Stabilizing Dynamical Systems with Fixed-Rate Feedback using Constrained Quantizers

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***Abstract*—The stabilization of unstable dynamical systems using rate-limited feedback links is investigated. In the scenario of a constant-rate link and a noise with unbounded support, the fundamental limit of communication is known, but no simple algorithm to achieve it exists. The main challenge in constructing an optimal scheme is to fully exploit the communication resources while occasionally signaling the controller that a special operation needs to be taken due to a large noise observation. In this work, we present a simple and explicit algorithm that stabilizes the dynamical system and achieves the fundamental limits of communication. The new idea is to use a *constrained quantizer* in which certain patterns of sequences are avoided throughout the quantization process. These patterns are preserved to signal the controller that a zoom-out operation should be initiated due to large noise observation. We show that the constrained quantizer has a negligible effect on the rate, so it achieves the fundamental limit of communication. Specifically, the rate-optimal algorithm is shown to stabilize any β-moment of the state if the noise has a bounded absolute $(\beta + \epsilon)$-moment for some $\epsilon > 0$ regardless of the other noise characteristics.**

***Index Terms*—Dynamical systems, fixed-rate communication, stability.**

I. Introduction

We study a hybrid control-communication system which consists of a scalar dynamical system whose observer and controller are separated by a communication link (Fig. 1). The aim is to stabilize the dynamical system while consuming the least communication resources. This leads to an inherent trade-off between the dynamical plant actuation and the communication resources that are allocated to stabilize the dynamical system. The setting is motivated by the benefits of a joint design for the communication and control components rather than the classical separation between these two themes. The aforementioned setting was widely investigated in the control literature and it represents one of the simplest networked control settings [1]–[11].

For the case of noiseless communication, allocation of communication resources conforms to either an average or a fixed rate constraint. In [12], sufficient and necessary conditions for the stabilizability of a dynamical plant with Gaussian noise were derived for the case of time-varying link rates. The more practical scenario of fixed-rate communication link was investigated in [13], where the authors proposed a (non-optimal) algorithm that uses a special *overflow symbol*. In [14],

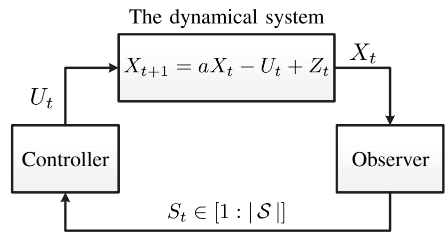

Fig. 1. A dynamical system with actuation a and fixed-rate communication of $|\mathcal{S}|$ symbols.

the authors improved this construction to be with an optimal rate, but the choice of the code parameters was not explicitly detailed or optimized over.

In this paper, we construct a new and simple algorithm for the case of fixed-rate communication link and noise with unbounded support (e.g., Gaussian noise) with bounded absolute moment. Standard approach to construct algorithms for this setting is an alternation between zoom-in and zoom-out procedures. The zoom-in mode is used for most of the time, where the observed noise is *small*, so that the observer-controller pair can maintain an interval they believe the system state lies in. In the rare events of a large noise observation a zoom-out mode is performed to increase the interval until the state is contained in it. The main challenge in constructing an algorithm for the case of unbounded noise support and fixed-rate communication link stems from signaling the controller to transit between these two modes of operation. That is, the communication link should be fully exploited in the zoom-in mode while reserving sufficient communication resources to signal that a zoom-out should be initiated. In this paper, we resolve this challenge by combining the idea of constrained coding in coding theory [15] and uniform quantizers to stabilize dynamical systems.

Our main contribution is the introduction of constrained quantizers. We construct a simple quantizer that avoids certain patterns of symbols. This allows the observer to use these patterns to signal the controller the zoom-in operation should be switched to a zoom-out operation. For any dynamical system, we provide an explicit, rate-optimal that stabilizes the dynamical system. Thus, reserving (short) patterns of sequences results in no loss of optimality. As in [7], [12], [14], the stabilizability of the algorithm relies on the system actuation and the bounded moments of the noise rather than

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its entire density function.

II. The Setting and Main Result

A scalar dynamical system is given by

$$X_{t+1} = aX_t - U_t + Z_t,$$

where $a \geq 1$ is the system actuation, U_t is a control action and Z_t is an independent and identically distributed noise (disturbance). We assume that Z_t has an α-bounded absolute moment, i.e.,

$$\mathbb{E}[|Z_t|^\alpha] \leq \rho_\alpha < \infty.$$

At time t, a (fully) observer has access to the current state X_t and its past occurrences, i.e., $X_1, \dots X_{t-1}$. Based on this information, the observer transmits a symbol $S_t \in [1 : |\mathcal{S}|]$ to the controller. The controller observes S_t and chooses a control action $U_t \in \mathbb{R}$ based on all past transmissions, i.e., $S_1, \dots, S_t$ [1]. The setting is described in Fig. 1.

A dynamical system is said to be *β-stable* if there exists a sequence of observer-controller mappings such that

$$\limsup_{t\to\infty} \mathbb{E}[|X_t|^\beta] < \infty. \quad (1)$$

The fundamental limit of communication for a dynamical system is the minimal M such that there exists an observer-controller pair that stabilize the dynamical system. In [14], it was established that, if $\beta < \alpha$, the minimal (optimal) rate is

$$M^* = \lfloor a \rfloor + 1. \quad (2)$$

The following theorem summarizes our main result.

Theorem 1 (Optimal algorithm)**.** *For any dynamical system, if $\beta < \alpha$, there exists a simple and rate-optimal algorithm (Algorithm 2) such that the dynamical system is β-stable.*

The theorem will be proven by constructing the algorithm in Section III and analyze the resulted performance in Section IV.

III. Algorithms

This section contains the algorithm to stabilize the scalar dynamical system. We will first present the main idea behind the algorithm and its main element, the constrained quantizer. Then, we will present our main algorithm.

A. Zoom-in/Zoom-out

The algorithm is composed of a repeating procedure to successively refine the controller's knowledge on the dynamical system state. Each procedure begins with an interval that is known to all parties, and corresponds to the controller's belief on the state. The observer quantizes the state on the belief interval and transmits the result to the controller. The controller then takes an action to minimize the next state, and the belief interval is updated at both parties according to a pre-determined rule.

[1]The observer and the controller mappings can be interpreted as a pair of encoder-decoder mappings with causal operation and with stability objective instead of the objective of minimizing the probability of error.

Ideally, if the state always lies in the belief interval that has a contraction property, the system is stable. The following simple claim provides a relation between the system actuation and the number of quantization points.

Lemma 1 (Maximal quantization error)**.** *Assume that $|x| \leq c$ and $Q(x)$ is a uniform quantizer over $[-c, c]$ with $|\mathcal{S}|$ levels. Then, there exists a controller action $U : \mathcal{S} \to \mathbb{R}$ such that*

$$\max_{|x|\leq c} |ax - U(Q(x))| \leq \frac{a}{|\mathcal{S}|} c.$$

Sketch of proof. Choose $U = a\hat{x}$, where $\hat{x}$ is the center of the quantization cell. □

Lemma 1 verifies that a *zoom-in* is possible when there is no noise. The lemma shows that if $a < |\mathcal{S}|$, then the quantization error can have a contraction property. That is, if $|x| \leq c$, a proper choice of the control action will shift the new state to lie in $[-c', c']$ where $c' < c$. Even if the noise has a bounded effect, the belief interval can still have a contraction property by adding a constant term that corresponds to the noise support.

However, for noise with unbounded support, there are cases where the additive noise will deviate the state from the belief interval drastically. In such cases, the belief interval must be increased in order to *catch* the system state, known as the zooming-out mode.

The crucial point in constructing an optimal algorithm is to maintain an optimal-rate zooming-in regime, while occasionally signaling the controller that a zooming-out operation should be initiated. The idea behind our algorithm is to compromise between these two regimes by operating in (small) blocks. In particular, every l times, the system state will be quantized into a sequence of l symbols from $[1 : M^*]$ that will be transmitted consecutively in the next l transmissions. The controller takes a non-zero control every l transmissions. When working in blocks, in order to have a contraction property, by Lemma 1, one should choose

$$a^l < |\mathcal{S}^l|,$$

where $|\mathcal{S}^l|$ represents the number of quantization levels every l times. In the next section, we show that a proper choice of $|\mathcal{S}^l|$ enables one to combine the zoom-out signaling without losing the optimal-rate policy during the zooming-in mode.

B. Constrained quantization

The idea behind the *constrained quantizer* comes from constrained coding in information theory [15]. Constrained coding generally refers to scenarios where certain patterns of sequences are not transmitted since they are prone to fatal errors in communication and storage models. A well-known family of constraints that are used in practice are the run-length limited (RLL) constraints. The (d, k)-RLL constraint implies that the minimal and maximal number of zeros (between two $'1'$s) are d and k, respectively.

In the context of quantization, we use the constrained quantizer as a uniform quantizer on a specified interval into

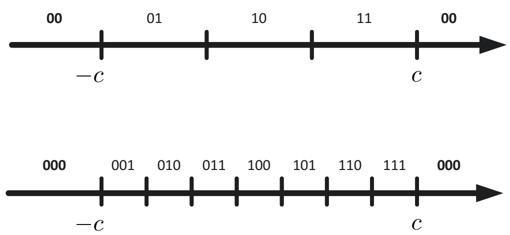

Fig. 2. An $(0, l-1)$-RLL constrained quantizer with $M = 2$ and $l = 2, 3$.

a sequence of l symbols (from $[1 : M^*]$) such that certain patterns are avoided. These avoided sequences will be preserved for the case where the state falls outside of the quantization range to signal the controller that zoom-out is needed. A simple instance of a constrained sequence is the l-zeros sequence, corresponds to the $(0, l-1)$-RLL constraint, and is described in Fig. 2.

Definition 1 (Constrained quantizer)**.** For fixed l and $|\mathcal{S}^l| = (\lfloor a \rfloor + 1)^l - 1$, the constrained quantizer is defined as

$$Q_C(x, c, l) = \begin{cases} \left(\left\lfloor \frac{(x+c)|\mathcal{S}^l|}{2c} + 1 \right\rfloor \right)_{\lfloor a \rfloor + 1} & \text{if } x \in [-c, c) \\ 0^l & \text{otherwise} \end{cases} \tag{3}$$

where $(n)_m$ is the representation of n with radix m and 0^l is the zeros sequence of length l. If the output sequence is shorter than l, then it is concatenated with zeros.

Note that if $x \in [-c, c)$, then $\left\lfloor \frac{(x+c)|\mathcal{S}^l|}{2c} + 1 \right\rfloor \in [1 : |\mathcal{S}^l|]$. In other words, the effective number of quantization levels is $|\mathcal{S}^l| = (\lfloor a \rfloor + 1)^l - 1$, and the function $Q_C(\cdot)$ is surjective.

C. Coding scheme for bounded noise support

For the purpose of studying the relation between the actuation parameter a and the length of the constrained quantizer outputs, we first present a simpler algorithm for the case of bounded noise, i.e., $|Z_t| \leq \Delta$ (w.p. 1). The optimal algorithm appears below as Algorithm 1, and its optimality is shown in the following proposition. Note that due to the definition of

Algorithm 1 Algorithm for dynamical systems with bounded noise

Inputs: state $|x| < c, l, \Delta$
procedure
  $s^l \leftarrow Q_C(x, c, l)$ ▷ *Quantization*
  $S_i \leftarrow s_i$ for $i = 1, \dots, l$ ▷ *Transmission*
  $U_i \leftarrow 0$ for $i = 1, \dots, l-1$
  $U_l \leftarrow U(s^l)$ ▷ *Control action*
  $c \leftarrow \frac{a^l}{|\mathcal{S}^l|} c + \Delta \sum_{i=0}^{l-1} a^i$ ▷ *Interval update*
end procedure

$Q_C(\cdot)$, Algorithm 1 uses the symbols $[1 : M^*]$ and transmits at the optimal rate. The following proposition reveals the connection between the constrained patterns length and the system actuation.

Proposition 1. For any dynamical system with bounded noise, Algorithm 1 stabilizes the dynamical system if

$$a < \sqrt[l]{(\lfloor a \rfloor + 1)^l - 1}. \tag{4}$$

Note that for any a, there exists l such that (4) holds. The value of l is typically small; for instance, if $a < 2$, constrained quantizers with lengths $l = 3, 4, 5$ stabilize dynamical systems with $a < 1.91, 1.97, 1.99$, respectively.

Remark 1 (Effective parameters)**.** *The condition in* (4) *can be written as* $a^l < (\lfloor a \rfloor + 1)^l - 1$. *The operational interpretation of* a^l *is the l-times actuation, while* $(\lfloor a \rfloor + 1)^l - 1$ *is the number of quantization levels every* l *times.*

Proof of Proposition 1. The l-steps dynamical system can be written as

$$X_l = a^l X_0 - U_l + \tilde{Z}_l, \tag{5}$$

where $\tilde{Z}_l = \sum_{i=0}^{l-1} a^{l-i} Z_i$. From Lemma 1, since $a^l < |\mathcal{S}^l| = (\lfloor a \rfloor + 1)^l - 1$, we have that $x \in [-c, c]$ by the end of each procedure. Moreover, the belief interval satisfies $c \to \frac{\Delta \sum_{i=0}^{l-1} a^i}{1 - \frac{a^l}{|\mathcal{S}^l|}} < \infty$, so the dynamical system is stable. □

D. Main algorithm

In this section, we present our main algorithm for the case of unbounded noise.

Main elements:

1) *The quantizer*: a constrained quantizer with $|\mathcal{S}^l|$ levels. The input to the quantizer is a state x and a constant c that corresponds to an interval $[-c, c]$. The constrained quantizer then outputs a sequence $s^l = Q_C(x, c, l)$ (Eq. (3)).
2) *The controller:* Given $s^l \neq 0^l$, it chooses a control action $U(s^l)$, as in Lemma 1.
3) *Zoom-out function*: When $|x| > c$, both the observer and the controller increase c gradually by multiplying c with $P > 1$ until $|x| < c$.
4) *Interval update:* By the end of each procedure, the belief interval is updated with design parameters r, Δ that will be specified later.

Algorithm 2 Optimal algorithm using constrained quantizers

Input: c_0, l, r, Δ
$[x, c] \leftarrow$ Zoom-Out(x, c_0, P)
procedure
  $s^l \leftarrow Q_C(x, c, l)$ ▷ *Quantization*
  $S_i \leftarrow s_i$, for $i = 1, \dots, l$ ▷ *Transmission*
  $U_i \leftarrow 0$, for $i = 1, \dots, l-1$
  if $s^l \neq 0^l$ **then**
    $U_l \leftarrow U(s^l)$ ▷ *Control action*
    $c \leftarrow r \cdot c + \Delta$ ▷ *Interval update*
  else
    $[x, c] \leftarrow$ Zoom-Out(x, c, P)
  end if
end procedure

The following theorem provides the conditions for which the stability is guaranteed with Algorithm 2.

Theorem 2. *Any dynamical system with* $\mathbb{E}[|X_0|^\alpha] \leq \rho_\alpha$ *is* β*-stable using Algorithm 2 if*

$$\begin{aligned} &\beta < \alpha \\ &P > a^{\frac{\alpha}{\alpha-\beta}}, \\ &\Delta^\alpha > \left(\frac{\ln(P^\beta)}{1 - \frac{a^\alpha}{P^{\alpha-\beta}}} \frac{a^{\alpha l}\rho_\alpha}{(1-a)^\alpha} \right) 2^{\beta-1}. \end{aligned} \tag{6}$$

Remark 2 (Non-optimal code). *A (non-optimal) code with rate* $M = \lfloor a \rfloor + 1 + N$ *can be directly constructed for any integer* N*. The motivation may be to limit the states deviation during the block, i.e., to decrease the value of* l*. The corresponding equation to be solved is then* $a < \sqrt[l]{(\lfloor a \rfloor + 1 + N)^l - 1}$.

Remark 3 (The role of each parameter:). *We provide the reasoning for the restriction on each of design parameter in* (6).

1) *The block length* l *should be large enough so that the loss caused by the constrained quantizer is negligible.*
2) *The contraction rate parameter,* r*, cannot be smaller than* $\frac{a^l}{|\mathcal{S}^l|}$ *due to Lemma 1.*
3) *The additive update parameter* Δ *is made to compensate on the additive noise* Z.
4) *The zoom-out parameter* P *should, intuitively, be greater than* a *since the intervals increase must be greater than the system actuation. However, it turns out from our analysis that* P *is also controlled by the noise moments and the stability criteria.*

IV. Proof of Theorem 2

In this section, we prove Theorem 2.

A. Technical result

The following lemma is stated independently of our setting and will be utilized throughout the proof of our main result.

Lemma 2 (Bounded sums-moment). *Let* Z_i *be random variables with* $\mathbb{E}[|Z_i|^\alpha] \leq \rho_\alpha$ *and* $a > 1$*. Then, for any* $\beta \leq \alpha$,

$$\mathbb{E}\left[|\sum_{j=0}^{i} a^{-j} Z_j|^\beta \right] \leq \rho_\alpha \left(\frac{1-a^{-i}}{1-a^{-1}} \right)^\beta.$$

Note that the upper bound is an increasing function of i. The proof appears in Section IV-C.

B. Proof of main result

The algorithm is composed of varying-lengths procedures, where the procedure ends at a random stopping time τ when $|X_\tau| < C_\tau$. The main idea of the proof is to analyze the stability of the state at the procedure ends. Then, it will be easy to conclude the stability of the state within a procedure.

Technically, the proof is comprised of three lemmas; the first corresponds to the first procedure where the encoder and the controller cannot assume $|X_0| < C_0$. The second lemma is an inductive step to show that if $|X_0| < C_0$, then the state moment by the end of the procedure has a contraction property. The third lemma is to show that the stability within a procedure.

We first analyze the stability by the end of the first procedure.

Lemma 3 (Initial zoom- out). *Let* c_0 *be a constant and* X_0 *be a random variable with* α*-bounded moment. Let* τ *be the stopping time when* $|X_\tau| < P^\tau c_0$*. If* $P > a^{\frac{\alpha}{\alpha-\beta}}$*, then*

$$\mathbb{E}[|X_\tau|^\beta] < \infty.$$

The proof of Lemma 3 appears in Section IV-C.

Remark 4. *The zooming-out phase can be improved by signaling one of the varying exponents* $\{P^0, P^1, \ldots, P^{\lfloor a \rfloor}\}$*. In this case, the condition in Lemma 3 is weaken to* $P > a^{\frac{\alpha}{\lfloor a \rfloor(\alpha-\beta)}}$.

Since we showed that $\mathbb{E}[|X_\tau|^\beta] < \infty$, we assume without loss of generality that $|X_0| \leq C_0$, and analyze the repetitive procedure of our algorithm.

Lemma 4 (Inductive stability). *Assume that* $|X_0| < C_0$*. Then, there exist* $\gamma < 1$ *and* $K > 0$ *such that*

$$\mathbb{E}[|C_{l+\tau}|^\beta] \leq \gamma\, \mathbb{E}[|C_0|^\beta] + K, \tag{7}$$

if

$$\begin{aligned} &P > a^{\frac{\alpha}{\alpha-\beta}} \\ &r \geq \frac{a^l}{(\lfloor a \rfloor + 1)^l - 1} \\ &\Delta^\alpha > \left(\frac{\ln(P^\beta)}{1 - \frac{a^\alpha}{P^{\alpha-\beta}}} \frac{a^{\alpha l}\rho_\alpha}{(1-a)^\alpha} \right) 2^{\beta-1}. \end{aligned} \tag{8}$$

By definition, we know that $|X_{l+\tau}| < C_{l+\tau}$. Therefore, Lemma 4 shows that the moment of the states by the end of the has a bounded β-moment.

The last lemma is a simple consequence of the stability at the procedures ends.

Lemma 5 (Intermediate stability). *Let* $\mathbb{E}[|X_{l+\tau}|^\beta] < \infty$*. Then,*

$$\mathbb{E}[|X_{l+i}|^\beta] \leq \mathbb{E}[|X_{l+\tau}|^\beta] + \frac{\rho_\alpha}{(1-a)^\alpha}, \tag{9}$$

for all $i = 1, \ldots, \tau$.

Proof of Lemma 5. The states evolution can be written as

$$X_{l+\tau} = a^{\tau-i} X_{l+i} + \sum_{j=0}^{\tau-i-1} a^{\tau-i-j-1} Z_{l+i+j}.$$

Then, noting that $a^{i-\tau} < 1$, applying Jensen's inequality and Lemma 2 gives

$$\mathbb{E}[|X_{l+i}|^\beta] \leq 2^{\beta-1}\, \mathbb{E}[|X_{l+\tau}|^\beta] + 2^{\beta-1} \frac{\rho_\alpha}{(1-a^{-1})^\beta}.$$

□

We are now ready to show the proof of the main result.

Proof of Theorem 1. The proof is a combination of the lemmas above. First, by lemma 3, the moment by the end of the first procedure the state is bounded. By Lemma 4, the limiting moments by the end of the procedures satisfy $\lim_{k\to\infty}\mathbb{E}[|X_{l+\bar{\tau}_k}|^\beta] \leq \frac{K}{1-\gamma}$, where $\bar{\tau}_k = \sum_{i=1}^k (l+\tau_k)$. Finally, Lemma 5 gives that stability by the procedure ends implies a stability within a procedure. □

C. Proof of Lemmas $2-4$

Proof of Lemma 2. We use the Minkowski's inequality repeatedly along with $\mathbb{E}[|Z_j|^\beta] \leq \rho_\alpha$:

$$\begin{aligned}\mathbb{E}\left[|\sum_{j=0}^{i} a^{-j}Z_j|^\beta\right]^{\frac{1}{\beta}} &\leq \mathbb{E}[|a^{-i}Z_i|^\beta]^{\frac{1}{\beta}} + \mathbb{E}\left[|\sum_{j=1}^{i-1} a^{-j}Z_j|^\beta\right]^{\frac{1}{\beta}} \\ &\leq a^{-i}\rho_\alpha^{\frac{1}{\beta}} + \mathbb{E}\left[|\sum_{j=1}^{i-1} a^{-j}Z_j|^\beta\right]^{\frac{1}{\beta}} \\ &\leq \rho_\alpha^{\frac{1}{\beta}}\left(\sum_{j=0}^{i} a^{-j}\right).\end{aligned}$$

□

Proof of Lemma 3 - Initial zoom-out. We use Holder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ to bound

$$\begin{aligned}\mathbb{E}[|X_\tau|^\beta] &= \mathbb{E}[a^{\beta\tau}|X_0 + \sum_{j=0}^{\tau} a^{-j}Z_j|^\beta] \\ &\leq \mathbb{E}[(a^{\beta\tau})^q]^{\frac{1}{q}}\,\mathbb{E}[|X_0 + \sum_{j=0}^{\tau} a^{-j}Z_j|^{\beta p}]^{\frac{1}{p}}. \qquad (10)\end{aligned}$$

We will show that each of the expected values is bounded. From Lemma 2, the expected value over the sum converges if $\beta p \leq \alpha$. Thus, we choose $p=\frac{\alpha}{\beta}$ and $q=\frac{\alpha}{\alpha-\beta}$.

For the first expectation in (10), we use the tail-sum formula and Markov inequality on $\Pr[\tau \geq i]$. First, note for the following inclusion,

$$\{\tau \geq i\} \subseteq \left\{c_0 P^i < |a^i(X_0 + \sum_{j=0}^{i} a^{-j}Z_j)|\right\}.$$

By this inclusion, Markov inequality and Lemma 2,

$$\Pr[\tau \geq i] \leq \frac{\mathbb{E}[|X_0 + \sum_{j=0}^{\infty} a^{-j}Z_j|^\alpha]}{c_0^\alpha\left(\frac{P}{a}\right)^{i\alpha}}.$$

The first expectation in (10) can now be bounded as

$$\begin{aligned}\mathbb{E}[(a^{\beta q})^\tau] &= \ln(a^{\beta q})\sum_{i=1}^{\infty}(a^{\beta q})^i \Pr[\tau \geq i] \\ &\leq \ln(a^{\beta q})\frac{\rho_\alpha}{c_0^\alpha(1-a)^\alpha}\sum_{i=1}^{\infty}(a^{\beta q})^i\left(\frac{a}{P}\right)^{i\alpha}.\end{aligned}$$

Recall that $q=\frac{\alpha}{\alpha-\beta}$, so a necessary condition for the sum convergence is $P > a^{\frac{\alpha}{\alpha-\beta}}$. □

Proof of Lemma 4 - Inductive step. Recall that we need to show that if the initial state lies in $X_0 \in [-C_0, C_0]$ almost surely, then the expected moment by the end of the procedure is bounded. Denote the stopping time of the procedure as τ, and consider

$$\begin{aligned}\mathbb{E}[|X_{l+\tau}|^\beta] &\leq \mathbb{E}[C_{l+\tau}^\beta] \\ &= \mathbb{E}[(rC_0+\Delta)^\beta P^{\tau\beta}] \\ &= \mathbb{E}[(rC_0+\Delta)^\beta\,\mathbb{E}[P^{\tau\beta}|C_0]]. \qquad (11)\end{aligned}$$

Let us show that $\mathbb{E}[P^{\tau\beta}|C_0]$ has a uniform upper bound that decays with growing Δ:

$$\begin{aligned}\mathbb{E}[P^{\tau\beta}|C_0] &= \ln(P^\beta)\sum_{i=1}^{\infty}(P^\beta)^i\Pr[\tau \geq i] \\ &\leq \ln(P^\beta)\sum_{i=1}^{\infty}(P^\beta)^i\left(\frac{a}{P}\right)^{i\alpha}\frac{\mathbb{E}[|\sum_j a^{-j}Z_j|^\alpha]}{\Delta^\alpha} \\ &= \left(\frac{\ln(P^\beta)}{1-\frac{a^\alpha}{P^{\alpha-\beta}}}\frac{\rho_\alpha}{(1-a)^\alpha}\right)\frac{1}{\Delta^\alpha}, \qquad (12)\end{aligned}$$

where the inequality follows from the Markov's inequality and the inclusion

$$\begin{aligned}&\{\tau \geq i\} \\ &\subseteq \{(rC_0+\Delta)P^i < |X_{l+i}|\} \\ &\subseteq \left\{(rC_0+\Delta)P^i < a^i(|a^l X_0 - U_l| + a^l\sum_{j=0}^{l+i} a^{-j}|Z_j|)\right\} \\ &\subseteq \left\{\Delta\left(\frac{P}{a}\right)^i < a^l\sum_{j=0}^{l+i} a^{-j}|Z_j|\right\},\end{aligned}$$

where the last inclusion follows from $r \geq \frac{a^l}{M}$.

Finally, we combine (11) and (12),

$$\mathbb{E}[|X_{l+\tau}|^\beta] \leq \left(\frac{\ln(P^\beta)}{1-\frac{a^\alpha}{P^{\alpha-\beta}}}\frac{\rho_\alpha}{(1-a)^\alpha}\right)\frac{1}{\Delta^\alpha}\mathbb{E}[(rC_0+\Delta)^\beta]. \qquad (13)$$

By Jensen's inequality, we can conclude that

$$\begin{aligned}&\mathbb{E}[|X_{l+\tau}|^\beta] \\ &\leq \left(\frac{\ln(P^\beta)}{1-\frac{a^\alpha}{P^{\alpha-\beta}}}\frac{a^{\alpha l}\rho_\alpha}{(1-a)^\alpha}\right)\frac{1}{\Delta^\alpha}(\mathbb{E}[2^{\beta-1}(rC_0)^\beta] + 2^{\beta-1}\Delta^\beta). \qquad (14)\end{aligned}$$

It is now easy to see that the coefficient of $\mathbb{E}[C_0^\beta]$ is controlled by Δ^α and can be made less than 1. □

V. Conclusions and Future Work

We presented a rate-optimal and explicit algorithm for stabilizing dynamical systems with fixed-rate feedback and noise with unbounded support. The proposed algorithm can be (optimally) extended to vector dynamical systems using time-sharing arguments (e.g., [14]) and to the case of noisy observations by employing a Kalman filter prior to the proposed algorithm (e.g., [12]). An interesting question under investigation is whether the constrained quantizer can be utilized in the case of multiple (noisy) observers.

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