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The Minimal Directed Information Needed to Improve the LQG Cost

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***Abstract*— We study a linear quadratic Gaussian (LQG) control problem, in which a noisy observation of the system state is available to the controller. To lower the achievable LQG cost, we introduce an extra communication link from the system to the controller. We investigate the trade-off between the improved LQG cost and the consumed communication (information) resources that are measured with the conditional directed information. The objective is to minimize the directed information over all encoding-decoding policies subject to a constraint on the LQG cost. The main result is a semidefinite programming formulation for the optimization problem in the finite-horizion scenario where the dynamical system may have time-varying parameters. This result extends the seminal work by Tanaka et al., where the direct noisy measurement of the system state at the controller is assumed to be absent. As part of our derivation to show the optimality of an encoder that transmits a Gaussian measurement of the state, we show that the presence of the noisy measurements at the encoder can not reduce the minimal directed information, extending a prior result of Kostina and Hassibi to the vector case. Finally, we show that the results in the finite-horizon case can be extended to the infinite-horizon scenario when assuming a time-invariant system, but possibly a time-varying policy. We show that the solution for this optimization problem can be realized by a time-invariant policy whose parameters can be computed explicitly from a finite-dimensional semidefinite program.**

I. Introduction

Networked control systems share an inherent tension between the control performance and the resources that are allocated to communicate by different nodes of the system. Despite the great advances on many interesting questions on this theme, for instance, data rate theorems for stabilizability of dynamical systems [1]–[7], there are still fundamental questions that remain open. One such question is the benefit of communication resources to the control cost [8]–[12]. In this paper, we study this fundamental question on a simple topology consisting of the classical Linear Quadratic Gaussian (LQG) setting with a single communication link.

The networked control setting investigated in this paper (Fig. 1) aims to reduce the control cost below some value at the expense of communication resources. The communication link introduced between an encoder and a decoder (co-located with the controller) serves as an information pipeline to the controller that also has an access to the noisy measurements of the system state. Based on its (full) observation of the the state, the encoder transmits information to reduce the LQG cost below some desired target cost. One may also

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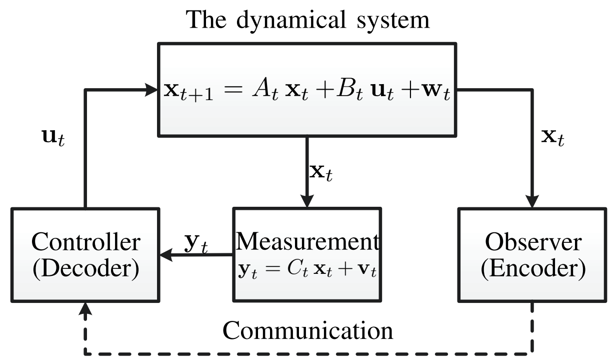

Fig. 1. The LQG setting with noisy measurement $\mathbf{y}_t$. The control performance (the quadratic cost) is improved using the dashed line which denotes a communication link from a fully observer to the controller.

view this setting as the standard rate-constrained LQG setting [13], but with an additional side information that is available to the controller (the measurement $\mathbf{y}_t$) [14]. The objective of this paper is to characterize the minimal communication resources subject to a constraint on the control performance.

The communication (information) resources are measured with the conditional directed information. The directed information is suitable for scenarios where the operations of the involved units are sequential, e.g., channels with feedback in communication [15] and the causal rate distortion function in the context of control problems [8], [11], [14]. Also, both the encoder and the controller are sequential mappings and the directed information serves as a lower bound to the operational variable-length (prefix) coding problem [8], [16] (See also Section V). The control performance in our setting is measured as the quadratic cost function on the state and control signals. The optimization problem is formulated for two scenarios corresponding to the finite-horizon and infinite-horizon regimes.

For the finite-horizon problem, the general case of time-varying dynamical systems is investigated and solved by formulating the optimization problem as a convex optimization problem. The optimization problem has a semidefinite programming (SDP) form (more precisely, `max log-det` form) and can be implemented using standard solvers even for *large* horizons. We also show that the solution to the optimization problem can be realized by three design steps: determining the controller gains and finding the solution for the convex optimization problem that can be done offline, and a standard Kalman filter based on the observations. For the infinite-horizon problem where the dynamical system matrices are time-invariant, we show that the optimization problem can be also formulated as an SDP with the op-

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timization variables being only two positive semidefinite matrices of finite dimensions. Additionally, we show that the optimal encoding policy is a simple, time-invariant Gaussian measurement of the state.

Our results generalize the work by Tanaka et al. [13], which introduced the SDP approach for solving control-communication problems [17]. Specifically, we investigate the setting in Fig. 1 with two kind of input signals received by the controller, while [13] assumed that the noisy measurement of the system state is absent ($\mathbf{y}_t = 0$ in Fig. 1). Thus, the controller in our setting combines both the communication link information and the noisy measurement of the state. Our results also extend the explicit solution in [14] for the scalar case of Fig. 1 in the infinite-horizon regime.

Two key changes in the SDP formulation compared to [13] are the objective function that includes a new term due to the study of conditional directed information rather than the directed information in [13], and a new linear matrix inequality (LMI) constraint which represents the reduction in the error covariance due to the quality of the noisy measurements. Additionally, the structure of the optimal policy cannot be shown directly and therefore, we study a relaxed optimization problem where the noisy measurements (that are available to the controller) are also available to the encoder. We then show that even in this relaxed scenario, the optimal encoder signaling is a memoryless Gaussian measurement of the state. Thus, the knowledge of the noisy measurements at the encoder can not further reduce the minimal communication resources.

The remainder of this paper is organized as follows. Section II introduces the setting and problem definition. Section III presents our main contributions. In Section IV, we present our numerical examples. Proofs are omitted from this paper due to space constraints.

II. The Setting and Problem Definition

A linear dynamical system is given by

$$\mathbf{x}_{t+1} = A_t\,\mathbf{x}_t + B_t\,\mathbf{u}_t + \mathbf{w}_t \quad t \geq 1,$$

where $\mathbf{w}_t \sim \mathcal{N}(0, W_t)$ are mutually independent. The initial state $\mathbf{x}_1$ is distributed according to $P_{1|0}$ and is independent of $\mathbf{w}_t$. A noisy measurement of the state is available to the controller,

$$\mathbf{y}_t = C_t\,\mathbf{x}_t + \mathbf{v}_t,$$

with $\mathbf{v}_t \sim \mathcal{N}(0, V_t)$. For a fixed time-horizon T, the LQG quadratic cost is defined as

$$\begin{aligned} J(\mathbf{x}^{T+1}, \mathbf{u}^T) &= \sum_{t=1}^{T} \mathbf{x}_{t+1}^*\, Q_t\, \mathbf{x}_{t+1} + \mathbf{u}_t^*\, R_t\, \mathbf{u}_t \\ &\triangleq \sum_{t=1}^{T} \mathbb{E}[\|\mathbf{x}_{t+1}\|_{Q_t}^2 + \|\mathbf{u}_t\|_{R_t}^2], \end{aligned} \tag{1}$$

with $Q_t \succeq 0$ and $R_t \succ 0$.

The objective is to design a system such that the LQG cost does not exceed a cost target denoted by Γ. Obviously, if the measurements $\mathbf{y}_t$ are sufficient to attain an LQG cost below Γ, then the classical solution for the LQG problem is satisfactory. However, our interest is in scenarios where the optimal LQG cost exceeds Γ. To reduce the LQG cost below Γ, we introduce a communication/information link between an encoder that has access to the state $\mathbf{x}_t$ and a decoder that is co-located with the controller (See Fig. 1).

We use the causal conditioning notation to represent the encoder as a set of random mappings

$$P(\mathbf{f}^T \,||\, \mathbf{x}^T, \mathbf{u}^{t-1}) \triangleq \prod_{t=1}^{T} P(\mathbf{f}_t \,|\, \mathbf{f}^{t-1}\, \mathbf{x}^t), \tag{2}$$

where $\mathbf{f}_t$ is the encoding variable. Similarly, the decoder (controller) is a collection of random mappings:

$$P(\mathbf{u}^T \,||\, \mathbf{f}^T, \mathbf{y}^T) \triangleq \prod_{t=1}^{T} P(\mathbf{u}_t \,|\, \mathbf{u}^{t-1}, \mathbf{f}^t, \mathbf{y}^t). \tag{3}$$

From the construction, the encoder-decoder pair satisfies at all times

$$\begin{aligned} &P(\mathbf{u}_t, \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{u}^{t-1}, \mathbf{x}^t, \mathbf{y}^t) \\ &\quad = P(\mathbf{u}_t \,|\, \mathbf{u}^{t-1}, \mathbf{f}^t, \mathbf{y}^t) P(\mathbf{f}_t |\, \mathbf{x}^t, \mathbf{f}^{t-1}). \end{aligned} \tag{4}$$

The overall joint distribution can be summarized with:

$$\begin{aligned} &P(\mathbf{x}_t, \mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t |\, \mathbf{x}^{t-1}, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}, \mathbf{u}^{t-1}) \\ &\quad = P(\mathbf{y}_t, \mathbf{x}_t \,|\, \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) P(\mathbf{u}_t, \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{u}^{t-1}, \mathbf{x}^t, , \mathbf{y}^t), \end{aligned}$$

The conditional directed information [18], [19], between the encoder and the controller conditioned on the noisy measurements, is given by

$$I(\mathbf{x}^T \to \mathbf{f}^T \,||\, \mathbf{y}^T) = \sum_{t=1}^{T} I(\mathbf{x}^t; \mathbf{f}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^t), \tag{5}$$

where $I(X;Y|Z)$ is the mutual information between X and Y conditioned on Z.

The objective of this paper is to solve the optimization problem:

$$\begin{aligned} &\min I(\mathbf{x}^T \to \mathbf{f}^T || \, \mathbf{y}^T) \\ &\text{s.t. } J(\mathbf{x}^{T+1}, \mathbf{u}^T) \leq \Gamma, \end{aligned} \tag{6}$$

where the minimum is over policies of the form (4).

When the measurement $\mathbf{y}_t$ is absent, the optimization problem in (6) simplifies to the directed information $I(\mathbf{x}^T \to \mathbf{f}^T)$ that was investigated in [13]. To see that the conditional directed information is the *correct* information measure, assume that the t-th element in the conditional directed information satisfies:

$$I(\mathbf{x}^t; \mathbf{f}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^t) = I(\mathbf{x}_t; \mathbf{f}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^t). \tag{7}$$

Then, the right hand side of (7) extracts the discrepancy in the state uncertainty at the controller with and without the encoding variable $\mathbf{f}_t$, i.e., $I(\mathbf{x}_t; \mathbf{f}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^t) = h(\mathbf{x}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^t) - h(\mathbf{x}_t \,|\, \mathbf{f}^t, \mathbf{y}^t)$. The presence of the conditioning on $\mathbf{y}^t$ in both terms reflects the fact that $\mathbf{f}_t$ is costly while $\mathbf{y}_t$ is already available to the controller and thus does not incur a communication cost. In the sequel, we will formalize these arguments in Theorems 1 and 2. We will also show in Theorem 2 a relation between the optimal conditional directed information and Kalman filtering theory with two independent measurements.

III. Main results

This section presents our main results. First, we provide a simple structure for the optimal policy in Theorem 1. Then, we present basic definitions of Kalman filtering theory to formulate the directed information in such terms. We then provide a semidefinite programming formulation of the optimization problem and thereafter present the optimal system design. Finally, Section III-D includes the formulation and the solution of the infinite-horizon problem.

A. Optimal policy structure

The first result is the optimal structure of the observer (encoder) and controller (decoder) policies.

Theorem 1 (Optimal policy structure): The optimal policy that achieves (6) has the following structure

$$\begin{aligned} \mathbf{f}_t &= D_t\,\mathbf{x}_t + \mathbf{m}_t, \\ \mathbf{u}_t &= -K_t\,\mathbb{E}[\mathbf{x}_t \,|\, \mathbf{f}^t, \mathbf{y}^t], \end{aligned} \tag{8}$$

where $\mathbf{m}_t \sim \mathcal{N}(0, M_t)$ and D_t are optimization variables, and K_t is a constant (Eq. (16)). The proof appears in Appendix **??**.

The theorem simplifies significantly the maximization domain from the general policy in (4) to the set $\{(D_t, M_t)\}_{t\geq 1}$. The result reveals that $\mathbf{f}_t$ has the role of reducing the communication resources by forming a linear Gaussian measurement of the state, also termed a *virtual sensor* [13]. Note that in our problem definition there are no assumptions on the structure such as linear, Gaussian or memoryless mappings. The control signal $\mathbf{u}_t$ is the standard LQG certainty equivalence controller. Thus, the separation between the control gain and the estimation is preserved in our setting.

Theorem 1 recovers the optimal structure reported in [13] for the case when $\mathbf{y}_t$ is absent. The extension of [13] to our setting is not straightforward, and involves a study of a relaxed optimization problem where, at time t, the vector $\mathbf{y}^t$ is also available to the encoder. For this relaxed optimization problem, we show that the optimal policy is of the form (8). A by product of our analysis is the following claim

Lemma 1: The knowledge of the measurements $\mathbf{y}^t$ at the encoder does not reduce the optimal directed information control problem (Eq. (6)).

Lemma 1 extends the result in [14, Th. 8] applicable to scalar and time-invariant systems to the case of vector systems with time-varying system characteristics. As remarked in [14], the availability of side information at the encoder also does not change the classical Wyner-Ziv Gaussian rate distortion function, e.g., [20].

B. Kalman filter with two observers

As is evident from the optimal structure in Theorem 1, the encoding function $\mathbf{f}_t$ is a noisy observation of the system state. Thus, the optimal system has a structure of standard LQG setting with two observations with independent noises. However, for the purpose of optimizing the communication resources, $\mathbf{f}_t$ has a cost, while $\mathbf{y}_t$ is a natural occurrence in our control scenario. In this section, we provide short preliminaries on Kalman filtering and then present the conditional directed information in terms of Kalman filtering.

By standard convention, we denote the error covariance matrices with respect to both measurements $\mathbf{y}_t$ and $\mathbf{f}_t$ as follows

$$\begin{aligned} P_{t|t-1} &\triangleq \mathrm{Cov}(\mathbf{x}_t - \mathbb{E}[\mathbf{x}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^{t-1}, \mathbf{u}^{t-1}]) \\ P_{t|t} &\triangleq \mathrm{Cov}(\mathbf{x}_t - \mathbb{E}[\mathbf{x}_t \,|\, \mathbf{f}^{t}, \mathbf{y}^{t}, \mathbf{u}^{t-1}]) \end{aligned} \tag{9}$$

Since $\mathbf{y}_t$ and $\mathbf{f}_t$ have different roles, we also define an intermediate error covariance matrix corresponding to the prediction error after observing $\mathbf{y}_t$ only:

$$\overline{P}_{t|t-1} \triangleq \mathrm{Cov}(\mathbf{x}_t - \mathbb{E}[\mathbf{x}_t \,|\, \mathbf{f}^{t-1}, \mathbf{y}^{t}, \mathbf{u}^{t-1}]). \tag{10}$$

The following technical lemma formalizes several relations between the error covariances.

Lemma 2 (Error covariances): Let $P_{1|0}$ be the covariance matrix of X_0. Then, for fixed (D_t, M_t), the error covariance matrices satisfy the recursions

$$\begin{aligned} (\overline{P}_{t|t-1})^{-1} &= P_{t|t-1}^{-1} + \mathrm{SNR}_t^Y \\ P_{t|t} &= (I - L_t^F D_t)\overline{P}_{t|t-1} \\ P_{t|t-1} &= A_{t-1} P_{t-1|t-1} A_{t-1}^{\mathrm{T}} + W_{t-1} \\ P_{t|t}^{-1} &= (P_{t|t-1}^{+})^{-1} + \mathrm{SNR}_t^F, \end{aligned} \tag{11}$$

where $L_t^F = \overline{P}_{t|t-1} D_t^{\mathrm{T}} (D_t \overline{P}_{t|t-1} D_t^{\mathrm{T}} + M_t)^{-1}$, $\mathrm{SNR}_t^F = D_t^{\mathrm{T}} M_t^{-1} D_t$ and $\mathrm{SNR}_t^Y = C_t^{\mathrm{T}} V_t^{-1} C_t$.

The identities are standard in Kalman filtering theory and their proofs are omitted. In the following, we present an alternative characterization of the conditional directed information.

Theorem 2: Given a policy with the optimal structure (Th. 1), the conditional directed information can be written as

$$I(\mathbf{x}^T \to \mathbf{f}^T \,||\, \mathbf{y}^T) = \sum_{t=1}^{T} \log\det(I - L_t^F D_t). \tag{12}$$

By Lemma 2, $(I - L_t^F D_t)$ is the multiplicative term of the error reduction when computing the error covariance matrix $P_{t|t}$ from $\overline{P}_{t|t-1}$ by adding the encoding variable $\mathbf{f}_t$. Therefore, Theorem 2 reveals that the conditional directed information measures the reduction in error covatiance with respect to $\mathbf{f}_t$ only. Despite the simple representation of the objective function in (12), it is not clear whether it can be formulated as a convex problem as its inverse includes a product of two optimization variables $(I - L_t^F D_t)^{-1} = I + \overline{P}_{t|t-1}\,\mathrm{SNR}_t^F$. Therefore, we proceed to show our main result on the convex formulation of the optimization problem in (6).

Theorem 3 (SDP formulation): For a fixed $P_{1|0}$, the optimization problem (6) can be formulated as

$$
\begin{aligned}
&\inf_{\{P_{t|t},\Pi_t\}_{t=1}^T} \quad \Lambda - \frac{1}{2}\sum_{t=1}^{T-1} \log\det(I + (A_t P_{t|t} A_t^{\mathrm{T}} + W_t)\,\mathrm{SNR}_t^Y) \\
&-\frac{1}{2}\sum_{t=1}^{T} \log\det \Pi_t \\
&\text{s.t.} \quad \mathrm{Tr}(\Phi_1 P_{1|0}) + \sum_{t=1}^{T} \mathrm{Tr}\left(\Theta_t P_{t|t}\right) + \mathrm{Tr}(S_t W_t) \le \Gamma, \\
&\begin{bmatrix} P_{t|t} - \Pi_t & P_{t|t} A_t^{\mathrm{T}} \\ A_t P_{t|t} & A_t P_{t|t} A_t^{\mathrm{T}} + W_t \end{bmatrix} \succeq 0, \Pi_t \succ 0, \quad t < T \\
&P_{T|T} = \Pi_T \succeq 0, \\
&\Omega_t \succeq 0,\ t = 1,\dots,T \text{ (Eq. (13) below)},
\end{aligned} \tag{14}
$$

where the constant matrices $\Phi_1 = S_0 - Q_0$ and Θ_t are obtained from (16), and the constant Λ is given by

$$
\Lambda = -\frac{1}{2}\log\det(P_{1|0}^{-1} + \mathrm{SNR}_1^Y) + \frac{1}{2}\sum_{t=1}^{T-1}\log\det W_t. \tag{15}
$$

The optimization problem in Theorem 3 is a convex optimization problem with respect to the set of $(P_{t|t}, \Pi_t)$, and can be solved using standard solvers [21], [22]. With some abuse of terminology, we refer to the formulation in Theorem 3 as an SDP but, indeed, it is the log-barrier function of the standard SDP formulation. The auxiliary optimization variable Π_t is introduced to convert the objective function into a standard form. In the special case $C_t = \mathrm{SNR}_t^Y = 0$, Theorem 3 recovers the formulation in [13, Th. 1]. Note that in this case the constraints on Ω_t boil down to $(A_{t-1}P_{t-1|t-1}A_{t-1}^{\mathrm{T}} + W_{t-1}) - P_{t|t} \succeq 0$ and $P_{1|0} \succeq P_{1|1}$ and the objective terms $\log\det(I + (A_t P_{t|t} A_t^{\mathrm{T}} + W_t)\,\mathrm{SNR}_t^Y) = 0$. The proof of Theorem 3 appears in Appendix **??**.

C. System design

In this section, we construct a three-steps realizable policy using the results from the previous section.

1) The controller gain: The controller is independent of the observations characteristics. Using the backward Riccati recursion, we compute the control gains

$$
\begin{aligned}
S_t &= \begin{cases} A_t^{\mathrm{T}} S_{t+1} A_t - A_t^{\mathrm{T}} S_{t+1} B_t K_t + Q_t & \text{if} \quad t < T, \\ Q_T & \text{if} \quad t = T. \end{cases} \\
K_t &= (B_t^{\mathrm{T}} S_{t+1} B_t + R_t)^{-1} B_t^{\mathrm{T}} S_{t+1} A_t \\
\Theta_t &= K_t^{\mathrm{T}} (B_t^{\mathrm{T}} S_{t+1} B_t + R_t) K_t.
\end{aligned} \tag{16}
$$

2) Covariance matrices: Given Θ_t for $t = 1,\dots,T$, one obtains the set $\{P_{t|t}\}_{t\ge 1}$ by solving the convex optimization problem in Theorem 3. By Lemma 2, one computes

$$
\mathrm{SNR}_t^F = P_{t|t}^{-1} - (A_t P_{t|t-1} A_t^{\mathrm{T}} + W_t)^{-1} - \mathrm{SNR}_t^Y .
$$

Using the SVD decomposition, SNR_t^F is written as $D_t^{\mathrm{T}} M_t^{-1} D_t$.

3) Kalman filter: Define the Kalman gain:

$$
L_t = P_{t|t-1} H_t^{\mathrm{T}} (H_t P_{t|t} H_t^{\mathrm{T}} + N_t)^{-1},
$$

where $H_t \triangleq \begin{bmatrix} C_t \\ D_t \end{bmatrix}$, $N_t \triangleq \begin{bmatrix} V_t & 0 \\ 0 & M_t \end{bmatrix}$.

The Kalman filter update is done in two steps:

$$
\begin{aligned}
\hat{x}_{t+1|t} &= A_t \hat{\mathbf{x}}_t + B_t\, \mathbf{u}_t \\
\hat{x}_t &= \hat{\mathbf{x}}_{t|t-1} + L_t \begin{bmatrix} \mathbf{y}_t - C_t \hat{\mathbf{x}}_{t|t-1} \\ \mathbf{f}_t - D_t \hat{\mathbf{x}}_{t|t-1} \end{bmatrix},
\end{aligned} \tag{17}
$$

where the control signal is $\mathbf{u}_t = -K_t \hat{\mathbf{x}}_t$.

D. The infinite-horizon setting

For the infinite-horizon problem, we study time-invariant systems, i.e., $A_t = A$, $B_t = B$, $W_t = W$, $C_t = C$, $V_t = V$. It is also assumed that the pair (A, B) is stabilizable.

The optimization problem is defined as:

$$
\begin{aligned}
&\inf \limsup_{T\to\infty} \frac{1}{T} (\mathbf{x}^T \to \mathbf{f}^T || \, \mathbf{y}^T) \\
&\text{s.t.} \ \limsup_{T\to\infty} \frac{1}{T} J(\mathbf{x}^{T+1}, \mathbf{u}^T) \le \Gamma.
\end{aligned} \tag{18}
$$

The structure of the solution is similar to the one in Theorem 3.

1) Controller gain: Let $\bar{S}$ be the unique stabilizing solution for the Riccati equation

$$
A^{\mathrm{T}} S A - S - A^{\mathrm{T}} S B (B^{\mathrm{T}} S B + R)^{-1} B^{\mathrm{T}} S A + Q = 0.
$$

The controller gain is computed as

$$
\begin{aligned}
K &= (B^{\mathrm{T}} \bar{S} B + R)^{-1} B^{\mathrm{T}} \bar{S} A \\
\Theta &= K^{\mathrm{T}} (B^{\mathrm{T}} \bar{S} B + R) K.
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1 &\triangleq \begin{bmatrix} P_{1|0} - P_{1|1} & P_{1|0} C_1^{\mathrm{T}} \\ C_1 P_{1|0} & C_1 P_{1|0} C_1^{\mathrm{T}} + V_1 \end{bmatrix} \\
\Omega_t &\triangleq \begin{bmatrix} (A_{t-1}P_{t-1|t-1}A_{t-1}^{\mathrm{T}} + W_{t-1}) - P_{t|t} & (A_{t-1}P_{t-1|t-1}A_{t-1}^{\mathrm{T}} + W_{t-1}) C_t^{\mathrm{T}} \\ C_t (A_{t-1}P_{t-1|t-1}A_{t-1}^{\mathrm{T}} + W_{t-1}) & C_t (A_{t-1}P_{t-1|t-1}A_{t-1}^{\mathrm{T}} + W_{t-1}) C_t^{\mathrm{T}} + V_t \end{bmatrix} \quad \text{for} \quad t = 2,\dots,T.
\end{aligned} \tag{13}
$$

2) The optimization problem: Given Θ and $\bar{S}$, the following SDP is solved.

$$\begin{aligned}
&\inf_{P,\Pi} \ \frac{1}{2}\log\det W - \frac{1}{2}\log\det(I + \mathrm{SNR}^Y(APA^T + W)) \\
&\quad - \frac{1}{2}\log\det \Pi \\
\text{s.t.} \quad & \mathrm{Tr}\left(\Theta P\right) + \mathrm{Tr}\left(W\bar{S}\right) \leq \Gamma, \\
& \begin{bmatrix} P - \Pi & PA^{\mathrm{T}} \\ AP & APA^{\mathrm{T}} + W \end{bmatrix} \succeq 0, \quad \Pi \succ 0. \\
& \begin{bmatrix} APA^{\mathrm{T}} + W - P & (APA^{\mathrm{T}} + W)C^{\mathrm{T}} \\ C(APA^{\mathrm{T}} + W) & C(APA^{\mathrm{T}} + W)C^{\mathrm{T}} + V \end{bmatrix} \succeq 0.
\end{aligned} \tag{19}$$

3) Time-invariant policy: From the optimization problem, the optimal P is used to compute

$$\mathrm{SNR}^F = P^{-1} - (APA^{\mathrm{T}} + W)^{-1} - \mathrm{SNR}^Y, \tag{20}$$

and by the singular value decomposition, we obtain $\mathrm{SNR}^F = D^{\mathrm{T}} M^{-1} D$.

Theorem 4: If (A, B) is stabilizable and (A, Θ) is detectable, the optimal value of the infinite-horizon optimization problem (18) is equal to the optimal value of the SDP in (19). Moreover, the time-invariant policy in (20) achieves the optimal value.

The proof of Theorem 4 follows by studying the limiting optimization problem for the finite-horizon in Theorem 3. The proof is omitted from this paper.

IV. Numerical examples

A. Side information reduces the minimal directed information

In this section, we present a numerical example to show that side information reduces the minimal directed information. We set the matrices A, B, W, Q, R to be the same as those in [13, Sec. V]. In addition, we choose $C = I$ and $V = \frac{1}{\rho} I$ with $\rho > 0$, then $\mathrm{SNR}^Y = \rho I$. For each $\rho = 0.1$, $\rho = 1$ and $\rho = 10$, we solve (19) for each LQG cost constraint Γ in the range $\Gamma \in [30, 90]$ and we plot the optimal value of (19) as a function of Γ in Fig. 2. The case without side information studied in [13] can be equivalently viewed as the case with $\rho = 0$. As one can see, for any fixed Γ, the minimal conditional directed information decreases as ρ (the signal-to-noise ratio of the side information) increases.

As expected, for any fixed Γ, the minimal conditional directed information decreases as ρ (the signal-to-noise ratio of the side information) increases. The red vertical line corresponds to the minimal cost that can be attained with clean observation available at the controller. In the high-cost regime, one can note that the curve $\rho = 0$ does not converge to zero since the system is not stabilizable without the communication resources. On the other hand, even for low SNR values, the minimal directed information converges to zero since the system is stabilizable by having $\mathbf{y}_t$ alone. It is also interesting to note the information gain due to the presence of $\mathbf{y}_t$ varies with cost.

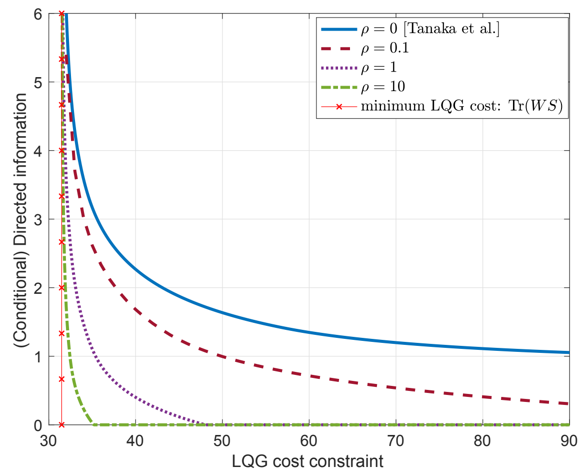

Fig. 2. The trade-offs between the conditional directed information and the LQG cost when the SNR of the side information varies.

B. Scalar systems

For scalar systems, the solution to (19) without the noisy measurement at the controller ($C = V = 0$) is

$$\frac{1}{2}\log\left(A^2 + \frac{W\Theta}{\Gamma - W\bar{S}}\right), \quad \forall\, \Gamma > W\bar{S}, \tag{21}$$

where $\bar{S}$ is the unique solution to the Riccati equation and can be solved in closed-form as

$$\bar{S} = \frac{(A^2 + B^2 - 1) + \sqrt{(A^2 + B^2 - 1)^2 + 4B^2}}{2B^2}. \tag{22}$$

In the following result, we provide a closed-form for the scalar problem.

Corollary 1: When A, B, W, C, V are scalars, $Q = R = 1$ and $|A| > 1$, the optimal value of the optimization (19) is

$$\begin{aligned}
&\frac{1}{2}\log\left(A^2 + \frac{W\Theta}{\Gamma - W\bar{S}}\right) \\
&\quad - \frac{1}{2}\log\left(1 + \mathrm{SNR}^Y\left(W + \frac{A^2(\Gamma - W\bar{S})}{\Theta}\right)\right),
\end{aligned} \tag{23}$$

when $W\bar{S} < \Gamma \leq W\bar{S} + \Theta P^\star$; and is 0 when $\Gamma > W\bar{S} + \Theta P^\star$, where $P^\star$ is the unique positive solution to the quadratic equation

$$A^2\, \mathrm{SNR}^Y\, P^2 + (1 - A^2 + \mathrm{SNR}^Y\, W)P - W = 0 \tag{24}$$

and $\bar{S}$ is given in (22) and $\Theta = \frac{(AB\bar{S})^2}{1 + B^2\bar{S}}$.

By comparing (21) and (23), the information gain due to the presence of the noisy measurements at the controller is the non-negative expression

$$\frac{1}{2}\log\left(1 + \mathrm{SNR}^Y\left(W + \frac{A^2(\Gamma - W\bar{S})}{\Theta}\right)\right).$$

Note that the gain is an increasing function of SNR^Y. Also, the gain is upper bounded by (21) which is achieved with equality when $\Gamma = W\bar{S} + \Theta P^\star$ since $P^\star$ satisfies $1 + \mathrm{SNR}^Y(W + A^2 P^\star) = A^2 + \frac{W}{P^\star}$ (see Eq. (24)).

V. Conclusions

In this paper, we formulated and solved an optimization problem for the LQG setting with an additional communication link. The main result is that the optimization problem minimizing the conditional directed information has a standard convex form.

An interesting research direction is to relate the information-theoretic framework of the directed information to the operational problem similar to [9], [16]. Specifically, while the directed information serves as a lower bound for the operational problem with variable-length coding, we are interested in constructing a coding scheme that will achieve the rate of the directed information up to some additive term that represents the loss due to the causal operations.

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