Optimal Control With Maximum Cost Performance Measure

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Abstract— **Optimal control of discrete-time systems with a less common performance measure is investigated, in which the cost function to be minimized is the maximum, instead of the sum, of a cost per stage over the control time. Three control scenarios are studied under a finite-horizon, a discounted infinite-horizon, and an undiscounted infinite-horizon performance measure. For each case, the Bellman equation is derived by direct use of dynamic programming, and the necessary and sufficient conditions for an optimal control are established around this equation. A motivating example on optimal control of dc-dc buck power converters is presented.**

I. Introduction

The most common performance measure in the theory of optimal control is the sum of a cost per stage (or running cost) over the control horizon. This paper investigates a class of optimal control problems with an alternative performance measure defined as the maximum, rather than the sum, of the cost per stage over the control horizon. Similar to other optimal control problems, the control horizon can be finite or infinite, the cost function can be discounted or undiscounted, and the system model can be continuous or discrete in time. The focus of this paper is on discrete-time systems.

Optimal control with the maximum cost per stage optimality criteria has been extensively studied for the continuous-time systems [1]–[9]. Prior work on this topic includes application of both Pontryagin's maximum principle [1]–[3] and dynamic programming [4]–[6] to finite-horizon problems, and the use of dynamic programming for discounted [7] and undiscounted [8] infinite-horizon problems. In particular, application of dynamic programming has led to development of the Hamilton-Jacobi-Bellman (HJB) equations for all these scenarios, and the value function has been constructed as the viscosity solution to these partial differential equations. In [9], an approximate solution to the HJB equation for a discounted, infinite-horizon problem was proposed, and this solution was interpreted as the value function for an associated discrete-time problem.

With the exception of [9] which establishes a link between continuous- and discrete-time domains, very few results have been reported on optimal control of discrete-time systems with maximum cost performance measure (one of those few [10] is discussed later in Remark 3). Therefore, this paper aims to fill this gap by contributing a formal analysis of the problem using dynamic programming and its related familiar techniques. An optimal control problem involving a discrete-time system and a maximum cost performance measure is addressed for finite and infinite control horizons, and discounted and undiscounted cost functions. For each case, the Bellman functional equation is developed and both necessary and sufficient conditions for an optimal control policy are established in terms of this equation.

The main analysis of this paper begins with the study of the infinite-horizon discounted problem, and then continues with the undiscounted problem as the limit of the discounted case. It is shown that the Bellman equation for the discounted problem admits a unique solution, while the solution is not unique for the undiscounted case. This non-uniqueness of solution is also observed in continuous-time problems [7], and is an obstacle to establish sufficient conditions of optimality. Yet, the analysis of this paper shows that the value function for the undiscounted problem is that specific solution to the Bellman equation which is the continuation of the solution to the discounted problem.

To demonstrate the practical significance of these results, an example on optimal control of dc-dc buck power converters is presented. The main emphasis of this example is to formulate a practical feedback design problem within the framework of this paper, rather than detailed solution of the problem, which is beyond the scope of this short paper.

II. Problem Statement

Suppose $\mathcal{D}$ is a domain in $\mathbb{R}^n$ and $\mathcal{U}$ is any subset of $\mathbb{R}^m$. Let $f(\cdot) : \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ be a continuous function and consider the state-space equation

$$x_{t+1} = f(x_t, u_t), \quad t = 0, 1, 2, \ldots \tag{1}$$

with a known initial state x_0. Here, $x_t \in \mathcal{D}$ is the state vector and u_t is a control vector in the control set $\mathcal{U}$. This system is controlled under the state feedback

$$u_t = \mu_t(x_t), \quad t = 0, 1, 2, \ldots \tag{2}$$

utilizing the feedback law $\mu_t(\cdot) : \mathcal{D} \to \mathcal{U}$, $t = 0, 1, 2, \ldots$. The control performance is measured by a maximum cost criterion defined in terms of the cost per stage $c(\cdot) : \mathcal{D} \times \mathcal{U} \to \mathbb{R}$ and the terminal cost $c_f(\cdot) : \mathcal{D} \to \mathbb{R}$.

For the discrete-time state-space equation (1), the following optimal control problems are considered in this paper:

P_1. Let T be a fixed integer and define the cost function

$$J = \max\left\{ c_f(x_T), \max_{t=0,1,\ldots,T-1} c(x_t, u_t) \right\} \tag{3}$$

for the state and control of the state-space equation (1). Determine an optimal control policy

$$\pi_T^* = \left\{ \mu_0^*(\cdot), \mu_1^*(\cdot), \ldots, \mu_{T-1}^*(\cdot) \right\} \tag{4}$$

to minimize this cost function subject to the dynamical system (1) under the state feedback (2).

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P_2. Let $0 < \alpha < 1$ be a constant discount rate and define the discounted cost function

$$J = \sup_{t \geqslant 0} \alpha^t c(x_t, u_t). \tag{5}$$

Determine an optimal control policy π_∞^* to minimize this cost function subject to the dynamical system (1) under the state feedback (2).

P_3. Determine an optimal control policy π_∞^* to minimize the undiscounted cost function

$$J = \sup_{t \geqslant 0} c(x_t, u_t) \tag{6}$$

subject to (1) under the state feedback (2).

Among the three problems above, P_3 is the most difficult to address, and indeed, the primary motivation for this work. The reason for emphasis on this specific problem is its potential to tackle a class of problems not possible to formulate in terms of an infinite-horizon cost function of the standard form

$$J = \sum_{t=0}^{\infty} c(x_t, u_t). \tag{7}$$

For many applications, the infinite sum in this cost function is not convergent, since the cost per stage $c(x_t, u_t)$ does not tend to 0 as $t \to \infty$. For the same cost per stage, however, the cost function (6) can still exist and be well-defined. An example of such applications is presented in Section IV.

III. Bellman Equation and Optimal Control

Problems P_1 through P_3 are addressed in this section with the same order they are stated in Section II. The solution to P_1 is a straightforward application of dynamic programming, and is briefly discussed in Section III-A as a point of departure. In Section III-B, problem P_2 is considered, its Bellman equation is derived, and a recursive construction of the solution to this equation is presented. Problem P_3 is treated as a limiting case of P_2 (i.e., $\alpha \uparrow 1$) in Section III-C.

A. Finite-Horizon Cost Function

The solution to problem P_1 is derived simply by application of dynamic programming, as stated in Proposition 1 below.

Proposition 1: Let the cost per stage $c(\cdot)$ and the terminal cost $c_f(\cdot)$ in the cost function (3) be continuous and consider the sequence of value functions $J_k^*(\cdot) : \mathcal{D} \to \mathbb{R}$ generated by

$$J_k^*(x) = \inf_{u \in \mathcal{U}} \max \left\{ c(x, u), J_{k+1}^*(f(x, u)) \right\} \tag{8}$$

recursively for $k = T-1, T-2, \ldots, 0$ with $J_T^*(\cdot) = c_f(\cdot)$. For any control sequence $u_0, u_1, \ldots u_{T-1}$ applied to the dynamical system (1) with the initial state x_0, define the cost function J according to (3). Then, $J_0^*(x_0)$ is a lower bound of J, which is attained by an optimal control policy of the form (4) with the feedback law $\mu_t^*(\cdot)$, $t = 0, 1, \ldots, T-1$ defined as

$$\mu_t^*(x) \in \arg\min_{u \in \mathcal{U}} \max \left\{ c(x, u), J_{t+1}^*(f(x, u)) \right\}, \tag{9}$$

provided that the minimum on the right-hand side exists.

Proof: Taking $J_k^*(\cdot)$ as the optimal cost-to-go, the proof is a straightforward application of dynamic programming. ∎

Remark 1: In Proposition 1 (and in the remainder of this paper), the optimal feedback law is not necessarily unique. In fact, any member in the set of minimizers of the right-hand side of (9) can be a valid optimal feedback law.

B. Discounted Infinite-Horizon Cost Function

The solution to problem P_2 is derived in three steps from the principle of optimality. First, the Bellman equation associated to this problem is introduced in Proposition 2, and it is shown that the value function of the problem must necessarily solve this equation. Next, Proposition 3 verifies the uniqueness of the solution to this equation and proposes a recursive construction of this solution based on the concept of contraction mapping. Finally, Theorem 1 presents the optimal control policy for P_2 and shows this policy is stationary.

Proposition 2: In the cost function (5), let $0 < \alpha < 1$ and assume that the cost per stage $c(\cdot)$ is nonnegative and bounded above by $\bar{c} < \infty$, that is

$$0 \leqslant c(x, u) \leqslant \bar{c} < \infty, \quad (x, u) \in \mathcal{D} \times \mathcal{U}. \tag{10}$$

Let $u_0, u_1, u_2, \ldots \in \mathcal{U}$ be any control sequence applied to the dynamical system (1) with the initial state $x_0 = x \in \mathcal{D}$, and define the value function $V(\cdot) : \mathcal{D} \to [0, \bar{c}]$ as

$$V(x) = \inf_{u_0, u_1, u_2, \ldots \in \mathcal{U}} \sup_{t \geqslant 0} \alpha^t c(x_t, u_t), \quad x \in \mathcal{D}. \tag{11}$$

Then, $V(\cdot)$ exists and is the uniform limit of the sequence of functions $V_k(\cdot) : \mathcal{D} \to [0, \bar{c}]$ generated recursively by

$$V_{k+1}(x) = \inf_{u \in \mathcal{U}} \max \{ c(x, u), \alpha V_k(f(x, u)) \} \tag{12}$$

for $k = 0, 1, 2, \ldots$ with the initial value $V_0(\cdot) = 0$. Moreover, this limit necessarily solves the Bellman equation

$$V(x) = \inf_{u \in \mathcal{U}} \max \{ c(x, u), \alpha V(f(x, u)) \}, \quad x \in \mathcal{D}. \tag{13}$$

Proof: For every $k = 1, 2, 3, \ldots$, the inequality

$$\sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x_t, u_t) \leqslant \sup_{t \geqslant 0} \alpha^t c(x_t, u_t) \tag{14a}$$

trivially holds. Also, the boundedness assumption (10) implies

$$\begin{aligned} \sup_{t \geqslant 0} \alpha^t c(x_t, u_t) &\leqslant \max \left\{ \sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x_t, u_t), \bar{c} \alpha^k \right\} \\ &\leqslant \sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x_t, u_t) + \bar{c} \alpha^k, \end{aligned} \tag{14b}$$

where the second inequality is concluded from the fact that

$$\max \{a_1, a_2\} \leqslant a_1 + a_2, \quad a_1, a_2 \geqslant 0.$$

The pair of inequalities (14) bound the value function $V(\cdot)$ as

$$V_k(x) \leqslant V(x) \leqslant V_k(x) + \bar{c} \alpha^k, \quad x \in \mathcal{D}, \tag{15}$$

where $V_k(\cdot)$, $k = 1, 2, 3, \ldots$ is explicitly given by

$$V_k(x) = \inf_{u_0, u_1, \ldots, u_{k-1} \in \mathcal{U}} \sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x_t, u_t).$$

By a method similar to the proof of Lemma 1 in Appendix, it is show that $V_k(\cdot)$, $k = 1, 2, 3, \ldots$ can be recursively generated by (12) starting with $V_0(\cdot) = 0$. Setting $k \to \infty$ in (15) implies that $V(\cdot)$ exists and is the uniform limit of the sequence of functions $V_k(\cdot)$. As the value function (11) exists and is well-defined, it must necessarily solve the Bellman equation (13) by the principle of optimality, as shown in Lemma 1 of Appendix. ■

The next proposition proves that the Bellman equation (13) admits a unique solution, and generalizes the recursive scheme of Proposition 2 for construction of this solution.

Proposition 3: Under the assumptions of Proposition 2, the Bellman equation (13) admits a unique solution $V(\cdot)$ for each fixed $0 < \alpha < 1$, and this solution is the uniform limit of the sequence of scalar functions $V_k(\cdot)$, $k = 1, 2, 3, \ldots$ recursively generated by (12) starting with an arbitrary $V_0(\cdot) : \mathcal{D} \to [0, \bar{c}]$.

Proof: Let $\mathcal{V}$ be the set of all functions $V(\cdot) : \mathcal{D} \to [0, \bar{c}]$ and measure the distance between two members of this set by

$$d(V(\cdot), V'(\cdot)) = \sup_{x \in \mathcal{D}} |V(x) - V'(x)|. \tag{16}$$

Define the mapping $T[\cdot] : \mathcal{V} \to \mathcal{V}$ as

$$T[V(\cdot)](x) = \inf_{u \in \mathcal{U}} \max\{c(x,u), \alpha V(f(x,u))\}. \tag{17}$$

It is shown in Lemma 2 of Appendix that $T[\cdot]$ is a contraction map on the complete metric space $\mathcal{V}$. Hence, the Banach fixed-point theorem [11, pp. 2-3] implies that $T[\cdot]$ has a unique fixed point satisfying $V(\cdot) = T[V(\cdot)]$. Moreover, this fixed point is the uniform limit of the sequence of functions generated by the recursive equation $V_{k+1}(\cdot) = T[V_k(\cdot)]$ with any arbitrary initial function $V_0(\cdot) \in \mathcal{V}$. ■

Theorem 1 below presents the solution of P_2 by introducing an optimal control policy that minimizes the cost function (5) subject to the dynamical system (1).

Theorem 1: Assume that the cost per stage $c(\cdot)$ in the cost function (5) is nonnegative and bounded above by $\bar{c} < \infty$, and that $0 < \alpha < 1$. Then, the unique solution $V(\cdot)$ to the Bellman equation (13) establishes a lower bound $J \geqslant V(x_0)$ on the cost function (5) for each initial state $x_0 \in \mathcal{D}$. Moreover, if the feedback law

$$\mu^*(x) \in \arg\min_{u \in \mathcal{U}} \max\{c(x,u), \alpha V(f(x,u))\} \tag{18}$$

exists for every $x \in \mathcal{D}$, this lower bound is achievable by the stationary optimal control policy

$$\pi^*_\infty = \{\mu^*(\cdot), \mu^*(\cdot), \mu^*(\cdot), \ldots\}$$

when applied to the dynamical system (1) according to (2).

Proof: The first statement of theorem is simply concluded from Propositions 2 and 3. To prove the second statement, let

$$J_{\mu^*}(x) = \lim_{T \to \infty} \sup_{0 \leqslant t \leqslant T} \alpha^t c(x^*_t, \mu^*(x^*_t))$$

denote the cost of feedback law (18), in which x^*_t is generated recursively from

$$x^*_{t+1} = f(x^*_t, \mu^*(x^*_t)), \quad t = 0, 1, 2, \ldots$$

with the initial state $x^*_0 = x$. By Proposition 2, $J_{\mu^*}(\cdot)$ is lower bounded by the value function $V(\cdot)$, and it is shown next that it is also upper bounded by $V(\cdot)$, implying that $J_{\mu^*}(\cdot) = V(\cdot)$.

In order to show $J_{\mu^*}(\cdot) \leqslant V(\cdot)$, consider the inequality

$$\begin{aligned}
&\sup_{0 \leqslant t \leqslant T} \alpha^t c(x^*_t, \mu^*(x^*_t)) \\
&\qquad \leqslant \max\left\{\sup_{0 \leqslant t \leqslant T} \alpha^t c(x^*_t, \mu^*(x^*_t)), \alpha^{T+1} V\left(x^*_{k+1}\right)\right\} \\
&\qquad = V(x)
\end{aligned} \tag{19}$$

and take the limit of left-hand side as $T \to \infty$. The equality in (19) can be verified by T times application of (13) and (18) via the recursive procedure

$$\begin{aligned}
&\max\left\{\sup_{0 \leqslant t \leqslant k} \alpha^t c(x^*_t, \mu^*(x^*_t)), \alpha^{k+1} V\left(x^*_{k+1}\right)\right\} \\
&\quad = \max\Bigg\{\sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x^*_t, \mu^*(x^*_t)), \\
&\qquad\qquad \alpha^k \max\{c(x^*_k, \mu^*(x^*_k)), \alpha V(f(x^*_k, \mu^*(x^*_k)))\}\Bigg\} \\
&\quad = \max\left\{\sup_{0 \leqslant t \leqslant k-1} \alpha^t c(x^*_t, \mu^*(x^*_t)), \alpha^k V(x^*_k)\right\}
\end{aligned} \tag{20}$$

for $k = T, T-1, \ldots, 1$, and one time direct application of (13) and (18) for $k = 0$. ■

The following proposition presents sufficient conditions for existence of the feedback law (18).

Proposition 4: Let the assumptions of Theorem 1 hold, and in addition, assume $c(\cdot)$ is continuous and $\mathcal{U}$ is compact. Then, the value function $V(\cdot)$ is continuous and the optimal feedback law (18) exists.

Proof: It is shown by induction that the functions $V_k(\cdot)$ generated by (12) are continuous for all $k = 0, 1, 2, \ldots$. First at $k = 0$, continuity of $V_0(\cdot) = 0$ is trivial. Assume that $V_k(\cdot)$ is continuous. Then, since $c(\cdot)$ and $f(\cdot)$ are continuous, the function under the $\inf$ operator on the right-hand side of (12) is continuous in u, and therefore, the infimum over the compact set $\mathcal{U}$ is replaced by minimum. Then, by Berge's maximum theorem [12, p. 570], $V_{k+1}(\cdot)$ is continuous.

By Proposition 2, the sequence of functions $V_k(\cdot)$ uniformly converges to $V(\cdot)$, and since $V_k(\cdot)$ is continuous for every k, the uniform limit theorem [12, p. 54] implies that $V(\cdot)$ is also continuous. Then, the minimum on the right-hand side of (18) exists over the compact set $\mathcal{U}$ and is attained by some $\mu^*(x)$. ■

C. Undiscounted Infinite-Horizon Cost Function

The undiscounted problem P_3 is treated in this section as the limiting case of its discounted counterpart P_2 when $\alpha \uparrow 1$. It is shown that the key results of Section III-B for the discounted problem P_2 can be extended to the undiscounted case P_3, albeit under more restrictive assumptions. First, Proposition 5 shows that the solution to the discounted Bellman equation (13) has a limit as $\alpha \uparrow 1$, and proposes a recursive construction for this limit. Next, Proposition 6 presents a set of assumptions under which this limit resolves the Bellman equation associated with problem P_3. Under these assumptions, Proposition 7 provides a solution for the undiscounted problem P_3. The disadvantage of this proposition is that its conditions are not straightforward to verify. Hence, Theorem 2 slightly narrows down the scope of this proposition to restate it with easily verifable conditions, namely, continuity of $c(\cdot)$ and compactness of $\mathcal{U}$.

Proposition 5: Assume that the cost per stage $c(\cdot)$ satisfes the boundedness condition (10), and denote the unique solution to the Bellman equation (13) by $V_\alpha(\cdot)$ for each $0 < \alpha < 1$. Then, for each fxed $x \in \mathcal{D}$, the limit

$$W(x) \triangleq \lim_{\alpha \uparrow 1} V_\alpha(x), \quad x \in \mathcal{D} \tag{21}$$

exists. Moreover, the function $W(\cdot)$ denoting this limit can be equivalently expressed as the limit of the sequence of functions $W_k(\cdot) : \mathcal{D} \to [0, \bar{c}]$, $k = 1, 2, 3, \ldots$ generated recursively by

$$W_{k+1}(x) = \inf_{u \in \mathcal{U}} \max\{c(x,u), W_k(f(x,u))\} \tag{22}$$

starting from the initial value $W_0(\cdot) = 0$.

Proof: For each fxed $x \in \mathcal{D}$, $V_\alpha(x)$ is increasing in α as indicated by (11) and $c(\cdot) \geqslant 0$. Moreover, $V_\alpha(x)$ is bounded above by $\bar{c}$ over $\alpha \in (0,1]$, which implies the limit (21) exists in the pointwise sense.

To prove the second statement, consider the sequence of functions in Proposition 2 generated recursively by (12) with the initial value $V_0(\cdot) = 0$. To emphasize the dependence of these functions on α, they are denoted by $V_{\alpha,k}(\cdot)$ in this proof. Then, by Proposition 2, the limit (21) can be expressed as

$$W(x) = \lim_{\alpha \uparrow 1} \lim_{k \to \infty} V_{\alpha,k}(x).$$

However, $V_{\alpha,k}(x)$ is bounded above by $\bar{c} < \infty$ and increasing in both α and k, so that the order of limits can be interchanged [both double limits are equal to $\sup_{k=1,2,3,\ldots,\, \alpha \in (0,1]} V_{\alpha,k}(x)$]. The proof is completed noting that

$$W(x) = \lim_{k \to \infty} \lim_{\alpha \uparrow 1} V_{\alpha,k}(x) = \lim_{k \to \infty} W_k(x).$$

■

The next proposition establishes suffcient conditions under which $W(\cdot)$ in (21) is the value function for problem P_3, and as a consequence, solves its associated Bellman equation.

Proposition 6: Let the assumptions of Proposition 5 hold and construct the sequence of functions $W_k(\cdot)$, $k = 1, 2, 3, \ldots$ by recursion of (22) starting with $W_0(\cdot) = 0$. Assume that the minimizer

$$\bar{\mu}_k(x) \in \arg\min_{u \in \mathcal{U}} \max\{c(x,u), W_k(f(x,u))\} \tag{23}$$

exists for every $k = 0, 1, 2, \ldots$ and every $x \in \mathcal{D}$. Then, $W(\cdot)$ defned by the limit (21) is the value function

$$W(x) = \inf_{u_0, u_1, u_2, \ldots \in \mathcal{U}} \sup_{t \geqslant 0} c(x_t, u_t), \quad x \in \mathcal{D} \tag{24}$$

for the undiscounted cost function (6) and necessarily resolves the Bellman equation

$$W(x) = \inf_{u \in \mathcal{U}} \max\{c(x,u), W(f(x,u))\}, \quad x \in \mathcal{D}, \tag{25}$$

but is not its unique solution.

Proof: Denote the value function on the right-hand side of (24) by $W'(\cdot)$. The defnition of $V_\alpha(\cdot)$ in (11) implies that for each fxed $x \in \mathcal{D}$, the value of $V_\alpha(x)$ and its pointwise limit $W(x) = \lim_{\alpha \uparrow 1} V_\alpha(x)$ cannot exceed the value $W'(x)$, that is $W(x) \leqslant W'(x)$.

Using a recursive procedure similar to (20) and $T+1$ times application of (22) and (23), it can be shown that

$$W_{T+1}(x) = \sup_{0 \leqslant t \leqslant T} c(\bar{x}_t, \bar{\mu}_{T-t}(\bar{x}_t)),$$

where $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_T$ are generated by the dynamical system

$$\bar{x}_{t+1} = f(\bar{x}_t, \bar{\mu}_{T-t}(\bar{x}_t)), \quad t = 0, 1 \ldots, T-1$$
$$\bar{x}_0 = x.$$

Taking the limit of both sides as $T \to \infty$ results in

$$\lim_{T \to \infty} \sup_{0 \leqslant t \leqslant T} c(\bar{x}_t, \bar{\mu}_{T-t}(\bar{x}_t)) = W(x).$$

However, the limit on the left-hand side is the value of the cost function (6) under some control policy not necessary optimal, so it upper bounds the value function, i.e., $W'(x) \leqslant W(x)$.

Since $W'(\cdot)$ is both upper and lower bounded by $W(\cdot)$, it is concluded that $W'(\cdot) = W(\cdot)$. Moreover, as $W(\cdot)$ is a value function, it necessarily solves the Bellman equation (25) by the principle of optimality shown in Lemma 1 of Appendix. Yet, this solution is not unique, as the constant function $W(\cdot) = \bar{c}$ also solve this Bellman equation. ■

The next proposition combines the results of Propositions 5 and 6 into a solution for the undiscounted problem P_3.

Proposition 7: Suppose the cost per stage $c(\cdot)$ in the cost function (6) is nonnegative and bounded above by $\bar{c}$. Let $V_\alpha(\cdot)$ be the unique solution to the Bellman equation (13) and defne the scalar function $W(\cdot)$ by its limit (21) as $\alpha \uparrow 1$. Then, $W(\cdot)$ establishes a lower bound $J \geqslant W(x_0)$ on the cost function (6) for each initial state $x_0 \in \mathcal{D}$. Moreover, if the feedback law

$$\mu^*(x) \in \arg\min_{u \in \mathcal{U}} \max\{c(x,u), W(f(x,u))\} \tag{26}$$

exists for $x \in \mathcal{D}$ and $W(\cdot)$ solves the Bellman equation (25), for instance under the assumptions of Proposition 6, this lower bound is attained by the stationary optimal control policy

$$\pi^*_\infty = \{\mu^*(\cdot), \mu^*(\cdot), \mu^*(\cdot), \ldots\} \tag{27}$$

when applied to the dynamical system (1) according to (2).

Proof: For any $x_0 \in \mathcal{D}$, the value function on the right-hand side of (24) is a lower bound of the cost function (6). In addition, the defnition of $V_\alpha(\cdot)$ in (11) implies that $V_\alpha(x_0)$ and its limit $W(x_0) = \lim_{\alpha \uparrow 1} V_\alpha(x_0)$ cannot exceed the value function of (6). Hence, it is concluded that $J \geqslant W(x_0)$. The rest of this proof closely parallels the proof of Theorem 1. ■

Proposition 7 characterizes the optimal control policy on the condition that $W(\cdot)$ solves the Bellman equation (25). In the rare case that $W(\cdot)$ is available in explicit form, this condition can be directly examined. In the absence of such explicit form, the existence of (23) in Proposition 6 can be examined, which is yet not straightforward. By introducing the new assumption that $\mathcal{U}$ is a compact set, the existence of (23) can be concluded. With this new assumption, the following theorem presents the solution to problem P_3 as the main result of this paper.

Theorem 2: Let $\mathcal{U}$ be a compact set and assume the cost per stage $c(\cdot)$ in the cost function (6) is continuous, nonnegative, and bounded above by $\bar{c}$. Let $V_\alpha(\cdot)$ be the unique solution to the Bellman equation (13) and define the value function $W(\cdot)$ by its limit (21) as $\alpha \uparrow 1$. Then, this value function establishes a lower bound $J \geqslant W(x_0)$ on the cost function (6) for each initial state $x_0 \in \mathcal{D}$. Moreover, if the feedback law (26) exists for every $x \in \mathcal{D}$, this lower bound is attained by the stationary optimal control policy (27) when applied to the dynamical system (1) via the state feedback (2).

Proof: Similar to the proof of Proposition 4, it is shown by induction that the scalar functions $W_k(\cdot)$ generated by (22) are continuous. Therefore, the minimum on the right-hand side of (23) exist over the compact set $\mathcal{U}$, which is achieved by the minimizer $\bar{\mu}_k(x)$. Then, Proposition 7 completes the proof. ■

Corollary 1: Let $\mathcal{D}$ and $\mathcal{U}$ be compact sets and assume that the cost per stage $c(\cdot)$ in the cost function (6) is continuous and nonnegative. Then, the results of Theorem 2 hold identically.

Proof: As $c(\cdot)$ is continuous, it has a maximum over the compact set $\mathcal{D} \times \mathcal{U}$, which is taken as its upper bound $\bar{c}$. ■

Remark 2: In the second statement of Theorem 2, attaining the minimum cost is conditioned on the existence of minimum over $\mathcal{U}$ in (26). At first glance, this may seem an unnecessary condition for a compact $\mathcal{U}$. However, this condition is indeed required, since continuity of $W(\cdot)$ is not proven in this paper. Even though $W(\cdot)$ is the limit of the sequence of continuous functions $W_k(\cdot)$, it does not necessarily inherit their continuity under the pointwise convergence proven in this paper.

Remark 3: An existence theorem in [10] states that under a set of assumptions implied by those in Corollary 1, some open-loop control exists (without explicit construction) to minimize the cost function (6). The assumptions of this theorem slightly differ from those of Proposition 4 by exchanging boundedness of $c(\cdot)$ with closeness of $\mathcal{D}$, compactness of the set of initial states, and existence of some control to keep (5) bounded.

IV. Optimal Control of Buck Power Converters

Consider the dc-dc power converter in Fig. 1(a) consisting of a linear RLC circuit and an electronic switch S to connect or disconnect a voltage supply V_0 to the circuit. The switching objective is to keep the output voltage V_C as close as possible to a fixed setpoint $0 \leqslant r \leqslant V_0$. The RLC circuit acts as a low-pass filter that attenuates the high frequencies caused by switching, and generates an output voltage V_C consisting of an average voltage $\bar{V}_C$ and small ripples around this average, as shown in Fig. 1(b). For a high performance power converter, an optimal feedback control is needed to minimize the amplitude of the ripples and maintain the average voltage $\bar{V}_C$ as close as possible to the setpoint r, despite the disturbances caused by variations in the load resistor R or the supply voltage V_0.

To establish a feedback loop, the capacitor voltage V_C and the inductor current I_L (i.e., the state of the RLC circuit) are uniformly sampled with a period T_s to send them to a feedback controller. This controller decides the state of switch S at the beginning of each sampling period and maintains it unchanged during that period of time. The sampled-data description of the power converter is given by the linear state-space equations

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t, \end{aligned}$$

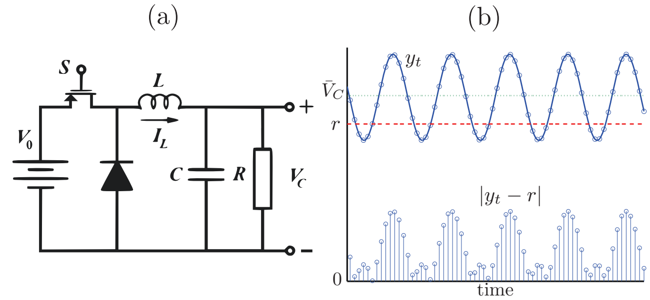

Fig. 1. Buck dc-dc power converter: (a) circuit diagram; (b) illustrative example of the output voltage V_C versus time (solid line), the setpoint r (dashed line), and the average voltage $\bar{V}_C$ (dotted line). The markers on the top of (b) illustrate the sampled output y_t and those on the bottom represent the error $|y_t - r|$.

where $x_t \in \mathbb{R}^2$ is the state vector consisting of the samples of the capacitor voltage and the inductor current, $u_t \in \mathcal{U} = \{0, 1\}$ is the control input representing the binary state of the switch, and the matrices A and B are expressed as

$$A_c = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}; \quad A = e^{A_c T_s}; \quad B = \int_0^{T_s} e^{A_c \tau} \begin{bmatrix} 0 \\ \frac{V_0}{L} \end{bmatrix} d\tau.$$

The eigenvalues of A are inside the unit circle since the RLC circuit is stable. The scalar output y_t represents the samples of the output (capacitor) voltage, and therefore, $C = [1 \quad 0]$.

The goal is to design an optimal feedback law $\mu^*(\cdot)$ to map the state vector x_t into the binary control variable u_t such that the output y_t stays as close as possible to the setpoint r over an infinite horizon. This is mathematically stated as minimization of the error $|y_t - r|$ over $t = 0, 1, 2, \ldots$, formally represented by an infinite-horizon optimal control problem with the cost per stage $c(x, u) = |Cx - r|$.

Except for the trivial cases of $r = 0$ and $r = V_0$, which are achieved by keeping the switch S only off ($u_t = 0$) or only on ($u_t = 1$), for any other value of r and under any control, the output y_t either contains ripples or is a constant not equal to r, so that the error $|y_t - r|$ cannot settle at 0. Since the cost per stage $c(x_t, u_t)$ does not vanish as $t \to \infty$, the infinite sum performance measure (7) will diverge. On the other hand, the maximum cost (6) is convergent over an infinite horizon, and formulates an optimal control problem meaningfully aimed at minimizing the maximum error. The resulting optimal control policy decides at each time t which of the controls $\mu^*(x_t) = 0$ or $\mu^*(x_t) = 1$ minimizes the cost-to-go $\sup_{s \geqslant t} |y_s - r|$.

Since A is a stable matrix and $\mathcal{U}$ is compact, it can be shown with some efforts that an arbitrarily large compact set $\mathcal{D}$ exists such that $(x, u) \in \mathcal{D} \times \mathcal{U}$ implies $Ax + Bu \in \mathcal{D}$. For such compact $\mathcal{D}$ and a compact $\mathcal{U} = \{0, 1\}$, Corollary 1 is applied to construct an optimal feedback law in three steps. First, the discounted Bellman equation

$$V_\alpha(x) = \min_{u \in \{0,1\}} \max\{|Cx - r|, \alpha V_\alpha(Ax + Bu)\} \quad (28)$$

is solved for $V_\alpha(\cdot)$ for each fixed $0 < \alpha < 1$. Next, the value function $W(\cdot)$ is obtained as the limit $W(x) = \lim_{\alpha \uparrow 1} V_\alpha(x)$. Finally, the optimal feedback law $\mu^*(\cdot)$ is determined as

$$\mu^*(x) \in \arg\min_{u \in \{0,1\}} \max\{|Cx - r|, W(Ax + Bu)\}. \quad (29)$$

The solution to the Bellman equation (28) is beyond the scope of this short paper and typically relies on approximate methods and numerical techniques such as those in [13] and references therein.

The optimal feedback law (29) can be expressed as

$$\mu^*(x) \in \begin{cases} 0, & g_0(x) < g_1(x) \\ 0|1, & g_0(x) = g_1(x) \\ 1, & g_0(x) > g_1(x), \end{cases} \tag{30}$$

where $g_i(x) = \max\{|Cx - r|, W(Ax + Bi)\}, i = 0, 1$. For any x satisfying $g_0(x) = g_1(x)$, the optimal control can take either values of 0 or 1, leading to multiple optimal feedback laws with the same cost values. In practice, only one of these multiple laws is chosen based on a specif c rule, for instance

$$\mu^*(x) = \begin{cases} 0, & W(Ax + B) - W(Ax) \geqslant 0 \\ 1, & W(Ax + B) - W(Ax) < 0. \end{cases} \tag{31}$$

It is straightforward to verify that (31) is a legitimate instance of (30) by noting that $W(Ax + B) - W(Ax) \;\; >, <, = \;0$ imply the inequalities $g_0(x) \leqslant, \geqslant, = g_1(x)$, respectively.

V. Conclusion

Optimal control of discrete-time systems under a maximum cost performance measure was studied based on the concept of dynamic programming. The performance measure was def ned as the maximum of a discounted or undiscounted cost per stage over a f nite or inf nite control horizon. For each of these cases, necessary and suff cient conditions for an optimal control were presented by developing their associated Bellman equation. A practical example on optimal control of power converters was brief y discussed to demonstrate the results of this paper.

Appendix

Lemma 1 (principle of optimality): Fix $0 < \alpha \leqslant 1$ and for the cost per stage $c(\cdot)$ def ne the value function $V(\cdot)$ by (11) as stated in Proposition 2. Then, if this function exists and is well-def ned, it necessarily solves the Bellman equation (13).

Proof: The proof is given by the sequence of operations

$$\begin{aligned} V(x) &= \inf_{u_0, u_1, u_2, \ldots \in \mathcal{U}} \sup_{t \geqslant 0} \alpha^t c(x_t, u_t) \\ &= \inf_{u_0, u_1, u_2, \ldots \in \mathcal{U}} \max \left\{ c(x_0, u_0), \sup_{t \geqslant 1} \alpha^t c(x_t, u_t) \right\} \\ &= \inf_{u_0 \in \mathcal{U}} \max \left\{ c(x_0, u_0), \inf_{u_1, u_2, \ldots \in \mathcal{U}} \sup_{t \geqslant 1} \alpha^t c(x_t, u_t) \right\} \\ &= \inf_{u_0 \in \mathcal{U}} \max\{c(x_0, u_0), \alpha V(x_1)\} \\ &= \inf_{u_0 \in \mathcal{U}} \max\{c(x_0, u_0), \alpha V(f(x_0, u_0))\} \\ &= \inf_{u \in \mathcal{U}} \max\{c(x, u), \alpha V(f(x, u))\}. \end{aligned}$$

■

Lemma 2: The set $\mathcal{V}$ of functions $V(\cdot) : \mathcal{D} \to [0, \bar{c}\,]$ paired with the uniform metric (16) is a complete metric space over which, $T[\cdot]$ in (17) is a contraction map satisfying

$$d(T[V(\cdot)], T[V'(\cdot)]) \leqslant \alpha d(V(\cdot), V'(\cdot)) \tag{32}$$

for every $V(\cdot), V'(\cdot) \in \mathcal{V}$.

Proof: Compactness of $\mathcal{V}$ is proven in [14, p. 42]. The proof of (32) relies on the sequence of operations

$$\begin{aligned} T[V(\cdot)](x) &= T[V'(\cdot) + V(\cdot) - V'(\cdot)](x) \\ &\leqslant T[V'(\cdot) + |V(\cdot) - V'(\cdot)|](x) \\ &= \inf_{u \in \mathcal{U}} \max\{c(x, u), \alpha V'(f(x, u)) \\ &\qquad + \alpha |V(f(x, u)) - V'(f(x, u))|\} \\ &\leqslant \inf_{u \in \mathcal{U}} \max\{c(x, u), \alpha V'(f(x, u)) \\ &\qquad + \alpha d(V(\cdot), V'(\cdot))\} \\ &\leqslant T[V'(\cdot)](x) + \alpha d(V(\cdot), V'(\cdot)), \end{aligned}$$

where the last inequality is concluded from

$$\max\{a_1, a_2 + a_3\} \leqslant \max\{a_1, a_2\} + a_3, \quad a_3 \geqslant 0.$$

This result is then rewritten as the inequality

$$T[V(\cdot)](x) - T[V'(\cdot)](x) \leqslant \alpha d(V(\cdot), V'(\cdot)).$$

Since $V(\cdot)$ and $V'(\cdot)$ can be interchanged in this inequality, it is concluded that

$$\left| T[V(\cdot)](x) - T[V'(\cdot)](x) \right| \leqslant \alpha d(V(\cdot), V'(\cdot)).$$

By taking the supremum of the left-hand side of this inequality over $x \in \mathcal{D}$, condition (32) of contraction mapping is verif ed.

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