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Non-equilibrium statistical mechanical approach to the formation of non-Maxwellian electron distribution in space

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Abstract. Boltzmann-Gibbs (BG) entropy has additive and extensive properties, but for certain physical systems, such as those governed by long-range interactions – plasma or fully ionized gas being an example – it is speculated that the entropy must be non-additive and non-extensive. Because of the fact that Tsallis entropy possesses such characteristics, many spacecraft observations of charged particle distributions in space are interpreted with the conceptual framework based upon Tsallis statistical principles. This paper formulates the non-equilibrium statistical theory of space plasma, and it is shown that the steady state electrostatic turbulence in plasma coincides with the formation of non-Maxwellian electron distribution function known as the kappa distribution. The kappa distribution is equivalent to the q-Gaussian distribution in the Tsallis statistical theory, which represents the most probable state subject to Tsallis entropy. This finding represents an independent confirmation that the space plasma may indeed be governed by the Tsallis statistical principle.

1 Introduction

It is well known that Boltzmann-Gibbs (BG) entropy $S_{\rm BG} = k_{\rm B} \log W$, where W represents the number of all possible micro states of a system and $k_{\rm B}$ is the Boltzmann constant $k_{\rm B} = 1.3806503 \times 10^{-23}\,{\rm m^2\,kg\,s^{-2}\,K^{-1}}$, is additive and extensive. That is, if A and B represent two subsystems and $A+B$ the total system, then $S_{\rm BG}(A+B) = k_{\rm B}\ln(W_A W_B) = k_{\rm B}\ln W_A + k_{\rm B}\ln W_B = S_{\rm BG}(A) + S_{\rm BG}(B)$, which is the additive property. If we consider that the number of possible states behaves as $W(N) \propto w^N$ $(w > 1)$, where N represents the total number of particles, then we have $S_{\rm BG} = N k_{\rm B} \ln w \propto N$, which represents the extensive nature of BG entropy. Boltzmann-Gibbs entropy governs ideal gas or systems governed by short-range interactions, but for systems dictated by long-ranged interactions, such as (fully) ionized gas, or plasma, whose dynamics is governed by long-ranged electromagnetic force, it is

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reasonable to expect that non-additive and non-extensive thermo-statistical principle may characterize their macroscopic behavior.

Among possible generalizations of BG entropy is the model put forth by Tsallis [1,2], namely, $S_q = -k_{\rm B}\left(1-\sum_{i=1}^{W} p_i^q\right)/(1-q)$, where p_i is the probability of the system being at a particular micro state, satisfying $\sum_{i=1}^{W} p_i = 1$. For equal probability, of course, $p_i = 1/W$. Tsallis entropy is non-additive in that the entropy of total system differs from the sum of entropies for subsystems, $k_{\rm B}^{-1} S_q(A+B) = k_{\rm B}^{-1} S_q(A) + k_{\rm B}^{-1} S_q(B) + (1-q) k_{\rm B}^{-2} S_q(A) S_q(B)$. For $q = 1$ one recovers the additive property. It is also non-extensive. If we suppose that $W(N) = w^N$, then one has $k_{\rm B}^{-1} S_q(N) = (q-1)^{-1}\left[1-w^{-N(q-1)}\right]$, which is not proportional to N, hence, non-extensive.

For continuous systems, Tsallis entropy is expressed as

$$S_q = -\frac{k_{\rm B}}{1-q}\left(1-\int d\mathbf{x}\int d\mathbf{v}[f(\mathbf{v})]^q\right). \tag{1}$$

Upon minimizing the free energy, $F = U - TS_q$, where $U = \int d\mathbf{x}\int d\mathbf{v}\,(mv^2/2)\,f(\mathbf{v})$, is the total energy, that is, by solving for f that satisfies $\delta F/\delta f = 0$, we find that the most probably state is given by

$$f_q(v) \sim \left(1+\frac{(1-q)\,v^2}{v_T^2}\right)^{-1/(1-q)} \equiv \exp_q\left(-\frac{v^2}{v_T^2}\right), \tag{2}$$

where $v_T = (2k_{\rm B}T/m)^{1/2}$ is the Maxwellian thermal speed, T and m being the temperature and mass of the charged particles. The mathematical function $\exp_q(x) \equiv [1+(1-q)x]^{1-q}$ is known as the q-Gaussian or q-exponential function. It is a straightforward exercise to show that the most probable state subject to the continuous version of Boltzmann-Gibbs (BG) entropy, namely, $S_{\rm BG} = -k_{\rm B}\int d\mathbf{x}\int d\mathbf{v}\, f(\mathbf{v})\,\ln f(\mathbf{v})$, is the Gaussian (or Maxwell-Boltzmann) distribution, $f(v) \sim \exp\left(-v^2/v_T^2\right)$.

In the space physical context, spacecraft measurements of charged particle distributions near Earth orbit, which began in the 1960s, showed that the typical electron distribution function features a Gaussian distribution for low energy range while displaying non-Maxwellian supra-thermal "tail" component in the high energy domain [3–5]. Vasyliunas [6] introduced an empirical model distribution in order to fit the measurement,

$$f_\kappa(v) \sim \left(1+\frac{v^2}{\kappa v_T^2}\right)^{-(\kappa+1)}. \tag{3}$$

The phenomenological model is known as the kappa distribution in space physics literature.

An example is shown in Figure 1, where typical electron velocity distribution function measured in the near-Earth space environment is showed. The measurement is a result of composite data taken from spacecraft *WIND* and *STEREO* [7]. Observations are indicated with dots. The panel on the left represents a theoretical fitting where the measured solar wind electron velocity distribution is modeled with two Maxwellian functions and a kappa function. Specifically, the low energy component, known as the "core", is fitted with a Maxwellian model, $f_{\rm core}(v) \sim \exp(-v^2/v_{Tc}^2)$, the intermediate energy portion is fitted with another Maxwellian, called the "halo", namely, $f_{\rm halo}(v) \sim \exp(-v^2/v_{Th}^2)$, with higher temperature and lower density, and

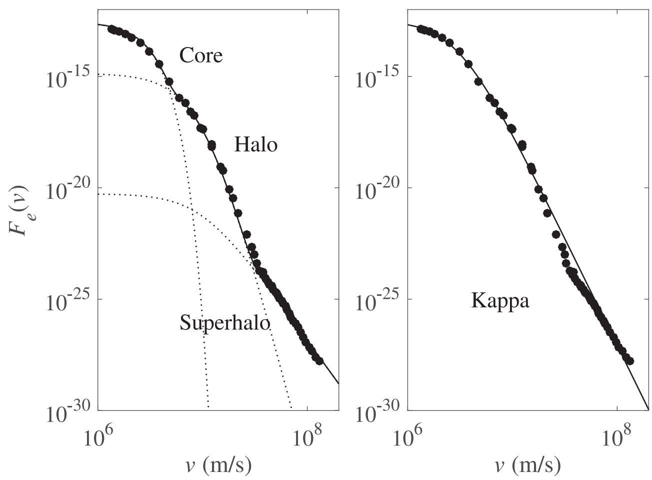

Fig. 1. Typical solar wind electron velocity distribution measured in the near-Earth region. The measurement is made by *WIND* spacecraft as well as by *STEREO* spacecraft. Observation is indicated with dots. Solid curves represent theoretical fittings. Left: Fitting with three-component, core, halo, and superhalo models. Right: Fitting with a single kappa model.

the highest energy velocity spectrum, known as the "superhalo", is fitted with a population of kappa distribution, $f_{\text{superhalo}}(v) \sim [1 + v^2/(\kappa v_{Th}^2)]^{-\kappa-1}$. On the right-hand panel, in contrast to the three population model, the measured distribution is fitted with a single kappa distribution.

The phenomenological kappa model thus proved to be quite useful in interpreting the data, but otherwise, it enjoyed no justification on the basis of first physical principles. However, later it was realized that f_κ is equivalent to the q-Gaussian. That is, if one simply interprets

$$\kappa = \frac{1}{1-q}, \qquad \text{or} \qquad \kappa = \frac{q}{1-q}, \tag{4}$$

then Vasyliunas' kappa model is equivalent to the most probable distribution according to Tsallis' theory. Note, however, that the first choice, namely, $\kappa = 1/(1-q)$, leads to $f \sim \left[1 + v^2/(\kappa v_T^2)\right]^{-\kappa}$, while the second identification $\kappa = q/(1-q)$ leads to $f \sim \left\{1 + v^2/[(\kappa+1)v_T^2]\right\}^{-\kappa+1}$, neither of which is exactly equivalent to the kappa model introduced by Vasyliunas, namely, $f \sim \left[1 + v^2/(\kappa v_T^2)\right]^{-\kappa+1}$, which is defined with both κ and $\kappa + 1$. Such a minor discrepancy withstanding, the importance is that the success of kappa distribution could be understood in the framework of non-extensive statistical concept, and this realization has prompted an explosion of interest in the space physics community [8–11].

Perhaps it is appropriate to note before we move on to the main discussion that the non-extensive statistical framework is not the only conceptual justification for non-Maxwellian distribution in space. Models based upon the combined collisional relaxation and wave-particle interaction had been put forth in the literature, e.g., see references [12–14]. In many respects, however, these works can be viewed as belonging to a class of models, namely, formation of non-Maxwellian distribution by means of wave-particle interaction, which will be presented in the main body of this paper.

In the model presented in references [15,16], which has received critical re-examination in the literature [17–19], the origin of non-Maxwellian distribution is explained by invoking the pervasively observed compressible low frequency turbulence in the solar wind. Reference [20], on the other hand, put forth a mechanism that involves superposition of stochastic processes in order to explain the pervasive non-Maxwellian distribution in the heliosphere. Their idea is essentially the same as the superstatistics model put forth by Beck and Cohen [21].

The superstatistics theory deserves an in-depth look. The non-Maxwellian distribution may emerge when a collection of charged particles undergo random walk in the background of varying temperature field. An example may be that of solar source region from which energetic charged particles emerge. In a layer below solar source the particles may survey many regions of differing temperatures. The temperature profile may be modeled by a power law, $(T/T_0)^{1-k/2}$. We may temper the strict power law behavior with an exponential factor in order to avoid divergences for infinitely large or small T,

$$\left(\frac{T}{T_0}\right)^{1-k/2} e^{-T_0/(2T)}. \tag{5}$$

This results in the chi-square distribution for the inverse temperature, $\beta = 1/(k_{\rm B}T)$. If we convolve or superpose the Gaussian velocity distribution $\exp(-mv^2/2T)$ with the above temperature distribution, hence, the superstatistics, then the result is the kappa distribution [21],

$$\int_0^\infty d\beta\, F(\beta) e^{-\beta\varepsilon} = \left(1 + \frac{\beta_0 \varepsilon}{\kappa}\right)^{-\kappa},$$

$$F(\beta) = \frac{1}{(1/\kappa)^\kappa \beta_0 \Gamma(\kappa)} \left(\frac{\beta}{\beta_0}\right)^{\kappa-1} \exp\left(-\frac{\kappa\beta}{\beta_0}\right), \tag{6}$$

where $\varepsilon = mv^2/2$. Energetic charged particles traveling through a vast region within the heliosphere may also survey differing regions of temperature, hence, exhibit superstatistics behavior [20].

The focus of the main body of the present paper is to deal with quasi steady state electrostatic turbulence generated by a weak electron beam propagating in the background plasma with uniform temperature field. In such a situation non-Maxwellian distribution emerges from nonlinear wave-particle interaction. The kinetic approach to the formation of non-Maxwellian distribution, or the approach based upon the non-equilibrium statistical mechanics, is an alternative way of understanding how such distributions may form. The method may be complimentary to the conceptual approach based upon non-extensive statistical mechanics, since the time asymptotic state of a turbulent plasma may correspond to the non-extensive statistical equilibrium state. However, the formalism to be discussed subsequently assumes uniform background temperature field so that the superstatistical mechanism is an additional process.

In the Appendix we will overview the non-equilibrium statistical mechanics of plasmas, or equivalently, the plasma kinetic theory. However, the discussion will be brief. More detailed in-depth formalism may be found in the present author's recent monograph [22]. The theory overviewed in the present paper deals with electrostatic turbulence generated in the plasma, which when fully developed, leads to the formation of non-Maxwellian electron distribution function. In the subsequent sections, we will begin the discussion with basic equations whose derivation is briefly overviewed in the Appendix.

2 Plasma turbulence and non-Maxwellian electron distribution

In plasma physics the problem of energetic electron beam interacting with a background plasma is well known. Such a wave-particle interaction process leads to what is known as the "bump-in-tail" instability, which excites electrostatic turbulence that involves Langmuir waves. Theoretical and numerical analyses of the bump-in-tail instability in the quasi linear regime are well known in the plasma physics literature [23–27]. In references [28–31] numerical studies of bump-in-tail instability were extended beyond the quasi linear regime, to weak turbulence regime, where the basic equations derived in the Appendix are solved. These equations are the particle kinetic equation that governs the time evolution of electron distribution function $f_e(\mathbf{v},t)$,

$$\frac{\partial f_e}{\partial t} = \frac{\pi e^2}{m_e^2} \int d\mathbf{k}\, \frac{\mathbf{k}}{k^2} \cdot \frac{\partial}{\partial \mathbf{v}} \sum_{\sigma=\pm 1} \delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}) \left(\frac{m_e}{4\pi^2}\, \sigma\omega_{\mathbf{k}}^L\, f_e + I_{\mathbf{k}}^{\sigma L}\, \mathbf{k}\cdot \frac{\partial f_e}{\partial \mathbf{v}} \right), \tag{7}$$

which is taken from (A.33), where contribution from binary collisional processes are omitted. For bump-in-tail instability, collective modes dominate the electric field perturbation. Consequently, non-collective fluctuations, which are intimately related to the collisional process, become unimportant. In the above e and m_e stand for unit electric charge and electron mass, respectively, $\omega_{\mathbf{k}}^L = \omega_{pe}\left(1 + \frac{3}{2}\, k^2 \lambda_{De}^2\right)$ represents the dispersion relation satisfied by high frequency electrostatic wave in the plasma known as the Langmuir wave, $\lambda_{De} = T_e^{1/2}/(4\pi n e^2)^{1/2}$ being the Debye length, $\omega_{pe} = (4\pi n e^2/m_e)^{1/2}$ being the plasma oscillation frequency, T_e being the electron temperature, and $I_{\mathbf{k}}^{\sigma L}$ denotes the spectral electric field intensity associated with the Langmuir wave, $E_{\mathbf{k},\omega}^2 = \sum_{\sigma=\pm 1} I_{\mathbf{k}}^{\sigma L} \delta(\omega - \sigma\omega_{\mathbf{k}}^L)$. The symbol $\sigma = \pm 1$ denotes the sign of the wave phase and group velocities.

The wave kinetic equations for Langmuir and ion-sound wave intensities are also derived in the Appendix – see (A.27)–(A.31),

$$\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} &= \frac{\pi\sigma\omega_{\mathbf{k}}^L\,\omega_{pe}^2}{k^2} \int d\mathbf{v}\; \delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}) \left(\frac{m_e}{4\pi^2}\, \sigma\omega_{\mathbf{k}}^L\, f_e + I_{\mathbf{k}}^{\sigma L}\, \mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}} \right) \\
&\quad + 2\sigma\omega_{\mathbf{k}}^L \sum_{\sigma',\sigma''=\pm 1} \int d\mathbf{k}'\; V_{\mathbf{k},\mathbf{k}'}^{\sigma L} \Big[\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} \\
&\quad - \left(\sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L} \right) I_{\mathbf{k}}^{\sigma L} \Big] \\
&\quad + \frac{\pi\sigma\omega_{\mathbf{k}}^L e^2}{m_e^2\omega_{pe}^2} \sum_{\sigma'=\pm 1} \int d\mathbf{k}' \int d\mathbf{v}\; \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\, k'^2}\, \delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] \\
&\quad \times \left[\frac{m_e}{m_i}\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma L}\, (\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}} + \frac{n e^2}{\pi\omega_{pe}^2} \left(\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L} - \sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}}^{\sigma L} \right) (f_e + f_i) \right], \\
\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} &= \frac{\pi\mu_{\mathbf{k}}\sigma\omega_{\mathbf{k}}^L\omega_{pe}^2}{k^2} \int d\mathbf{v}\; \delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v}) \Big[\frac{m_e}{4\pi^2}\, \sigma\omega_{\mathbf{k}}^L (f_e + f_i) \\
&\quad + I_{\mathbf{k}}^{\sigma S}\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} \left(f_e + \frac{m_e}{m_i}\, f_i \right) \Big] + \sigma\omega_{\mathbf{k}}^L \sum_{\sigma',\sigma''=\pm 1} \int d\mathbf{k}'\; V_{\mathbf{k},\mathbf{k}'}^{\sigma S} \\
&\quad \times \Big[\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} - \left(\sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L} \right) I_{\mathbf{k}}^{\sigma S} \Big] .
\end{aligned} \tag{8}$$

For L mode wave equation the first term on the right-hand side represents linear wave-particle interaction between the electrons and Langmuir wave; the second term

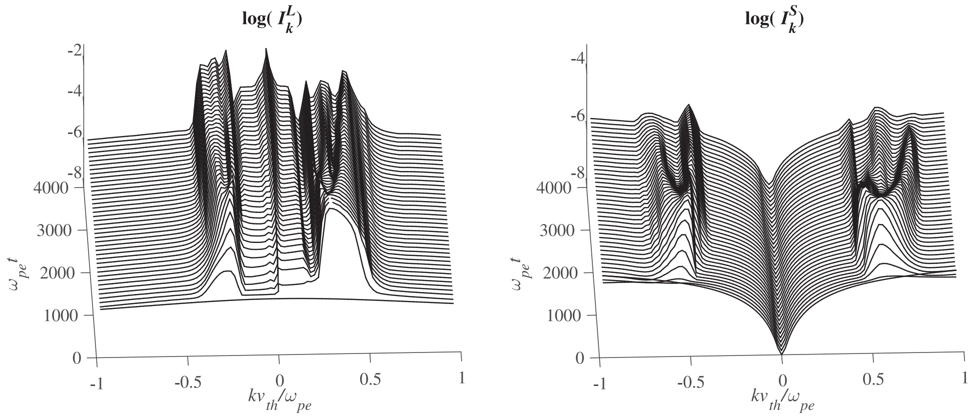

Fig. 2. Nonlinear progression of Langmuir wave [left] and ion-sound wave [right] turbulence in the dynamic spectral representation, that is, intensity (I_k) versus wave number (k) and time (t) space.

describes nonlinear wave-wave processes; the third term describes nonlinear wave-particle interactions. Here, f_i represents the Maxwellian ion distribution function. The ion sound mode, whose dispersion relation is given by $\omega_{\mathbf{k}}^S = kc_S$, obeys a similar wave kinetic equation as that of L mode. Various objects which appear in the wave kinetic equations (8) are defined by (A.28) and (A.30).

Among the findings according to numerical studies, particularly those of references [28,30], is that the long time evolution of the electron distribution function, initially modeled by a Maxwellian plus a shifted Maxwellian,

$$f_e(\mathbf{v},0) = \left(1 - \frac{n_b}{n_0}\right) \exp\left(-\frac{v^2}{v_{T0}^2}\right) + \frac{n_b}{n_0} \exp\left(-\frac{(\mathbf{v}-\mathbf{V}_b)^2}{v_{Tb}^2}\right), \tag{9}$$

where n_b and n_0 denote the beam and the background number densities, respectively; $v_{T0} = \sqrt{2T_0/m_e}$ and $v_{Tb} = \sqrt{2T_b/m_e}$ are their respective thermal spreads, T_0 and T_b their temperatures, respectively; and $\mathbf{V}_b$ represents the initial velocity for the beaming electrons, is such that the asymptotic distribution involves the generation of suprathermal tail population, which is superficially reminiscent of observed distribution in space.

An example is shown in the next couple of figures. In obtaining the numerical result, we assumed $n_b/n_0 = 10^{-2}$, $T_b = T_0$, $V_b/v_{T0} = 4$, and the dimensionless plasma parameter of $g = n(\lambda_{De})^{-1} = 10^{-3}$ is adopted for one dimensional system. Note that this is a one dimensional plasma parameter. The three dimensional value should be roughly $g_{3D} = n\lambda_{De}^{-3} \propto 10^{-9}$, which is typical of the solar wind near Earth orbit. The normalization of Langmuir wave spectral energy density is $gI_{\mathbf{k}}^{\sigma L}/(8m_e v_{Te}^2)$. Figure 2 shows the time development of wave intensities. The left-hand panel plots the dynamic spectrum of Langmuir wave intensity, where positive k region corresponds to the forward propagating Langmuir wave ($\sigma = 1$), while the negative k region designates the backward propagating L mode ($\sigma = -1$). We combined the two modes into a single figure, plotting I_k^{+L} over positive k range, and I_k^{-L} in $k < 0$ space. Actual numerical computation was done over $k > 0$ space for both I_k^{+L} and I_k^{-L}. For early time periods between $t = 0$ and $\omega_{pe}t = 200$, or so, the Langmuir wave dynamics is simply dictated by the exponential growth and subsequent quasilinear saturation in the positive k range, which corresponds to the quasi linear development of bump-in-tail instability [23–27]. During this stage we begin to see the growth associated with

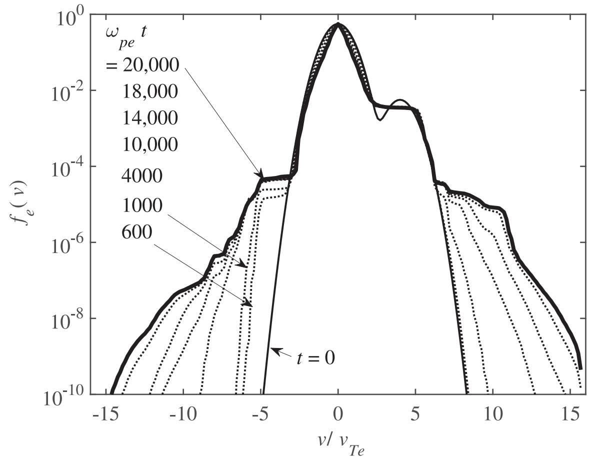

Fig. 3. Development of energetic tail during the nonlinear mode-coupling stage.

the backward Langmuir waves ($k < 0$ region). This is the result of combined three-wave decay process and scattering of forward L mode off thermal ions [28,29]. It is also seen that Langmuir waves near $k \sim 0$ slowly but steadily grow in intensity. This is known as the Langmuir condensation effect. Nonlinear mode coupling processes involve multiple back-and-forth mode coupling interactions, which continue on well beyond the quasi linear saturation phase.

The right-hand panel of Figure 2 corresponds to the dynamic spectrum of ion-sound mode turbulence. In the early stage, between $t = 0$ and $\omega_{pe}t = 200$ or so, no ion-sound mode is apparent above the initial noise level. However, around the time when the backward Langmuir wave begins to appear, it can be see that, first, the forward-propagating S mode wave becomes slightly enhanced, followed by the backward ($k < 0$) ion-sound mode waves. The production of S mode turbulence is owing to the decay process. It is important to note that the ion-sound turbulence is a transient phenomenon, since over long time period, it is seen that the S mode wave intensity gradually settles down back toward the initial noise level.

The Langmuir condensation is responsible for the acceleration of small amount of electrons to form an energetic tail population. This is because for long wavelength mode the resonant velocity $v_{\rm res} \approx \omega_{pe}/k$, can become very high. This is the origin of suprathermal electron population. Figure 3 displays the long-time evolution of electron distribution function. Observe the formation of energetic tail population in the suprathermal energy range. Reference [32] confirmed this findings by means of particle-in-cell (PIC) simulation. This has led the present author to seek the time asymptotic solution of the equations of weak turbulence theory (7) and (8), in order to prove that Vasyliunas' kappa distribution indeed characterizes the asymptotic state of the Langmuir turbulence [33].

In the steady-state we ignore contributions from S mode. This is because the generation of S mode is a transient phenomenon, as seen in Figure 2. Reference [33] also shows that the three-wave processes are largely ignorable, which can be understood from intuition as well. The time-asymptotic state represents a situation where electrons and Langmuir waves exchange momenta and energies but wave-wave interaction only involves momentum and energy exchanges among the waves themselves. Hence, they do not contribute to the steady-state turbulence. From (7) and (8) it is seen that the right-hand side of the particle equation and linear wave-particle term in the L mode equation share a common factor,

$$\left(\frac{m_e}{4\pi^2}\,\sigma\omega_{\mathbf{k}}^L f_e + I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right).$$

This means that a suitable choice of Langmuir wave intensity $I_{\mathbf{k}}^{\sigma L}$ will lead to a steady-state electron distribution f_e, which together will make the above factor vanish. Conversely, a judicious model for f_e will lead to a steady-state spectrum of Langmuir turbulence $I_{\mathbf{k}}^{\sigma L}$, which together will satisfy the condition for vanishing factor specified above.

In short, there is an infinite class of solutions $(f_e, I_{\mathbf{k}}^{\sigma L})$ that satisfy the steady-state particle and linear wave equations. Of such an infinite class of solutions, reference [33] chose the kappa electron velocity distribution and its associated Langmuir fluctuation spectral intensity,

$$\begin{aligned} f_e(v) &= \frac{m_e^{3/2}}{(2\pi T_e)^{3/2}}\frac{\Gamma(\kappa+1)}{\left(\kappa-\frac{3}{2}\right)^{\frac{3}{2}}\Gamma\left(\kappa-\frac{1}{2}\right)}\left(1+\frac{m_e v^2}{2\left(\kappa-\frac{3}{2}\right)T_e}\right)^{-\kappa-1}, \\ I_L(k) &= \frac{T_e}{4\pi^2}\frac{\kappa-\frac{3}{2}}{\kappa+1}\left(1+\frac{m_e\omega_{pe}^2}{2\left(\kappa-\frac{3}{2}\right)k^2T_e}\right). \end{aligned} \tag{10}$$

However, it is obvious that while this choice is convenient, it is by no means unique. In order to address the uniqueness, one must consider the nonlinear wave-particle interaction between the electrons and Langmuir turbulence. As discussed in reference [33], the nonlinear part of the wave kinetic equation is given by the following under the assumption of isotropic Langmuir turbulence intensity, $I_{\mathbf{k}}^{+L} = I_{\mathbf{k}}^{-L} \equiv I_L(\mathbf{k})$:

$$\begin{aligned} \left.\frac{\partial I_L(\mathbf{k})}{\partial t}\right|_{\rm nl} = &-\frac{\pi}{\omega_{pe}^2}\frac{e^2}{m_e^2}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\,\delta[\omega_{\mathbf{k}}^L-\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] \\ &\times\left(\frac{ne^2}{\pi\omega_{pe}}\left[\omega_{\mathbf{k}'}^L I_L(\mathbf{k})-\omega_{\mathbf{k}}^L I_L(\mathbf{k}')\right]f_i\right. \\ &\left.-\frac{m_e}{m_i}\,\omega_{pe}I_L(\mathbf{k}')I_L(\mathbf{k})(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial\mathbf{v}}\right). \end{aligned} \tag{11}$$

Then reference [33] proceeded to show that the steady-state solution is given by

$$I_L(k) = \frac{T_i}{4\pi^2}\left(1+\frac{m_e\omega_{pe}^2}{2(\kappa-\frac{3}{2})k^2T_e}\right), \tag{12}$$

which can be reconciled with $I_L(k)$ defined in (10) if we identify

$$\kappa = \frac{9}{4} = 2.25, \qquad T_i = T_e\,\frac{\kappa-\frac{3}{2}}{\kappa+1}. \tag{13}$$

Such a reconciliation would not have been possible if we chose any other electron distribution f_e than the kappa distribution. Consequently, this proves that the kappa model defined in (10) is the only solution for steady state Langmuir turbulence.

This finding strongly implies that the space plasma may be ruled by non-extensive thermostatics. Recall that the kappa distribution is equivalent to the q-Gaussian, which corresponds to the most probable state in Tsallis thermostatics theory. The steady-state Langmuir turbulence and its associated kappa distribution are consistent with actual observations made by artificial spacecraft. For suprathermal velocity range, $v \gg v_{Te}$, the kappa electron distribution (10) behaves as an inverse power law distribution, $f_e \sim v^{-6.5}$ since $\kappa \approx 9/4 = 2.25$. Recall that the solar wind electrons are customarily modeled by a combination of Maxwellian core, suprathermal halo, and superhalo – see Figure 1. Observation made in the near Earth space shows that superhalo electrons behave as $f_e \sim v^{-5.0}$ to $v^{-8.7}$ with average behavior $f_e^{\rm obs} \sim v^{-6.69}$ [7], which agrees quite well with theoretical prediction of $f_e \sim v^{-6.5}$. Reference [34] investigated the properties of solar wind halo electrons by modeling them with the kappa distribution. They analyzed *Helios*, *Cluster*, and *Ulysses* spacecraft data, and found that the κ parameter decreases from $\sim$9 near 0.3 AU (1 AU or an Astronomical Unit being the distance between Sun and Earth) to $\sim$4 near 1 AU (near Earth), to $\sim$2.25 near $\sim$5 AU (near Jovian orbit). This seems to imply that the radially expanding solar wind evolves into the quasi equilibrium state, where the distinction between halo and superhalo electrons disappears, and the κ index approaches closer and closer to the theoretically predicted value.

3 Alternative approach to dynamic equilibrium for space plasma

Thus far, we have largely reviewed the recent findings regarding the steady-state electrostatic turbulence that generates non-Maxwellian (or to be specific, kappa) distribution of electrons in space. Such a state, however, may not be in true equilibrium, since for long time scale, non-collective oscillations (or fluctuations) that naturally arise in thermal plasma may not be ignored. Such fluctuations lead to collisional relaxation, such that the governing equation for the particles must include the influence of collisions. In Appendix, we have derived the particle kinetic equation (A.33) that includes both collective and non-collective fluctuations. Consequently, the genuine steady state electron velocity distribution must be discussed on the basis of (A.33) rather than (7). Upon expressing (A.33) in spherical velocity coordinate and assuming a priori that f_e is isotropic, we obtain

$$\frac{\partial f_e}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2\left(A_v + A_v^c\right) f_e\right] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\left(D_{vv} + D_{vv}^c\right)\frac{\partial f_e}{\partial v}\right), \tag{14}$$

where

$$\begin{aligned}
A_v &= \frac{e^2\omega_{pe}^2}{m_e v^2}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}, \\
D_{vv} &= \frac{4\pi^2 e^2\omega_{pe}^2}{m_e^2 v^3}\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\, I_L(k), \\
A_v^c &= \frac{4\pi n e^4 \ln\Lambda}{m_e^2}\frac{2}{v_{Te}^2}\left(G(x_e) + \frac{T_e}{T_i}\, G(x_i)\right), \\
D_{vv}^c &= \frac{4\pi n e^4 \ln\Lambda}{m_e^2}\frac{G(x_e) + G(x_i)}{v}, \\
x_e &= \frac{v}{v_{Te}}, \qquad x_i = \frac{v}{v_{Ti}}, \qquad \Lambda = 4\pi n\lambda_{De}^3, \\
G(x) &= \frac{\mathrm{erf}(x) - (2/\sqrt{\pi})\, x\, e^{-x^2}}{2x^2}.
\end{aligned} \tag{15}$$

In the above we took the approach of treating the collisional processes via Rosenbluth potential approximation [35].

The steady-state solution can be obtained as follows:

$$
\begin{aligned}
f_e &= \text{const}\,\exp\left(-\int dv\,\frac{A_v + A_v^c}{D_{vv} + D_{vv}^c}\right)\\
&= C\exp\left(-\int dv\,\frac{v\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k} + \frac{m_e v^3}{T_e}\,\ln\Lambda\left(G(x_e) + \frac{T_e}{T_i}\,G(x_i)\right)}{\dfrac{4\pi^2}{m_e}\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\,I_L(k) + v^2\,\ln\Lambda\,[G(x_e) + G(x_i)]}\right),
\end{aligned}
\tag{16}
$$

where C represents the normalization constant. If we ignore the contribution from collective fluctuations, that is, the k integral terms in the numerator and denominator, then we have

$$
f_e = C\exp\left(-\frac{m_e}{T_e}\int dv v\,\frac{G(x_e) + \dfrac{T_e}{T_i}\,G(x_i)}{G(x_e) + G(x_i)}\right) = C\exp\left(-\frac{m_e v^2}{2T}\right),
\tag{17}
$$

where we have assumed $T_e = T_i = T$ in the second equality, which is true for thermal equilibrium.

On the other hand, if we ignore the collisional part dictated by $\ln\Lambda$, then we have

$$
f_e = C\exp\left(-\frac{m_e}{4\pi^2}\int dv v\,\frac{\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}}{\displaystyle\int_{\omega_{pe}/v}^{\infty}\frac{dk}{k}\,I_L(k)}\right).
\tag{18}
$$

If we take the form of $I_L(k)$ given by (12), and formally define the divergent integral quantity,

$$
\mathscr{H} \equiv \int_{\omega_{pe}/v}^{\infty}\frac{dk}{k},
\tag{19}
$$

as a quasi constant, which was what was done in reference [33], then we have the desired kappa distribution defined in (10). However, the quantity $\mathscr{H}$ is not only divergent in a formal sense, but also is a function of v. We thus re-examine the steady state particle distribution (18) in more detail in this section.

Let us consider the intensity $I_L(k)$, which is conveniently re-expressed as

$$
\begin{aligned}
I_L(k) &= \frac{T_e}{4\pi^2}\,a\left(1 + \frac{k_0^2}{k^2}\right),\\
a &= \frac{\kappa - \frac{3}{2}}{\kappa + 1}, \qquad \kappa = \frac{m_e\omega_{pe}^2}{k_0^2 T_e}\,\mathscr{H} + \frac{3}{2}.
\end{aligned}
\tag{20}
$$

Then inserting this to (18), we arrive at

$$f_e = C \exp\left(-\frac{m_e}{aT_e}\int dvv\,\frac{1}{1+\dfrac{1}{\mathscr{H}(v)}\dfrac{k_0^2 v^2}{2\omega_{pe}^2}}\right),$$

$$\mathscr{H}(v) = \int_{\omega_{pe}/v}^{\infty}\frac{dk}{k} \to \begin{cases} 0, & v \to 0 \\ \int_0^\infty dk/k, & v \to \infty \end{cases}. \tag{21}$$

In the limit of small v, since $\mathscr{H}(v)$ approaches zero, f_e becomes quasi constant. For large v one may replace $\mathscr{H}(v)$ by

$$\int_{k_{\min}}^{k_{\max}}\frac{dk}{k} = \ln\frac{k_{\max}}{k_{\min}} \equiv r, \tag{22}$$

and thus we have a kappa like behavior. Here, r may be equivalent to $\ln(4\pi n\lambda_{De}^3)$, or it may be defined in a more general way. For the moment, we treat it as a free parameter. In between $v \sim 0$ and large v, the matter becomes somewhat complex. One way to treat $\mathscr{H}(v)$ is to approximate it by r except for small v, by introducing a cutoff function that approaches 0 as v approaches 0,

$$\mathscr{H}(v) \to S(v)\, r, \tag{23}$$

where $S(v) \to 0$ for $v \to 0$ and $S(v) \to 1$ for finite v. If we take this approach then we have

$$f_e = C \exp\left(-\frac{m_e}{aT_e}\int dvv\,\frac{S(v)\,r}{S(v)\,r + \dfrac{k_0^2 v^2}{2\omega_{pe}^2}}\right). \tag{24}$$

As a specific example of the factor $S(v)$, let us model it by

$$S(v) = \tanh^2\frac{m_e v^2}{2T_e}. \tag{25}$$

This function preserves the required behavior $S(v) \to 0$ for $v \to 0$ and $S(v) \to 1$ for finite v. In general, the formal solution (24) does not enjoy closed form manipulation of indefinite velocity integral. However, one may proceed to construct the solution by means of numerical integration.

Plotted in Figure 4 is the numerically computed distribution function, which may be considered as the generalized kappa model, as a function of normalized velocity $u = v/v_{Te}$, for various values of input parameter r. For relatively high values of r, such as $r = 5$ and higher, the model resembles Maxwellian distribution (the Maxwellian model, f_{Max} is plotted with blue dotted curve as a reference). For $r = 1$, the model becomes virtually identical to the kappa distribution, f_κ, shown with red dots. For low value of r, the velocity power law spectrum becomes harder. We have shown one particular case of $r = 0.5$.

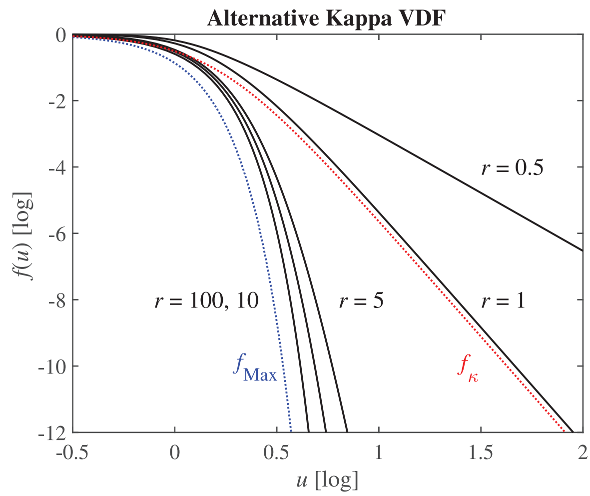

Fig. 4. Generalized kappa distribution (24) as a function of normalized velocity $u = v/v_{Te}$.

Returning to (16), we expect the general solution to behave as follows:

$$f_e = \begin{cases} C \exp\left(-\dfrac{m_e}{T_e}\displaystyle\int dv v\,\dfrac{G(x_e)+(T_e/T_i)G(x_i)}{G(x_e)+G(x_i)}\right), & v < v_T, \\ C \exp\left(-\dfrac{m_e}{4\pi^2}\displaystyle\int dv v\,\dfrac{\mathscr{H}}{\displaystyle\int_{\omega_{pe}/v}^{\infty}\dfrac{dk}{k}\,I_L(k)}\right), & v > v_T. \end{cases} \tag{26}$$

In short, we expect the collisional processes to dominate the core part of the electron distribution, while the suprathermal range will be dominated by collective processes. This may offer a natural explanation for why the solar wind electron distribution appears to be composed of the Maxwellian "core" plus non-Maxwellian "tail".

4 Conclusions

To conclude the present paper in which we overviewed the theory of origin of non-Maxwellian electron distribution in space plasma, we have approached the problem from the perspective of non equilibrium statistical mechanics. Energetic charged particles are constantly spewed out from the Sun into interplanetary space. The steady stream of energetic electrons interact with the pre-existing population of background electrons in space, which leads to the excitation of collective instability. The high frequency electrostatic turbulence thus generated is called the Langmuir turbulence. As the expanding solar plasma reaches the near Earth region in space and even farther out, the Langmuir turbulence reaches the steady state. The electron velocity distribution function corresponding to such a quasi stationary turbulent state is characterized by a non-Maxwellian feature, including the kappa distribution.

The kinetic theory of plasma turbulence, which is systematically formulated from the non equilibrium statistical method is overviewed in the Appendix. According to such a theory, the formation of non-Maxwellian (or kappa) electron distribution function can be discussed on a rigorous basis. The finding that the quasi stationary state of Langmuir turbulence coincides with the formation of kappa distribution strongly implies the existence of an inter relationship between the non-extensive statistical description of plasma and the steady state theory of Langmuir turbulence. Both descriptions share a common feature in that the equilibrium distribution function corresponds to the kappa distribution, or equivalently, the q-Gaussian distribution. On this basis, it is reasonable to assume that the underlying statistical principle that governs the space plasma is none other than Tsallis statistical theory.

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Appendix A: Overview of non-equilibrium statistical mechanics of plasmas

A.1 General formulation

The kinetic theory for plasmas can be found in the standard literature, which includes the present author's recent monograph [22], so the overview will be brief, see e.g., [36–39]. The plasma is a collection of fully ionized gas governed by classical Newtonian dynamics. It is convenient to consider an N-body probability distribution function in phase space $(\mathbf{r},\mathbf{p})$, called the Klimontovich function, $N_a(\mathbf{r},\mathbf{p},t)$, defined by [36]

$$N_a(\mathbf{r},\mathbf{p},t)=\sum_{j=1}^{N}\delta\left[\mathbf{r}-\mathbf{r}_j^a(t)\right]\delta\left[\mathbf{p}-\mathbf{p}_j^a(t)\right], \tag{A.1}$$

where $\mathbf{r}_j^a(t)$ and $\mathbf{p}_j^a(t)$ are exact particle orbits for the jth particle of species a ($e_a=-e$ for electrons and $e_a=e$ for ions), $\mathbf{v}_j^a(t)=\dot{\mathbf{r}}_j^a(t)$, $\dot{\mathbf{p}}_j^a(t)=e_a\mathbf{E}(\mathbf{r},t)+(e_a/c)\mathbf{v}\times\mathbf{B}(\mathbf{r},t)$. Here, $e_a=-e$ for the electrons and e for the protons, e being the unit electric charge, and c is the speed of light in vacuo. The electric and magnetic field vectors, $\mathbf{E}$ and $\mathbf{B}$, satisfy Maxwell's equation. The kinetic equation for the Klimontovich function is equivalent to the Liouville equation. The one-particle distribution function, $f_a(\mathbf{r},\mathbf{p},t)$, is the ensemble averaged Klimontovich function, $f_a(\mathbf{r},\mathbf{p},t)=\langle N_a(\mathbf{r},\mathbf{p},t)\rangle$. If we assume field-free environment and impose electrostatic approximation, then the basic equations are

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+e_a\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}}\right)N_a=0,\\ \nabla\cdot\mathbf{E}-4\pi\sum_a e_a\int d\mathbf{p}N_a=0.\end{aligned} \tag{A.2}$$

It is useful to consider the Klimontovich function describing the phase space evolution of free particles (ideal gas) that do not interact with each other,

$$N_a^0(\mathbf{r},\mathbf{p},t) = \sum_{j=1}^{N} \delta\left[\mathbf{r} - \mathbf{r}_j^{a0}(t)\right]\delta\left[\mathbf{p} - \mathbf{p}_j^{a0}(t)\right], \tag{A.3}$$

where $\mathbf{r}_j^{a0}(t) = \mathbf{r}^{a0} + \mathbf{v}_i^a t$ and $\mathbf{p}_j^{a0}(t) = \mathbf{p}_i^a$ are exact orbits of free streaming particles satisfying, $\dot{\mathbf{p}}_j^{a0}(t) = 0$ and $\mathbf{v}_j^{a0}(t) = \dot{\mathbf{r}}_j^{a0}(t)$. The corresponding Klimontovich equation for free particles is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) N_a^0 = 0. \tag{A.4}$$

The plasma is a fully ionized gas in which collective interaction dominates the dynamics. As such, it is preferable to remove effects that arise from purely non-interacting particle behavior. Consequently, let us subtract (A.4) from (A.2), which results in

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(N_a - N_a^0\right) + e_a\mathbf{E}\cdot\frac{\partial N_a}{\partial \mathbf{p}} &= 0,\\ \nabla\cdot\mathbf{E} - 4\pi\sum_a e_a\int d\mathbf{p} N_a &= 0. \end{aligned} \tag{A.5}$$

We denote the deviation of the Klimontovich functions N_a and N_a^0 from their averages $f_a = \langle N_a\rangle = \langle N_a^0\rangle$ (here, we have made an assumption that the ensemble average of N_a and N_a^0 are approximately equal) by $\delta N_a = N_a - \langle N_a\rangle$ and $\delta N_a^0 = N_a^0 - \langle N_a^0\rangle$, that is, δN_a and δN_a^0 denote the fluctuations. We assume random phases so that ensemble averages of δN_a and δN_a^0 are zero. Since the medium is free of average field the electric field is only made of fluctuations, $\mathbf{E}(\mathbf{r},t) = \delta\mathbf{E}(\mathbf{r},t)$. Then (A.5) can be re-expressed as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) &= -e_a\frac{\partial}{\partial \mathbf{p}}\cdot\langle\delta\mathbf{E}\delta N_a\rangle,\\ \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\left(\delta N_a - \delta N_a^0\right) + e_a\delta\mathbf{E}\cdot\frac{\partial f_a}{\partial \mathbf{p}} &= -e_a\frac{\partial}{\partial \mathbf{p}}\cdot\left(\delta\mathbf{E}\delta N_a - \langle\delta\mathbf{E}\delta N_a\rangle\right),\\ \nabla\cdot\delta\mathbf{E} &= 4\pi\sum_a e_a\int d\mathbf{p}\,\delta N_a. \end{aligned} \tag{A.6}$$

In this equation δN_a^0 represents the "source" term for the inhomogeneous nonlinear differential equation for δN_a. We are not interested in δN_a^0 per se, but rather in the ensemble average of the product $\langle\delta N_a^0(\mathbf{r},\mathbf{p},t)\delta N_a^0(\mathbf{r}',\mathbf{p}',t')\rangle$, that is, the two-body correlation function for the fluctuations of free-streaming Klimontovich functions. We may compute this quantity directly from definition (A.3),

$$\langle\delta N_a^0(\mathbf{r},\mathbf{p},t)\delta N_b^0(\mathbf{r}',\mathbf{p}',t')\rangle = \delta_{ab}\delta[\mathbf{r} - \mathbf{r}' - \mathbf{v}(t-t')]\delta(\mathbf{p} - \mathbf{p}')f_a(\mathbf{r},\mathbf{p},t). \tag{A.7}$$

Upon writing the electrostatic field in terms of the potential, $\delta\mathbf{E}_{\mathbf{k},\omega} = i\mathbf{k}\delta\phi_{\mathbf{k},\omega}$, under the assumption of spatially uniform average background, we may express the relevant equations in terms of spectral representation,

$$
\begin{aligned}
\frac{\partial f_a(\mathbf{v})}{\partial t} &= ie_a \int dq\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}}\langle \phi_{-q} N_q^a(\mathbf{p})\rangle, \\
N_q^a(\mathbf{p}) &= N_q^{a0}(\mathbf{p}) + \mathbf{k}\cdot\mathbf{g}_q f_a(\mathbf{p})\,\phi_q + \int dq'\,\mathbf{k}'\cdot\mathbf{g}_q\left(\phi_{q'}N^a_{q-q'}(\mathbf{p}) - \langle\phi_{q'}N^a_{q-q'}(\mathbf{p})\rangle\right), \\
\phi_q &= \sum_a \frac{4\pi e_a}{k^2}\int d\mathbf{p}\, N_q^a(\mathbf{p}),
\end{aligned} \tag{A.8}
$$

where we have introduced short hand notations,

$$
\begin{aligned}
q &\equiv (\mathbf{k},\omega), \\
\mathbf{g}_q &= \mathbf{g}^a_{\mathbf{k},\omega} = -\frac{e_a}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}\frac{\partial}{\partial\mathbf{p}},
\end{aligned} \tag{A.9}
$$

and have omitted δ for the perturbed quantities. The spectral representation of the source fluctuation (A.7) is given by

$$
\langle \delta N_a^0(\mathbf{p})\delta N_b^0(\mathbf{p}')\rangle_q = (2\pi)^{-3}\delta_{ab}\delta(\mathbf{p}-\mathbf{p}')\delta(\omega - \mathbf{k}\cdot\mathbf{v}) f_a(\mathbf{p}). \tag{A.10}
$$

We solve the nonlinear equation for N_a^a by iterative means, $N_q^a = N_q^{a(1)} + N_q^{a(2)} + N_q^{a(3)} + \cdots$, where each order in the perturbative expansion is of the similar magnitude with the electric field perturbation of the same order, $\mathscr{O}\left(N_q^{a(n)}\right) \propto \mathscr{O}\left(\phi_q^n\right)$. It is straightforward to show that the iterative solution is given order by order,

$$
\begin{aligned}
N_q^{a(1)} &= N_q^{a0} + \mathbf{k}\cdot\mathbf{g}_q^a f_a \phi_q, \\
N_q^{a(2)} &= \int dq'\mathbf{k}'\cdot\mathbf{g}_q^a(\mathbf{k}-\mathbf{k}')\cdot\mathbf{g}^a_{q-q'} f_a\left[\phi_{q'}\phi_{q-q'} - \langle\phi_{q'}\phi_{q-q'}\rangle\right], \\
N_q^{a(3)} &= \int dq'\int dq''\mathbf{k}'\cdot\mathbf{g}_q^a\mathbf{k}''\cdot\mathbf{g}^a_{q-q'}(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\cdot\mathbf{g}^a_{q-q'-q''} f_a \\
&\quad\times\left[\phi_{q'}\phi_{q''}\phi_{q-q'-q''} - \phi_{q'}\langle\phi_{q''}\phi_{q-q'-q''}\rangle - \langle\phi_{q'}\phi_{q''}\phi_{q-q'-q''}\rangle\right],
\end{aligned} \tag{A.11}
$$

where we have kept the effects of source fluctuation only in the leading order term.

Inserting the net solution to the wave equation while symmetrizing various terms with respect to the dummy integral variable, we have

$$
\begin{aligned}
\epsilon(q)\phi_q &= \sum_a \frac{4\pi e_a}{k^2}\int d\mathbf{p} N_q^{a0}(\mathbf{p}) \\
&\quad + \sum_{\substack{q_1\ q_2 \\ (q_1+q_2=q)}}\!\!\sum \frac{ik_1k_2}{k}\chi^{(2)}(q_1|q_2)\left[\phi_{q_1}\phi_{q_2} - \langle\phi_{q_1}\phi_{q_2}\rangle\right] \\
&\quad + \sum_{\substack{q_1\ q_2\ q_3 \\ (q_1+q_2+q_3=q)}}\!\!\sum\sum \frac{k_1k_2k_3}{k}\bar{\chi}^{(3)}(q_1|q_2|q_3)\left[\phi_{q_1}\phi_{q_2}\phi_{q_3} - \phi_{q_1}\langle\phi_{q_2}\phi_{q_3}\rangle - \langle\phi_{q_1}\phi_{q_2}\phi_{q_3}\rangle\right],
\end{aligned} \tag{A.12}
$$

where we have defined various response functions,

$$\epsilon(q) = 1 + \chi(q) = 1 + \sum_a \chi_a(q) = \sum_a \frac{4\pi e_a^2}{k^2} \int d\mathbf{p} \frac{\mathbf{k}\cdot\partial f_a/\partial\mathbf{p}}{\omega - \mathbf{k}\cdot\mathbf{v} + i0}, \tag{A.13}$$

$$\begin{aligned}\chi^{(2)}(q_1|q_2) &= \sum_a \chi_a^{(2)}(q_1|q_2) = \sum_a \frac{-ie_a}{2} \frac{4\pi e_a^2}{k_1 k_2 |\mathbf{k}_1 + \mathbf{k}_2|} \\ &\times \int d\mathbf{p} \frac{1}{\omega_1 + \omega_2 - (\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{v} + i0} \\ &\times \left[\mathbf{k}_1 \cdot \frac{\partial}{\partial\mathbf{p}} \left(\frac{\mathbf{k}_2\cdot\partial f_a/\partial\mathbf{p}}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v} + i0}\right) + \mathbf{k}_2 \cdot \frac{\partial}{\partial\mathbf{p}} \left(\frac{\mathbf{k}_1\cdot\partial f_a/\partial\mathbf{p}}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v} + i0}\right)\right],\end{aligned} \tag{A.14}$$

$$\begin{aligned}\bar{\chi}^{(3)}(q_1|q_2|q_3) &= \sum_a \bar{\chi}_a^{(3)}(q_1|q_2|q_3) = \sum_a \frac{(-i)^2 e_a^2}{2} \frac{4\pi e_a^2}{k_1 k_2 k_3 |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|} \\ &\times \int d\mathbf{p} \frac{1}{\omega_1 + \omega_2 + \omega_3 - (\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\cdot\mathbf{v} + i0} \\ &\times \mathbf{k}_1 \cdot \frac{\partial}{\partial\mathbf{p}} \left\{ \frac{1}{\omega_2 + \omega_3 - (\mathbf{k}_2 + \mathbf{k}_3)\cdot\mathbf{v} + i0} \right. \\ &\times \left. \left[\mathbf{k}_2 \cdot \frac{\partial}{\partial\mathbf{p}} \left(\frac{\mathbf{k}_3\cdot\partial f_a/\partial\mathbf{p}}{\omega_3 - \mathbf{k}_3\cdot\mathbf{v} + i0}\right) + \mathbf{k}_3 \cdot \frac{\partial}{\partial\mathbf{p}} \left(\frac{\mathbf{k}_2\cdot\partial f_a/\partial\mathbf{p}}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v} + i0}\right)\right]\right\}.\end{aligned} \tag{A.15}$$

The definitions and notations of the various dielectric susceptibilities are consistent with [37].

From (A.12) it is possible to obtain the equation for the spectral electric field energy density fluctuation $\langle E^2\rangle_q = \langle k^2\phi^2\rangle_q$. This is done by first multiplying ϕ_{-q} to (A.12) and taking the ensemble average. Then replacing q by $-q$ in (A.12), we may also multiply $N_{-q}^{a0}(\mathbf{p})$ and taking the average. The result is

$$\begin{aligned}0 &= \epsilon(q)\langle E^2\rangle_q - i\int dq' \chi^{(2)}(q'|q-q') k k' |\mathbf{k}-\mathbf{k}'| \langle \phi_{q'}\phi_{q-q'}\phi_{-q}\rangle \\ &-2\int dq' \bar{\chi}^{(3)}(q'|-q'|q) \langle E^2\rangle_{q'} \langle E^2\rangle_q \\ &-\sum_a \int d\mathbf{p} \frac{(4\pi e_a)^2}{(2\pi)^3 k^2 \epsilon^*(q)} \delta(\omega - \mathbf{k}\cdot\mathbf{v}) f_a(\mathbf{p}) \\ &+i\sum_a \int dq' \int d\mathbf{p} \frac{(4\pi e_a)\chi^{(2)*}(q'|q-q')}{k\epsilon^*(q)} k' |\mathbf{k}-\mathbf{k}'| \langle \phi_{-q'}\phi_{-q+q'} N_q^{a0}(\mathbf{p})\rangle.\end{aligned} \tag{A.16}$$

Note that we use summation and integral over $q = (\mathbf{k}, \omega)$ interchangeably in the present paper, that is, $\sum_q = \int dq = \int d\mathbf{k} \int d\omega$.

Equation (22) is not closed since it contains third-order cumulants, $\langle \phi_{q'}\phi_{q-q'}\phi_{-q}\rangle$ and $\langle \phi_{-q'}\phi_{-q+q'} N_q^{a0}(\mathbf{p})\rangle$. These quantities may be constructed from (A.12) by ignoring the third-order nonlinearity at the outset. The three-body cumulants are zero if nonlinear terms are neglected, since the linear solutions are plane waves, hence, all odd moments vanish upon taking the ensemble average. Thus, if we write the perturbed field as the sum of plane-wave solution plus nonlinear correction, $\phi_q = \phi_q^{(0)} + \phi_q^{(1)}$, where $\phi_q^{(0)}$ satisfies $\epsilon(q)\phi_q^{(0)} = 0$, then we obtain

$$\phi_{q_1}^{(1)} = \frac{1}{k_1^2\epsilon(q_1)}\sum_{q''} ik_1k''|\mathbf{k}_1 - \mathbf{k}''|\chi^{(2)}(q''|q_1 - q'')\left[\phi_{q''}^{(0)}\phi_{q_1-q''}^{(0)} - \langle\phi_{q''}^{(0)}\phi_{q_1-q''}^{(0)}\rangle\right] + \frac{1}{k_1^2\epsilon(q_1)}\sum_a 4\pi e_a \int d\mathbf{v} N_{q_1}^{a0}(\mathbf{p}). \tag{A.17}$$

The quantity $\langle\phi_{q'}\phi_{q-q'}\phi_{-q}\rangle$, can be constructed by successively making use of (A.17) for each of $\phi_{q'}$, $\phi_{q-q'}$, and ϕ_{-q}, $\langle\phi_{q'}\phi_{q-q'}\phi_{-q}\rangle = \langle\phi_{q'}^{(1)}\phi_{q-q'}^{(0)}\phi_{-q}^{(0)}\rangle + \langle\phi_{q'}^{(0)}\phi_{q-q'}^{(1)}\phi_{-q}^{(0)}\rangle + \langle\phi_{q'}^{(0)}\phi_{q-q'}^{(0)}\phi_{-q}^{(1)}\rangle + \cdots$. Then we omit the superscript (0) on the right-hand side at the end. We also make use of the symmetry property, $\chi^{(2)}(-q_1|-q_2) = -\chi^{(2)*}(q_1|q_2)$, in order to simplify various coupling coefficients, and decompose the four-body cumulants as products of two-body cumulants while ignoring irreducible components, thereby closing the hierarchy of correlations,

$$\begin{aligned}\langle\phi_{q_1}\phi_{q_2}\phi_{q_3}\phi_{q_4}\rangle = \delta(q_1+q_2+q_3+q_4)\,[&\langle\phi_{q_1}\phi_{q_2}\rangle\langle\phi_{q_3}\phi_{q_4}\rangle\delta(q_1+q_2)\\ &+\langle\phi_{q_1}\phi_{q_3}\rangle\langle\phi_{q_2}\phi_{q_4}\rangle\delta(q_1+q_3)\\ &+\langle\phi_{q_1}\phi_{q_4}\rangle\langle\phi_{q_2}\phi_{q_3}\rangle\delta(q_1+q_4)]\,.\end{aligned} \tag{A.18}$$

This closure scheme is the simplest, which in the theory of neutral fluid turbulence, is known as the *quasi-normal closure*.

After some tedious but otherwise straightforward algebraic manipulations, we obtain

$$\begin{aligned}\langle\phi_{q'}\phi_{q-q'}\phi_{-q}\rangle = 2ikk'|\mathbf{k}-\mathbf{k}'|&\left(\frac{\chi^{(2)}(q|-q+q')}{k'^2\epsilon(q')}\langle\phi^2\rangle_{q-q'}\langle\phi^2\rangle_q\right.\\ &\left.+\frac{\chi^{(2)}(q|-q')}{|\mathbf{k}-\mathbf{k}'|^2\epsilon(q-q')}\langle\phi^2\rangle_{q'}\langle\phi^2\rangle_q - \frac{\chi^{(2)*}(q'|q-q')}{k^2\epsilon^*(q)}\langle\phi^2\rangle_{q'}\langle\phi^2\rangle_{q-q'}\right)\\ &+\sum_a 4\pi e_a\int d\mathbf{p}\left(\frac{\langle\phi_{q-q'}\phi_{-q}N_{q'}^{a0}(\mathbf{p})\rangle}{k'^2\,\epsilon(q')}\right.\\ &\left.+\frac{\langle\phi_{q'}\,\phi_{-q}\,N_{q-q'}^{a0}(\mathbf{p})\rangle}{|\mathbf{k}-\mathbf{k}'|^2\,\epsilon(q-q')} + \frac{\langle\phi_{q'}\,\phi_{q-q'}\,N_{-q}^{a0}(\mathbf{p})\rangle}{k^2\,\epsilon^*(q)}\right).\end{aligned} \tag{A.19}$$

It is evident that we need to further evaluate the remaining third-order cumulants $\langle\phi_{q-q'}\phi_{-q}N_{q'}^{a0}(\mathbf{p})\rangle$, $\langle\phi_{q'}\phi_{-q}N_{q-q'}^{a0}(\mathbf{p})\rangle$, $\langle\phi_{q'}\phi_{q-q'}N_{-q}^{a0}(\mathbf{p})\rangle$, and $\langle\phi_{-q'}\phi_{-q+q'}N_{q}^{a0}(\mathbf{p})\rangle$. These quantities are but special cases of a generic form $\langle\phi_{q_1}\phi_{-q_1+q_2}N_{-q_2}^{a0}(\mathbf{p})\rangle$. We proceed to evaluate this quantity by making use of (A.12) in order to evaluate ϕ_{q_1} and $\phi_{-q_1+q_2}$ successively. The result is

$$\begin{aligned}\langle\phi_{q_1}\phi_{-q_1+q_2}N_{-q_2}^{a0}(\mathbf{p})\rangle = \frac{8\pi e_a i}{(2\pi)^3k_1k_2|\mathbf{k}_1-\mathbf{k}_2|\epsilon(q_2)}&\left(\frac{\chi^{(2)}(q_2|q_1-q_2)}{\epsilon(q_1)}\langle E^2\rangle_{q_1-q_2}\right.\\ &\left.+\frac{\chi^{(2)}(-q_1|q_2)}{\epsilon(-q_1+q_2)}\langle E^2\rangle_{q_1}\right)\delta(\omega_2-\mathbf{k}_2\cdot\mathbf{v})f_a(\mathbf{p}).\end{aligned} \tag{A.20}$$

Identifying $q_1 = q - q'$ and $q_2 = -q'$, we may obtain the expression for $\langle\phi_{q-q'}\phi_{-q}N_{q'}^{a0}(\mathbf{p})\rangle$. Making the identification for $q_1 = q'$ and $q_2 = -q + q'$, we have $\langle \phi_{q'}\phi_{-q}N^{a0}_{q-q'}(\mathbf{v})\rangle$. Likewise, setting $q_1 = q'$ and $q_2 = q$ leads to $\langle \phi_{q'}\phi_{q-q'}N^{a0}_{-q}(\mathbf{v})\rangle$. Finally, identifying $q_1 = -q'$ and $q_2 = -q$ yields the expression for $\langle \phi_{-q'}\phi_{-q+q'}N^{a0}_{q}(\mathbf{v})\rangle$. In this way, the contributions from all the necessary third-order cumulants to (A.19) can be obtained. The result is the nonlinear spectral balance equation,

$$\begin{aligned}
0 &= \epsilon(q)\langle E^2\rangle_q - \sum_a \frac{(4\pi e_a)^2}{(2\pi)^3 k^2 \epsilon^*(q)} \int d\mathbf{p}\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}) f_a(\mathbf{p}) \\
&- 2\int dq' \frac{|\chi^{(2)}(q'|q-q')|^2}{\epsilon^*(q)} \langle E^2\rangle_{q'}\langle E^2\rangle_{q-q'} \\
&+ 2\int dq' \left[\{\chi^{(2)}(q'|q-q')\}^2 \left(\frac{\langle E^2\rangle_{q-q'}}{\epsilon(q')} + \frac{\langle E^2\rangle_{q'}}{\epsilon(q-q')}\right) - \bar{\chi}^{(3)}(q'|-q'|q)\langle E^2\rangle_{q'}\right]\langle E^2\rangle_q \\
&+ \sum_a \int dq' \frac{2(4\pi e_a)^2}{(2\pi)^3 k'^2 |\epsilon(q')|^2} \left(\frac{\{\chi^{(2)}(q'|q-q')\}^2}{\epsilon(q-q')}\langle E^2\rangle_q - \frac{|\chi^{(2)}(q'|q-q')|^2}{\epsilon^*(q)}\langle E^2\rangle_{q-q'}\right) \\
&\times \int d\mathbf{p}\, \delta(\omega' - \mathbf{k}'\cdot\mathbf{v}) f_a(\mathbf{p}) \\
&+ \sum_a \int dq' \frac{2(4\pi e_a)^2}{(2\pi)^3 |\mathbf{k}-\mathbf{k}'|^2 |\epsilon(q-q')|^2} \left(\frac{\{\chi^{(2)}(q'|q-q')\}^2}{\epsilon(q')}\langle E^2\rangle_q \right. \\
&\left. - \frac{|\chi^{(2)}(q'|q-q')|^2}{\epsilon^*(q)}\langle E^2\rangle_{q'}\right) \int d\mathbf{p}\, \delta[\omega - \omega' - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] f_a(\mathbf{p}). \qquad \text{(A.21)}
\end{aligned}$$

For more detailed discussions on the derivation of this result, the readers are referred to the author's recent monograph [22].

If we take the real part of this equation while ignoring nonlinear terms, then we obtain the dispersion relation, $\mathrm{Re}\epsilon(q) = 0$. By taking the imaginary part we obtain the evolution equation for wave amplitude, that is, wave kinetic equation. However, in order to complete the formulation for wave kinetic equation, we must introduce the slow-time derivative associated with the angular frequency, which is implicit in the present formalism. In short, we apply the following prescription to the linear response function leads to the formal wave kinetic equation:

$$\epsilon(q)\langle E^2\rangle_q \to \left(\epsilon(q) + \frac{i}{2}\frac{\partial\epsilon(q)}{\partial\omega}\frac{\partial}{\partial t}\right)\langle E^2\rangle_q. \qquad \text{(A.22)}$$

Formal particle kinetic equation in (A.8) can be easily obtained if make use of the first order perturbed distribution $N^{q(1)}_q$ in (A.11),

$$\begin{aligned}
\frac{\partial f_a}{\partial t} &= \pi e_a^2 \int d\mathbf{k} \int d\omega \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega - \mathbf{k}\cdot\mathbf{v}) \\
&\times \left[\mathrm{Im}\,\frac{1}{2\pi^3 k \epsilon^*(\mathbf{k},\omega)} f_a + \langle \delta E^2\rangle_{\mathbf{k},\omega}\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial f_a}{\partial\mathbf{p}}\right)\right]. \qquad \text{(A.23)}
\end{aligned}$$

A.2 Wave kinetic equation for collective eigenmodes

The linear dispersion relation $\mathrm{Re}\,\epsilon(\mathbf{k},\omega)\langle\delta E^2\rangle_{\mathbf{k},\omega} = 0$ determines the relation between ω and $\mathbf{k}$, that is, $\omega = \omega^\alpha_{\mathbf{k}}$. This means that we may express the electric field fluctuation corresponding to the eigenmode intensity by

$$\langle \delta E^2 \rangle_{\mathbf{k}\omega} = \sum_{\sigma=\pm 1} \sum_{\alpha=L,S} I_{\mathbf{k}}^{\sigma\alpha} \delta(\omega - \sigma\omega_{\mathbf{k}}^{\alpha}), \tag{A.24}$$

where $I_{\mathbf{k}}^{\sigma\alpha}$ is the intensity for each eigenmode, $\alpha = L, S$ denoting the Langmuir (L) and ion-sound S (or ion-acoustic) modes, respectively. Here, $\sigma = \pm 1$ signifies the wave propagation direction, forward and backward, with respect to some reference axis. The wave dispersive properties for Langmuir and ion acoustic waves are well known, $\omega_{\mathbf{k}}^L = \omega_{pe}\left(1 + \frac{3}{2} k^2 \lambda_{De}^2\right)$ and $\omega_{\mathbf{k}}^S = kc_S$, respectively, where $\omega_{pe}^2 = 4\pi n_e e^2/m_e$ is the square of plasma frequency, $\lambda_{De}^2 = T_e/(4\pi n_e e^2)$ is the square of Debye length. Ion thermal speed is defined by $v_{Ti} = (2T_i/m_i)^{1/2}$.

Substituting (A.24) to the spectral balance equation (A.21), taking the imaginary part with the prescription for including the slow time derivative – see (A.22), it is possible to derive the wave kinetic equation for collective eignmodes excited in a plasma [22,37–39]. The result is given by

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t} &= 2\gamma_{\mathbf{k}}^{\sigma\alpha} I_{\mathbf{k}}^{\sigma\alpha} + S_{\mathbf{k}}^{\sigma\alpha} - \int d\mathbf{k}' \left(N_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha} I_{\mathbf{k}'}^{\sigma'\beta} I_{\mathbf{k}}^{\sigma\alpha} + P_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha} \right) \\ &\quad - \int d\mathbf{k}' \, \frac{M_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha}}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} \left(\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma} I_{\mathbf{k}}^{\sigma\alpha}}{\epsilon'(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta})} + \frac{I_{\mathbf{k}'}^{\sigma'\beta} I_{\mathbf{k}}^{\sigma\alpha}}{\epsilon'(\mathbf{k}-\mathbf{k}', \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})} - \frac{I_{\mathbf{k}'}^{\sigma'\beta} I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} \right),\end{aligned} \tag{A.25}$$

where

$$\begin{aligned}\gamma_{\mathbf{k}}^{\sigma\alpha} &= -\frac{\mathrm{Im}\,\epsilon(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})}, \qquad \epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha}) = \frac{\partial \mathrm{Re}\,\epsilon(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})}{\partial \sigma\omega_{\mathbf{k}}^{\alpha}}, \\ S_{\mathbf{k}}^{\sigma\alpha} &= \sum_{a=e,i} \frac{4e_a^2}{k^2[\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})]^2} \int d\mathbf{p}\, \delta(\sigma\omega_{\mathbf{k}}^{\alpha} - \mathbf{k}\cdot\mathbf{v}) f_a(\mathbf{p}), \\ M_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha} &= 4\pi \sum_{\sigma',\sigma''=\pm 1} \sum_{\beta,\gamma=L,S} \chi^{(2)}(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta} | \mathbf{k}-\mathbf{k}', \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})|^2 \\ &\quad \times \delta(\sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma}), \\ N_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha} &= \frac{4\mathrm{Im}}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} \sum_{\sigma'=\pm 1} \sum_{\beta=L,S} \left(\mathscr{P} \frac{2\{\chi^{(2)}(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta} | \mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta})\}^2}{\epsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta})} \right. \\ &\quad \left. - \bar{\chi}^{(3)}(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta} | -\mathbf{k}', -\sigma'\omega_{\mathbf{k}'}^{\beta} | \mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha}) \right), \\ P_{\mathbf{k},\mathbf{k}'}^{\sigma\alpha} &= \sum_{a=e,i} \frac{16e_a^2}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} \sum_{\sigma'=\pm 1} \sum_{\beta=L,S} \frac{|\chi^{(2)}(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta} | \mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta})|^2}{|\mathbf{k}-\mathbf{k}'|^2 |\epsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta})|^2} \\ &\quad \times \left(\frac{I_{\mathbf{k}}^{\sigma\alpha}}{\epsilon(\mathbf{k}', \sigma'\omega_{\mathbf{k}'}^{\beta})} - \frac{I_{\mathbf{k}'}^{\sigma'\beta}}{\epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} \right) \int d\mathbf{p}\, \delta[\sigma\omega_{\mathbf{k}}^{\alpha} - \sigma'\omega_{\mathbf{k}'}^{\beta} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}] f_a(\mathbf{p}).\end{aligned} \tag{A.26}$$

References [22,38,39] further discuss the reduction of formal wave kinetic equation (A.26) by approximately calculating the various response functions in explicit forms that lend themselves to either numerical or analytical treatment. The result is summarized as follows:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} &= \left(\frac{\partial}{\partial t}\bigg|_{\mathrm{em.}} + \frac{\partial}{\partial t}\bigg|_{\mathrm{decay}} + \frac{\partial}{\partial t}\bigg|_{\mathrm{sc.}}\right) I_{\mathbf{k}}^{\sigma L},\\ \frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} &= \left(\frac{\partial}{\partial t}\bigg|_{\mathrm{em.}} + \frac{\partial}{\partial t}\bigg|_{\mathrm{decay}}\right) I_{\mathbf{k}}^{\sigma S},\end{aligned} \tag{A.27}$$

where "em.", "decay", and "sc." denote linear wave-particle (or spontaneous and induced emissions), nonlinear wave-wave (or three-wave decay), and nonlinear wave-particle (or spontaneous and induced scattering) processes. Each process is defined explicitly as follows: The spontaneous and induced emissions processes are specified by

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t}\bigg|_{\mathrm{em.}} &= \frac{4\pi e^2}{m_e k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(n_e^2 e^2\, f_e + \pi\sigma\omega_{\mathbf{k}}^L\,\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}} I_{\mathbf{k}}^{\sigma L}\right),\\ \frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t}\bigg|_{\mathrm{em.}} &= \frac{\mu_{\mathbf{k}} 4\pi e^2}{m_e k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})\\ &\quad\times\left[\mu_{\mathbf{k}} n_e^2 e^2 (f_e + f_i) + \pi\sigma\omega_{\mathbf{k}}^L\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\left(f_e + \frac{m_e}{m_i} f_i\right) I_{\mathbf{k}}^{\sigma S}\right],\\ \mu_{\mathbf{k}} &= k^3\lambda_{De}^3\sqrt{\frac{m_e}{m_i}}\left(1 + \frac{3T_i}{T_e}\right)^{1/2}.\end{aligned} \tag{A.28}$$

The induced and spontaneous decay processes are described by

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t}\bigg|_{\mathrm{decay}} &= \sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^{\sigma L}\Big(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\\ &\quad - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\mu_{\mathbf{k}-\mathbf{k}'}\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma L}\Big),\\ \frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t}\bigg|_{\mathrm{decay}} &= \sigma\mu_{\mathbf{k}}\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^{\sigma S}\Big(\sigma\mu_{\mathbf{k}}\,\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\\ &\quad - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L} I_{\mathbf{k}}^{\sigma S} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma S}\Big),\end{aligned} \tag{A.29}$$

where

$$\begin{aligned}V_{\mathbf{k},\mathbf{k}'}^{\sigma L} &= \frac{\pi}{2}\frac{e^2}{T_e^2}\sum_{\sigma',\sigma''=\pm 1}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S),\\ V_{\mathbf{k},\mathbf{k}'}^{\sigma S} &= \frac{\pi}{4}\frac{e^2}{T_e^2}\sum_{\sigma',\sigma''=\pm 1}\frac{[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2 k'^2|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L).\end{aligned} \tag{A.30}$$

Finally, the induced and spontaneous scattering processes, which only affects L mode, is given by

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t}\bigg|_{\mathrm{sc.}} &= \sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2 k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]\\ &\quad\times\left[\frac{1}{4n_e^2 m_i}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}} I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L} - \frac{e^4\lambda_{De}^4}{T_e^2}\left(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L}\right)(f_e + f_i)\right].\end{aligned} \tag{A.31}$$

A.3 Particle kinetic equation

In the particle kinetic equation both collective eigenmodes and non collective fluctuations contribute. Non collective fluctuations are spontaneously emitted by thermal plasma. Consequently, the electric field spectrum (A.24) that enters the formal particle equation (A.23) must be given by

$$\langle \delta E^2 \rangle_{\mathbf{k}\omega} = \sum_{\sigma=\pm 1} \sum_{\alpha=L,S} I_{\mathbf{k}}^{\sigma\alpha} \delta(\omega - \sigma\omega_{\mathbf{k}}^{\alpha}) + \frac{2}{\pi} \frac{1}{k^2 |\epsilon(\mathbf{k},\omega)|^2} \sum_a n_a e_a^2 \int d\mathbf{v}\, \delta(\omega - \mathbf{k}\cdot\mathbf{v}) f_a(\mathbf{v}). \tag{A.32}$$

The resulting equation, which was discussed in reference [40], is given by

$$\begin{aligned} \frac{\partial f_a(\mathbf{v})}{\partial t} = & \sum_b \frac{2 n_b e_a^2 e_b^2}{m_a^2} \int d\mathbf{k} \int d\mathbf{v}' \frac{k_i k_j}{k^4} \frac{\delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')}{|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} \\ & \times \left(\frac{\partial}{\partial v_j} - \frac{m_a}{m_b} \frac{\partial}{\partial v_j'} \right) f_a(\mathbf{v}) f_b(\mathbf{v}') \\ & + \frac{\pi e_a^2}{m_a^2} \sum_{\sigma=\pm 1} \sum_{\alpha=L,S} \int d\mathbf{k} \left(\frac{\mathbf{k}}{k} \cdot \frac{\partial}{\partial \mathbf{v}} \right) \delta(\sigma\omega_{\mathbf{k}}^{\alpha} - \mathbf{k}\cdot\mathbf{v}) \\ & \times \left[\frac{\pi m_a f_a(\mathbf{v})}{2\pi^3 k \epsilon'(\mathbf{k}, \sigma\omega_{\mathbf{k}}^{\alpha})} + I_{\mathbf{k}}^{\sigma\alpha} \left(\frac{\mathbf{k}}{k} \cdot \frac{\partial f_a(\mathbf{v})}{\partial \mathbf{v}} \right) \right]. \end{aligned} \tag{A.33}$$

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