $p \in \mathbb{R}_k[x]$ can be expressed in the monomial basis as $p(x) = \sum_{|\alpha| \leq k} p_\alpha x^\alpha = \sum_{|\alpha| \leq k} p_\alpha \cdot (x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ where α ranges over vectors of non-negative integers such that $|\alpha| = \sum_{i=1}^n \alpha_i \leq k$, and we denote $\text{vec}(p) = (p_\alpha)_{|\alpha| \leq k}$ as the vector of coefficients of p. Given a vector of real numbers $y = (y_\alpha)$ indexed by α, we define the linear functional $L_y : \mathbb{R}_k[x] \to \mathbb{R}$ as $L_y(p) := \sum_\alpha p_\alpha y_\alpha$ Note that, when the entries of y are moments of a measure μ defined as $y_\alpha = \int x^\alpha \, d\mu(x)$, then $\langle \mu, p \rangle = \int \left(\sum_\alpha p_\alpha x^\alpha\right) d\mu = L_y(p)$ If $|\alpha| \leq 2k$, the *moment matrix*, $M_k(y)$, defined as $[M_k(y)]_{\alpha\beta} = y_{(\alpha+\beta)}$. Given any polynomial $h \in \mathbb{R}_l[x]$ with $l < k$, the *localizing matrix*, $M_k(h, y)$, is defined as $[M_k(h,y)]_{\alpha\beta} = \sum_{|\gamma| \leq l} h_\gamma y_{(\gamma+\alpha+\beta)}$.

A. LMI Relaxations and SOS Approximations

A sequence of SDPs approximating (P) can be obtained by replacing constraints on measures with constraints on moments. Since h_i and H_i are polynomials, the objective function of (P) can be written using linear functionals as $\sum_{i \in \mathcal{I}} L_{y_{\mu^i}}(h_i) + \sum_{i \in \mathcal{I}} L_{y_{\mu_T^i}}(H_i)$, where y_{μ^i} and $y_{\mu_T^i}$ are the sequence of moments of μ^i and μ_T^i, respectively. The equality constraints in (P) can be approximated by an infinite-dimensional linear system, which is obtained by restricting to polynomial test functions: $v_i(t,x) \in \mathbb{R}[t,x]$, for any $i \in \mathcal{I}$. The positivity constraints in (P) can be replaced with semidefinite constraints on moment and localizing matrices, which guarantees the existence of Borel measures defined on proper domains [40, Theorem 3.8].

A finite-dimensional SDP is then obtained by truncating the degree of moments and polynomial test functions to $2k$. Let $\Xi_{\mathcal{I}} = \coprod_{i \in \mathcal{I}} \mu^i$, $\Xi_{\mathcal{E}} = \coprod_{e \in \mathcal{E}} \mu^{S_e}$, $\Xi_T = \coprod_{i \in \mathcal{I}} \mu_T^i$, and $\Xi = \Xi_{\mathcal{I}} \cup \Xi_{\mathcal{E}} \cup \Xi_T$. Let $(y_{k,\xi})$ be the sequence of moments truncated to degree $2k$ for each $(\xi, i) \in \Xi$, and let $\mathbf{y}_k$ be a vector of all the sequences $(y_{k,\xi})$. The equality contraints in (P) can then be approximated by a finite-dimensional linear system $A_k(\mathbf{y}_k) = b_k$. Define the k-th relaxed SDP representation of (P), denoted (P_k), as

$$\begin{aligned}
\inf_{\mathbf{y}_k} \quad & \sum_{i\in\mathcal{I}} L_{y_{k,\mu^i}}(h_i) + \sum_{i\in\mathcal{I}} L_{y_{k,\mu_T^i}}(H_i) && (P_k)\\
\text{s.t.} \quad & A_k(\mathbf{y}_k) = b_k, \\
& M_k(y_{k,\xi}) \succeq 0 && \forall (\xi, i) \in \Xi,\\
& M_{k_{X_{i_j}}}(h_{X_{i_j}}, y_{k,\mu^i}) \succeq 0 && \forall (j,i) \in \coprod_{i\in\mathcal{I}} \{1,\cdots,n_{X_i}\},\\
& M_{k_{U_{i_j}}}(h_{U_j}, y_{k,\mu^i}) \succeq 0 && \forall (j,i) \in \{1,\cdots,n_U\}\times\mathcal{I},\\
& M_{k_{S_{e_j}}}(h_{e_j}, y_{k,\mu^{S_e}}) \succeq 0 && \forall (j,e) \in \coprod_{e\in\mathcal{E}} \{1,\cdots,n_e\},\\
& M_{k_{T_{i_j}}}(h_{T_{i_j}}, y_{k,\mu_T^i}) \succeq 0 && \forall (j,i) \in \coprod_{i\in\mathcal{I}} \{1,\cdots,n_{T_i}\},\\
& M_{k-1}(h_\tau, y_{k,\xi}) \succeq 0 && \forall (\xi, i) \in \Xi_{\mathcal{I}} \bigcup \Xi_{\mathcal{E}},
\end{aligned}$$

where the infimum is taken over $\mathbf{y}_k$; $h_\tau = t(T-t)$, $k_{X_{i_j}} = k - \lceil \deg(h_{X_{i_j}})/2 \rceil$, $k_{U_{i_j}} = k - \lceil \deg(h_{U_{i_j}})/2 \rceil$, $k_{S_{e_j}} = k - \lceil \deg(h_{e_j})/2 \rceil$, $k_{T_{i_j}} = k - \lceil \deg(h_{T_{i_j}})/2 \rceil$, and $\succeq$ denotes positive semidefiniteness of matrices.

The dual of (P_k) is a Sums-of-Squares (SOS) program denoted by (D_k) for each $k \in \mathbb{N}$, which is obtained by first restricting the optimization space in (D) to the polynomial functions with degree truncated to $2k$ and by then replacing the non-negativity constraints in (D) with SOS constraints. For notational convenience, we let x_i be the indeterminate that corresponds to X_i. Define $Q_{2k}(h_{T_{i_1}}, \cdots, h_{T_{i_{n_{T_i}}}}) \subset \mathbb{R}_{2k}[x_i]$ to be the set of polynomials $l \in \mathbb{R}_{2k}[x_i]$ expressible as $l = s_0 + \sum_{j=1}^{n_{T_i}} s_j h_{T_{i_j}}$ for some polynomials $\{s_j\}_{i=0}^{n_{T_i}} \subset \mathbb{R}_{2k}[x_i]$ that are sums of squares of other polynomials. Every such polynomial is non-negative on X_{T_i}. Similarly, we define $Q_{2k}(h_\tau, h_{X_{i_1}}, \cdots, h_{X_{i_{n_{X_i}}}}, h_{U_1}, \cdots, h_{U_{n_U}}) \subset \mathbb{R}_{2k}[t, x_i, u]$, and $Q_{2k}(h_\tau, h_{(i,i')_1}, \cdots, h_{(i,i')_{n_{(i,i')}}}) \subset \mathbb{R}_{2k}[t, x_i]$ for each $i \in \mathcal{I}$ and $(i, i') \in \mathcal{E}$. Therefore k-th relaxed SDP representation of (D), denoted (D_k) is given as

$$\begin{aligned}
\sup_{v_i} \quad & \sum_{i\in\mathcal{I}} \langle \mu_0^i, v_i(0,\cdot)\rangle && (D_k)\\
\text{s.t.} \quad & \mathcal{L}_i v_i + h_i \in Q_{2k}(h_\tau, h_{X_{i_1}}, \cdots, h_{X_{i_{n_{X_i}}}}, h_{U_1}, \cdots, h_{U_{n_U}}) && \forall i \in \mathcal{I},\\
& -v_i(T,\cdot) + H_i \in Q_{2k}(h_{T_{i_1}}, \cdots, h_{T_{i_{n_{T_i}}}}) && \forall i \in \mathcal{I},\\
& v_{i'} \circ \tilde{R}_{(i,i')} - v_i \in Q_{2k}(h_\tau, h_{(i,i')_1}, \cdots, h_{(i,i')_{n_{(i,i')}}}) && \forall (i,i') \in \mathcal{E},
\end{aligned}$$

where the supremum is taken over $v_i \in \mathbb{R}_{2k}[t, x]$ for all $i \in \mathcal{I}$. Using Slater's condition [41, Chapter 5.3.2] and noting that (D_k) is bounded from below, we can prove that the pair of problems are well-posed:

Theorem 21. *For each $k \in \mathbb{N}$, if (P_k) is feasible, then there is no duality gap between (P_k) and (D_k).*

Next, we describe how to extract a polynomial control law from the solution of (P_k). Given moment sequences truncated to $2k$, we want to find an appropriate feedback control law $u_{k,i}^*$ in each mode $i \in \mathcal{I}$ with components $[u_{k,i}^*]_j \in \mathbb{R}[t, x]$, such that the analogue of (28) is satisfied, i.e.,

$$\begin{aligned}
\int_{[0,T]\times X_i} t^{\alpha_0} x^\alpha \cdot [u_{k,i}^*(t,x)]_j \int_U d\mu_k^{i*}(t,x,u) \\
= \int_{[0,T]\times X_i} t^{\alpha_0} x^\alpha \cdot \int_U [u]_j \, d\mu_k^{i*}(t,x,u)
\end{aligned} \tag{33}$$

for all $i \in \mathcal{I}$, $j \in \{1, \cdots, m\}$, and $(\alpha_0, \alpha) \in \mathbb{N} \times \mathbb{N}^{n_i}$ satisfying $\sum_{l=0}^n \alpha_l \le k$, $\alpha_l \ge 0$. Here μ_k^{i*} is any measure whose truncated moments match $y_{\mu_i}^*$. In fact, when constructing a polynomial control law from the solution of (P_k), these linear equations written with respect to the coefficients of $[u_{k,i}^*]_j$ are expressible in terms of the optimal solution y_{k,μ^i}^*. To see this, define the (t,x)-moment matrix of y_{k,μ^i}^* as $\left[M_k^{(t,x)}(y_{k,\mu^i}^*)\right]_{(\alpha_0,\alpha)(\beta_0,\beta)} = L_{y_{k,\mu^i}^*}(t^{\alpha_0+\beta_0} x^{\alpha+\beta} u^{\mathbf{0}})$ for all $i \in \mathcal{I}$, $\mathbf{0} \in \{0\}^m$ and $(\alpha_0, \alpha), (\beta_0, \beta) \in \mathbb{N} \times \mathbb{N}^{n_i}$ satisfying $\sum_{l=0}^n \alpha_l \le k$, $\alpha_l \ge 0$, $\sum_{l=0}^n \beta_l \le k$, $\beta_l \ge 0$. Also define a vector b_k^j as $\left[b_k^j(y_{k,\mu^i}^*)\right]_\alpha = L_{y_{k,\mu^i}^*}(t^{\alpha_0} x^\alpha [u]_j)$ for all $j \in \{1, \cdots, m\}$, and $(\alpha_0, \alpha) \in \mathbb{N} \times \mathbb{N}^{n_i}$ satisfying $\sum_{l=0}^n \alpha_l \le k$, $\alpha_l \ge 0$. Direct calculation shows (33) is equivalent as the following linear system of equations:

$$M_k^{(t,x)}(y_{k,\mu^i}^*) \operatorname{vec}([u_{k,i}^*]_j) = b_k^j(y_{k,\mu^i}^*) \tag{34}$$

To extract the coefficients of the controller, one needs only to compute the pseudoinverse of $M_k^{(t,x)}(y_{k,\mu^i}^*)$.

B. Convergence of Relaxed Problems

Next, we prove the convergence of the approximations:

Theorem 22. *Let p_k^* and d_k^* denote the infimum of (P_k) and supremum of (D_k), respectively. Then $\{p_k^*\}_{k=1}^\infty$ and $\{d_k^*\}_{k=1}^\infty$ converge monotonically from below to the optimal value of (P) and (D).*

Proof: This can be proved using a technique adopted in the proof of [42, Theorem 4.2]. We first establish a lower bound of d_k^* by finding a feasible solution to (D_k) for some k, and then show that there exists a convergent subsequence of $\{d_k^*\}_{k=1}^\infty$, by arguing the lower bound can be arbitrarily close to d^* for large enough k. Using Theorem 21, we only need to prove $\{d_k^*\}_{k=1}^\infty$ converges monotonically from below to d^*. Note that the higher the relaxation order k, the looser the constraint set of the optimization problem (D_k), so $\{d_k^*\}_{k=1}^\infty$ is non-decreasing.

Suppose $v \in C^1([0,T] \times \mathcal{D})$ is feasible in (D). For every $\epsilon > 0$ and $i \in \mathcal{I}$, let $\tilde{v}_i(t,x) := v_i(t,x) + \epsilon t - (1+T)\epsilon$. Therefore, $\mathcal{L}_i \tilde{v}_i = \mathcal{L}_i v_i + \epsilon$, $\tilde{v}_i(T,x) = v_i(T,x) - \epsilon$, and it follows that $\coprod_{i\in\mathcal{I}} \tilde{v}_i$ is strictly feasible in (D) with a margin at least ϵ. Since $[0,T] \times X_i$ and X_i are compact for every $i \in \mathcal{I}$, and by an extension to the Stone-Weierstrass Theorem that allows for the simultaneous uniform approximation of a function and its derivatives by a polynomial [43], we are guaranteed the existence of polynomials $\hat{v}_i$, such that $\|\hat{v}_i - \tilde{v}_i\|_\infty < \epsilon$, and $\|\mathcal{L}_i \hat{v}_i - \mathcal{L}_i \tilde{v}_i\|_\infty < \epsilon$ for any $i \in \mathcal{I}$. By Putinar's Positivstellensatz [40, Theorem 2.14], those polynomials are strictly feasible for (D_k) for a sufficiently large relaxation order k, therefore $d_k^* \ge \sum_{i\in\mathcal{I}} \hat{v}_i(0,x_0) \ge \sum_{i\in\mathcal{I}} \tilde{v}_i(0,x_0) - |\mathcal{I}|\epsilon$, where $|\mathcal{I}|$ is the number of elements in $\mathcal{I}$. Also, since $\tilde{v}_i(0,x_0) = v_i(0,x_0) - (1+T)\epsilon$, we have $d_k^* > \sum_{i\in\mathcal{I}} v_i(0,x_0) - (1+T+|\mathcal{I}|)\epsilon = d^* - (1+T+|\mathcal{I}|)\epsilon$, where $1 + T + |\mathcal{I}| < \infty$ is a constant. Using the fact that d^* is non-decreasing and bounded above by d, we know $\{d_k^*\}_{k=1}^\infty$ converges to d from below. ■

Then, by applying [42, Theorem 4.5], one can prove:

Theorem 23. *Let $\{y_{k,\xi}^*\}_{(\xi,i)\in\Xi}$ be an optimizer of (P_k), and let $\{\mu_k^{i*}\}_{i\in\mathcal{I}}$ be a set of measures such that the truncated* *moments of μ_k^{i*} match y^*_{k,μ^i} for each $i \in \mathcal{I}$. For each $k \in \mathbb{N}$, let $u^*_{k,i}$ denote the controller constructed by* (34)*, and $\tilde{u}_i$ be the optimal control law in mode $i \in \mathcal{I}$ from Theorem 18, then there exists a subsequence $\{k_l\}_{l\in\mathbb{N}} \subset \mathbb{N}$ such that for all $i \in \mathcal{I}$, $v_i \in C^1([0,T] \times X_i)$, and $j \in \{1,\cdots,m\}$, $\int_{[0,T]\times X_i} v_i(t,x)[u^*_{k_l,i}]_j(t,x)\, d\mu^{i*}_{t,x;k_l}(t,x)$ converges to $\int_{[0,T]\times X_i} v_i(t,x)\,[\tilde{u}_i(t,x)]_j\, d\mu^{i*}_{t,x}(t,x)$ as $l \to \infty$.*

VI. Examples

This section illustrates the performance of our approach using several examples. Our algorithm is implemented using MOSEK [44]. The trajectory is obtained by plugging the computed, saturated control law back into the system dynamics in each mode and simulating forward using a standard ODE solver with event detection. To provide a thorough comparison, all examples are also solved with the method proposed in [23], [25] by fixing the sequence of transitions and optimizing over each mode. Since the optimal sequence is not known a priori, this method is then applied over all feasible sequences of bounded total length. In addition, all examples are solved either analytically or using GPOPS-II [45] by iterating through a finite set of possible transitions. Notice that in this latter instance we fix the sequence of transition in each GPOPS-II call and provide an initial guess. All experiments are performed on an Intel Xeon, 144 core, 2.40 GHz, 1056 GB RAM machine. Our code and detailed description of the examples are available online at https://github.com/pczhao/hybridOCP.git.

A. Hybridized Double Integrator

We first consider a double integrator with states $x = (x_1, x_2) \in \mathbb{R}^2$ and input $u \in [-1,1]$. We hybridize this system by dividing the domain into two parts $X_1 = [0.5, 2] \times [-1,1]$ and $X_2 = [-1, 0.5] \times [-1,1]$ and with transitions only from mode 1 to mode 2 with an identity reset map between them. The guard is defined as $\{0.5\} \times \{[-1, -10^{-3}] \cup [10^{-3}, 1]\}$. We solve a Linear Quadratic Regulator (LQR) problem, where the goal is to drive the system towards $(0,0)$ while minimizing the control action. The problem is setup according to Table I. Note that Assumptions 2-5 are satisfied. Our results, which are summarized in Table II, are compared to those generated by [25] with degree of relaxation be $2k = 12$ when applied to finite mode sequences of total length 2. Table II also describes the results generated by a standard LQR solver which does not treat the problem as hybrid. This latter result is treated as ground truth. The proposed method is able to generate tight lower bounds and the optimal sequence of transitions even when degree of relaxation is low ($2k = 6$).

B. Dubins Car Model with Shortcut Path

The next example illustrates our algorithm can work with different dimensions in each mode. Consider a planar Dubins Car model with the states $x = (x_1, x_2, x_3) \in [-1,1] \times [-1,1] \times [-\pi/2, \pi/2]$ representing the 2D position and heading angle, and the inputs $u = (v, \omega) \in [10^{-3}, 1] \times [-3, 3]$ representing the linear and angular velocity. We hybridize this system by dividing the domain into two parts along the line $x_2 = 0$ and defining an identity reset map. Note that only transitions from the mode where x_2 is greater than or equal to zero to the mode where x_2 is less than or equal to zero are permitted. We also add to the system another 1-dimensional mode with dynamics $\dot{x} = -v$, where $x \in [-1,1]$ and $v \in [10^{-3}, 2]$. We connect this mode with the other two modes by defining $S_{(1,3)} = [-1,1] \times \{1\} \times \left([-\pi/2, -10^{-3}] \cup [10^{-3}, \pi/2]\right)$, $R_{(1,3)}(x) = 1$, $S_{(3,2)} = \{-1\}$, and $R_{(3,2)} = (0.6, -0.8, 0)$. We are interested in solving an optimal control problem where the goal is to get to the target position as quickly as possible. To solve this free final time problem, we modify HLE by substituting $\delta_T \otimes \mu_T^i$ with μ_T^i whose support is in $[0,T] \times X_T$, for all $i \in \mathcal{I}$. (P) and (D) can be modified accordingly. Notice that by treating the measure associated with the time-varying target set as a guard measure without any associated reset map, we can extend Theorems 9 and 12 to show that (P) can solve the free final time problem [23, Remark 2.1]. The optimal control problem is defined in Table I so that Assumptions 2-5 are satisfied.

Notice the transition sequences "1-2" and "1-3-2" are both feasible in this instance according to our guard definition, but direct calculation shows that we arrive at the target point in less time by taking the "shortcut path" in mode 3. This problem is solved using our algorithm with degrees of relaxation $2k = 6$, $2k = 8$, and $2k = 10$. As comparison, we also solve the problem using the method presented in [25] with degree of relaxation $2k = 10$ by applying it to each possible feasible mode sequence that has a maximum length 3, and treat the analytically computed optimal control as ground truth. The results are compared in Table II. Our algorithm is able to pick the transition sequence "1-3-2" and approximate the true optimal solution even when $2k = 6$.

C. SLIP Model

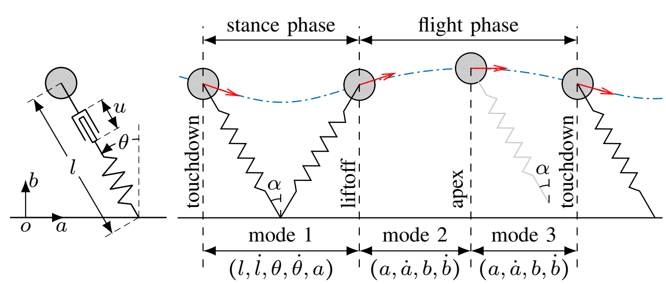


Fig. 2: An illustration of the SLIP model (left) and its hybrid modes (right)

The Spring-Loaded Inverted Pendulum (SLIP) is a model that describes the center-of-mass dynamics of animals and has been used to perform control synthesis for legged robots [46]. We may simulate the system numerically, but the optimal control problem is still difficult to solve if the sequence of transition is not known beforehand. We focus on the active SLIP model (Fig. 2a), which is an actuated mass-spring physical system, modeled as a point mass, M, a mass-less spring leg with stiffness k and length l, and a mass-less actuator u. The behavior of such a system can be fully characterized using 8 variables: leg length l, leg angle θ, horizontal displacement a, vertical displacement b, and their time derivatives (denoted as $\dot{l}$, $\dot{\theta}$, $\dot{a}$, and $\dot{b}$, respectively). The system states in each of the 3 hybrid modes are defined as shown in Fig. 2b. The github repo describes the physical parameters, dynamics, guards, and reset maps. To ensure that we satisfy Assumptions 2-5, the guard at touch-down is satisfied when $\dot{b} \leq -10^{-3}$, and the guard at lift-off is satisfied when $\dot{l} \geq 10^{-3}$.

We fix the initial condition, and consider the following two hybrid optimal control problems for the active SLIP: In the first problem, we maximize the vertical displacement b up to time $T = 2.5$. In stance phase, the 1st-order Taylor approximation $b = l\cos(\theta) \approx l$ is used; In the second problem, we define a constant-speed reference trajectory $a(t) = vt - 0.5$ in the horizontal coordinate, then try to follow this trajectory with active SLIP up to time $T = 3$. The optimal control problems are defined according to Table I. Note that these problems are defined such that the optimal transition sequences are different in each instance, and some modes are visited multiple times.

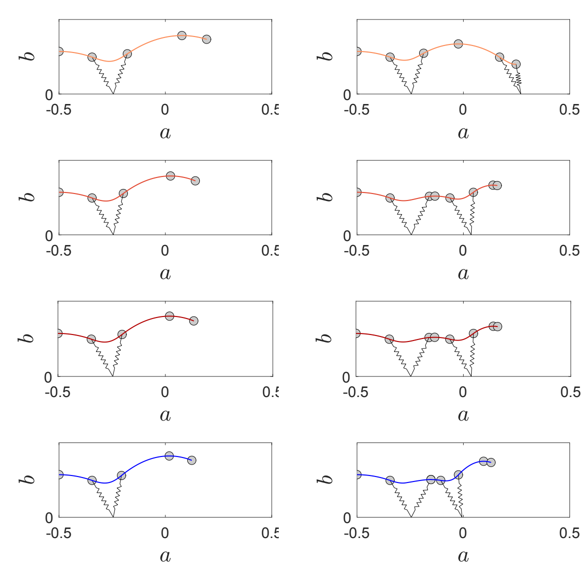
The optimization problems are solved by our algorithm with degrees of relaxation $2k = 4$, $2k = 6$, and $2k = 8$. For the sake of comparison, the same problems are also solved using the method presented in [25] and GPOPS-II for all possible, feasible mode sequences of maximum total length 12. The results are compared in Fig.TABLE 3 and Table II. The proposed method is able to generate the optimal sequence of transitions even at low relaxation degrees (e.g. $2k = 6$) while other methods have to search through all possible sequences. In particular, the proposed method takes an order of magnitude less time to find the optimal sequence of transitions on both examples when compared to GPOPS-II.

TABLE I: The setup for each example problem.

	Mode	$i = 1$	$i = 2$	$i = 3$
Double Integrator LQR	h_i	$x_1^2 + x_2^2 + 20u^2$	$x_1^2 + x_2^2 + 20u^2$	N/A
	H_i	0	0	
	x_0	$(1, 1)$	N/A	
	X_{T_i}	$[0.5{+}10^{-3}, 2] \times [-1, 1]$	X_2	
	T	5 or 15		
Dubins Car	h_i	1	1	1
	H_i	0	0	0
	x_0	(-0.8,0.8,0)	N/A	N/A
	X_{T_i}	N/A	$\{0.8\} \times \{-0.8\} \times [-\pi/2, \pi/2]$	N/A
	T	3		
SLIP Max jump	h_i	$-x_1$	$-x_3$	$-x_3$
	H_i	0	0	0
	x_0	N/A	N/A	(-0.5,0.3,0.2,0)
	X_{T_i}	$\{x \in X_1 \mid x_1 \leq l_0 - 10^{-3}\}$	$\{x \in X_2 \mid x_4 \geq 10^{-3}\}$	$\{x \in X_3 \mid x_3 \geq l_0 \cos(\alpha) + 10^{-3}\}$
	T	2.5		
SLIP Track speed	h_i	$(vt - 0.5 - x_5)^2$	$(vt - 0.5 - x_1)^2$	$(vt{-}0.5{-}x_1)^2$
	H_i	0	0	0
	x_0	N/A	N/A	(-0.5,0.3,0.2,0)
	X_{T_i}	$\{x \in X_1 \mid x_1 \leq l_0 - 10^{-3}\}$	$\{x \in X_2 \mid x_4 \geq 10^{-3}\}$	$\{x \in X_3 \mid x_3 \geq l_0 \cos(\alpha) + 10^{-3}\}$
	T	3		

TABLE II: Numerical results for the proposed algorithm on each example.

		Computation time	Cost from optimization	Cost from simulation
Double Integrator LQR $T = 5$	$2k = 6$	3.2004[s]	24.9496	24.9908
	$2k = 8$	9.4318[s]	24.9496	24.9908
	$2k = 12$	252.8047[s]	24.9496	24.9914
	[25], $2k = 12$	326.1610[s]	24.9496	24.9905
	Ground truth	N/A	24.9503	N/A
Double Integrator LQR $T = 15$	$2k = 6$	3.1583[s]	26.1993	26.3557
	$2k = 8$	9.8637[s]	26.1993	26.3644
	$2k = 12$	219.8932[s]	26.1994	26.3710
	[25], $2k = 12$	295.1562[s]	26.1993	26.3694
	Ground truth	N/A	26.2033	N/A
Dubin's Car	$2k = 6$	67.6682[s]	1.5640	1.5748
	$2k = 8$	956.6177[s]	1.5646	1.5718
	$2k = 10$	$1.0654{\times}10^4$[s]	1.5648	1.5708
	[25], $2k = 10$	$2.6259{\times}10^4$[s]	1.5648	1.5708
	Ground truth	N/A	1.5651	N/A
SLIP Max Jump	$2k = 4$	45.1598[s]	-0.6962	-0.5525
	$2k = 6$	584.8139[s]	-0.5815	-0.5474
	$2k = 8$	$7.7398{\times}10^3$[s]	-0.5776	-0.5545
	[25], $2k = 8$	$2.1225{\times}10^5$[s]	-0.5737	-0.5728
	GPOPS-II	792.9885[s]	-0.5735	N/A
SLIP Track Speed	$2k = 4$	40.7036[s]	0.0534	0.2250
	$2k = 6$	565.7164[s]	0.1417	0.1813
	$2k = 8$	$1.0263{\times}10^4$[s]	0.1523	0.1825
	[25], $2k = 8$	$2.2373{\times}10^5$[s]	0.1592	0.1718
	GPOPS-II	673.5100[s]	0.1626	N/A



(a) Maximizing vertical displacement

(b) Tracking constant speed $v = 0.1$

Fig. 3: An illustration of the performance of our algorithm on the active SLIP model. The blue lines are the optimal control computed by GPOPS-II by iterating through all the possible transition sequences, and the red lines of various saturation are controls generated by our method. As the saturation increases the corresponding degree of relaxation increases between $2k = 4$ to $2k = 6$ to $2k = 8$. Fig. 3a shows trajectories that maximize vertical displacement, where the optimal solution goes through 3 transitions; Fig. 3b shows trajectories that track $v = 0.1$, where the optimal solution goes through 6 transitions.

VII. Discussion

This paper proposes a convex approach for solving hybrid optimal control problems by relating the trajectories of hybrid systems to the solutions of a system of linear equations over measures. The hybrid optimal control problem is then formulated as an infinite-dimensional LP that does not require pre-specifying the sequence of possible transitions. A sequence of provably convergent SDPs to this LP are constructed to approximate the optimal cost from below and synthesize the optimal control law. Though it does not require pre-specifying the sequence of transitions of the hybrid system, the proposed method can be difficult to apply when the state space dimension is high, since the number of decision variables in the SDP grows exponentially with the state space dimension.

Appendix A

Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a pointwise bounded vector field, such that $F(t,\cdot)$ is Lipschitz for all $t \in [0,T]$. Consider a non-homogeneous PDE $\partial_t \mu_{t,x} = \sigma - \eta - D_x \cdot (F\mu_{t,x})$, where $\mu_{t,x}, \sigma, \eta \in \mathcal{M}([0,T]\times\mathbb{R}^n)$. Applying integration by parts and Lemma 8, this PDE becomes:

$$\int_{[0,T]\times\mathbb{R}^n} \partial_t v(t,x) d\mu_{x|t}(x) dt = \int_{[0,T]\times\mathbb{R}^n} v(t,x)\, d\big(\eta(t,x) + - \sigma(t,x)\big) + \int_{[0,T]\times\mathbb{R}^n} \nabla_x v(t,x)\cdot F d\mu_{x|t}(x)\, dt \tag{35}$$

for any $v \in C^1([0,T]\times\mathbb{R}^n)$. To establish a relationship between F and this PDE, let Φ satisfy (42) with $\bar{F}_i^\epsilon$ replaced by F. Since F is pointwise bounded and $F(t,\cdot)$ is Lipschitz for all $t \in [0,T]$, the solutions of the ODE are unique [47, Theorem 5.3]. By differentiating the identity $\Phi_i(t,s,\Phi_i(s,\tau,z)) = \Phi_i(t,\tau,z)$ with respect to s, we can show that $\Phi(t,\cdot,\cdot)$ is a solution to $\frac{d}{ds}\Phi_i(t,s,x) + \nabla_x \Phi_i(t,s,x)\cdot F(s,x) = 0$. This leads to:

Corollary 24. *Let $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be pointwise bounded and suppose $F(t,\cdot)$ is Lipschitz for all $t \in [0,T]$. Let σ and η satisfy* (35)*, and let Φ be the a.e. solution to the ODE with vector field F, then for any $w \in L^1(\mathbb{R}^n)$, $\int_{[0,T]\times\mathbb{R}^n} w(\Phi(T,s,x))\, d(\sigma(s,x) - \eta(s,x)) = 0$.*

Proof: The result for $w \in C_b^1(\mathbb{R}^n)$ follows by substituting $v(s,x) := w(\Phi(T,s,x))$ and $\frac{d}{ds}\Phi_i(t,s,x) + \nabla_x\Phi_i(t,s,x)\cdot F(s,x) = 0$ into (35). Since $C_b^1(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ [37, Corollary 4.2.2], the statement is true for all $w \in L^1(\mathbb{R}^n)$. ∎

We can now establish a relationship between $\mu_{x|t}$ and Φ:

Theorem 25. *Let $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be pointwise bounded and suppose $F(t,\cdot)$ is Lipschitz for all $t \in [0,T]$. Given $\sigma,\eta \in \mathcal{M}_+([0,T]\times\mathbb{R}^n)$, the solution to* (35) *is given by $\mu_{x|t} = \Phi(t,\cdot,\cdot)_\#\,(\sigma-\eta)$ for almost every $t \in [0,T]$, where $\Phi(t,\cdot,\cdot) : [0,t]\times\mathbb{R}_n \to \mathbb{R}^n$ is defined in* (42) *with $\bar{F}_i^\epsilon$ replaced by F.*

Proof: We first verify $\mu_{x|t} = \Phi(t,\cdot,\cdot)_\#\,(\sigma-\eta)$ satisfies (35). We need to check the equality only on test functions of the form $\psi(t)w(x)$. We substitute $\mu_{x|t} = \Phi(t,\cdot,\cdot)_\#\,(\sigma-\eta)$ into the left-hand side of (35) and show it is equal to the right-hand side of (35):

$$\int_0^T \dot{\psi}(t)\int_{\mathbb{R}^n} w(x)\, d\mu_{x|t}(x)\, dt$$

$$= \int_{[0,T]\times\mathbb{R}^n} \Big(\int_s^T \dot{\psi}(t) w(\Phi_i(t,s,x))dt\Big) d\big(\sigma^i(s,x) - \eta^i(s,x)\big) \tag{36}$$

$$= \int_{[0,T]\times\mathbb{R}^n} \Big(\psi(T)w(\Phi(T,s,x)) - \psi(s)w(\Phi(s,s,x)) + - \int_s^T \psi(t)\frac{d}{dt}w(\Phi(t,s,x))\, dt\Big)\, d(\sigma(s,x) - \eta(s,x)) \tag{37}$$

$$= \int_{[0,T]\times\mathbb{R}^n} \psi(s)w(x)\, d(\eta(s,x) - \sigma(s,x)) + - \int_0^T \psi(t)\int_{[0,t]\times\mathbb{R}^n} \nabla_x w(\Phi_i(t,s,x))\cdot F(t,\Phi(t,s,x))\, d(\sigma(s,x) - \eta(s,x))\, dt \tag{38}$$

$$= \int_{[0,T]\times\mathbb{R}^n} \psi(s)w(x)\, d(\eta(s,x) - \sigma(s,x)) + - \int_0^T \psi(t)\langle \mu_{x|t}, \nabla_x w\cdot F\rangle\, dt \tag{39}$$

where (36) follows from Fubini's Theorem; (37) follows from integration by parts; (38) follows from Corollary 24 and Fubini's Theorem; (39) follows from $\mu_{x|t} = \Phi(t,\cdot,\cdot)_\#\,(\sigma-\eta)$. As a result, $\mu_{x|t} = \Phi(t,\cdot,\cdot)_\#\,(\sigma-\eta)$ is a solution to (35). To show the solution is unique dt-almost everywhere, suppose there exists measures $\mu_{x|t,1}$, $\mu_{x|t,2} \in \mathcal{M}_+(\mathbb{R}^n)$ defined for $t \in [0,T]$ that satisfy (35). Let $\mu_{x|t,3} := \mu_{x|t,1} - \mu_{x|t,2} \in \mathcal{M}(\mathbb{R}^n)$, then $\int_0^T\int_{\mathbb{R}^n} (\partial_t v(t,x) + \nabla_x v(t,x)\cdot F)\, d\mu_{x|t,3}\, dt = 0$, which has the zero measure as a solution. Using the proof of [48, Lemma 3], such $\mu_{x|t,3}$ is defined uniquely dt-a.e. Therefore $\mu_{x|t,3}$ is zero for a.e. $t \in [0,T]$, which proves the result. ∎

Appendix B

In this section we prove Theorem 9.

Proof: This proof consists of several steps: in Step 1, we use a family of mollifiers parameterized by ϵ to smooth the vector field and all relevant measures and establish a relationship between the smooth measures using the solution to the smooth vector field via Theorem 25; in Step 2, we prove that all trajectories that satisfy this smooth vector field and enter the domain, eventually leave the domain, and vice versa; in Steps 3 and 4, we prove a connection between the time at which each trajectory enters and leaves; since Steps 2-4 are all proven for the "smoothed" versions of the vector field and measures, in Step 5 we prove that there exists a limiting measure as the parameter controlling smoothness, ϵ, goes to zero; in Step 6, we prove that this limit satisfies (b); in Step 7, we prove (a) when the vector field is continuous; in Step 8, we approximate the discontinuous vector field with a sequence of smooth functions and bound the approximation error; in Step 9, we prove (a) for arbitrary bounded vector fields.

Step 1 (Regularization). We first mollify $\mu_{x|t}^i$, σ^i, and η^i with respect to the space variable using a family of strictly positive mollifiers $\{\theta_\epsilon\} \subset C^\infty(\mathbb{R}^{n_i})$ with unit mass, zero mean, and uniformly bounded second moment, obtaining smooth measures $\mu_{x|t;\epsilon}^i := \mu_{x|t}^i * \theta_\epsilon$, $\sigma_\epsilon^i := \sigma^i * \theta_\epsilon$, and $\eta_\epsilon^i := \eta^i * \theta_\epsilon$. We also define a smooth vector field $\bar{F}_i^\epsilon$ by

$$\bar{F}_i^\epsilon(t,\cdot) := \begin{cases} \frac{\bar{F}_i(t,\cdot)\mu^i_{x|t} * \theta_\epsilon}{\mu^i_{x|t} * \theta_\epsilon}, & \text{if } \|\mu^i_{x|t}\| > 0; \\ 0, & \text{if } \|\mu^i_{x|t}\| = 0. \end{cases} \tag{40}$$

Notice the smooth vector field $\bar{F}_i^\epsilon$ is pointwise bounded: Let $M < +\infty$ be a pointwise bound for $\bar{F}_i$, then

$$\left|\bar{F}_i^\epsilon(t,x)\right| \le \frac{M\mu^i_{x|t} * \theta_\epsilon}{\mu^i_{x|t} * \theta_\epsilon} \le M\frac{\mu^i_{x|t} * \theta_\epsilon}{\mu^i_{x|t} * \theta_\epsilon} = M \tag{41}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^{n_i}$. By applying Young's convolution inequality, one can prove $\|\mu^i_{x|t;\epsilon}\| \le \|\mu^i_{x|t}\|$, $\|\sigma^i_\epsilon\| \le \|\sigma^i\|$, and $\|\eta^i_\epsilon\| \le \|\eta^i\|$.

Such $\mu^i_{x|t;\epsilon}$ is a solution of (22) with respect to $\bar{F}_i^\epsilon$, σ^i_ϵ, and η^i_ϵ. Since $\bar{F}_i^\epsilon$ is pointwise bounded and $\bar{F}_i^\epsilon(t,\cdot)$ is Lipschitz, Theorem 25 implies that $\mu^i_{x|t;\epsilon} = \Phi^\epsilon_i(t,\cdot,\cdot)_\#\left(\sigma^i_\epsilon - \eta^i_\epsilon\right)$ for a.e. $t \in [0,T]$, where $\Phi^\epsilon_i(t,s,x)$ satisfies:

$$\Phi^\epsilon_i(t,s,x) = x + \int_s^t \bar{F}_i^\epsilon(\tau, \Phi^\epsilon_i(\tau,s,x))\,d\tau, \quad 0 \le s \le t \le T \tag{42}$$

The function $\Phi^\epsilon_i(\cdot,s,x)$ can be extended to $[0,T]$ (as opposed to $[s,T]$) due to the regularity of $\bar{F}_i^\epsilon$. Denote the extended version as $\hat{\Phi}^\epsilon_i(\cdot,s,x) \in \Gamma_i$ for any $(s,x) \in [0,T] \times \mathbb{R}^{n_i}$. The space of all such functions is denoted as $\Gamma^\epsilon_i := \{\hat{\Phi}^\epsilon_i(\cdot,s,x) \mid (s,x) \in [0,T] \times \mathbb{R}^{n_i}\} \subset \Gamma_i$ endowed with the subspace topology. It follows by the existence and uniqueness theorem for ODE that the evaluation map $e_t(0,T,\cdot)$ restricted to Γ^ϵ_i is an isomorphism for any $t \in [0,T]$. Define $\Psi^\epsilon : (t,x) \mapsto \hat{\Phi}^\epsilon(\cdot,t,x)$ from $[0,T] \times \mathbb{R}^{n_i}$ to Γ^ϵ_i, and also a projection map $\pi^1 : (s,x) \mapsto s$ from $[0,T] \times \mathbb{R}^{n_i}$ to $[0,T]$. Define

$$\begin{aligned} \rho^{i,+}_\epsilon &:= \left(\pi^1 \times \Psi^\epsilon\right)_\# \sigma^i_\epsilon \in \mathcal{M}_+([0,T] \times \Gamma^\epsilon_i), \\ \rho^{i,-}_\epsilon &:= \left(\pi^1 \times \Psi^\epsilon\right)_\# \eta^i_\epsilon \in \mathcal{M}_+([0,T] \times \Gamma^\epsilon_i). \end{aligned} \tag{43}$$

Step 2 (Marginals of $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$). This step shows that all trajectories that enter the domain via σ^i_ϵ leave through η^i_ϵ by proving that the γ-marginals of $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$ are equal. Since ρ^{i+}_ϵ and ρ^{i-}_ϵ are finite measures and $\mathbb{R} \times \Gamma^\epsilon_i$ is Radon separable metric space, using [36, Theorem 5.3.1], the measures $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$ can be disintegrated as

$$\begin{aligned} d\rho^{i,+}_\epsilon(s,\gamma) &= d\rho^{i,+}_{s|\gamma;\epsilon}(s)\,d\rho^{i,+}_{\gamma;\epsilon}(\gamma), \\ d\rho^{i,-}_\epsilon(\tau,\gamma) &= d\rho^{i,-}_{\tau|\gamma;\epsilon}(\tau)\,d\rho^{i,-}_{\gamma;\epsilon}(\gamma), \end{aligned} \tag{44}$$

where $\rho^{i,+}_{s|\gamma;\epsilon}$ and $\rho^{i,-}_{\tau|\gamma;\epsilon}$ are probability measures for all $\gamma \in \operatorname{spt}(\rho^{i,+}_{\gamma;\epsilon})$ and $\gamma \in \operatorname{spt}(\rho^{i,-}_{\gamma;\epsilon})$, respectively. We next show the γ-marginals are equal. Let $w \in L^1(\mathbb{R}^{n_i})$ be arbitrary. Notice

$$0 = \int_{[0,T]\times\mathbb{R}^{n_i}} w(\Phi^\epsilon_i(T,s,x))\,d\left(\sigma^i_\epsilon(s,x) - \eta^i_\epsilon(s,x)\right) \tag{45}$$

$$= \int_{[0,T]\times\Gamma^\epsilon_i} w(e_T(0,T,\gamma))\,d\left(\rho^{i,+}_\epsilon(s,\gamma) - \rho^{i,-}_\epsilon(s,\gamma)\right) \tag{46}$$

$$= \int_{\Gamma^\epsilon_i} w(e_T(0,T,\gamma))\,d\left(\rho^{i,+}_{\gamma;\epsilon}(\gamma) - \rho^{i,-}_{\gamma;\epsilon}(\gamma)\right) \tag{47}$$

where (45) follows from Corollary 24; (46) follows from definition of Ψ^ϵ and (43); (47) follows from (44). Since $e_T(0,T,\cdot)$ is an isomorphism and $w \in L^1(\mathbb{R}^{n_i})$ is arbitrary, $\rho^{i,+}_{\gamma;\epsilon} = \rho^{i,-}_{\gamma;\epsilon}$. For convenience, we denote them both by $\rho^i_{\gamma;\epsilon}$.

Step 3 (Construct $\rho^i_{\epsilon,\delta}$). We now want to combine $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$ to generate a measure $\rho^i_\epsilon \in \mathcal{M}_+([0,T] \times [0,T] \times \Gamma^\epsilon_i)$ that describes the trajectories that evolve in the domain as well as their entering and exiting time. Such a measure can be defined by pushing forward $\rho^{i,+}_\epsilon$ through a map that associates entering and exiting times. However, such a map may not be well defined; for example, two trajectories can enter the domain at the same time but leave at different times. To address such issues, we mollify the t-component and define a sequence of measures $\rho^i_{\epsilon,\delta}$ first, and then define ρ^i_ϵ as the limit of this sequence as $\delta \downarrow 0$ which is done in Step 4. Let $\{\theta_\delta\} \subset C^\infty(\mathbb{R})$ be a family of smooth mollifiers with unit mass and zero mean, and define $\rho^{i,+}_{s|\gamma;\epsilon,\delta} := \rho^{i,+}_{s|\gamma;\epsilon} * \theta_\delta$ and $\rho^{i,-}_{\tau|\gamma;\epsilon,\delta} = \rho^{i,-}_{\tau|\gamma;\epsilon} * \theta_\delta$. We further define measures $\rho^{i,+}_{\epsilon,\delta}, \rho^{i,-}_{\epsilon,\delta} \in \mathcal{M}_+(\mathbb{R} \times \Gamma^\epsilon_i)$ as $d\rho^{i,+}_{\epsilon,\delta}(s,\gamma) := d\rho^{i,+}_{s|\gamma;\epsilon,\delta}(s)\,d\rho^{i,+}_{\gamma;\epsilon}(\gamma)$ and $d\rho^{i,-}_{\epsilon,\delta}(\tau,\gamma) := d\rho^{i,-}_{\tau|\gamma;\epsilon,\delta}(\tau)\,d\rho^{i,-}_{\gamma;\epsilon}(\gamma)$. For a.e. $t \in [0,T]$ and any non-negative $w \in L^1(\mathbb{R}^{n_i})$:

$$0 \le \langle \mu^i_{x|t;\epsilon}, w \rangle \tag{48}$$

$$= \int_{[0,t]\times\mathbb{R}^{n_i}} w(\Phi^\epsilon_i(t,s,x))\,d\left(\sigma^i_\epsilon(s,x) - \eta^i_\epsilon(s,x)\right) \tag{49}$$

$$= \int_{\Gamma^\epsilon_i} w(e_t(0,T,\gamma))\left(\rho^{i,+}_{s|\gamma;\epsilon}([0,t]) + \\ - \rho^{i,-}_{\tau|\gamma;\epsilon}([0,t])\right) d\rho^i_{\gamma;\epsilon}(\gamma), \tag{50}$$

where (48) follows from the fact that $\mu^i_{x|t;\epsilon}$ is an unsigned measure; (49) follows by substituting in $\mu^i_{x|t;\epsilon} = \Phi^\epsilon_i(t,\cdot,\cdot)_\#\left(\sigma^i_\epsilon - \eta^i_\epsilon\right)$; (50) follows from (43) and (44).

Equivalently, given any Borel set $E_\Gamma \subset \Gamma^\epsilon_i$,

$$\int_{E_\Gamma} \left(\rho^{i,+}_{s|\gamma;\epsilon}([0,t]) - \rho^{i,-}_{\tau|\gamma;\epsilon}([0,t])\right) d\rho^i_{\gamma;\epsilon}(\gamma) \ge 0. \tag{51}$$

Since the functions $t \mapsto \rho^{i,+}_{s|\gamma;\epsilon}([0,t])$ and $t \mapsto \rho^{i,-}_{\tau|\gamma;\epsilon}([0,t])$ are absolutely continuous, $\rho^{i,+}_{s|\gamma;\epsilon}([0,t]) \ge \rho^{i,-}_{\tau|\gamma;\epsilon}([0,t])$ is satisfied for all $t \in [0,T]$ for all $\gamma \in \operatorname{spt}(\rho^i_{\epsilon,\Gamma})$. Using the definition of convolution and Fubini's theorem, one can prove a similar result for the mollified measures $\rho^{i,+}_{s|\gamma;\epsilon,\delta}$ and $\rho^{i,-}_{s|\gamma;\epsilon,\delta}$, i.e. $d\rho^{i,+}_{s|\gamma;\epsilon,\delta}((-\infty,t]) \ge \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}((-\infty,t])$ for all $\gamma \in \operatorname{spt}(\rho^i_{\gamma;\epsilon})$.

Since $\rho^{i,+}_{s|\gamma;\epsilon,\delta}$ and $\rho^{i,-}_{\tau|\gamma;\epsilon,\delta}$ are smooth non-negative measures, the functions $t \mapsto \rho^{i,+}_{s|\gamma;\epsilon,\delta}((-\infty,t])$ and $t \mapsto \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}((-\infty,t])$ are continuous and non-decreasing. Also, $0 \le \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}((-\infty,t]) \le \rho^{i,+}_{s|\gamma;\epsilon,\delta}((-\infty,t]) \le \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}(\mathbb{R}) = 1$, where the last equality follows because $\rho^{i,-}_{\tau|\gamma;\epsilon,\delta}$ is a probability measure; by the Mean Value Theorem, for any $\gamma \in \operatorname{spt}(\rho^i_{\gamma;\epsilon})$ there exists a function $r_\gamma : \mathbb{R} \to \mathbb{R}$ such that $r_\gamma(t) \ge t$ and $\rho^{i,+}_{s|\gamma;\epsilon,\delta}((-\infty,t]) = \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}((-\infty,r_\gamma(t)])$ for every $\gamma \in \operatorname{spt}(\rho^i_{\gamma;\epsilon})$. Moreover, the function r_γ is strictly increasing and therefore invertible, i.e., there exists a function $r_\gamma^{-1} : \mathbb{R} \to \mathbb{R}$ such that $r_\gamma(r_\gamma^{-1}(t)) = r_\gamma^{-1}(r_\gamma(t)) = t$. Using Step 2, $\rho^{i,+}_{s|\gamma;\epsilon,\delta}((-\infty,t]) = \rho^{i,-}_{\tau|\gamma;\epsilon,\delta}((-\infty,r_\gamma(t)])$ can be written as

$$\int_{\mathbb{R}\times E_\Gamma} \mathbb{1}_{(-\infty,t]}(s)\,d\rho^{i,+}_{\epsilon,\delta}(s,\gamma) = \int_{\mathbb{R}\times E_\Gamma} \mathbb{1}_{(-\infty,r_\gamma(t)]}(\tau)\,d\rho^{i,-}_{\epsilon,\delta}(\tau,\gamma) \tag{52}$$

for any $t \in \mathbb{R}$ and any Borel subset $E_\Gamma \subset \Gamma^\epsilon_i$.

We now abuse notation and define a map $r : \mathbb{R} \times \operatorname{spt}(\rho^i_{\gamma;\epsilon}) \to \mathbb{R}$ by letting $r(s,\gamma) := r_\gamma(s)$ for all $\gamma \in \operatorname{spt}(\rho^i_{\gamma;\epsilon})$, and also projection maps $\pi^1 : (s,\gamma) \in \mathbb{R} \times \Gamma^\epsilon_i \mapsto s \in \mathbb{R}$, $\pi^2 : (s,\gamma) \in \mathbb{R} \times \Gamma^\epsilon_i \mapsto \gamma \in \Gamma^\epsilon_i$. We can then define a measure $\rho^i_{\epsilon,\delta} \in \mathcal{M}_+(\mathbb{R} \times \mathbb{R} \times \Gamma^\epsilon_i)$ as $\rho^i_{\epsilon,\delta} = \left(\pi^1 \times r \times \pi^2\right)_\# \rho^{i,+}_{\epsilon,\delta}$. Notice for any triplet $(s,\tau,\gamma) \in \operatorname{spt}(\rho^i_{\epsilon,\delta})$ we know $s \le \tau$ since $r_\gamma(t) \ge t$.

We now establish the relationship between the marginals of $\rho^i_{\epsilon,\delta}$ and the measures $\rho^{i,+}_{\epsilon,\delta}$ and $\rho^{i,-}_{\epsilon,\delta}$. We use variables $(s,\tau,\gamma)\in\mathbb{R}\times\mathbb{R}\times\Gamma^\epsilon_i$ to denote any point in $\mathrm{spt}(\rho^i_{\epsilon,\delta})$. Since $\pi^1\times\pi^2$ is identity map, the (s,γ)-marginal of $\rho^i_{\epsilon,\delta}$ is equal to $\rho^{i,+}_{\epsilon,\delta}$. To show the (τ,γ)-marginal of $\rho^i_{\epsilon,\delta}$ is equal to $\rho^{i,-}_{\epsilon,\delta}$, it is then sufficient to show $\int_{\mathbb{R}\times\mathbb{R}\times E_\Gamma}\mathbb{1}_{(-\infty,t]}(\tau)d\rho^i_{\epsilon,\delta}(s,\tau,\gamma)=\int_{\mathbb{R}\times E_\Gamma}\mathbb{1}_{(-\infty,t]}(\tau)d\rho^{i,-}_{\epsilon,\delta}(\tau,\gamma)$ holds for all $t\in\mathbb{R}$ and all Borel subsets $E_\Gamma\subset\Gamma^\epsilon_i$. The equation is true because

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{R}\times E_\Gamma}\mathbb{1}_{(-\infty,t]}(\tau)\,d\rho^i_{\epsilon,\delta}(s,\tau,\gamma)\\ &=\int_{\mathbb{R}\times E_\Gamma}\mathbb{1}_{(-\infty,r_\gamma^{-1}(t)]}(s)\,d\rho^{i,+}_{\epsilon,\delta}(s,\gamma) \quad (53)\\ &=\int_{\mathbb{R}\times E_\Gamma}\mathbb{1}_{(-\infty,t]}(\tau)\,d\rho^{i,-}_{\epsilon,\delta}(\tau,\gamma), \quad (54)\end{aligned}$$

where (53) follows by the definition of $\rho^i_{\epsilon,\delta}$ and because r_γ is strictly monotonic and therefore $r_\gamma(s)\in(-\infty,t]$ if and only if $s\in(-\infty,r_\gamma^{-1}(t)]$; (54) follows by substituting in (52) and from the fact that r_γ is invertible;

Step 4 (Properties of the limiting measure of $\{\rho^i_{\epsilon,\delta}\}_\delta$). We now show that the limit of $\rho^i_{\epsilon,\delta}$ exists as $\delta\downarrow 0$ and that for this limiting measure $\mu^i_{x|t;\epsilon}=(e_t)_\#\rho^i_\epsilon$ for a.e. $t\in[0,T]$. We also show that specific marginals of this limiting measure are equal to $\rho^{i,+}_\epsilon$ and $\rho^{i,-}_\epsilon$ and that for any (s,τ,γ) in the support of this limiting measure, $s\le\tau$. To prove this condition, we use the notion of tightness of measures [49, pp. 605-606]:

Integral Condition for Tightness: Let X be a separable metric space. A family $\mathcal{K}\subset\mathcal{M}_+(X)$ is tight if and only if there exists a function $\Theta:X\to[0,+\infty]$ whose sublevel sets are compact in X such that $\sup_{\mu\in\mathcal{K}}\int_X\Theta(x)\,d\mu(x)$ is finite.

Tightness Criterion: Let X, X_1, X_2 be separable metric spaces and let $r^i:X\to X_i$, $i=1,2$ be continuous maps such that the product map $r:r^1\times r^2:X\to X_1\times X_2$ is proper. Let $\mathcal{K}\subset\mathcal{M}_+(X)$ be such that $\mathcal{K}_i:=r^i_\#(\mathcal{K})$ is tight in $\mathcal{M}_+(X_i)$ for $i=1,2$. Then also $\mathcal{K}$ is tight in $\mathcal{M}_+(X)$. Notice the statement also holds for finitely many maps by induction.

Choosing maps r^1, r^2 defined on $\mathbb{R}\times\mathbb{R}\times\Gamma^\epsilon_i$ as $r^1:(s,\tau,\gamma)\mapsto(s,\gamma)\in\mathbb{R}\times\Gamma^\epsilon_i$ and $r^2:(s,\tau,\gamma)\mapsto\tau\in\mathbb{R}$. Notice that $r=r^1\times r^2$ is an isomorphism and therefore proper. The family $\{r^1_\#\rho^i_{\epsilon,\delta}\}_\delta$ is given by $\{\rho^{i,+}_{\epsilon,\delta}\}_\delta$ which are tight by definition, and the family $\{r^2_\#\rho^i_{\epsilon,\delta}\}_\delta$ is given by the first marginal of $\{\rho^{i,-}_{\epsilon,\delta}\}_\delta$ which are also tight. Applying the tightness criterion, the family $\{\rho^i_{\epsilon,\delta}\}_\delta$ is tight, and therefore narrowly sequentially relatively compact according to Prokhorov Compactness Theorem. Let ρ^i_ϵ be any limit of the family $\{\rho^i_{\epsilon,\delta}\}$ as $\delta\downarrow 0$. Since the (s,γ)-marginal of $\rho^i_{\epsilon,\delta}$ is equal to $\rho^{i,+}_{\epsilon,\delta}$ and the (τ,γ)-marginal of $\rho^i_{\epsilon,\delta}$ is equal to $\rho^{i,-}_{\epsilon,\delta}$, we let $\delta\downarrow 0$ and therefore the (s,γ)-marginal of ρ^i_ϵ is equal to $\rho^{i,+}_\epsilon$ and the (τ,γ)-marginal of ρ^i_ϵ is equal to $\rho^{i,-}_\epsilon$, i.e.,

$$\begin{aligned}\int_{[0,T]\times[0,T]\times\Gamma^\epsilon_i}\varphi(s,\gamma)\,d\rho^i_\epsilon(s,\tau,\gamma)&=\int_{[0,T]\times\Gamma^\epsilon_i}\varphi(s,\gamma)\,d\rho^{i,+}_\epsilon(s,\gamma)\\ \int_{[0,T]\times[0,T]\times\Gamma^\epsilon_i}\varphi(\tau,\gamma)\,d\rho^i_\epsilon(s,\tau,\gamma)&=\int_{[0,T]\times\Gamma^\epsilon_i}\varphi(\tau,\gamma)\,d\rho^{i,-}_\epsilon(\tau,\gamma)\end{aligned}\quad (55)$$

for all $\varphi\in L^1(\mathbb{R}\times\Gamma_i)$.

Let $(s,\tau,\gamma)\in\mathrm{spt}(\rho^i_\epsilon)$ be arbitrary. To show $s\le\tau$, let $\varphi'\in C_b(\mathbb{R}^2)$ be such that $\mathrm{spt}(\varphi')\subset\{(s,\tau)\in\mathbb{R}^2\mid s>\tau\}$. Since $\int_{\mathbb{R}\times\mathbb{R}\times\Gamma_i}\varphi'(s,\tau)\,d\rho^i_{\epsilon,\delta}(s,\tau,\gamma)=0$ for all δ, it follows from narrow convergence that $\int_{[0,T]\times[0,T]\times\Gamma_i}\varphi'(s,\tau)\,d\rho^i_\epsilon(s,\tau,\gamma)=0$. Since $\mathbb{1}_{\{(s,\tau)\in[0,T]^2|s>\tau+\Delta\}}$ is a limit point of such functions φ' with respect to $L_1(\rho^i_\epsilon;\mathbb{R})$ for any $\Delta>0$ [37, Corollary 4.2.2], ρ^i_ϵ is supported on (s,τ,γ) such that $s\le\tau$.

For a.e. $t\in[0,T]$ and any $w\in L^1(\mathbb{R}^{n_i})$,

$$\begin{aligned}&\int_{\mathbb{R}^{n_i}}w(x)\,d\mu^i_{x|t;\epsilon}(x)\\ &=\int_{[0,t]\times[0,T]\times\Gamma^\epsilon_i}w(e_t(s,T,\gamma))\,d\rho^i_\epsilon(s,\tau,\gamma)+\\ &\qquad-\int_{[0,t]\times[0,t]\times\Gamma^\epsilon_i}w(e_t(\tau,T,\gamma))\,d\rho^i_\epsilon(s,\tau,\gamma) \quad (56)\\ &=\int_{[0,t]\times[0,t]\times\Gamma^\epsilon_i}(w(e_t(s,T,\gamma))-w(e_t(\tau,T,\gamma)))\,d\rho^i_\epsilon(s,\tau,\gamma)+\\ &\qquad+\int_{[0,t]\times(t,T]\times\Gamma^\epsilon_i}w(e_t(s,T,\gamma))\,d\rho^i_\epsilon(s,\tau,\gamma) \quad (57)\\ &=0+\int_{[0,t]\times[t,T]\times\Gamma^\epsilon_i}w(e_t(s,\tau,\gamma))\,d\rho^i_\epsilon(s,\tau,\gamma)+\\ &\qquad-\int_{[0,t]\times\{t\}\times\Gamma^\epsilon_i}w(e_t(0,T,\gamma))\,d\rho^i_\epsilon(s,\tau,\gamma), \quad (58)\end{aligned}$$

where (56) follows from $\mu^i_{x|t;\epsilon}=\Phi^\epsilon_i(t,\cdot,\cdot)_\#\left(\sigma^i_\epsilon-\eta^i_\epsilon\right)$, (43), and (55); (57) follows by splitting the domain of integration; Since $e_t(t_1,T,\cdot)=e_t(0,T,\cdot)$ and $e_t(t_1,T,\cdot)=e_t(t_1,t_2,\cdot)$ for all $0\le t_1\le t\le t_2\le T$, the first term of (57) is zero because the integrand is zero, (58) follows by adding and subtracting $[0,t]\times\{t\}\times\Gamma^\epsilon_i$ to the domain of integration. Since $\rho^i_\epsilon([0,t]\times\{t\}\times\Gamma^\epsilon_i)$ is non-zero for at most countably many t's (otherwise ρ^i_ϵ would not be bounded), $\mu^i_{x|t;\epsilon}=(e_t)_\#\rho^i_\epsilon$ for a.e. $t\in[0,T]$.

Step 5 (Tightness of the family $\{\rho^i_\epsilon\}_\epsilon$). We show that the limit of ρ^i_ϵ exists as $\epsilon\downarrow 0$. To begin, choose maps r^1, r^2, r^3 defined in $[0,T]\times[0,T]\times\Gamma_i$ as $r^1:(s,\tau,\gamma)\mapsto s\in[0,T]$, $r^2:(s,\tau,\gamma)\mapsto\tau\in[0,T]$, and $r^3:(s,\tau,\gamma)\mapsto\gamma\in\Gamma_i$. Observe that $r=r^1\times r^2\times r^3$ is the identity map and therefore proper. The family $\{r^1_\#\rho^i_\epsilon\}_\epsilon$ and $\{r^2_\#\rho^i_\epsilon\}_\epsilon$ are given by the first marginals of σ^i_ϵ and η^i_ϵ, respectively, which are tight and are independent of ϵ. To establish a similar result for $r^3_\#\rho^i_\epsilon$, let $\Theta:\Gamma_i\to\mathbb{R}\bigcup\{+\infty\}$ as $\Theta(\gamma)=\|\gamma\|$ if $|\dot\gamma(t)|\le M$ a.e., and $\Theta(\gamma)=+\infty$ otherwise. We next show this function Θ satisfies the requirement of the integral condition for tightness. Let $S:=\{\gamma\in\Gamma_i\mid\Theta(\gamma)\le C\}$. Since any sequence $\{\gamma_n\}\subset S$ is uniformly bounded and equicontinuous, S is precompact according to Arzela-Ascoli Theorem. To show S is closed, let $\{\gamma_n\}$ be a convergent sequence in S, and by definition $\dot\gamma_n\to\dot\gamma$ in $L^1([0,T])$. There is a subsequence of $\dot\gamma_n$ that converges pointwise a.e. to $\dot\gamma$ [50, Proposition 2.29], therefore $|\dot\gamma(t)|\le M$ a.e., which implies that the set S is closed. Notice

$$\begin{aligned}&\int_{\Gamma_i}\Theta(\gamma)\,d\left(r^3_\#\rho^i_\epsilon\right)(\gamma)=\int_{[0,T]\times\Gamma^\epsilon_i}\Theta(\gamma)\,d\rho^{i,+}_\epsilon(s,\gamma) \quad (59)\\ &=\int_{[0,T]\times\mathbb{R}^{n_i}}\left(\left|\hat\Phi^\epsilon_i(0,s,x)\right|+\int_0^T\left|\dot{\hat\Phi}^i_\epsilon(t,s,x)\right|dt\right)d\sigma^i_\epsilon(s,x) \quad (60)\\ &\le\int_{[0,T]\times\mathbb{R}^{n_i}}\left(\left|\hat\Phi^\epsilon_i(s,s,x)\right|+\int_0^s\left|\bar F^\epsilon_i(\hat\Phi^i_\epsilon(t,s,x))\right|dt+\right.\\ &\qquad\left.+\int_0^T\left|\bar F^\epsilon_i(\hat\Phi^i_\epsilon(t,s,x))\right|dt\right)d\sigma^i_\epsilon(s,x) \quad (61)\\ &\le\int_{[0,T]\times\mathbb{R}^{n_i}}|x|\,d\sigma^i_\epsilon(s,x)+2MT\|\sigma^i_\epsilon\| \quad (62)\\ &\le\int_{[0,T]\times\mathbb{R}^{n_i}}(|x|^2+1)\,d\sigma^i_\epsilon(s,x)+2MT\|\sigma^i_\epsilon\| \quad (63)\end{aligned}$$

$$\leq \int_{[0,T]\times X_i}\int_{\mathbb{R}^{n_i}}|x+y|^2\theta_\epsilon(y)dyd\sigma_\epsilon^i(s,x)+(1+2MT)\|\sigma^i\| \quad (64)$$

$$=\int_{[0,T]\times X_i}|x|^2\, d\sigma^i(s,x) + \left(\int_{\mathbb{R}^{n_i}}|y|^2\theta_\epsilon(y)\,dy\right)\|\sigma^i\| + \\ + \int_{[0,T]\times X_i}\int_{\mathbb{R}^{n_i}}2x^Ty\theta_\epsilon(y)dyd\sigma^i(s,x)+(1+2MT)\|\sigma^i\|, \quad (65)$$

where (59) follows from (55); (60) follows from (43) and (42); (61) follows from triangle inequality; (62) follows from (41); (63) is true because $|x|^2+1 \geq |x|$ for all $x \in \mathbb{R}^{n_i}$, and σ_ϵ^i is nonnegative; (64) follows from the definition of convolution and $\|\sigma_\epsilon^i\| \leq \|\sigma^i\|$; Since σ^i is bounded and X_i is compact therefore $|x|^2$ is bounded for all $x \in X_i$, the first and last term in (65) are bounded. Because θ_ϵ is assumed to have zero mean and bounded second moment, the second term in (65) is bounded and the third term in (65) is zero. As a result, the left hand side of (59) is bounded. Using the integral condition for tightness, $\{r^3_{\#}\rho_\epsilon^i\}_\epsilon$ is tight, and $\{\rho_\epsilon^i\}_\epsilon$ is tight via the tightness criterion.

Step 6 (Part (b)). We prove the limit of ρ_ϵ^i as ϵ goes to zero satisfies Part (b). Using the Prokhorov Compactness Theorem, the family ρ_ϵ^i is narrowly sequentially relatively compact. Choose a narrowly convergent sequence in $\{\rho_\epsilon^i\}_\epsilon$ and define its limit by $\rho^i \in \mathcal{M}_+([0,T]\times[0,T]\times\Gamma_i)$. For a.e. $t \in [0,T]$ and all $w \in C_b(\mathbb{R}^{n_i})$, it follows from $\mu^i_{x|t;\epsilon} = (e_t)_\#\rho^i_\epsilon$ that

$$\int_{\mathbb{R}^{n_i}} w(x)\, d\mu^i_{x|t;\epsilon}(x) = \int_{[0,T]\times[0,T]\times\Gamma_i} w(e_t(s,\tau,\gamma))\, d\rho^i_\epsilon(s,\tau,\gamma). \quad (66)$$

Since e_t is continuous, $w\circ e_t \in C_b([0,T]\times[0,T]\times\Gamma_i)$. We then pass to the limit $\epsilon \downarrow 0$ on both sides of (66) to obtain $\int_{X_i} w(x)\, d\mu^i_{x|t}(x) = \int_{[0,T]\times[0,T]\times\Gamma_i} w(e_t(s,\tau,\gamma))\, d\rho^i(s,\tau,\gamma)$ for a.e. $t\in[0,T]$. Since $C_b(\mathbb{R}^{n_i})$ is dense in $L^1(\mathbb{R}^{n_i})$ [37, Corollary 4.2.2], $\mu^i_{x|t} = (e_t)_\#\rho^i$ for a.e. $t\in[0,T]$.

Step 7 (Part (a) with continuous vector field). Using a similar argument in Step 4, we may show $s \leq \tau$ for any triplet $(s,\tau,\gamma) \in \mathrm{spt}(\rho^i)$. Moreover, it follows from $\mu^i_{x|t} = (e_t)_\#\rho^i$ that $\gamma(t) \in \mathrm{spt}((e_t)_\#\rho^i) \subset X_i$ for a.e. $t \in [s,\tau]$. Since γ is absolutely continuous and X_i is compact, $\gamma(t)$ stays in X_i for all $t\in[s,\tau]$. To prove the rest of (a), we only need to show

$$\int_{[0,t]\times[t,T]\times\Gamma_i}\left|\gamma(t)-\gamma(s)-\int_s^t \bar{F}_i(\tau',\gamma(\tau'))d\tau'\right| d\rho^i(s,\tau,\gamma) = 0 \quad (67)$$

for all $t\in[0,T]$. Let $v \in C_b([0,T]\times X_i;\mathbb{R}^{n_i})$, then

$$\int_{[0,t]\times[t,T]\times\Gamma_i}\left|\gamma(t)-\gamma(s)-\int_s^t v(\tau',\gamma(\tau'))d\tau'\right|d\rho^i_\epsilon(s,\tau,\gamma)$$

$$\leq \int_{[0,t]\times[t,T]\times\Gamma_i}\int_s^t\left|\bar{F}^\epsilon_i(\tau',\gamma(\tau')) - v(\tau',\gamma(\tau'))\right|d\tau' d\rho^i_\epsilon(s,\tau,\gamma) \quad (68)$$

$$\leq \int_0^t\int_{[0,\tau']\times[\tau',T]\times\Gamma_i}\left|\bar{F}^\epsilon_i(\tau',\gamma(\tau')) - v(\tau',\gamma(\tau'))\right|d\rho^i_\epsilon(s,\tau,\gamma)d\tau' \quad (69)$$

$$= \int_0^t\int_{\mathbb{R}^{n_i}}\left|\bar{F}^\epsilon_i(\tau,x) - v(\tau,x)\right| d\mu^i_{x|\tau;\epsilon}(x)\, d\tau \quad (70)$$

$$\leq \int_{[0,T]\times\mathbb{R}^{n_i}}\left|\bar{F}_i(\tau,x)-v(\tau,x)\right| d\mu^i_{\tau,x}(\tau,x) + \\ + \Big(\sup_{\substack{\tau\in[0,T]\\ x\in\mathbb{R}^{n_i}}}|v^\epsilon(\tau,x)-v(\tau,x)|\Big)\|\mu^i_{\tau,x}\| \quad (71)$$

for any $t \in [0,T]$ where (68) follows by substituting in $\int_s^t \bar{F}^\epsilon_i(\tau',\gamma(\tau'))d\tau' = \gamma(t)-\gamma(s)$ and applying the triangle inequality for integrals; (69) follows by first applying Fubini's theorem to change the order of integration, and then relaxing the domain of integration (since ρ^i_ϵ is nonnegative); (70) follows from $\mu^i_{x|t;\epsilon} = (e_t)_\#\rho^i_\epsilon$ and a change of variables $\tau' = \tau$; in (71) we add and subtract $v^\epsilon(\tau,\cdot) := \frac{(v(\tau,\cdot)\,\mu^i_{x|\tau})*\theta_\epsilon}{\mu^i_{x|\tau;\epsilon}}$, and then apply the triangle inequality and [49, Lemma 3.9]. Since the family $\{\rho^i_\epsilon\}_\epsilon$ is tight and the integrand is a bounded continuous function, and v is uniformly continuous v^ϵ converges to v uniformly as $\epsilon\downarrow 0$, and the second term of (71) converges to 0, therefore for a.e. $t\in[0,T]$,

$$\int_{[0,t]\times[t,T]\times\Gamma_i}\left|\gamma(t)-\gamma(s)-\int_s^t v(\tau',\gamma(\tau'))d\tau'\right| d\rho^i(s,\tau,\gamma) \\ \leq \int_{[0,T]\times X_i}|\bar{F}_i(\tau,x)-v(\tau,x)|\, d\mu^i_{\tau,x}(\tau,x). \quad (72)$$

If $\bar{F}_i$ is uniformly continuous, let $v := \bar{F}_i$, and (67) follows.

Step 8 (Error bound of vector field approximation). When there is no regularity in $\bar{F}_i$ other than boundedness, we choose a sequence of continuous functions converging to $\bar{F}_i$ in $L^1(\mu^i_{t,x};\mathbb{R}^{n_i})$, and prove an error bound of the approximation: Let $\{v_k\}_{k\in\mathbb{N}} \subset C([0,T]\times X_i;\mathbb{R}^{n_i})$ be a sequence of continuous functions converging to $\bar{F}_i$ in $L^1(\mu^i_{t,x};\mathbb{R}^{n_i})$, whose existence is guaranteed by [37, Corollary 4.2.2]. Given any $t\in[0,T]$, we compute the following error between v_k and $\bar{F}_i$:

$$\int_{[0,t]\times[t,T]\times\Gamma_i}\int_s^t\left|v_k(\tau',\gamma(\tau'))-\bar{F}_i(\tau',\gamma(\tau'))\right| d\tau'\, d\rho^i(s,\tau,\gamma)$$

$$\leq \int_0^t\int_{[0,\tau']\times[\tau',T]\times\Gamma_i}\left|v_k(\tau',\gamma(\tau'))-\bar{F}_i(\tau',\gamma(\tau'))\right| d\rho^i(s,\tau,\gamma)\, d\tau' \quad (73)$$

$$= \int_{[0,T]\times X_i}\left|v_k(\tau,x)-\bar{F}_i(\tau,x)\right| d\mu^i_{\tau,x}(\tau,x), \quad (74)$$

where (73) follows by first applying Fubini's Theorem to change the order of integrations, and then relaxing the domain of integration (since ρ^i is nonnegative); (74) follows by substituting in $\mu^i_{x|t} = (e_t)_\#\rho^i$. Observe that as $k\to\infty$ this error goes to zero.

Step 9 (Condition (a) with bounded vector field). We may now combine Step 7 and Step 8 together and prove Part (a) in a more general setting. Using the results in Step 7 and Step 8, we obtain for any $t\in[0,T]$,

$$\int_{[0,t]\times[t,T]\times\Gamma_i}\left|\gamma(t)-\gamma(s)-\int_s^t\bar{F}_i(\tau',\gamma(\tau'))d\tau'\right| d\rho^i(s,\tau,\gamma) \\ \leq 2\int_{[0,T]\times X_i}\left|\bar{F}_i(\tau,x)-v_k(\tau,x)\right| d\mu^i_{\tau,x}(\tau,x), \quad (75)$$

where (75) follows by adding and subtracting the term $\int_s^t v_k(\tau',\gamma(\tau'))d\tau'$, applying the triangle inequality, and using the results in Step 7 and Step 8. When we let $k\to\infty$, (75) goes to zero, therefore Part (a) holds. ■

Appendix C

In this section we prove Theorem 12.

Proof: This proof consists of several steps: in Step 1 we show that trajectories defined in support of ρ^i and ρ^j satisfy the reset map for all $(i,j)\in\mathcal{E}$; in Step 2 we show trajectories in each mode can be connected to obtain hybrid trajectories that are defined on $[0,T]$; in Step 3 we prove that those hybrid trajectories are admissible by showing they all start from spt(μ_0^i) and end in spt(μ_T^i) thus proving (a); in Step 4 we define a measure ρ and prove that it satisfies (b) and (c); in Step 5 we prove (d) using (b) and (c).

Step 1 (Reset maps are satisfied). According to Corollary 10, it suffices to show $\sigma^j = \delta_0 \otimes \mu_0^j + \sum_{(i,j)\in\mathcal{E}} \tilde{R}_{(i,j)\#}\eta^i,\ \forall j \in \mathcal{I}$. This can be proved by using (19) and Assumption 2 to obtain $\tilde{R}_{(i,j)\#}\eta^i = \tilde{R}_{(i,j)\#}\mu^{S_{(i,j)}}$. As a result, all trajectories in the support of ρ^i are reinitialized to another trajectory in the support of ρ^j after it reaches the guard $S_{(i,j)}$; On the other hand, a trajectory can only start in mode i either from the given initial condition x_0 at time 0, or by transitioning from another mode j if $(j,i) \in \mathcal{E}$. We can therefore connect trajectories in each mode together to obtain hybrid trajectories.

Step 2 (Hybrid trajectories are defined on $[0,T]$). This step shows that all hybrid trajectories are defined on $[0,T]$. To prove this, we first show that there is a $\Delta t > 0$ such that $\tau - s \geq \Delta t$ for any $i \in \mathcal{I}$ and $(s,\tau,\gamma) \in$ spt(ρ^i), $\tau \neq T$. Let $(s,\tau,\gamma) \in$ spt(ρ^i) for some $i \in \mathcal{I}$, and let $0 \leq s \leq \tau < T$. According to Corollary 10, $\gamma(s) \in \{x_0\} \bigcup_{(i',i)\in\mathcal{E}} R_{(i',i)}(S_{(i',i)})$ and $\gamma(\tau) \in \bigcup_{(i,i')\in\mathcal{E}} S_{(i,i')}$. According to Definition 1 and Assumptions 2 and 5, $\gamma(s)$ and $\gamma(\tau)$ belong to disjoint compact sets (since the image of a compact set under a continuous map is compact) and therefore there exists a $d_i > 0$ such that $|\gamma(\tau) - \gamma(s)| \geq d_i$. Let $M_i > 0$ be a bound for $\bar{F}_i(t,x)$ over $[0,T] \times X_i$, and define $\Delta t := \min_{i\in\mathcal{I}}(d_i/M_i)$. Then it follows from the Fundamental Theorem of Calculus that $(\tau - s) \geq \Delta t$.

We can apply proof by contradiction to show all hybrid trajectories are defined on $[0,T]$. Let a hybrid trajectory be defined on a strict subinterval of $[0,T]$, then according to Corollary 10 its endpoints must belong to either S_e or $R_{e\#}\mu^{S_e}$ for some $e \in \mathcal{E}$. It then follows from Step 1 that its domain can always be extended by at least Δt due to transitioning from or to another point Notice it follows from the above discussion that for any $i \in \mathcal{I}$ and $(0,\tau,\gamma) \in$ spt(ρ^i), $\tau \geq \Delta t$. As a result, spt(μ^{S_e}) $\subset [\Delta t, T] \times S_e$ for all $e \in \mathcal{E}$. Then, as a result of Step 1, for all $e \in \mathcal{E}$, spt($\tilde{R}_{e\#}\mu^{S_e}$) $\subset [\Delta t, T] \times R_e(S_e)$.

Step 3 (Part (a)). For any triplet $(0,\tau,\gamma) \in$ spt(ρ^i), $(0,\gamma(0)) \in$ spt(σ^i) according to Corollary 10. It then follows from spt($\tilde{R}_{e\#}\mu^{S_e}$) $\subset [\Delta t,T] \times R_e(S_e)$ that $\gamma(0) \in$ spt(μ_0^i). Now suppose $(s,T,\gamma) \in$ spt(ρ^i) but $\gamma(T) \notin$ spt(μ_T^i). According to Corollary 10 and Step 1 γ is reinitialized to another trajectory γ' in some mode $i' \in \mathcal{I}$. As a result of Corollary 10, $(T,\gamma'(T)) \in$ spt($\sigma^{i'}$) $\bigcup$ spt($\eta^{i'}$), therefore as a result of Assumptions 2 and 4, $\gamma'(T) \in$ spt($\mu_T^{i'}$).

Step 4 (Part (b) and (c)). As a result of Step 3, there exists a measure $\rho \in \mathcal{M}_+(\mathcal{X})$ such that $(e_t^i)_\#\rho = (e_t)_\#\rho^i$ for all $t \in [0,T]$. Therefore, Part (b) follows from Theorem 9. To prove Part (c), notice $\rho(\mathcal{X}) = \sum_{i\in\mathcal{I}} \left((e_0^i)_\#\rho\right)(X_i) = \sum_{i\in\mathcal{I}} \rho^i(\{0\} \times [0,T] \times \Gamma_i)$. According to Corollary 10, $\int_{[0,T]\times[0,T]\times\Gamma_i} \mathbb{1}_{\{0\}}(s)\, d\rho^i(s,\tau,\gamma) = \int_{[0,T]\times X_i} \mathbb{1}_{\{0\}}(s)\, d\sigma^i(s,x) = \sigma^i(\{0\} \times X_i)$. If $\sum_{i\in\mathcal{I}} \mu_0^i(X_i) = 1$, then $\rho(\mathcal{X}) = \sum_{i\in\mathcal{I}} \sigma^i(\{0\} \times X_i) = \sum_{i\in\mathcal{I}} \mu_0^i(X_i) = 1$.

Step 5 (Part (d)). Let $A \times B$ be in the Borel σ-algebra of $[0,T] \times X_i$, then

$$\mu_{t,x}^i(A \times B) = \int_{\mathcal{X}_T} \int_0^T \mathbb{1}_{A\times B}(t,\gamma_i(t))\, dt\, d\rho(\gamma), \tag{76}$$

which follows by substituting in $(e_t^i)_\#\rho = \mu_{x|t}^i$ and applying Fubini's Theorem. Since ρ is a probability measure, $\sum_{i\in\mathcal{I}} \mu_{t,x}^i([0,T] \times X_i) = T$.

For all B in the Borel σ-algebra of X_i,

$$\mu_0^i(B) = \int_{[0,T]\times X_i} \mathbb{1}_{\{0\}\times B}(s,x)\, d\sigma^i(s,x) \tag{77}$$

$$= \int_{\mathcal{X}} \mathbb{1}_B(\gamma_i(0))\, d\rho(\gamma), \tag{78}$$

where (77) follows from definition of δ_0, from (19) and spt($\tilde{R}_{e\#}\mu^{S_e}$) $\subset [\Delta t,T] \times R_e(S_e)$; (78) follows from Corollary 10 and because $(e_t^i)_\#\rho = (e_t)_\#\rho^i$. Similarly, for all B in the Borel σ-algebra of X_{T_i}, $\mu_T^i(B) = \int_{\mathcal{X}_T} \mathbb{1}_B(\gamma_i(T))\, d\rho(\gamma)$. Since ρ is a probability measure, $\sum_{i\in\mathcal{I}} \mu_T^i(X_{T_i}) = 1$.

Finally, for all $(i,i') \in \mathcal{S}$ and $A \times B$ in the Borel σ-algebra of $[0,T] \times S_{(i,i')}$,

$$\mu^{S_{(i,i')}}(A \times B) = \int_{[0,T]\times X_i} \mathbb{1}_{A\times B}(\tau,x)\, d\eta^i(\tau,x) \tag{79}$$

$$= \int_{[0,T]\times[0,T]\times\Gamma_i} \mathbb{1}_{A\times B}(\tau,\gamma(\tau))\, d\rho^i(s,\tau,\gamma) \tag{80}$$

$$= \int_{[0,T]\times[0,T]\times\Gamma_i} \#\{(t,e_t(s,\tau,\gamma)) \in A \times B\}\, d\rho^i(s,\tau,\gamma) \tag{81}$$

$$= \int_{\mathcal{X}} \#\{t \in A \mid \lim_{\tau\to t^-} \gamma_i(\tau) \in B\}\, d\rho(\gamma), \tag{82}$$

where (79) follows from (19), Assumption 4, and the fact that $B \subset S_{(i,i')}$; (80) follows from Corollary 10; (81) follows from Assumption 2; (82) follows because $(e_t^i)_\#\rho = (e_t)_\#\rho^i$ and because all $\gamma_i \in \Gamma_i$ are absolutely continuous. From Step 2, each $\gamma \in$ spt(ρ) undergoes at most $\frac{T}{\Delta t}$ transitions, where Δt is defined as in Step 2. Therefore $\sum_{(i,i')\in\mathcal{E}} \#\{t \in [0,T] \mid \lim_{\tau\to t^-} \gamma_i(\tau) \in S_{(i,i')}\} \leq \frac{T}{\Delta t}$ for all $\gamma \in$ spt(ρ). Since ρ is a probability measure, $\sum_{e\in\mathcal{E}} \mu^{S_e}([0,T] \times S_e) \leq \frac{T}{\Delta t}$. ■

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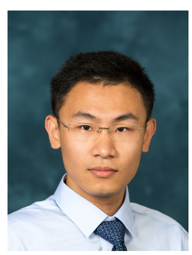

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