CIRCLE ACTIONS ON SYMPLECTIC FOUR-MANIFOLDS

TARA S. HOLM AND LIAT KESSLER

with an Appendix by TAIR PNINI

ABSTRACT. We complete the classification of Hamiltonian torus and circle actions on symplectic four-dimensional manifolds. Following work of Delzant and Karshon, Hamiltonian circle and 2-torus actions on any fixed simply connected symplectic four-manifold were characterized by Karshon, Kessler and Pinsonnault. What remains is to study the case of Hamiltonian actions on blowups of S^2-bundles over a Riemann surface of positive genus. These do not admit 2-torus actions. In this paper, we characterize Hamiltonian circle actions on them. We then derive combinatorial results on the existence and counting of these actions. As a by-product, we provide an algorithm that determines the g-reduced form of a blowup form. Our work is a combination of "soft" equivariant and combinatorial techniques, using the momentum map and related data, with "hard" holomorphic techniques, including Gromov-Witten invariants.

1. INTRODUCTION

In this paper we complete the classification of Hamiltonian torus and circle actions on symplectic four-dimensional manifolds. It is in this dimension that there are abundant examples (indeed, Gompf has shown that any finitely presented group may be the fundamental group of a four-dimensional symplectic manifold [7]), and yet powerful holomorphic techniques (J-holomorphic curves and Gromov-Witten invariants) are developed enough to make classification problems tractable.

A symplectic action of a $T^n = (S^1)^n$ on a symplectic manifold (M, ω) is **Hamiltonian** if it admits a **moment map**: an equivariant smooth map $\Phi\colon M \to \mathfrak{t}^*$ such that the components $\Phi^\xi = \langle \Phi, \xi \rangle$ satisfy Hamilton's equation $d\Phi^\xi = -\iota(\xi_M)\omega$, for all $\xi \in \mathfrak{t}$. Here ξ_M is the vector field that generates the action on M of the one-parameter subgroup $\{\exp(s\xi) \,|\, s \in \mathbb{R}\}$ of T. The actions we consider are effective and the manifolds are compact and connected. By the Convexity Theorem [2, 9], the image of the momentum map $\Delta = \Phi(M) \subset \mathfrak{t}^* \cong \mathbb{R}^n$ is a convex polytope. We say that two Hamiltonian T actions are **equivalent** if they differ by an equivariant symplectomorphism and a reparametrization of T. For an effective Hamiltonian action $T^n \circlearrowright M$, we are forced to have $\dim(T) \leq \frac{1}{2}\dim(M)$. Thus, if M is four-dimensional, the only tori that can act effectively are S^1 and T^2.

Delzant [6] has classified symplectic manifolds equipped with a Hamiltonian action of a torus of half the dimension in terms of their momentum polytopes, up to equivariant symplectomorphism. Building on work of Audin [3] and Ahara and Hattori [1], Karshon [13] has classified symplectic four-manifolds with Hamiltonian S^1 actions in terms of decorated graphs. In particular, the only four-manifolds that admit Hamiltonian S^1 actions are blowups of S^2-bundles over Riemann surfaces and $\mathbb{CP}^2$. [13, § 6].

Delzant's Theorem and Karshon's Theorem leave open questions about inequivalent actions on the same manifold. Explicitly, given a compact connected four-dimensional symplectic manifold (M, ω), what Hamiltonian actions does it admit? The difficulty lies in determining exactly which Hamiltonian spaces $S^1 \circlearrowright (M, \omega)$ have symplectomorphic underlying manifolds (M, ω). The question of characterizing Hamiltonian 2-torus and circle actions on simply connected symplectic four-manifolds was answered by Karshon, Kessler and Pinsonnault in [17, 18].

To complete the characterization of Hamiltonian actions on symplectic four manifolds, it remains to study the case of Hamiltonian actions on blowups of S^2-bundles over a Riemann surface of positive genus. These do not admit Hamiltonian 2-torus actions. In what follows, we characterize Hamiltonian circle actions on them, up to (possibly non-equivariant) symplectomorphism (Theorems 2.13 and 2.14). We prove that a reduced symplectic blowup of an irrational ruled symplectic four-manifold is compatible with all the Hamiltonian circle actions: there are no "exotic actions".

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We give an algorithm in pseudo-code in Appendix B, written by Tair Pnini, to count the (inequivalent) Hamiltonian circle actions on blowups of irrational ruled symplectic manifolds. We use the algorithm to derive results on the existence and number of such actions in Section 3. Along the way, we also provide an algorithm that determines the reduced form corresponding to a symplectic form on a blowup of a ruled symplectic manifold (Algorithm 6.5). The proofs combine "soft" combinatorial techniques and "hard" holomorphic techniques.

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2. Statement of Main Results

We now turn to the full statement of our main results. We begin by setting our notation.

2.1. **Notation.** Let (Σ, j) be a compact connected Riemann surface endowed with a complex structure j. Up to diffeomorphism, there are two S^2-bundles over Σ. We fix a smooth structure on the trivial bundle $\Sigma \times S^2$ and on the non-trivial bundle M_Σ. We equip $\Sigma \times S^2$ and M_Σ with a complex structure such that each fiber is a holomorphic sphere. We fix basepoints $* \in S^2$ and $* \in \Sigma$. For the trivial S^2-bundle $\Sigma \times S^2$ over Σ, we denote $F := [* \times S^2]$, $B := [\Sigma \times *]$, classes in the homology group $H_2(\Sigma \times S^2; \mathbb{Z})$. When we consider the non-trivial S^2-bundle $M_\Sigma \xrightarrow{\pi} \Sigma$, denote the homology class of the fiber by $F = [\pi^{-1}(*)] \in H_2(M_\Sigma; \mathbb{Z})$. For each ℓ, the trivial bundle admits a section $\sigma_{2\ell} : \Sigma \to \Sigma \times S^2$ whose image $\sigma_{2\ell}(\Sigma)$ has even self intersection number 2ℓ. Similarly, for each ℓ, the non-trivial bundle admits a section $\sigma_{2\ell+1} : \Sigma \to M_\Sigma$ whose image $\sigma_{2\ell+1}(\Sigma)$ has odd self intersection number $2\ell + 1$. We denote $B_n := [\sigma_n(\Sigma)] \in H_2$ over $\mathbb{Z}$. For every $n \in \mathbb{Z}$, we have the classes $B_n = B_{-n} + nF$.

For a non-negative integer k, denote by $(\Sigma \times S^2)_k$ the manifold obtained from the trivial S^2-bundle over Σ by k complex blowups at k distinct points, and by $(M_\Sigma)_k$ the manifold obtained from the non-trivial S^2-bundle over Σ by k complex blowups at k distinct points. Let $E_1, \ldots, E_k$ denote the homology classes of the exceptional divisors. We fix the labeling of the exceptional divisors.

Let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. We say that a vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ in $\mathbb{R}^{2+k}$ **encodes** a degree 2 cohomology class $\Omega \in H^2(M; \mathbb{R})$ if

- $\frac{1}{2\pi} \langle \Omega, F \rangle = \lambda_F$;
- $\frac{1}{2\pi} \langle \Omega, E_j \rangle = \delta_j$ for $j = 1, \ldots, k$; and
- Either
  - $\frac{1}{2\pi} \langle \Omega, B \rangle = \lambda_B$ when $M = (\Sigma \times S^2)_k$; or
  - $\frac{1}{2\pi} \left(\langle \Omega, B_{-1} \rangle + \frac{1}{2} \langle \Omega, F \rangle \right) = \lambda_B$ when $M = (M_\Sigma)_k$.

For $k \geq 2$, let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. we say that a cohomology class $\Omega \in H^2(M; \mathbb{R})$ encoded by a vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ is **in g-reduced form** or is **g-reduced** if

$$\delta_1 \geq \ldots \geq \delta_k \,, \quad \delta_1 + \delta_2 \leq \lambda_F \,, \tag{2.2}$$

and, if $g(\Sigma) = 0$ and $M = (\Sigma \times S^2)_k$,

$$\lambda_F \leq \lambda_B \,, \tag{2.3}$$

and, if $g(\Sigma) = 0$ and $M = (M_\Sigma)_k$,

$$\left(\frac{1}{2} \lambda_F + \delta_1 \right) \leq \lambda_B. \tag{2.4}$$

2.5. **Remark.** We can realize $(M_{S^2})_k$ as a complex blowup of the complex projective plane $\mathbb{CP}^2$, the class B_{-1} with the class E of the exceptional divisor, and the class B_1 with the class L of a line $\mathbb{CP}^1$ in

$\mathbb{CP}^2$. In this notation, $F = L - E$. A cohomology class on $(M_{S^2})_k$ encoded by $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ may be identified with a cohomology class on a $(k+1)$-fold blowup of $\mathbb{CP}^2$ encoded by $(\lambda; \delta, \delta_1, \dots, \delta_k)$, where

$$\lambda_F = \lambda - \delta \text{ and } \lambda_B = \frac{1}{2}\Big(\lambda + \delta\Big). \tag{2.6}$$

In this case, our definition of 0-**reduced** coincides with the notion of **reduced** given in [16, 17, 18]. That is, a class $\Omega \in H^2((M_{S^2})_k; \mathbb{R})$ is in 0-reduced form if and only if the vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ encoding it determines, by (2.6), a vector $(\lambda; \delta, \delta_1, \dots, \delta_k)$ which satisfies $\delta \geq \delta_1 \geq \ldots \geq \delta_k$ and $\delta + \delta_1 + \delta_2 \leq \lambda$. In the positive genus case, our notion of g-reduced does not coincide with the notion of reduced given in [22, 24]. An explanation of the difference is in Remark 2.16.

2.7. **Definition.** We assume $g(\Sigma) > 0$. A **blowup form** on $(\Sigma \times S^2)_k$ or $(M_\Sigma)_k$ is a symplectic form for which there exist disjoint embedded symplectic spheres (oriented by the symplectic form) in the homology classes $F, E_1, \dots, E_k$.

2.8. **Remark.** Blowing down symplectically along k disjoint ω-symplectic embedded spheres in $E_1, \dots, E_k$ in $M = (\Sigma \times S^2)_k$, or $M = (M_\Sigma)_k$, with a blowup form ω yields a symplectic manifold (M', ω') with an ω'-symplectic embedded sphere C in a simple homology class F with self intersection zero. Then, by [25, Proposition 4.1], there is a smooth S^2-fibration of the obtained manifold M' over a compact 2-manifold Σ' which is compatible with ω' (i.e., is nondegenerate on each fiber), and has a fiber equal to C. By homology considerations, $\Sigma' = \Sigma$, and the obtained manifold is a trivial S^2-bundle over Σ if $M = (\Sigma \times S^2)_k$ and a nontrivial S^2-bundle over Σ if $M = (M_\Sigma)_k$. Moreover, by [20, §6], the two quantities $[\omega']^2$ and $\langle [\omega'], F\rangle$ uniquely determine the symplectic form on M', up to isotopy; see [30, Example 3.6].

2.9. **Remark.** In Definition 2.7 we have restricted to case when Σ has positive genus. When $\Sigma = S^2$, Karshon and Kessler [16] have defined a **blowup form** on $(M_{S^2})_k$: it is a symplectic form for which there exist pairwise disjoint embedded symplectic spheres in L, E, $E_1, \dots, E_k$.

2.10. **Definition.** We say that two symplectic forms ω_1 and ω_2 on M are **equivalent** if there exists a diffeomorphism f of M that acts trivially on the homology $H_2(M)$ and such that $f^*\omega_2$ can be connected to ω_1 through a continuous path of symplectic forms.

The following lemma follows from the work of Gromov [8], McDuff [25], Lalonde-McDuff [20, Theorem 2.4] and McDuff-Salamon [26, Proposition 7.21]. We give the proof in the positive genus case in [11]; the proof in the genus 0 case is in [18].

2.11. **Lemma.** *The set of blowup forms on M is an equivalence class of symplectic forms.*

The set $\mathcal{E}(M)$ of exceptional classes in $H_2(M)$ is the same for all the blowup forms, as discussed in Lemma 4.5. An **exceptional sphere** in a symplectic four-manifold (M, ω) is an embedded ω-symplectic sphere of self intersection -1. A homology class $E \in H_2(M)$ is **exceptional** if it is represented by an exceptional sphere. For blowups of S^2-bundles over a Riemann surface Σ with a positive genus, there is an explicit identification of the classes in $\mathcal{E}$, by a holomorphic argument of Biran [4, Corollary 5.C].

$$\mathcal{E}(M) = \{E_1, \dots, E_k, F - E_1, \dots, F - E_k\}. \tag{2.12}$$

Pinsonnault [28, Proposition 3.13] showed that every Hamiltonian circle action on a k-fold blowup of a ruled symplectic four-manifold can be obtained from an action on some ruled symplectic manifold M_0 by a sequence of k equivariant blowups of some sizes. For blowups of irrational ruled symplectic manifolds, we use the identification of $\mathcal{E}(M)$ in order to derive, from the same holomorphic tools used in the non-constructive result of Pinsonnault, a constructive characterization, in which we identify the ruled manifold M_0 and the sizes of the equivariant blowups.

2.13. **Theorem.** *Let $k \geq 1$ be an integer, and let $M = (\Sigma \times S^2)_k$ (respectively $M = (M_\Sigma)_k$). Assume that the genus of Σ is positive. Let ω be a blowup form on M whose cohomology class is encoded by a vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$; if $k \geq 2$, assume that $[\omega]$ is in g-reduced form. Then every Hamiltonian circle action is equivalent to*

- ⓐ *one that is obtained from a Hamiltonian circle action on $\Sigma \times S^2$ (resp. M_Σ) with the symplectic form whose cohomology class is encoded by (λ_F, λ_B) by a sequence of S^1-equivariant symplectic blowups of sizes δ_1,δ_2, δ_3, …, δ_k, in this order,*

and to

- (b) *one that is obtained from a Hamiltonian circle action on M_Σ (resp. $\Sigma \times S^2$) with the symplectic form whose cohomology class is encoded by $(\lambda_F, \lambda_B + \frac{1}{2}\lambda_F - \delta_1)$ by a sequence of S^1-equivariant symplectic blowups of sizes $\lambda_F - \delta_1$, δ_2, ..., δ_k, in this order.*

Moreover, Theorem 2.13 holds if we replace "Hamiltonian circle action" with "action of a compact Lie group G that preserves ω and induces the identity morphism on $H_2(M)$." To complete the characterization we provide a description of the Hamiltonian circle actions on $\Sigma \times S^2$ and on M_Σ in Proposition 3.2.

Proof of Theorem 2.13. Consider an action on (M, ω) of a compact Lie group G that preserves ω and induces the identity morphism on $H_2(M)$. Let J_G be a G-invariant ω-tamed almost complex structure on M. The structure J_G is the almost complex structure associated by the polar decomposition to the invariant Riemann metric g obtained by averaging some Riemann metric g' along the action of G with respect to the Haar measure: $g(u, v) := \int_G g'(\sigma_{a*}u, \sigma_{a*}v)da$.

First assume $k \geq 2$. By the identification (2.12) of $\mathcal{E}(M)$ and the g-reduced assumption, the class E_k is of minimal symplectic area in $\mathcal{E}(M)$. Hence, by Lemma [28, Lemma 1.2], E_k is represented by an embedded J_G-holomorphic sphere C. To see that C is G-invariant, apply the positivity of intersections [27, Proposition 2.4.4], the fact that $E \cdot E = -1$, and the assumption that the action is trivial on $H_2(M)$ to get that for every $a \in G$, the J_G-holomorphic sphere aC equals the J_G-holomorphic sphere C.

Equivariantly blowing down along this sphere yields a G-action on $(\Sigma \times S^2)_{k-1}$ (or $(M_\Sigma)_{k-1}$) with the blow up form whose cohomology class is encoded by $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_{k-1})$ and is in g-reduced form, as assured by Lemma 4.12 (on blow downs).

We now let $k \geq 1$. By $k-1$ repeated applications of this blow down, we reduce the theorem to the following claim: There exist embedded G-invariant symplectic spheres C in E_1 and D in $F - E_1$ in $(\Sigma \times S^2)_1$ (resp. $(M_\Sigma)_1$), with the blowup form encoded by $(\lambda_F, \lambda_B; \delta_1)$, and blowing down equivariantly along C yields a G-action on $\Sigma \times S^2$ (resp. M_Σ) with the symplectic form encoded by (λ_F, λ_B), and blowing down along D yields a G-action on M_Σ (resp. $\Sigma \times S^2$) with the symplectic form encoded by $(\lambda_F, \lambda_B + \frac{1}{2}\lambda_F - \delta_1)$. This claim follows from the existence of embedded J_G-holomorphic spheres in E_1 and in $F - E_1$, stated in Lemma 4.6, which is the positive genus version of the genus zero [29, Lemma 2.2], and from Lemma 4.12 and Corollary 5.3 (on blow downs). □

Our second main result provides a unique representation for a blowup form on the k-fold blowup of a ruled symplectic four-manifold $\Sigma \times S^2$ or M_Σ.

2.14. **Theorem.** *Let $k \geq 2$ be an integer, and let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. Assume that the genus $g(\Sigma)$ is positive. Given a blowup form ω on M, there exists a unique blowup form ω' whose cohomology class is in g-reduced form such that $(M, \omega) \cong (M, \omega')$.*

In fact, we prove even more than is stated in Theorem 2.14. We include an algorithm, Algorithm 6.5, which puts a blowup form into its unique g-reduced form, by a composition of permutations of the δ_is and a map we denote the **Cremona transformation** (see Definition 5.7). The latter map is a positive genus version of the (usual) Cremona transformation on blowups of M_{S^2}; it is motivated by the combinatorial description of the transformation as interwining two ways to obtain the same Hamiltonian S^1-manifold, as shown in Figure 2.15.

Theorem 2.14 also provides a way to determine when two symplectic forms on the same manifold, on $(\Sigma \times S^2)_k$ or on $(M_\Sigma)_k$, are diffeomorphic: they must have the same g-reduced form. In the case when $g(\Sigma) = 0$ and $M = (M_\Sigma)_k$, the existence and uniqueness of a g-reduced form is proved in [16, Theorem 1.4].

2.16. **Remark.** Li-Li [22] and Li-Liu [24], in the context of relating the symplectic genus of an integral cohomology class to its minimal genus, define reduced forms for blowups of $S^2 \times \Sigma$ (with $g(\Sigma) > 0$) by different conditions. First, they work with homology classes with $\mathbb{Z}$ coefficients whereas we work with cohomology classes over $\mathbb{R}$. More significantly, their conditions translate (by Poincaré duality) to the conditions on an integral 2-cohomology class:

$$\lambda_B \geq \delta_1 \geq \ldots \geq \delta_k \geq 0.$$

By contrast, we require $\lambda_F \geq \delta_1 + \delta_2$, and neither require $\lambda_B \geq \delta_1$ nor positivity. Note that the vector $(3, 3; 2, 2)$ from Figure 2.15 is reduced in the sense of [22, 24]. The Cremona transformation turns it into

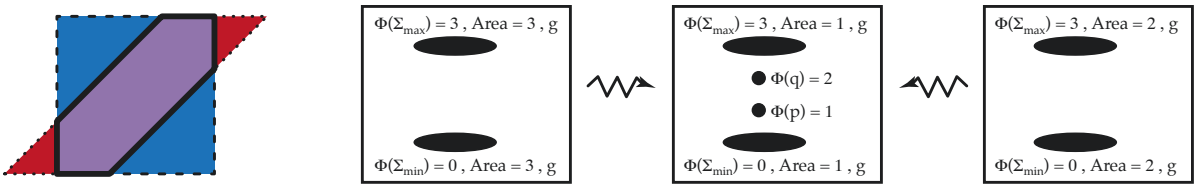

FIGURE 2.15. **On the left**, the same toric manifold, with the purple hexagonal moment image, can be seen in two ways as a 2-fold blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$, with different symplectic forms and blowup sizes. In this case, the Cremona transformation turns the vector $(3,3;2,2)$ (the blowup of the blue square) into $(3,2;1,1)$ (the blowup of the red parallelogram).
On the right we can see the S^1-manifold represented by the middle decorated graph as a blowup from the left $(3,3)$ with size 2 blowups, one each from the top and bottom; or from the right $(3,2)$ with size 1 blowups, one each from the top and the bottom. As before, the Cremona transformation turns the vector $(3,3;2,2)$ into $(3,2;1,1)$.

$(3,2;1,1)$, which encodes a blowup form diffeomorphic to $(3,3;2,2)$ that is g-reduced. The requirement $\lambda_F \geq \delta_1 + \delta_2$ is essential for the key claim that the class E_k is of minimal symplectic area in our proof of Theorem 2.13.

Li and Li [22, Lemma 3.4] prove that any class in H_2 over $\mathbb{Z}$ of nonnegative square is equivalent to a reduced class under the action of orientation-preserving diffeomorphisms; they provide an algorithm to find the reduced class using reflections along integral homology classes that are realized by orientation-preserving diffeomorphisms. In Li-Li's terminology, the Cremona transformation and the (δ_i, δ_j)-transpositions we use in Algorithm 6.5 correspond to reflections along $F - E_1 - E_2$ and along $E_i - E_j$, respectively. However Li and Li's algorithm uses other transformations which we do not allow, e.g., $-\mathbb{1}$ and the reflections along $r_i F + \varepsilon_i E_i$. Their reflections represent diffoemorphisms, but possibly not symplectomorphisms. Their algorithm might send a vector encoding (the dual of) a blowup form ω to a vector that cannot encode a blowup form, or to a vector encoding a blowup form ω' that is not diffeomorphic to ω. For example, Li and Li's algorithm sends the (g-reduced) vector $(6,1;2,1)$ by reflection over $F - E_1$ to $(4,1;1,0)$, which does not represent a blowup form. Their algorithm sends the (g-reduced) vector $(12,2;3,3)$ by reflections over $F - E_1$ and over $F - E_2$ to $(10,2;1,1)$. Those vectors represent manifestly different symplectic forms: the former admits no Hamiltonian circle actions, while the latter does admit a Hamiltonian circle action.

We conclude this section with a brief outline of the rest of the paper. In Section 3, we recall the notion of the decorated graph associated to a Hamiltonian circle action, and describe the effect of an equivariant symplectic blowup on the graph. We then derive from Theorem 2.13 results on the existence and number of Hamiltonian circle actions on a fixed symplectic four-manifold. Section 4 examines symplectic blowups of ruled symplectic four-manifolds and highlights the symplectic facts and techniques we need to prove the main theorems. We deduce from Li-Liu's characterization of symplectic forms [24] a necessary and sufficient condition for a vector to encode a cohomology class of a blowup form, and describe how these techniques can compute certain symplectic invariants like Gromov width and packing number (see Remarks 4.9 and 4.10). In Section 5, we use the correspondence between Hamiltonian circle actions and decorated graphs to deduce results on symplectic manifolds from combinatorial observations. Finally, we prove the existence portion of Theorem 2.14 in Section 6, and the uniqueness in Section 7.

3. Counting Hamiltonian circle actions on blowups of irrational ruled manifolds

Decorated graphs. If a circle action $S^1 \circlearrowright (M, \omega)$ is Hamiltonian, then there is a real-valued momentum map $\Phi : M \to \mathbb{R}$. This is a Morse-Bott function with critical set corresponding to the fixed points. When $\dim(M) = 4$, the critical set can only consist of isolated points and two-dimensional submanifolds. The latter can only occur at the extrema of Φ. To (M, ω, Φ) Karshon associates the following **decorated graph** [13, §2.1]. For each isolated fixed point p there is a vertex $\langle p \rangle$, labeled by the real number $\Phi(p)$. For each two dimensional component S of the fixed point set there is a **fat** vertex $\langle S \rangle$ labeled by two real numbers and one integer: the **momentum map label** $\Phi(S)$, the **area label** $\frac{1}{2\pi}\int_S \omega$, and the **genus** g of the surface S. A $\mathbf{Z}_k$**-sphere** is a gradient sphere in M on which S^1 acts with isotropy $\mathbf{Z}_k$. For each $\mathbf{Z}_k$-sphere containing two fixed points p and q, the graph has an edge connecting the vertices $\langle p \rangle$ and

$\langle q \rangle$ labeled by the integer k; the size $\frac{1}{2\pi}\int_S \omega$ of a $\mathbf{Z}_k$-sphere S is $1/k$ of the difference of the moment map values of its vertices. We note that vertical translations of the graph correspond to equivariant symplectomorphisms, and flips correspond to automorphisms of the circle.

Hamiltonian circle actions on irrational ruled symplectic manifolds. Karshon's results [13, Theorem 6.3 and Lemma 6.15] and [14, Table in Proof of Lemma 1] imply that Hamiltonian circle actions on a ruled symplectic four-manifolds over a closed Riemann surface of positive genus g correspond to decorated graphs that consist of two fat vertices with the same genus label g. Moreover, Karshon's results yield the following characterization of Hamiltonian circle actions on irrational ruled symplectic four-manifolds. See also [28, Corollary 3.12].

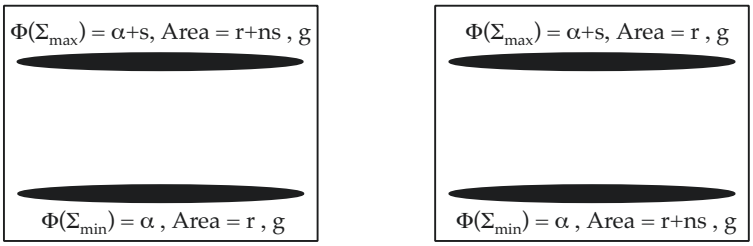

FIGURE 3.1. Decorated graphs with exactly two fat vertices, differing by a flip.

3.2. **Proposition.** *Let (M,ω) be a ruled symplectic manifold over a Riemann surface Σ of positive genus g. Up to equivariant symplectomorphisms and automorphisms of the circle, the number of Hamiltonian circle actions on (M,ω) is $\left\lceil \frac{\lambda_B}{\lambda_F} \right\rceil$ if $M = \Sigma \times S^2$ and $\left\lceil \frac{\lambda_B - \frac{1}{2}\lambda_F}{\lambda_F} \right\rceil$ if $M = M_\Sigma$.*

Moreover, there is a one-to-one correspondence between inequivalent Hamiltonian S^1-actions on (M,ω) and the nonnegative integers $0 \leq n < 2\frac{\lambda_B}{\lambda_F}$ such that

- *the integer n is even if $\pi\colon M \to \Sigma$ is the trivial bundle; n is odd if $\pi\colon M \to \Sigma$ is the non-trivial bundle.*
- *there exists $r \in \mathbb{R}^{>0}$ such that $\frac{1}{2}(r + (r + n\lambda_F)) = \lambda_B$.*

3.3. **Remark.** Note that the enumeration of the inequivalent circle actions on a ruled symplectic manifold over a Riemann surface Σ of positive genus g is analogous to the enumeration of the inequivalent toric actions on a ruled symplectic manifold over S^2 as in [14].

The effect of an S^1-equivariant blowup on the decorated graph. Recall that we can think of a symplectic blowup of size $\delta = r^2/2$ as cutting out an embedded ball of radius r and identifying the boundary to an exceptional sphere via the Hopf map. This carries a symplectic form that integrates on the sphere to $2\pi\delta$. For more details see [10] and [26, Section 7.1]. If the embedding of the ball is G-equivariant centered at a G-fixed point, then the G-action extends to the symplectic blowup, see details in [15]. If the action is Hamiltonian, its moment map naturally extends to the equivariant symplectic blowup.

In Figure 3.4, we describe all the possible effects of an S^1-equivariant symplectic blowup of a ruled manifold over a surface of positive genus, up to flips. If the surface is of genus 0, an S^1-equivariant symplectic blowup can have additional effects. For a complete description of Hamiltonian S^1-blowups of 4-manifolds, see [13, §6]. We observe that flipping commutes with S^1-equivariant symplectic blowup.

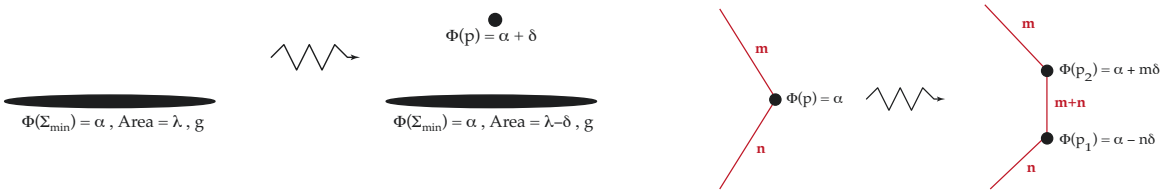

FIGURE 3.4. The effect on the decorated graph of an S^1-equivariant blowup of size δ: at a minimum surface on the left, and at an interior isolated fixed point on the right.

3.5. **Remark.** By [13, Proposition 7.2], a Hamiltonian S^1-space admits an S^1-equivariant blowup of size $\delta > 0$ centered at some fixed point if and only if one obtains a *valid* decorated graph after the blowup. That is, the (fat or not) vertices created in the blowup do not surpass the other pre-existing (fat or not) vertices in the same chain of edges, and the fat vertices after the blowup have positive size labels. For quantitative description, see [15, Lemma 3.3].

An algorithm to count the (non-equivalent) Hamiltonian circle actions on a k-fold blowup of an irrational ruled manifold. Let $k \geq 1$. Let Σ be a Riemann surface of positive genus. When the cohomology class of a blowup form $\omega = \omega_{\lambda_F,\lambda_B;\delta_1,\ldots,\delta_k}$ on $M = (\Sigma \times S^2)_k$ ($M = (M_\Sigma)_k$) is g-reduced, Theorem 2.13 reduces the question of counting the number of Hamiltonian circle actions on (M,ω) to a combinatorial one. It is enough to count the number of possible ways to get valid decorated graphs by k S^1-blowups of sizes $\delta_1,\ldots,\delta_k$ in descending order, starting from a decorated graph with two fat vertices, the top of size $\lambda_B - \frac{n}{2}\lambda_F$ and the bottom of size $\lambda_B + \frac{n}{2}\lambda_F$, for $0 \leq n < 2\frac{\lambda_B}{\lambda_F}$ that is even if $M = (\Sigma \times S^2)_k$ and odd if $M = (M_\Sigma)_k$. We count the resulting decorated graphs up to vertical translations and flips.

By Corollary 5.4, it is enough to count the Hamiltonian circle actions on symplectic k-fold blowups of $\Sigma \times S^2$. The counting algorithm was implemented by Tair Pnini. Let $\omega_{\lambda_F,\lambda_B;\delta_1,\ldots,\delta_k}$ be a blowup form on $(\Sigma \times S^2)_k$ whose cohomology class is in g-reduced form and is encoded by a vector $(\lambda_F,\lambda_B;\delta_1,\ldots,\delta_k)$. At step 0 the program creates the set of graphs with two fat vertices determined by λ_B,λ_F, corresponding to the even integers $0 \leq n < 2\frac{\lambda_B}{\lambda_F}$. At step i, for $1 \leq i \leq k$, it creates the set of graphs (up to equivalence) which may be obtained by performing a valid blowup of size δ_i on a graph in the set created at step $i-1$. The output of the program is the number of graphs in the set obtained at step k.

This is a "greedy" algorithm. We use data structures designed to reduce the number of tests for equivalence between pairs of graphs and to optimize the equivalence test. A detailed description of the data structures and the pseudo-code of the algorithm is given in Appendix B by Tair Pnini.

3.6. **Corollary.** *Let $k \geq 1$ and assume that we are performing k blowups of equal size: $\delta_1 = \ldots = \delta_k = \varepsilon$.*

- *Assume $2\varepsilon < \lambda_F$. The number of Hamiltonian circle actions on $((\Sigma \times S^2)_k, \omega_{\lambda_F,\lambda_B;\varepsilon,\ldots,\varepsilon})$ equals*

$$\left(\sum_{n=1}^{\left\lceil \frac{\lambda_B}{\lambda_F}\right\rceil -1}\left(\sum_{j=0}^{k}\delta_{j\varepsilon<(\lambda_B-n\lambda_F)}\delta_{(k-j)\varepsilon<(\lambda_B+n\lambda_F)}\right)\right)+\sum_{j=0}^{\left\lfloor \frac{k}{2}\right\rfloor}\delta_{j\varepsilon<\lambda_B}\delta_{(k-j)\varepsilon<\lambda_B}, \tag{3.7}$$

where

$$\delta_{a<b} = \begin{cases} 1 \text{ if } a < b \\ 0 \text{ if } a \geq b \end{cases}.$$

We count the possible ways to preform k blowups of size ε, each at a fat vertex, starting from one of the decorated graphs of two fat vertices listed in Proposition 3.2.

- *Assume $2\varepsilon = \lambda_F$. The number of Hamiltonian circle actions on $((\Sigma \times S^2)_k, \omega_{\lambda_F,\lambda_B;\varepsilon,\ldots,\varepsilon})$ equals*

$$(3.7) - \left(\sum_{n=1}^{\left\lceil \frac{\lambda_B}{\lambda_F}\right\rceil -1}\left(\sum_{j=0}^{k-2}\delta_{j\varepsilon<(\lambda_B-n\lambda_F)}\cdot\delta_{(k-2-j)\varepsilon<(\lambda_B+(n-1)\lambda_F)}\right)\right).$$

We have subtracted the equivalent actions, like those in Figure 3.9 where blowups of size $\frac{\lambda_F}{2}$ from the top or bottom fat vertices will end up at the same height, resulting in the same decorated graphs.

- *Similarly, if $2\varepsilon < \lambda_F$, the number of Hamiltonian circle actions on $((M_\Sigma)_k, \omega_{\lambda_F,\lambda_B;\varepsilon,\ldots,\varepsilon})$ equals the sum*

$$\sum_{n=0}^{\left\lceil \frac{\lambda_B-\frac{1}{2}\lambda_F}{\lambda_F}\right\rceil -1}\left(\sum_{j=0}^{k}\delta_{j\varepsilon<\left(\lambda_B-\frac{2n+1}{2}\lambda_F\right)}\delta_{(k-j)\varepsilon<\left(\lambda_B+\frac{2n+1}{2}\lambda_F\right)}\right). \tag{3.8}$$

If $2\varepsilon = \lambda_F$, *it equals*

$$(3.8)\quad -\left(\sum_{n=1}^{\left\lceil \frac{\lambda_B - \frac{1}{2}\lambda_F}{\lambda_F}\right\rceil - 1} \left(\sum_{j=0}^{k-2} \delta_{j\varepsilon < \left(\lambda_B - \frac{2n+1}{2}\lambda_F\right)} \cdot \delta_{(k-2-j)\varepsilon < \left(\lambda_B + \frac{2(n-1)+1}{2}\lambda_F\right)}\right)\right).$$

- *In particular, if* $k\varepsilon \geq 2\lambda_B$ *then there are no Hamiltonian circle actions.*

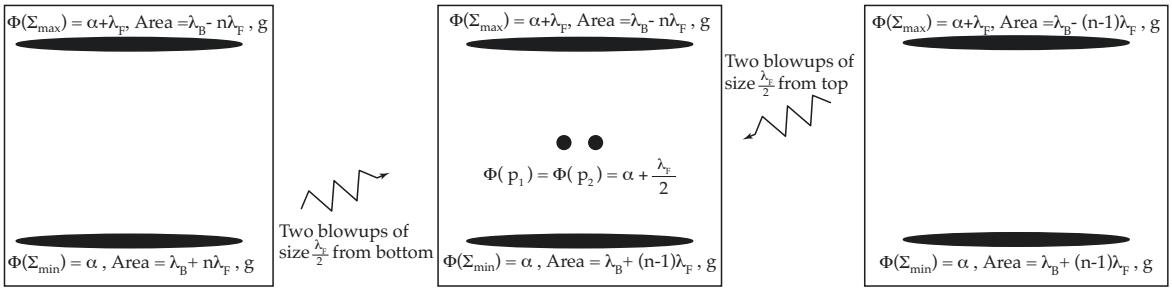

FIGURE 3.9. This figure shows coincidentally equivalent decorated graphs arising from blowups of different graphs.

For example, for $k \geq 2$ there are no Hamiltonian circle actions on $(\Sigma \times S^2)_k$ with a blowup form $\omega_{2,1;\frac{2}{k},\ldots\frac{2}{k}}$; by Lemma 4.8, such a blowup form exists. This lack of Hamiltonian actions also appears for equal blowups in the simply connected case; see for example [15].

3.10. **Corollary.** *Let* $k \geq 1$. *The number of Hamiltonian circle actions on* $((\Sigma \times S^2)_k, \omega_{\lambda_F,\lambda_B;\delta_1,\ldots,\delta_k})$ *is at most*

$$(3.11)\quad \left(\left\lceil \frac{\lambda_B}{\lambda_F}\right\rceil - \frac{1}{2}\right)(k+1)!$$

where $\lceil a \rceil$ *denotes the smallest integer greater than or equal to* a. *If*

- $\sum_{i=1}^{k} \delta_i < \lambda_F$, *and*
- $\sum_{i=1}^{k} \delta_i < \lambda_B - \frac{n}{2}\lambda_F$ *for every even* $n < 2\frac{\lambda_B}{\lambda_F}$, *and*
- *for every* j, *the sum* $\sum_{i=j+1}^{k} \delta_i < \delta_j$, *and moreover*
- *for every* j *and* s, *we have* $\sum_{i=1}^{s} F_{i+1}\delta_{j+i} < F_1\delta_j$, *where* $\{F_i\}$ *is the sequence of Fibonacci numbers,*

then the number of Hamiltonian circle actions on $((\Sigma \times S^2)_k, \omega_{\lambda_F,\lambda_B;\delta_1,\ldots,\delta_k})$ *equals* (3.11).

For example, if $\lambda_B \in \mathbb{N}$, $\lambda_F = 1$ and $\delta_i = \frac{1}{4^i}$ for $1 \leq i \leq k$, the number of Hamiltonian circle actions is $(\lambda_B - \frac{1}{2}) \cdot (k+1)!$. (By Lemma 4.8, there exists a blowup form on $(\Sigma \times S^2)_k$ whose cohomology class is encoded by this vector, for every $k \geq 1$.)

4. Blowups of ruled symplectic four-manifolds

Let Σ be a compact connected Riemann surface. Fix an integer $k \geq 0$. Let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. As a smooth manifold, M is obtained by k iterations of a connected sum with $\overline{\mathbb{CP}^2}$, $\mathbb{CP}^2$ equipped with the opposite orientation. In general, the connected sum construction $\widetilde{N} = N\#\overline{\mathbb{CP}^2}$ depends on a number of choices, but if $\widetilde{N}'$ is another connected sum obtained from different choices, then there exists a diffeomorphism from $\widetilde{N}$ to $\widetilde{N}'$ that respects the splitting $H_2(\widetilde{N}) = H_2(N) \oplus \mathbb{Z}E$ induced by the constructions. This follows from standard arguments in differential topology (see e.g., [5, Chapter 10]). In particular, if M is obtained from $\Sigma \times S^2$ or M_Σ by blowing up at k distinct points and M' is obtained by blowing up at a permutation of the same points, then the two manifolds are the same, up to relabeling the exceptional classes $E_1, \ldots, E_k$.

4.1. **Lemma.** *Fix a vector* $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ *and a permutation* $\sigma \in S_k$. *Suppose that there exists a blowup form* ω *on* M *whose cohomology class is encoded by this vector. Then there exists a blowup form* ω' *whose cohomology class is encoded by the vector* $(\lambda_F, \lambda_B; \delta_{\sigma(1)}, \ldots, \delta_{\sigma(k)})$ *and* (M, ω), *and* (M, ω') *are symplectomorphic.*

4.2. For any manifold M, an **almost complex structure** is an automorphism $J\colon TM \to TM$ such that $J^2 = -\mathbb{1}$. An almost complex structure J on M is **tamed** by a symplectic form ω if $\omega(u, Ju) > 0$ for all nonzero tangent vectors $u \in TM$. Let $\mathcal{J}_\tau(M,\omega)$ denote the set of almost complex structures J that are tamed by ω. The space $\mathcal{J}_\tau(M,\omega)$ is a nonempty contractible open subset of the space of all almost complex structures [26, Proposition 2.51].

A **(parametrized) J-holomorphic sphere** is a map $f\colon (\mathbb{CP}^1, j) \to (M, J)$ that satisfies the Cauchy-Riemann equations $df \circ j = J \circ df$ at every $p \in \mathbb{CP}^1$. An embedding is a one-to-one immersion which is a homeomorphism with its image. An **embedded J-holomorphic sphere** $C \subset M$ is the image of a J-holomorphic embedding $f\colon \mathbb{CP}^1 \to M$. If J is ω-tamed then such a C is an embedded ω-symplectic sphere. We will refer to the genus zero Gromov Witten invariant (with point constraints) $\mathrm{GW}\colon H_2(M) \to \mathbb{Z}$. For the precise definition, see [27]. Fixing a symplectic form ω, if $\mathrm{GW}(A) \neq 0$, then for generic ω-tamed almost complex structure J there exists a J-holomorphic sphere in the class A.

Because the first Chern class of a complex vector bundle does not change under a continuous deformation of the fibrewise complex structure, the first Chern class $c = c_1(TM, J)$ is the same for all $J \in \mathcal{J}_\tau(M,\omega)$. It follows from Lemma 2.11 that this first Chern class and the Gromov-Witten invariant are the same for all the blowup forms on $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. We denote the first Chern class and the Gromov Witten invariant associated to any blowup form on M by $c_1(TM)$ and GW.

4.3. **Facts.** We will make use of the following facts.

(1) The classes F, B, and $E_1, \dots, E_k$ form a basis of the homology group $H_2((\Sigma \times S^2)_k)$. The classes F, B_{-1} (B_1), and $E_1, \dots, E_k$ form a basis of $H_2((M_\Sigma)_k)$. Recall that we have classes $B_n = [\sigma_n(\Sigma)]$ in $H_2((\Sigma \times S^2)_k)$ when n is even and in $H_2((M_\Sigma)_k)$ when n is odd.

(2) The intersection numbers are given in the following table, where $i \neq j$.

(4.4)

	F	B	B_n	E_i	E_j
F	0	1	1	0	0
B	1	0	$\frac{n}{2}$	0	0
B_n	1	$\frac{n}{2}$	n	0	0
E_i	0	0	0	-1	0
E_j	0	0	0	0	-1

Note also that $B_n \cdot B_{-n} = 0$.

(3) For the first Chern class $c_1(TM)$ of a blowup form on M, $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$,

$$c_1(TM)(F) = 2, \;\; c_1(TM)(E_i) = 1 \;\forall 1 \leq i \leq k.$$

In $M = (\Sigma \times S^2)_k$,

$$c_1(TM)(B) = 2 - 2g(\Sigma).$$

In $M = (M_\Sigma)_k$,

$$c_1(TM)(B_{-1}) = 1 - 2g(\Sigma).$$

(4) The genus zero Gromov-Witten invariant $\mathrm{GW}(F)$ with respect to any blowup form is not zero.

The last item follows from [25, §4] and the fact that Gromov-Witten invariants are consistent under the natural inclusions

$$H_2((\Sigma \times S^2)_k) \to H_2((\Sigma \times S^2)_{k+1}) \text{ and } H_2((M_\Sigma)_k) \to H_2((M_\Sigma)_{k+1}),$$

as in [12, Theorem 1.4], [21, Proposition 3.5], and the explanation in [18, Appendix A].

Exceptional classes. We compile a number of results about exceptional classes on k-fold blowups of ruled symplectic four-manifolds. The first lemma follows from McDuff's "C_1 lemma" [25, Lemma 3.1], Gromov's compactness theorem [8, 1.5.B], the adjunction formula [27, Cor. E.1.7], and Lemma 2.11. Full details are in [18, Lemma 2.9].

4.5. **Lemma.** *Let $M = (\Sigma \times S^2)_k$ ($M = (M_\Sigma)_k$). Let $E \in H_2(M)$ be a homology class. Then the following are equivalent:*

(a) *There exists a blowup form ω on M such that the class E is represented by an embedded ω-symplectic sphere with self intersection -1.*
(b) (i) $c_1(TM)(E) = 1$;
(ii) $E \cdot E = -1$; *and*
(iii) *the genus zero Gromov-Witten invariant* $\mathrm{GW}(E) \neq 0$.
(c) *For every blowup form ω' on M, the class E is represented by an embedded ω'-symplectic sphere with self intersection -1.*

Lemma 4.5 guarantees that the set of exceptional classes is independent of the choice of a blowup form ω on M, and for every exceptional class E, we have $c_1(TM)(E) = 1$ and $\mathrm{GW}(E) \neq 0$, i.e., for generic ω-tamed almost complex structure J there exists a J-holomorphic sphere in the class E.

The next lemma guarantees the existence of a J-holomorphic sphere in exceptional classes for *every* ω-tamed J in the case of a single blowup of a ruled symplectic manifold over a Riemann surface of positive genus. It follows from the adjunction formula [27, Cor. E.1.7], the positivity of intersections [27, Proposition 2.4.4], and Gromov's compactness theorem [8, 1.5.B]. This is analogous to a result of Pinsonnault in the genus 0 case, see [29, Lemma 2.2], [18, Lemma 3.24]. We prove the positive genus case in Appendix A.

4.6. **Lemma.** *Let $M = (\Sigma \times S^2)_1$ or $M = (M_\Sigma)_1$. Assume that $g = g(\Sigma) > 0$. Let ω be a blowup form on M. Then for every $J \in \mathcal{J}_\tau(M, \omega)$ there exists an embedded J-holomorphic sphere in the class E_1 and there exists an embedded J-holomorphic sphere in the class $F - E_1$.*

The following lemma is a consequence of (2.12) and Facts 4.3 that we will need in the proof of the uniqueness theorem.

4.7. **Lemma.** *Let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. Assume that $g(\Sigma) > 0$.*

- *When $k = 0$, the class F is the unique class in $H_2(M; \mathbb{Z})$ satisfying*
  (1) *Its symplectic area with respect to a symplectic ruling of the S^2-bundle is positive;*
  (2) *Its self intersection number is zero; and*
  (3) *Its coupling with the first Chern class $c_1(TM)$ equals two.*
- *When $k \geq 1$, the class F is the unique class in $H_2(M; \mathbb{Z})$ satisfying*
  (1) *The intersection number of the class with every class in $\mathcal{E}(M)$ is zero; and*
  (2) *Its coupling with the first Chern class $c_1(TM)$ equals two.*

Proof. Denote $\hat{B} = B$ if $M = (\Sigma \times S^2)_k$ and $\hat{B} = B_1$ if $M = (M_\Sigma)_k$.

- In the case $k = 0$, a class $A \in H_2(M; \mathbb{Z})$ is written as $A = p\hat{B} + qF$ for $p, q \in \mathbb{Z}$. Assume that A satisfies the three conditions. Since $A \cdot A = 0$, we get, using Facts 4.3, that if $M = \Sigma \times S^2$ then $2pq = 0$, i.e., either $p = 0$ or $q = 0$, and if $M = M_\Sigma$ then $0 = p^2 + 2pq = p(p + 2q)$, i.e., either $p = 0$ or $p + 2q = 0$. If $M = \Sigma \times S^2$ and $q = 0$ then by the third property of A we have $2 = c_1(TM)(A) = (2 - 2g)p$. Similarly, if $M = M_\Sigma$ and $p + 2q = 0$ then $2 = c_1(TM)A = (2 - 2g)p + p + 2q = (2 - 2g)p$. Since g is a positive integer, the equality $2 = (2 - 2g)p$ holds only if $g = 2$ and $p = -1$, however, if $M = \Sigma \times S^2$ this (and $q = 0$) yield that $\omega(A) = -\omega(\hat{B}) < 0$ contradicting the first condition; if $M = M_\Sigma$ this (and $p + 2q = 0$) yield that $2q = 1$ contradicting the fact that q is an integer. We conclude that $p = 0$ hence $2q = c_1(TM)(A) = 2$, i.e., $q = 1$.
- Assume $k \geq 1$. Let $A \in H_2(M; \mathbb{Z})$. By item (1) in Facts 4.3, $A = p\hat{B} + qF - \sum_{i=1}^k r_i E_i$ for $p, q \in \mathbb{Z}$. If A satisfies the first condition, then, by (2.12) and item(2) in Facts 4.3, for all $1 \leq i \leq k$ we have $r_i = A \cdot E_i = 0$ hence $A = p\hat{B} + qF$, so $p = A \cdot (F - E_1) = 0$. Since $2 = c_1(TM)(A) = qc_1(TM)(F) = 2q$, we get $q = 1$.
  On the other hand, by (2.12) and Facts 4.3, the class F satisfies the conditions.

□

Necessary and sufficient conditions for a vector to encode a blowup form. We now turn to the question of when a vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \in \mathbb{R}^{2+k}$ encodes the cohomology class of a blowup form.

4.8. **Lemma.** *Assume that Σ is of positive genus. Let $k \geq 0$. A vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ encodes the cohomology class of a blowup form ω on $M = (\Sigma \times S^2)_k$ ($M = (M_\Sigma)_k$) if and only if*

- *the numbers* $\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k$ *are positive;*
- $\lambda_F > \delta_i$ *for all* i*; and*
- *the volume inequality* $\lambda_F\lambda_B - \frac{1}{2}(\delta_1^2 + \ldots + \delta_k^2) > 0$ *holds.*

Proof. If the vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ encodes the cohomology class of a blowup form ω, then the symplectic areas of the embedded ω-symplectic exceptional spheres representing the classes in $\mathcal{E}(M) = \{E_1, \ldots, E_k, F-E_1, \ldots, F-E_k\}$ and the symplectic volume of M are necessarily positive. This establishes the fact that $\delta_i > 0$, $\lambda_F > \delta_i$ and the volume inequality. The definition of a blowup form requires $\lambda_F > 0$, including for the case $k = 0$. Finally becuase the symplectic volume and λ_F are positive, $\lambda_B > 0$ as well.

The fact that the listed conditions on the vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ are sufficient to guarantee the existence of a blowup form follows from Li and Liu's characterization of symplectic forms with a standard canonical class on blowups of ruled symplectic manifolds [24, Theorem 3]. They prove that for any symplectic 4-manifold with $b^+ = 1$ (which includes all blowups of ruled surfaces), the symplectic cone of symplectic forms for which the first Chern class is the same as that of a blowup form (i.e., encoded as in item (3) of Facts 4.3) is described by

$$\mathcal{C}_0 = \left\{ \alpha \in H^2(M;\mathbb{R}) \;\middle|\; \begin{array}{c} \alpha \smile \alpha > 0 \textbf{ and} \\ \alpha(E) > 0 \;\forall \;\text{ “exceptional class” } E \text{ s.t. } c_1(TM)(E) = 1 \end{array} \right\}.$$

By “exceptional class” Li and Liu refer to a homology class E that is represented by a smoothly embedded sphere with self intersection -1. The facts that

- a class E is “exceptional” with $c_1(TM)(E) = 1$ if and only if $E \in \mathcal{E}(M)$; and
- every symplectic form with first Chern class as that of a blowup form is a blowup form

follow from results that are given in [24, Lemma 3.5 Part 2], and [23, Theorem A]; see also the explanation in [16, Section 6]. Thus, any cohomology class encoded by a vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ satisfying the conditions of the lemma is in $\mathcal{C}_0$ and so is the cohomology class of some blowup form ω. We will call $\mathcal{C}_0$ the **standard symplectic cone**. □

4.9. **Remark.** Note that Lemma 4.8 may be used to determine the **Gromov width** of a symplectic 4-manifold $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$ with $g(\Sigma) > 0$ and $k \geq 0$. To do so, we must determine for which positive numbers α there is a symplectic embedding of a ball of capacity α into M. This is equivalent to determining if there is a symplectic blowup of M of size α. We start with a vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \in \mathbb{R}^{k+2}_{>0}$ that encodes a blowup form on M. Lemma 4.8 tells us that there exists a blowup of M of size α so long as $0 < \alpha < \lambda_F$ and the volume inequality

$$\lambda_F\lambda_B - \frac{1}{2}\left(\alpha^2 + \sum_{i=1}^{k} {\delta_i}^2\right) > 0.$$

is satisfied. The Gromov width of M is the sup of all such α. Hence, the Gromov width will be determined by λ_F and the volume of M.

4.10. **Remark.** Lemma 4.8 may also be used to determine the **packing number** of a symplectic 4-manifold $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$ with $g(\Sigma) > 0$ and $k \geq 0$. Following [4, Definition 2.C], the packing number of (M, ω) is

$$P_{(M,\omega)} = 1 + \max\{N \in \mathbb{N} \mid \text{there does not exist a full packing of } M \text{ by } N \text{ equal balls}\}.$$

For such M with a blowup form encoded by the vector $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$, it is straight forward to show that

$$P_{(M,\omega)} = \left\lceil \frac{2 \cdot \left(\lambda_F\lambda_B - \frac{1}{2}\sum_{i=1}^{k} {\delta_i}^2\right)}{\lambda_F^2} \right\rceil.$$

In the special case when $k = 0$ and $M = S^2 \times \Sigma$, this is Biran's result [4, Corollary 5.C].

Symplectic blow down. By Weinstein's tubular neighborhood theorem [31], a neighborhood of an exceptional sphere is symplectomorphic to a neighborhood of the exceptional divisor in a standard blowup of $\mathbb{C}^2$. We can then **blow down** along C and get a symplectic manifold whose symplectic blowup is naturally isomorphic to (M, ω).

4.11. **Lemma** (Uniqueness of blow-downs)**.** *Let (M,ω) be a closed symplectic four-manifold. Let C_1 and C_2 be embedded ω-symplectic spheres of self intersection -1. Assume that C_1 and C_2 are in the same homology class. Let (N_1,ω_1) and (N_2,ω_2) be symplectic four-manifolds that are obtained by blowing down (M,ω) along C_1 and C_2, respectively. Then there is a symplectomorphism between (N_1,ω_1) and (N_2,ω_2) that induces the identity map on the second homology with respect to the decompositions $H_2(M) = H_2(N_i) \oplus \mathbb{Z}[C_i]$.*

For a proof of the uniqueness of blow downs in four dimensions, see [19, Lemma A.1].

As a result of Lemma 2.11, Lemma 4.11, the definition of a blowup form, and Remark 2.8, we get the following lemma.

4.12. **Lemma.** *Let $M = (\Sigma \times S^2)_k$ (or $M = (M_\Sigma)_k$), let ω be a blowup form on M, and let the vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_{k-1}, \delta_k)$ be the one encoding the cohomology class $[\omega]$. Then*

1. *there exists a blowup form $\overline{\omega}$ on $(\Sigma \times S^2)_{k-1}$ (resp. $(M_\Sigma)_{k-1}$) whose cohomology class is encoded by the vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_{k-1})$;*
2. *for every embedded ω-symplectic sphere in the class E_k, blowing down along it yields a manifold that is symplectomorphic to $((\Sigma \times S^2)_{k-1}, \overline{\omega})$ (resp. $((M_\Sigma)_{k-1}, \overline{\omega})$).*

5. Hamiltonian circle actions on ruled symplectic four-manifolds with one or two blowups

In this section we use the correspondence between Hamiltonian circle actions and decorated graphs to deduce results on symplectic manifolds from combinatorial observations. We first consider Hamiltonian S^1-actions on a symplectic manifold obtained from a ruled symplectic manifold by a single blowup.

5.1. **Lemma.** *Assume that $0 < \epsilon < \min\{r, s\}$.*

1. *There is a Hamiltonian S^1-action on $(\Sigma \times S^2)_1$ with the blowup form encoded by $(s, r; \varepsilon)$, for which there exists an equivariant sphere C in the class E_1 and an equivariant sphere D in the class $F - E_1$, and blowing down along C yields a Hamiltonian S^1-action on $\Sigma \times S^2$ with the compatible symplectic form encoded by (s, r), and blowing down along D yields a Hamiltonian S^1-action on M_Σ with the compatible symplectic form encoded by $(s, r + \frac{1}{2}s - \varepsilon)$.*
2. *There is a Hamiltonian S^1-action on $(M_\Sigma)_1$ with the blowup form encoded by $(s, r; \varepsilon)$, for which there exists an equivariant sphere C in the class E_1 and an equivariant sphere D in the class $F - E_1$, and blowing down along C yields a Hamiltonian S^1 action on M_Σ with the compatible symplectic form encoded by (s, r), and blowing down along D yields a Hamiltonian S^1-action on $\Sigma \times S^2$ with the compatible symplectic form encoded by $(s, r + \frac{1}{2}s - \varepsilon)$.*

Proof. To each case of the lemma, it suffices to inspect the decorated graphs given in Figure 5.2.

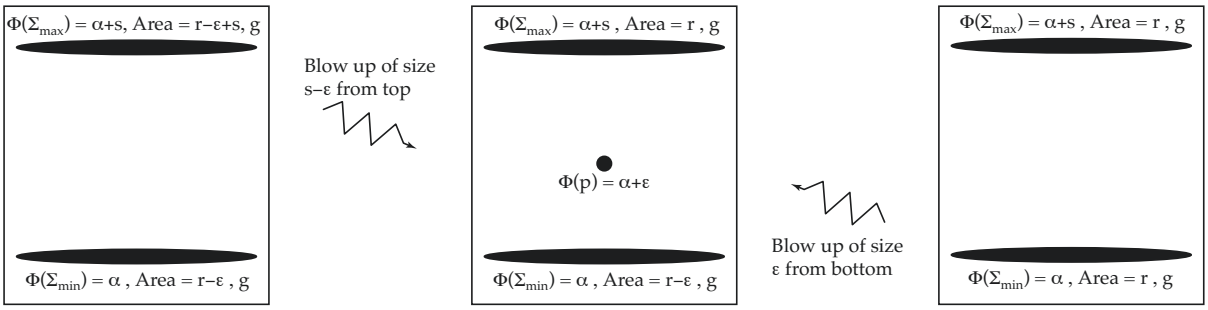

FIGURE 5.2. A circle blow up of size $s - \varepsilon$ at the maximal surface of $(M_\Sigma, \omega_{s,r+\frac{s}{2}-\varepsilon})$, on the left, is isomorphic to a circle blow up of size ε at the minimal surface of $(\Sigma \times S^2, \omega_{s,r})$, on the right.

□

Together, Lemma 5.1 and Lemma 4.11 allow us to prove the following result.

5.3. **Corollary.** *Let $M = (\Sigma \times S^2)_1$ (resp. $M = (M_\Sigma)_1$) with a blowup form ω encoded by $(\lambda_F, \lambda_B; \delta)$. For every embedded ω-symplectic sphere in the class $F - E_1$, blowing down along it yields a symplectic manifold that is symplectomorphic to M_Σ (resp. $\Sigma \times S^2$) with the symplectic ruling encoded by $(\lambda_F, \lambda_B + \frac{1}{2}\lambda_F - \delta)$.*

Proof. By Lemma 5.1, for some blowup form on M there is an embedded symplectic sphere in the exceptional class $F - E_1$ such that blowing down along it yields the stated symplectic manifold. By Lemma 4.5, for every blowup form on M there exists a symplectic sphere in $F - E_1$, and by Lemma 2.11, blowing down along it has the same effect on homology. The blow down is well-defined by Lemma 4.11, and hence yields the desired symplectic manifold. □

5.4. **Corollary.** *Let $k \geq 1$. The symplectic manifold $(M_\Sigma)_k$ (resp. $(\Sigma \times S^2)_k$) with a blowup form ω with $[\omega]$ encoded by $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ is symplectomorphic to $(\Sigma \times S^2)_k$ (resp. $(M_\Sigma)_k$) with a blowup form ω' with $[\omega']$ encoded by $(\lambda_F, \lambda_B + \frac{1}{2}\lambda_F - \delta_1; \lambda_F - \delta_1, \delta_2, \ldots, \delta_k)$. Moreover if $k \geq 2$ and $[\omega]$ is in g-reduced form, then so is $[\omega']$.*

We now continue our study by looking at the effect of two blowups on a decorated graph. In Figures 5.5 and 5.6 we describe two different Hamiltonian blowups of different ruled symplectic manifolds that yield the same symplectic manifold with Hamiltonian S^1-action. Assume that $0 < \varepsilon_1, \varepsilon_2 < \min\{r, s\}$.

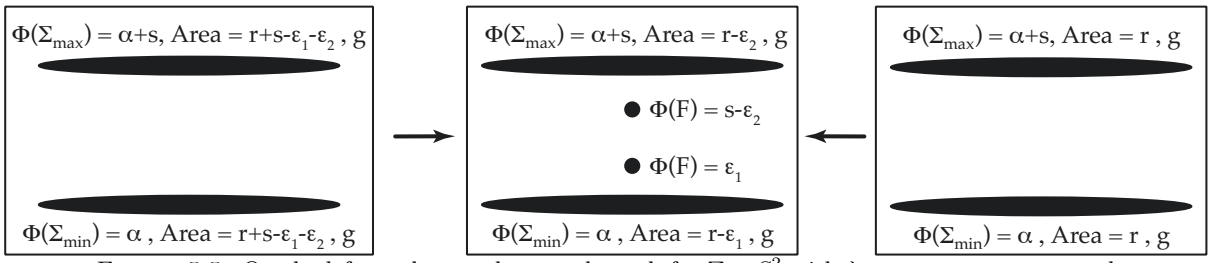

FIGURE 5.5. On the left, we have a decorated graph for $\Sigma \times S^2$ with $\lambda_B = r + s - \varepsilon_1 - \varepsilon_2$ and $\lambda_F = s$. From the left to the center, we take two blow-ups of $\Sigma \times S^2$ of sizes $s - \varepsilon_2$ from the bottom and $s - \varepsilon_1$ from the top.
On the right, we have a decorated graph for $\Sigma \times S^2$ with $\lambda_B = r$ and $\lambda_F = s$. From the right to the center, we take two blow-ups of $\Sigma \times S^2$ of sizes ε_1 from the bottom and ε_2 from the top.

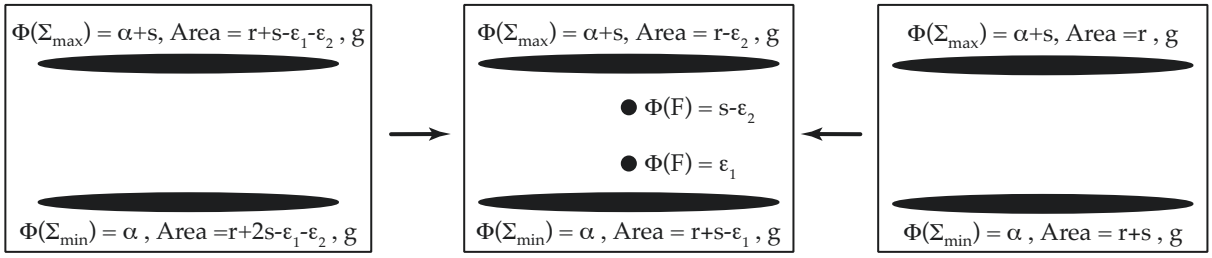

FIGURE 5.6. **On the left,** we have a decorated graph for M_Σ with $\lambda_B = r + \frac{s}{2} + s - \varepsilon_1 - \varepsilon_2$ and $\lambda_F = s$. From the left to the center, we take two blow-ups of M_Σ of sizes $s - \varepsilon_2$ from the bottom and $s - \varepsilon_1$ from the top.
On the right, we have a decorated graph for M_Σ with $\lambda_B = r + \frac{s}{2}$ and $\lambda_F = s$. From the right to the center, we take two blow-ups of M_Σ of sizes ε_1 from the bottom and ε_2 from the top.

Consequently, we get an analogue of the **Cremona transformation**, adjusted for the positive genus case.

5.7. **Definition.** Let $k \geq 2$. For a vector $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$, let

$$\mathbf{defect}(v) := \delta_1 + \delta_2 - \lambda_F.$$

We define

$$\mathbf{cremona}(v) := (\lambda'_F, \lambda'_B; \delta'_1, \ldots, \delta'_k)$$

by setting

$$\begin{aligned} \lambda'_F &:= \lambda_F \\ \lambda'_B &:= \lambda_B - \mathbf{defect}(v) \\ \delta'_j &:= \begin{cases} \delta_1 - \mathbf{defect}(v) = \lambda_F - \delta_2 & \text{if } j = 1 \\ \delta_2 - \mathbf{defect}(v) = \lambda_F - \delta_1 & \text{if } j = 2 \\ \delta_j & \text{if } 3 \leq j \leq k. \end{cases} \end{aligned}$$

5.8. **Corollary.** *Let $k \geq 2$, and consider a k-fold blowup $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$ of a ruled symplectic manifold, and $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$. Let $(\lambda'_F, \lambda'_B; \delta'_1, \ldots, \delta'_k) =$ **cremona**(v). Suppose that there exists a blowup form $\omega_{\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k}$ whose cohomology class is encoded by v. Then there exists a blowup form $\omega_{\lambda'_F, \lambda'_B; \delta'_1, \ldots, \delta'_k}$ whose cohomology class is encoded by* **cremona**(v). *Moreover, $(M, \omega_{\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k})$ and $(M, \omega_{\lambda'_F, \lambda'_B; \delta'_1, \ldots, \delta'_k})$ are symplectomorphic.*

Proof. First assume $k = 2$. In Figures 5.5 and 5.6 we show that there exists a blowup form ω on the manifold $M^2 := (\Sigma \times S^2)_2$ or on $M^2 := (M_\Sigma)_2$, such that there is a diffeomorphism $\psi \colon M^2 \to M^2$ with the following properties. The induced map ψ_* on H_2 maps

$$F \mapsto F;\ E_1 \mapsto F - E_2;\ E_2 \mapsto F - E_1;\ F - E_1 \mapsto E_2;\ F - E_2 \mapsto E_1;\ \hat{B} \mapsto \hat{B} + F - E_1 - E_2,$$

where $\hat{B} = B$ if $M^2 = (\Sigma \times S^2)_2$, and $\hat{B} = B_{-1}$ if $M^2 = (M_\Sigma)_2$. Moreover $\psi^*\omega$ is a blowup form.

In particular, there exists a blowup form on M^2 such that the pullback by ψ is a blowup form. By Lemma 2.11, it follows that for **every** blowup form, its pullback by ψ is a blowup form. Because of how ψ acts on homology, if ω is a symplectic form that is encoded by the vector $(\lambda_F, \lambda_B; \delta_1, \delta_2)$, then $\psi^*\omega$ is a symplectic form that is encoded by the vector $(\lambda'_F, \lambda'_B; \delta'_1, \delta'_2)$. The case $k \geq 3$ then follows by the uniqueness of symplectic blowups of given sizes. □

6. Existence of g-reduced form

In this section we prove the existence part of Theorem 2.14. That is, given a k-fold blowup of a ruled symplectic manifold over a surface of positive genus, $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$, we show that there exists a blowup form on M whose cohomology class is g-reduced. We achieve this by providing an algorithm that starts with a vector encoding the cohomology class of a blowup form on a k-fold blowup of an irrational ruled manifold, and returns a symplectomorphic blowup form which is g-reduced.

6.1. **Definition.** Let $k \geq 2$. The **Cremona move** on $\mathbb{R}^{2+k}$ is the composition of the following two maps:

(1) first the map
$$v \mapsto \begin{cases} \textbf{cremona}(v) & \text{if defect}(v) > 0 \\ v & \text{otherwise.} \end{cases}; \text{ followed by}$$
(2) the map **sort** : $(\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \mapsto (\lambda_F, \lambda_B; \delta_{i_1}, \ldots, \delta_{i_k})$, which permutes the last k entries such that $\delta_{i_1} \geq \ldots \geq \delta_{i_k}$.

Let $k \geq 2$. It is straight forward to check that the Cremona move satisfies the following properties.

6.2. **Lemma.** *Let $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ be a vector in $\mathbb{R}^{2+k}$. The set of real numbers δ'_i that occur among the last k entries in vectors $v' = (\lambda'_F, \lambda'_B; \delta'_1, \ldots, \delta'_k)$ that can be obtained from v by finitely many iterations of the Cremona move is a nonempty subset of the finite set $\{\delta_1, \ldots, \delta_k, \lambda_F - \delta_1, \ldots, \lambda_F - \delta_k\}$, and in particular has no accumulation points.*

6.3. **Lemma.** (1) *The Cremona move preserves the standard symplectic cone*

$$\mathcal{C}_0 = \left\{ (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \in \mathbb{R}^{2+k}_{>0} \;\middle|\; \lambda_F - \delta_i > 0 \text{ for all } i \text{ and } \lambda_F\lambda_B - \frac{1}{2}\sum_{i=1}^{k} {\delta_i}^2 > 0 \right\}.$$

(2) *If the vector $v' = (\lambda_F, \lambda'_B; \delta'_1, \ldots, \delta'_k)$ is obtained from $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k)$ by the Cremona move and v' is different from v, then $\delta'_i \leq \delta_i$ for all i, and for at least one i we have $\delta'_i < \delta_i$.*
(3) *A vector $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \in \mathcal{C}_0$ is fixed by the Cremona move if and only if it satisfies equation* (2.2).

Sketch of Proof. Let $v = (\lambda_F, \lambda_B; \delta_1, \ldots, \delta_k) \in \mathcal{C}_0$ and $v' = (\lambda_F, \lambda'_B; \delta'_1, \ldots, \delta'_k)$ the vector obtained from v by the Cremona move. In proving part (1), we note that both the "volume" $\lambda_F\lambda_B - \frac{1}{2}\sum_{i=1}^{k} {\delta_i}^2$ and λ_F are preserved by the Cremona move. Hence, since the "volume" of v' and $\lambda'_F = \lambda_F$ are positive and all other terms in the "volume" are nonpositive, we get that $\lambda'_B > 0$ as well. Also, by Lemma 6.2 and since $v \in \mathcal{C}_0$, for all i, the numbers δ'_i and $\lambda_F - \delta'_i$ are positive as well. Part (2) and part (3) are immediate from the definition of the Cremona move. □

6.4. **Notation.** Let $g > 0$. A vector in $\mathbb{R}^{2+k}$ ($k \geq 2$) with all positive entries encodes a g-reduced cohomology class if it satisfies equation (2.2); we will call such a vector **g-reduced**.

Assume $k \geq 2$ and $g > 0$. We will now use the Cremona move to obtain a g-reduced vector.

Algorithm 6.5 — CremonaReduce.
∘ Input a vector $\mathbf{v} = (\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ in the standard symplectic cone in $\mathbb{R}^{k+2}$
∘ Output a **g-reduced** vector $(\lambda'_F, \lambda'_B; \delta'_1, \dots, \delta'_k)$ in the standard symplectic cone in $\mathbb{R}^{k+2}$

```
sort(v)
while defect(v) > 0 do
    set v = sort(cremona(v))
end while loop

return v
```

6.6. **Lemma.** *The algorithm* CremonaReduce *terminates after finitely many steps.*

Proof. We prove this by contradiction. Suppose that this process does not terminate for some vector v in the standard symplectic cone. Then we get an infinite sequence $v^{(n)}$, for any $n \in \mathbb{N}$, of vectors in the standard symplectic cone in $\mathbb{R}^{2+k}$. That is, $v^{(1)} = \mathbf{sort}(v)$ and $v^{(n+1)} = \mathbf{sort}\left(\mathbf{cremona}\left(v^{(n)}\right)\right)$. By part (2) of Lemma 6.3, for some $1 \leq i_0 \leq k$, the sequence $\delta_{i_0}^{(n)}$ has an infinite subsequence $\delta_{i_0}^{(n_\ell)}$ that is strictly decreasing. Moreover the numbers $\delta_{i_0}^{(n_\ell)}$ are all bounded below, as they are positive. Thus, the subsequence $\delta_{i_0}^{(n_\ell)}$ has an accumulation point, contradicting Lemma 6.2. □

6.7. **Remark.** Lemma 6.6 implies that Algorithm 6.5 terminates. By item (3) of Lemma 6.3 it produces a g-reduced vector. Moreover, by Corollary 4.1 and Corollary 5.8, if the input vector encodes the cohomology class of a blowup form ω on $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$, then the output vector encodes the cohomology class of a blowup form on M that is diffeomorphic to ω. This completes the proof of the existence portion of Theorem 2.14.

6.8. **Remark.** In the case when $g(\Sigma) = 0$, i.e., $\Sigma = S^2$, if we apply Algorithm 6.5 to a vector $(\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ encoding a blowup form on $M = (S^2 \times S^2)_k$ or $M = (M_{S^2})_k$, the output vector $(\lambda'_F, \lambda'_B; \delta'_1, \dots, \delta'_k)$ may encode a blowup-up form that is not o-reduced. The trouble is that the condition

$$\begin{cases} \lambda'_F \leq \lambda'_B & \text{if } M = (S^2 \times S^2)_k; \text{ or} \\ (\frac{1}{2}\lambda'_F + \delta'_1) \leq \lambda'_B & \text{if } M = (M_{S^2})_k, \end{cases}$$

may not hold. In particular, Algorithm 6.5 is **not** the push forward of the algorithm in [16, §2.16] to obtain a 0-reduced form given a blowup form on $(M_{S^2})_k$ under the map

$$(\lambda; \delta, \delta_1, \dots, \delta_k) \mapsto (\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$$

as in (2.6).

7. Uniqueness of the g-reduced form

Our goal in this section is to prove the uniqueness part of Theorem 2.14. Let $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$; assume that $g(\Sigma) > 0$. For any blowup form ω the set of exceptional classes of minimal area in (M, ω) only depends on the cohomology class $[\omega]$. We denote this set by

$$\mathcal{E}^v_{\min},$$

where $v \in \mathbb{R}^{2+k}$ is the vector that encodes the cohomology class $[\omega]$. If $k \geq 2$ and $[\omega]$ is in g-reduced form, then by (2.12), $E_k \in \mathcal{E}^v_{\min}$. We proceed to identify *all* the elements of $\mathcal{E}^v_{\min}$.

7.1. **Notation.** Let $v = (\lambda_F, \lambda_B; \delta_1, \dots, \delta_k)$ be the vector encoding the cohomology class of a blowup form ω on $M = (\Sigma \times S^2)_k$ or $M = (M_\Sigma)_k$. If $k \geq 2$, assume that $[\omega]$ is in g-reduced form. In particular, $\delta_1 \geq \ldots \geq \delta_k$, and the numbers λ_F, $\lambda_B, \delta_1, \dots, \delta_k$ are positive.

We denote by j_v the smallest nonnegative integer j such that $\delta_{j+1} = \ldots = \delta_k$. Thus, either $j_v = 0$ and $\delta_1 = \ldots = \delta_k$, or $1 \leq j_v \leq k-1$ and

$$\delta_1 \geq \ldots \geq \delta_{j_v} > \delta_{j_v+1} = \ldots = \delta_k.$$

As an immediate corollary of the identification of the exceptional classes in (2.12), and the definition of a g-reduced form, we have the following proposition.

7.2. **Proposition.** *Let (M,ω) and v be as in Notation 7.1. Assume that $g(\Sigma)>0$.*

- *In the case of one blowup, $k=1$, we have the following possibilities.*
  (1) *Suppose that $\delta_1 < \frac{1}{2}\lambda_F$. Then $\mathcal{E}^v_{\min} = \{E_1\}$, so $\sharp\mathcal{E}^v_{\min} = k$.*
  (2) *Suppose that $\delta_1 > \frac{1}{2}\lambda_F$. Then $\mathcal{E}^v_{\min} = \{F-E_1\}$, so $\sharp\mathcal{E}^v_{\min} = k$.*
  (3) *Suppose that $\delta_1 = \frac{1}{2}\lambda_F$. Then $\mathcal{E}^v_{\min} = \{E_1, F-E_1\}$, so $\sharp\mathcal{E}^v_{\min} = 2k$.*
- *For $k\geq 2$ blowups, we have the following possibilities.*
  (1) *Suppose that $\delta_1 \leq \frac{1}{2}\lambda_F$.*
    (a) *Suppose that $\delta_k < \frac{1}{2}\lambda_F$. Then $\mathcal{E}^v_{\min} = \{E_{j_v+1},\dots,E_k\}$, so $\sharp\mathcal{E}^v_{\min}\leq k$.*
    (b) *Suppose that $\delta_k = \frac{1}{2}\lambda_F$. Then $\mathcal{E}^v_{\min} = \bigcup_{i=1}^k \{E_i\}\cup\bigcup_{i=1}^k\{F-E_i\}$, so $\sharp\mathcal{E}^v_{\min} = 2k$.*
  (2) *Suppose that $\delta_1 > \frac{1}{2}\lambda_F$.*
    (a) *Suppose that $\lambda_F-\delta_1 > \delta_k$. Then $\mathcal{E}^v_{\min} = \{E_{j_v+1},\dots,E_k\}$ and $j_v>0$, so $\sharp\mathcal{E}^v_{\min}\leq k-1$.*
    (b) *Suppose that $\lambda_F-\delta_1=\delta_k$. Then $\mathcal{E}^v_{\min} = \{F-E_1,E_2,\dots,E_k\}$, so $\sharp\mathcal{E}^v_{\min}=k$.*

For the uniqueness of a g-reduced form, we will also need the following observations on symplectomorphic blowups of ruled manifolds.

7.3. **Lemma.** *Consider $M=(\Sigma\times S^2)_k$ or $M=(M_\Sigma)_k$. Assume that $g(\Sigma)>0$. Let ω and ω' be blowup forms on M whose cohomology classes are encoded by the vectors $v=(\lambda_F,\lambda_B;\delta_1,\dots,\delta_k)$ and $v'=(\lambda'_F,\lambda'_B;\delta'_1,\dots,\delta'_k)$, which are both g-reduced.*

Let $\varphi\colon (M,\omega)\to(M,\omega')$ be a symplectomorphism, and let $\varphi_*\colon H_2(M)\to H_2(M)$ be the induced map on the homology.

(1) *The isomorphism φ_* sends $\mathcal{E}(M)$ to itself.*
(2) *The isomorphism φ_* sends the set $\mathcal{E}^v_{\min}$ to the set $\mathcal{E}^{v'}_{\min}$. If $k\geq 2$ then*

$$\delta_k = \delta'_k. \tag{7.4}$$

If $k=1$ then

$$\min(\delta_1,\lambda_F-\delta_1) = \min(\delta'_1,\lambda'_F-\delta'_1). \tag{7.5}$$

(3) $\lambda_F=\lambda'_F$.
(4) $\lambda_B\lambda_F - \frac{1}{2}\sum_{i=1}^k {\delta_i}^2 = \lambda'_B\lambda'_F - \frac{1}{2}\sum_{i=1}^k {\delta'_i}^2$.

Proof. Item (1) and the first part of (2) follow immediately from the fact that φ carries ω to ω'. The second part of (2) follows from (2.12) and the "g-reduced" assumption. Item (3) follows from Lemma 4.7. Item (4) follows from the fact that the symplectomorphism preserves symplectic volume, because

$$\lambda_B\lambda_F - \frac{1}{2}\sum_{i=1}^k {\delta_i}^2 = \frac{1}{(2\pi)^2}\int_M \frac{\omega\wedge\omega}{2!}$$

and a similar formula with ω'. □

We now have all the ingredients to prove that the g-reduced form is unique.

Proof of the uniqueness part of Theorem 2.14. Recall that we have a manifold M that is a k-fold blowup, either $(M_\Sigma)_k$ or $(S^2\times\Sigma)_k$, where Σ has positive genus. Let ω and ω' be blowup forms on M whose cohomology classes are encoded by the vectors $v=(\lambda_F,\lambda_B;\delta_1,\dots,\delta_k)$ and $v'=(\lambda'_F,\lambda'_B;\delta'_1,\dots,\delta'_k)$, and, if $k\geq 2$, are both in g-reduced form. Suppose that (M,ω) and (M,ω') are symplectomorphic. We aim to show that $v=v'$.

The case enumeration refers to the case enumeration in Proposition 7.2.

Suppose that $k\geq 1$ and that the size of $\mathcal{E}^v_{\min}$ equals $2k$.
By item (2) of Lemma 7.3, the set $\mathcal{E}^{v'}_{\min}$ also has $2k$ elements. If $k=1$, by Proposition 7.2, both v and v' exhibit case (3). If $k\geq 2$, by Proposition 7.2, both v and v' exhibit case (1b). In either case, we have that $v=(\lambda_F,\lambda_B;\frac{\lambda_F}{2},\dots,\frac{\lambda_F}{2})$ and $v'=(\lambda'_F,\lambda'_B;\frac{\lambda'_F}{2},\dots,\frac{\lambda'_F}{2})$. From (7.4) we deduce that $\lambda_F=\lambda'_F$, hence by item (4) of Lemma 7.3, (and the fact that $\lambda_F>0$) also $\lambda_B=\lambda'_B$, and thus $v=v'$.

Suppose that the size of $\mathcal{E}^v_{\min}$ equals k.
By item (2) of Lemma 7.3, the set $\mathcal{E}^{v'}_{\min}$ also has k elements. If $k = 1$ then Proposition 7.2 says that the vector v also exhibits one of the cases (1) or (2), and the vector v' exhibits one of the cases (1) or (2). When $k \geq 2$, Proposition 7.2 implies that the vector v exhibits one of the cases (1a) with $j_v = 0$ or (2b); and the vector v' also exhibits one of the cases (1a) or (2b). For the purpose of what follows, if $k = 1$, we will use (1a) to refer to (1), and (2b) to refer to (2).

The classes in $\mathcal{E}^v_{\min}$ can be represented simultaneously by k pairwise disjoint embedded ω-symplectic spheres. Indeed, by Lemma 4.5 there exists an ω-tamed almost complex structure J such that each of the exceptional classes in $\mathcal{E}^v_{\min}$ is represented by an embedded J-holomorphic sphere. Since in cases (1a) and (2b), for each two classes in this set the intersection number is zero, we conclude, by positivity of intersections of J-holomorphic spheres [27, Proposition 2.4.4], that these J-holomorphic spheres are pairwise disjoint. By item (2) of Lemma 7.3, the symplectomorphism sends k pairwise disjoint embedded ω-symplectic spheres in the classes of $\mathcal{E}^v_{\min}$ to k pairwise disjoint embedded ω'-symplectic spheres in $\mathcal{E}^{v'}_{\min}$. Moreover, the symplectomorphism between (M, ω) and (M, ω') descends to a symplectomorphism between the blowdowns along k such spheres in $\mathcal{E}^v_{\min}$ and along the corresponding k spheres in $\mathcal{E}^{v'}_{\min}$.

If the vector is in case (1a), blowing down along the k spheres gives a manifold diffeomorphic to $S^2 \times \Sigma$ if $M = (S^2 \times \Sigma)_k$ and to M_Σ if $M = (M_\Sigma)_k$, by Lemma 4.12. If the vector is in case (2b), blowing down along the k spheres gives a manifold diffeomorphic to M_Σ if $M = (\Sigma \times S^2)_k$ and to $\Sigma \times S^2$ if $M = (M_\Sigma)_k$, by Lemma 4.12 and Corollary 5.3. Because the manifolds $S^2 \times \Sigma$ and M_Σ are not diffeomorphic, either both v and v' are in case (1a) or both are in case (2b). In either case, $\mathcal{E}^v_{\min} = \mathcal{E}^{v'}_{\min}$.

By item (3) in Lemma 7.3, $\lambda_F = \lambda'_F$. Moreover, if $k \geq 1$ and both vectors are in case (1a) of Theorem 7.2, then $v = (\lambda_F, \lambda_B; \delta, \dots, \delta)$ and $v' = (\lambda_F, \lambda'_B; \delta', \dots, \delta')$, and if $k \geq 1$ and both vectors are in case (2b), $v = (\lambda_F, \lambda_B; \lambda_F - \delta, \dots, \delta)$ and $v' = (\lambda_F, \lambda'_B; \lambda_F - \delta', \dots, \delta')$. In each of these cases, we conclude, by (7.4), that $\delta = \delta'$. Then, for $k \geq 0$, by Item (4) of Lemma 7.3 (and the fact that $\lambda_F > 0$) we get that $\lambda_B = \lambda'_B$ hence $v = v'$.

Suppose that $\mathbf{k \geq 2}$ and that the size of $\mathcal{E}^v_{\min}$ is smaller or equal to $k - 1$.
By item (2) of Lemma 7.3, the set $\mathcal{E}^{v'}_{\min}$ has size at most $k - 1$. Then by Proposition 7.2 each of the vectors v and v' either exhibits case (1a) or case (2a). So

$$\mathcal{E}^v_{\min} = \{E_{j_v+1}, \dots, E_k\} = \mathcal{E}^{v'}_{\min} \text{ where } 1 \leq j_v = k - \sharp\mathcal{E}^v_{\min}.$$

By (7.4), we get $\delta_i = \delta'_i$ for $j_v + 1 \leq i \leq k$.

Since ω and ω' are blowup forms, the pairwise disjoint exceptional classes $E_{j_v+1}, \dots, E_k$ can be represented simultaneously by pairwise disjoint embedded ω–symplectic or ω'–symplectic spheres. By Lemma 4.12, blowing down along these spheres in (M, ω) and in (M, ω') gives manifolds with a blowup form that is encoded by the vector $\hat{v} = (\lambda_F, \lambda_B; \delta_1, \dots, \delta_j)$ and the vector $\hat{v}' = (\lambda'_F, \lambda'_B; \delta'_1, \dots, \delta'_j)$. By "uniqueness of blowdown" (Lemma 4.11) the resulting manifolds are symplectomorphic. Because the cohomology classes encoded by vectors $\hat{v}$ and $\hat{v}'$ are in g-reduced form, we may proceed by induction. We note that the case $k = 0$ is included in the case where the size of $\mathcal{E}^v_{\min}$ equals k. □

Since Algorithm 6.5 and Lemma 6.6 prove that a g-reduced form always exists, see Remark 6.7, the proof of Theorem 2.14, the existence and uniqueness of a g-reduced form, is now complete.

Appendix A. Proof of Lemma 4.6

In the proof we will use the following corollary of the positivity of intersections of J-holomorphic curves, see [18, Proposition 2.3].

A.1. **Lemma.** *Let (M, ω) be a symplectic four-manifold. Let A and B be homology classes in $H_2(M)$ that are linearly independent (over $\mathbb{R}$). Suppose that $\mathrm{GW}(B) \neq 0$, that $c_1(TM)(A) \geq 1$, and that there exists an almost complex structure $J_0 \in \mathcal{J}_\tau(M, \omega)$ such that the class A is represented by a J_0-holomorphic sphere. Then the intersection number $A \cdot B$ is nonnegative.*

We now give a proof of Lemma 4.6.

Proof of Lemma 4.6. Assume that $g = g(\Sigma) > 0$. If $M = (\Sigma \times S^2)_1$ then (by (2.12)) there are two cases:

a. The class E_1 is of minimal area in $\mathcal{E}(M)$. In this case, denote

$$\hat{B} := B, \;\; \hat{E} := E_1.$$

b. Otherwise, the class $F - E_1$ is of minimal area in $\mathcal{E}(M)$. Denote

$$\hat{B} := B + F - E_1, \;\; \hat{E} := F - E_1.$$

If $M = (M_\Sigma)_1$, then there are also two cases:

a. The class $F - E_1$ is of minimal area in $\mathcal{E}(M)$. In this case, denote

$$\hat{B} := B_{-1} + F - E_1, \;\; \hat{E} := F - E_1.$$

b. Otherwise, the class E_1 is of minimal area in $\mathcal{E}(M)$. Denote

$$\hat{B} := B_1, \;\; \hat{E} := E_1.$$

Note that the classes $\hat{B}, F, \hat{E}$, form a $\mathbb{Z}$-basis to $H_2(M)$.

Let ω be a blowup form on M. For simplicity, we normalize ω such that $\frac{\omega(F)}{2\pi} = 1$, and we denote $u = \frac{\omega(\hat{B})}{2\pi}$ and $c = \frac{\omega(\hat{E})}{2\pi}$. Then, since both E_1 and $F - E_1$ are exceptional classes and by the definition of $\hat{E}$ and $\hat{B}$,

$$0 < c \leq \frac{1}{2},$$

and

$$u > 0.$$

Moreover, in case (b), $u > \frac{1}{2}$. Also, in case (a), $\hat{B} \cdot \hat{B} = 0$ and $c_1(TM)(\hat{B}) = 2 - 2g$ and in case (b), $\hat{B} \cdot \hat{B} = 1$ and $c_1(TM)(\hat{B}) = 3 - 2g$. In both cases, $\hat{B} \cdot F = 1$ and $\hat{B} \cdot \hat{E} = 0$.

Fix $J \in \mathcal{J}_\tau(M, \omega)$. To start, suppose that A is a homology class that is represented by a simple J-holomorphic sphere. Write

$$A = p\hat{B} + qF - r\hat{E},$$

with p, q, r integers. We claim that the coefficient p of $\hat{B}$ is nonnegative and that if this coefficient is zero then A is one of the classes F, $\hat{E}$, and $F - \hat{E}$.

Indeed, the adjunction inequality [27, Cor. E.1.7] says that

$$0 \leq 2 + A \cdot A - c_1(TM)(A).$$

In case (a), $A \cdot A = 2pq - r^2$ and $c_1(TM)(A) = (2 - 2g)p + 2q - r$, so we get

$$0 \leq 2(p-1)(q-1) - r(r-1) + 2gp \tag{A.2}$$

In case (b), $A \cdot A = p^2 + 2pq - r^2$ and $c_1(TM)(A) = (3 - 2g)p + 2q - r$, so we get

$$0 \leq 2(p-1)(q-1) - r(r-1) + p^2 + 2gp - p \tag{A.3}$$

Because A is represented by a J-holomorphic sphere for an ω-tamed J, the symplectic area of A is positive, i.e.,

$$0 < up + q - cr. \tag{A.4}$$

Also note that, because r is an integer, $r(r-1) \geq 0$ with equality only if $r = 0$ or $r = 1$.

By Lemma [28, Lemma 1.2], the minimal exceptional class $\hat{E}$ is represented by a simple J-holomorphic sphere, hence, and by the assumption on A and the positivity of intersections of J-holomorphic curves in almost complex four-manifolds [27, Proposition 2.4.4], we get that either $A = \hat{E}$, i.e. $p = 0$, $q = 1$, $r = 1$, or $r = A \cdot \hat{E} \geq 0$. Assume the latter case.

Assume that $p \leq 0$.

- In case (a), first assume $2q - r \geq 1$. Thus (since g is positive and $p \leq 0$) we have $c_1(TM)(A) = (2 - 2g)p + 2q - r \geq 2q - r \geq 1$, so, by Lemma A.1, we get $p = A \cdot F \geq 0$; hence $p = 0$ and, by (A.2), $-2q + 2 \geq r(r-1) \geq 0$, which yields, since $q \geq \frac{1}{2}(r+1) > 0$ and $r(r-1) \geq 0$, that either $q = 1$ and $r = 0$ or $q = 1$ and $r = 1$, i.e., either $A = F$ or $A = F - \hat{E}$. Now assume $2q - r < 1$,

i.e., $2q-1 \le r-1$. By (A.4), and since $-p$, r, c, u are all nonnegative, we have $q > cr - up \ge 0$. So $0 \le q-1$ and $0 \le 2(q-1) < (r-1)$. We deduce then that

$$\begin{aligned} 2(p-1)(q-1) - r(r-1) + 2gp & \\ &< 2(p-1)(q-1) - r(2(q-1)) + 2gp \\ &= 2(q-1)(p-1-r) + 2gp \le 0 \end{aligned}$$

contradicting (A.2).

- In case (b), first assume $p+2q-r \ge 1$. Thus $c_1(TM)(A) = (2-2g)p + p + 2q - r \ge p + 2q - r \ge 1$, so, by Lemma A.1, we get $p = A \cdot F \ge 0$; hence it follows from (A.3) as before that either $A = F$ or $A = F - \hat{E}$. Now assume $p + 2q - r < 1$, i.e., $p + 2q - 1 < r$. By (A.4), and since $-p$, r, c are all nonnegative and $u > \frac{1}{2}$, we have $0 < up + q - cr < \frac{1}{2}p + q$. So $0 \le p + 2q - 2 < r - 1$. We deduce then that

$$\begin{aligned} 2(p-1)(q-1) - r(r-1) + p^2 + 2gp - p = &\ (p-1)(2q-2+p) - r(r-1) + 2gp \\ <&\ (p-1)(2q-2+p) - r(2q-2+p) + 2gp \\ =&\ (p-1-r)(2q-2+p) + 2gp \le 0 \end{aligned}$$

contradicting (A.3).

Having proved the claim, we proceed to prove that there exists an embedded J-holomorphic sphere in the class $F - \hat{E}$. This will complete the proof.

By Lemma 4.5, we have $\mathrm{GW}(F - \hat{E}) \neq 0$, hence by Gromov's compactness theorem, we can write $F - \hat{E} = C_1 + \ldots + C_N$ where each C_i is represented by a simple J holomorphic sphere. We would like to show that $N = 1$; the simple J holomorphic sphere in the class $F - \hat{E} = C_1$ would then be embedded by the adjunction formula [27, Cor. E.1.7]. Write each C_i as a combination of $\hat{B}$, F, and $\hat{E}$. By the previous claim, all the coefficients of $\hat{B}$ are nonnegative. Because they sum to zero, they must all be equal to zero. Applying the previous claim again, every C_i is either F, or $\hat{E}$, or $F - \hat{E}$. In particular, all the coefficients of F are nonnegative integers. Because they sum to 1, the coefficient of F in all but one of the C_i-s are equal to zero, and in one of the C_i-s, say in C_1, the coefficient of F is equal to 1, so C_1 is either F or $F - \hat{E}$. Because the coefficients of $\hat{E}$ sum to -1, at least one of the C_i-s is $F - \hat{E}$. We conclude that $C_1 = F - \hat{E}$, and $N = 1$. □

Appendix B. Algorithm to Count Circle Actions

by Tair Pnini[1]

Introduction. This is a "greedy" algorithm. It starts by creating the set of Graphs determined by λ_F, λ_B. Then, at each stage BlowupGraph creates the set of Graphs (up to equivalence) which may be obtained by performing a blowup in the given size on a Graph from the set received in the previous step. The output of the program is the number of Graphs in the set at the final stage. In order to optimize the program, we use data structures designed to reduce the number of tests for equivalence between pairs of Graphs and to optimize the equivalence test.

Data Structure. For each Graph we arbitrarily define one of the two fat vertices to be the bottom fat vertex and the other to be the top fat vertex. In addition, we keep the distance between these two vertices and define it as the **Height** of the Graph, λ_F in this case. We relate to different paths between the two fat vertices as **Chains**. A Chain will be represented by a doubly linked list of rational positive numbers, consists of the serial representation of the path, built from vertices and edges that appear one by one alternately. Each vertex is encoded by its distance from the bottom fat vertex and each edge is encoded by its label. The list representing each Chain starts with the lowest non-fat vertex in the Chain and ends with the highest non-fat vertex in the Chain (since edges between a fat vertex and a non-fat vertex are all labeled 1, we do not need to store them). For each list we will keep its first and its last nodes. In addition, every Chain in the Graph will receive a unique serial ID number, so that in a Graph

[1]Dept. of Mathematics, Physics, and Computer Science, University of Haifa, at Oranim, Tivon 36006, Israel
E-mail address: tair.pnini@gmail.com

with n Chains the IDs would be from 0 to $(n-1)$. The Chains of a Graph are kept in an array so that each Chain is put in its ID number cell of the array.

We define two different sorting methods on Chains; SortByStart and SortByEnd. Sorting by start is a lexicogaphic sort in which each node of the list is considered a letter. Sorting by end is defined by the same principle, except that the lists is read backwards and the value encoding each vertex is its distance from the top fat vertex (equals the Height of the Graph minus its distance from the bottom). Notice that sorting the Chains of a Graph by end will give the same result as sorting the Chains of the flipped Graph by start. For each Graph we will keep two additional arrays, keeping the IDs of the Chains. Both arrays keep the IDs in order, according to the corresponding Chains sort; one according to the "sort by start" and the other according to the "sort by end". See Figure B.1.

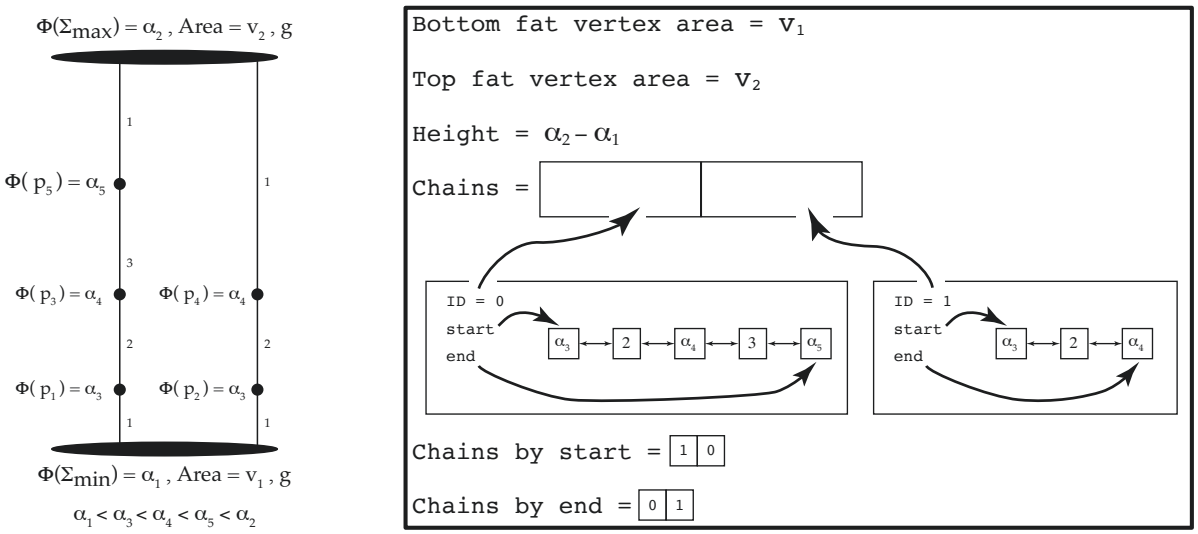

FIGURE B.1. **On the left** is a decorated graph. **On the right** is the corresponding data structure.

Another structure we use is referred to as **"Graphs Tree"**. It is a balanced binary search tree, storing a set of non-equivalent Graphs. Each node in the tree contains two fields - a key field and a field containing a linked list of non-equivalent Graphs of that key. The key of a Graph is encoding the areas of the two fat vertices in the Graph - the larger area first, and the number of the Chains in the Graph. Notice that two Graphs may be equivalent only if they have the same key, meaning that they are stored in the same Graphs Tree node. The operation of adding a Graph to the Graphs Tree would be done only if there is no equivalent Graph already kept in the Graphs Tree.

Algorithms. The main algorithm counts the number of non-equivalent Graphs that can be produced as blowups of sizes $\delta_1, \ldots, \delta_n$ from the Graphs determined by λ_B and λ_F. It calls on the remaining algorithms defined in this section.

Algorithm B.2 — CountGraphs $(\lambda_F, \lambda_B; \delta_1, \delta_2, \ldots, \delta_n)$.
∘ *Input*: Two positive real numbers λ_F, λ_B; and n positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$, in decreasing order.
∘ *Output*: The number of non-equivalent Graphs that can be created by performing the blowups sequentially, starting from the Graphs that are determined by λ_B and λ_F.

1: **Create** a Graphs Tree structure gt (as described in the data structures section) of the Graphs defined by λ_B, λ_F using Graphs$(\lambda_F, \lambda_B;)$.
2: **Create** a Graphs Tree structure t of all of the non-equivalent Graphs obtained from performing the blowups of sizes $\delta_1, \delta_2, \ldots, \delta_n$ sequentially, in every possible way, on each of the Graphs in gt, using BlowupSet$(gt, \delta_1, \delta_2, \ldots, \delta_n)$.
3: **Return** the number of Graphs in t.

Algorithm B.3 — GRAPHS $(\lambda_F, \lambda_B;)$
∘ *Input*: Two positive real numbers λ_F, λ_B
∘ *Output*: Graphs Tree structure gt, storing the Graphs defined by λ_B and λ_F

1: **for** $i = 0$ to $\left\lceil \frac{\lambda_B}{\lambda_F} \right\rceil - 1$
2: **Create** a Graph with the following data:
`bottom fat vertex area =` $\lambda_B + i\lambda_F$
`top fat vertex area =` $\lambda_B - i\lambda_F$
`Height =` λ_F
`Chains =` none
3: **Add** the Graph created to the Graphs Tree structure gt
4: **Return** gt

Algorithm B.4 — BLOWUPSET $(gt, \delta_1, \delta_2, \ldots, \delta_n)$
∘ *Input*: A Graphs Tree structure gt and n positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ in decreasing order.
∘ *Output*: Graphs Tree structure t, storing the non-equivalent Graphs that can be created by performing the blowups sequentially, starting from the Graphs in gt.

1: **for** $i = 1$ to n
2: **Define** t as a new empty Graphs Tree
3: **for** each Graph g in gt
4: **Create** all the Graphs that can be obtained by performing a blowup of size δ_i on g (perform the blowup in every possible way in g), using BLOWUPGRAPH(g, δ_i).
5: **Add** each Graph that have been created in the previous step to the Graphs Tree t using ADDGRAPH(g, t) (an algorithm that ensures that a Graph would be added if and only if there is no equivalent Graph in the Graphs Tree).
6: **Set** $gt = t$
7: **Return** gt

Algorithm B.5 — BlowupGraph(g, δ)
∘ *Input*: A Graph g and a positive number δ
∘ *Output*: A list *list* of all the Graphs can be obtained by performing a blowup of size δ on the Graph g

if a blowup can be performed at the bottom fat vertex
(that is, if $\delta <$ `bottomFatVertexArea`) **then**
Create a deep copy g' of Graph g. Let n be the number of Chains in g, make the size of the arrays in g' (Chains, ChainsByStart, ChainByEnd) be $n + 1$.
Subtract δ from the area of the bottom fat vertex.
Add a new Chain with one node δ to g'. Let n be the number of Chains in g, the ID of the new Chain will be n and a pointer to the new Chain will be placed in Chains[n].
Use binary search and the sorting methods defined in the data structure section to find the right place for the ID of the new Chain in the ChainsByStart and ChainsByEnd arrays and place it there (move other IDs to the right if needed to "make place").
Add g' to the *list*.
if a blowup can be performed on the top fat vertex ($\delta <$ `topFatVertexArea`) **then**
Do the same as when a blowup can be performed at the bottom fat vertex, except subtract δ from the top fat vertex area and the new Chain will have the node height $-\delta$
for every Chain *chain*
for every node v that represents a vertex in *chain*
if a monotone blowup can be made on the vertex **then**
Create a new Chain replacing to *chain* with a blowup on v, with the same ID.
Create a deep copy g' of the Graph g and replace the copy of *chain* with the new Chain that have been created.
Use binary search and the sorting methods described above, to find the right place for the ID of the new Chain in the ChainsByStart and ChainsByEnd arrays and place it there (move other IDs to the right if needed to "make place").
Add g' to *list*.
Return *list*

Algorithm B.6 — AddGraph(g, t)
∘ *Input*: A Graph g and a Graph Tree structure t
∘ *Output*: The Graph g would be added to the Tree structure t if there is no equivalent Graph to g in t

Get the node from t with the same key as the key of the Graph g.
if there is no such node in t **then**
    add a new node to t with the key of g.
    Set the list of Graphs of this node to contain g.
else
    for each Graph g_2 in the list of Graphs of the received node
        if g is equivalent to g_2, (to be checked using AreEquivalent(g, g_2)), **then**
        return
    add g to the list of Graphs in the node that matches to its key.

Algorithm B.7 — AreEquivalent(g_1, g_2)
∘ *Input*: A Graph g_1 and a Graph g_2 . Both g_1 and g_2 have the same key according to the Graphs Tree structure
∘ *Output*: True, if the two Graphs are equivalent and False otherwise

if g_1.bottomFatArea = g_1.topFatArea
    Return AreTheSame(g_1, g_2) or AreReflection$(g1, g2)$
else
    if (g_1.bottomFatArea=g_2.bottomFatArea)
        Return AreTheSame(g_1, g_2)
    else
        Return AreReflection(g_1, g_2)

Algorithm B.8 — AreTheSame(g_1, g_2)
∘ *Input*: A Graph g_1 and a Graph g_2. Both g_1 and g_2 have the same key according to the Graphs Tree structure. g_1 and g_2 fat vertices area are equals respectively.
∘ *Output*: True, if the two Graphs are exactly the same; False otherwise.

for $i = 0$ to the number of Chains in g_1 and g_2
    if not g_1.Chain[g_1.ChainsByStart[i]] equals g_2.Chain[g_2.ChainsByStart[i]]
        Return false
Return true

Algorithm B.9 — AreReflection(g_1, g_2)
∘ *Input*: Graph g_1 and a Graph g_2 . Both g_1 and g_2 have the same key according to the Graphs Tree structure. g_1 first fat area equals g_2 last fat area and vice versa.
∘ *Output*: True, if the two Graphs are a reflection of one another; False otherwise.

for $i = 0$ to the number of Chains in g_1 and g_2
    if not g_1.Chain[g_1.ChainsByStart[i]] is a
    reflection of g_2.Chain[g_2.ChainsByEnd[i]] (check node by node,
    one Chain from the start and the other from the end, labels should be
    equal and a vertex should be equal to the height of the Graph minus
    the corresponding one.) **then**
        Return false
Return true

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Dept. of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA
E-mail address: `tsh@math.cornell.edu`

Dept. of Mathematics, Physics, and Computer Science, University of Haifa, at Oranim, Tivon 36006, Israel
E-mail address: `lkessler@math.haifa.ac.il`