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# Circular flows via extended Tutte orientations

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This paper consists of two major parts. In the first part, the relations between Tutte orientations and circular flows are explored. Tutte orientation (modulo orientation) was first observed by Tutte for the study of 3-flow problem, and later extended by Jaeger for circular $(2+1/p)$-flows. In this paper, it is extended for circular $\lambda$-flows for all rational numbers $\lambda$. This theorem is one of the key tools in the second part of the paper. It was proved by Lovász et al. (2013) [8] that every $(6p+1)$-odd-edge-connected graph admits a circular $(2+1/p)$-flow. In the second part, this result is further extended to other odd integers $6p+3$ and $6p-1$ for any positive integer $p$. We show that (i) *every* $(6p-1)$*-odd-edge-connected graph admits a circular* $(2+\frac{2}{2p-1})$*-flow*, and (ii) *every* $(6p+3)$*-odd-edge-connected graph has flow index strictly less than* $2+1/p$. Both (i) and (ii) generalize some early results. For example, the case $p=1$ of (i) is the well known 4-flow theorem of Jaeger (1979) [4], and the case $p=1$ of (ii) is a recent result by Thomassen

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and the authors (2018) [7]. The proofs of (i) and (ii) provide (principally different) new approach to those previous results.


## 1. Introduction

Throughout this paper, we always let $k, q, p$ be positive integers, where $k \geq 2q$. An integer flow $(D, f)$ of a graph $G$ is called a *circular* $\frac{k}{q}$*-flow* if $q \leq |f(e)| \leq k - q$ for every edge $e \in E(G)$. As introduced in [2], the *flow index* of a graph $G$, denoted by $\phi(G)$, is the least rational number $\lambda$ such that $G$ admits a circular $\lambda$-flow.

The relation between flow and orientation was first observed by Tutte for 3-flow problem. And it was later generalized by Jaeger [5] that a graph admits a circular $(2 + \frac{1}{p})$-flow if and only if it has a modulo $(2p + 1)$-orientation, that is, an orientation such that, at each vertex, the indegree is congruent to outdegree modulo $2p + 1$. Now Jaeger's result is generalized for all rational number $k/q$ below, providing tools to study arbitrary circular flows via orientations. The case of odd $k$ is easier to state: *a graph $G$ admits a circular $k/q$-flow if and only if $(k - 2q)G$ has a modulo $k$-orientation.* Here $tG$ denotes the graph obtained from $G$ by replacing each edge with $t$ parallel edges. To state the theorem including the case of even $k$, we need the following definition concerning orientations with prescribed outdegrees.

**Definition 1.1.** For a graph $G$, an orientation $D$ of the extended graph $(k-2q)G$ is called a $(k, q)$*-extended-Tutte-orientation* (or $(k, q)$**-ETO** for short) with respect to $G$ if

$$d_D^+(v) \equiv -qd_G(v) \pmod{k}, \quad \forall v \in V(G). \tag{1}$$

In this paper, we establish a relation between this orientation and arbitrary circular $\frac{k}{q}$-flow as follows: *a graph admits a circular $\frac{k}{q}$-flow if and only if it has a $(k, q)$-ETO.* To explain the $(k, q)$-ETO more illustratively, we have the following detailed interpretation.

**Theorem 1.2.** *Let $G$ be a graph. The following statements are equivalent.*

*(i) $G$ admits a circular $\frac{k}{q}$-flow.*
*(ii) $G$ has a $(k, q)$-extended-Tutte-orientation $D$.*
*(iii) $(k - 2q)G$ admits an orientation $D$ such that*

$$d_D^+(v) - d_D^-(v) \equiv kd_G(v) \pmod{2k}, \quad \forall v \in V(G). \tag{2}$$

*In particular, for odd $k$, Eq. (2) is equivalent to*

$$d^+_D(v) - d^-_D(v) \equiv 0 \pmod{k}, \quad \forall v \in V(G), \tag{3}$$

*that is,* $(k-2q)G$ *admits a modulo* $k$*-orientation* $D$.

This relation would help us to find certain circular flows via orientations. In particular, the special case $q = 1$ and $k = 5$ of Theorem 1.2 is exactly a fact, observed by Jaeger [5], that if $3G$ admits a modulo 5-orientation, then the summation of flow values on corresponding parallel edges of $3G$ provides a nowhere-zero modulo 5-flow of $G$. Jaeger [5] further applied this fact to show that Tutte's 5-flow conjecture follows from a stronger conjecture that every 9-edge-connected graph admits a circular 5/2-flow. Jaeger [5] also proposed a more general circular flow conjecture that every $4p$-edge-connected graph admits a circular $(2 + \frac{1}{p})$-flow, where the $p = 1$ case is exactly Tutte's 3-flow conjecture. The weak version of this conjecture was solved by Thomassen [12] who showed edge connectivity $2(2p + 1)^2 + 2p + 1$ suffices. This was later improved to $6p$-edge-connected graphs by Lovász, Thomassen, Wu, Zhang [8]. Recently, Jaeger's conjecture was, however, disproved for $p \geq 3$ in [3], while Tutte's 3-flow and 5-flow conjectures remain open.

It was pointed out in some literature (cf. [6,8,18]) that the odd-edge-cuts play important role for flow problems. A graph is *$t$-odd-edge-connected* if each odd-edge-cut is of size at least $t$. The $(2 + \frac{1}{p})$-flow result of Lovász et al. [8] holds for odd edge connectivity as well.

**Theorem 1.3.** (Lovász et al. [8]) *For every* $(6p+1)$*-odd-edge-connected graph* $G$*, the flow index* $\phi(G) \leq 2 + \frac{1}{p}$.

How about $(6p - 1)$- or $(6p + 3)$-odd-edge-connected graphs? As applications of Theorem 1.2, we provide new upper bound of flow index for those graphs with given odd edge connectivity.

Jaeger's 4-flow theorem [4] in 1979 states that *every* 5*-odd-edge-connected graph admits a nowhere-zero* 4*-flow.* This result is generalized as follows.

**Theorem 1.4.** *For every* $(6p - 1)$*-odd-edge-connected graph* $G$*, the flow index* $\phi(G) \leq 2 + \frac{2}{2p-1}$.

Jaeger's pioneer work on 4-flows applied a very beautiful and elegant argument to find two even subgraph cover from a pair of edge-disjoint spanning trees and fundamental cycles (Tutte and Nash-Williams Theorem [15], [9] is applied). Our proof of Theorem 1.4, which provides a nowhere-zero modulo 4-flow, is in fact a pure orientation technique avoiding using even subgraphs (spanning trees). To the best of our knowledge, this is the first alternative method for Jaeger's 4-flow theorem.

By revisiting a recent result in [7], every 9-odd-edge-connected graph has flow index strictly less than 3. This result is also generalized as follows.

**Theorem 1.5.** *For every* $(6p+3)$*-odd-edge-connected graph* $G$*, the flow index* $\phi(G) < 2+\frac{1}{p}$.

The techniques in [7] refine arguments in [12,8] to find a strongly connected modulo 3-orientation under a similar setting. However, the method does not seem to working for arbitrary $p$ due to some technical obstructions. Here, with the aid of Theorem 1.2, we are able to not only provide a much simpler proof, but also extend to all natural numbers $p$.

In the next section, we will present our proof of Theorem 1.2. Then Theorems 1.4 and 1.5 are proved in Section 3, which consists of two parts: We first apply Theorem 1.2 to reduce them to certain orientation problems, and then apply the orientation theorem in [8] to find such a desired orientation if the graph has a sufficient odd edge connectivity. The proof of Theorem 1.5 also needs some special tricks of assigning pre-oriented edges. We end this paper with a few remarks on strongly connected orientations, contractible configurations and a summary of flow indices.

## 2. Circular flows via orientations

A classical theorem of Tutte [13] (see also [17]) converts modulo $k$-flows into integer $k$-flows, which preserves the flow value of each edge modulo $k$. The following is an easy consequence of Tutte's theorem, as observed in [7].

**Lemma 2.1.** ([7]) *Let $G$ be a graph and $k \geq 2q$ be two positive integers. The following are equivalent.*

*(a)* $\phi(G) \leq \frac{k}{q}$*;*
*(b) $G$ admits an integer flow $(D, f)$ with $q \leq |f(e)| \leq k - q$ for every edge $e \in E(G)$;*
*(c) $G$ admits a modulo $k$-flow $(D, f')$ with $q \leq |f'(e)| \leq k - q$ for every edge $e \in E(G)$.*

Let $G$ be a graph with orientation $D$. The deficiency of orientation $D$ at vertex $v \in V(G)$ is defined as the difference between outdegree and indegree. For a mapping $f$ from $E(G)$ to integers, its *deficiency* at vertex $v \in V(G)$, under orientation $D$, is denoted by $\partial_v(f) = \sum_{e\in E^+_D(v)} f(e) - \sum_{e\in E^-_D(v)} f(e)$.

Now we are ready to prove Theorem 1.2, in which we apply the $(k, q)$-ETO in $(k-2q)G$ to find a modulo $k$-flow of $G$ as in Lemma 2.1(c), and vice versa.

**Proof of Theorem 1.2.** Clearly, Eq. (2) implies Eq. (3). When $k$ is odd, the numbers $d^+_D(v) - d^-_D(v)$, $(k - 2q)d_G(v)$ and $d_G(v)$ all have the same parity, and so Eq. (3) implies that $d^+_D(v) - d^-_D(v) \equiv k \pmod{2k}$ if $d_G(v)$ is odd, and $d^+_D(v) - d^-_D(v) \equiv 0 \pmod{2k}$ if $d_G(v)$ is even. Hence Eq. (2) and Eq. (3) are equivalent for odd $k$. This verifies that Eq. (2) is equivalent to that $(k - 2q)G$ admits a modulo $k$-orientation for odd $k$.

Observe that, for each $v \in V((k - 2q)G)$,

$$\begin{aligned} & d^+_D(v) - d^-_D(v) \equiv kd_G(v) \pmod{2k} \\ \Leftrightarrow\ & 2d^+_D(v) \equiv kd_G(v) + d_D(v) \pmod{2k} \\ \Leftrightarrow\ & 2d^+_D(v) \equiv kd_G(v) + (k - 2q)d_G(v) \pmod{2k} \end{aligned}$$

$$\Leftrightarrow 2d_D^+(v) \equiv -2qd_G(v) \pmod{2k}$$
$$\Leftrightarrow d_D^+(v) \equiv -qd_G(v) \pmod{k}.$$

Hence a $(k,q)$-extended-Tutte-orientation as in Eq. (1) is equivalent to the orientation as in Eq. (2) of Theorem 1.2(iii).

Therefore, (ii) and (iii) in Theorem 1.2 are equivalent. Now it suffices to prove the equivalence of (i) and (iii). For each edge $e \in E(G)$, we always let $[e]$ denote the set of its corresponding $t$ parallel edges in $tG$. For technical reasons to handle modules, we shall divide the proof into two cases based on the parity of $k$.

**Case 1:** $k = 2s + 1$.

In this case, notice that a $(k,q)$-ETO $D$ in $(k-2q)G$ as in Eq. (1) is exactly equivalent to a modulo $k$-orientation of $(k-2q)G$ as discussed above. We proceed to prove the following.

**"(iii)⇒(i)":** Assume that $(k-2q)G$ has a modulo $k$-orientation $D$. This modulo $k$-orientation can be also viewed as a modulo $k$-flow with each edge having flow value one. By taking the summation of flow value of $[e]$ in $D((k-2q)G)$ for each edge $e \in E(G)$, we can obtain a modulo $k$-flow $(D', f_1)$ in $G$, where $|f_1(e)|$ is odd and satisfies $|f_1(e)| \le k-2q$ for any edge $e \in E(G)$. We may, by possibly reversing the direction of some edges, choose an orientation $D'$ of $G$ such that the values of $f_1$ are taken in $\{(k-2q), (k-2q-2), \ldots, 3, 1\}$. Note that $(D', f_1)$ is precisely a modulo $k$-flow of $G$ since $D$ is a modulo $k$-orientation of $(k-2q)G$.

Let $f' = sf_1$ modulo $k$, where $s = \frac{k-1}{2}$. Then $(D', f')$ remains a modulo $k$-flow of $G$ as $(D', f_1)$ does. We claim that the values of $f'$ are taken in $\{q, \ldots, k-q\}$ modulo $k$. In fact, for any edge $e \in E(G)$ with $f_1(e) = k - 2i$ $(q \le i \le s)$, we have

$$f'(e) \equiv sf_1(e) \equiv s(k-2i) \equiv -2si \equiv i \pmod{k}.$$

Hence $f'(e) \in \{q, \ldots, s\} \subseteq \{q, \ldots, k-q\}$. Therefore, $(D', f')$ is a modulo $k$-flow of $G$ with $q \le f'(e) \le k-q$ for every edge $e \in E(G)$. By Lemma 2.1, $G$ admits a circular $\frac{k}{q}$-flow and (i) holds.

**"(i)⇒(iii)":** As in Lemma 2.1, let $(D', f')$ be a modulo $k$-flow of $G$ with $q \le f'(e) \le k-q$ for every edge $e \in E(G)$. Set $f_1 = (k-2)f'$ modulo $k$. Then $(D', f_1)$ is also a modulo $k$-flow of $G$ as $(D', f')$ does. Moreover, for each edge $e \in E(G)$, we have $q \le f'(e) \le k-q$ and, therefore,

$$f_1(e) \equiv (k-2)f'(e) \equiv -2f'(e) \equiv k - 2f'(e) \pmod{k}.$$

This shows $f_1(e) \in \{\pm(k-2q), \pm(k-2q-2), \ldots, \pm 3, \pm 1\}$. By reversing the direction of some edges if necessary, we may further obtain a modulo $k$-flow $(D_2, f_2)$ such that $f_2(e) \in \{k-2q, k-2q-2, \ldots, 3, 1\}$ for each edge $e \in E(G)$.

We now construct a modulo $k$-orientation $D$ of $(k-2q)G$ as in Eq. (3) from $(D_2, f_2)$. For each edge $e = (u, v) \in D_2(G)$ (where the flow value $f_2(e)$ is odd and satisfies $1 \le f_2(e) \le k - 2q$), we first orient $f_2(e)$ edges of $[e]$ in $(k-2q)G$ from $u$ to $v$, and then orient the remaining $\frac{1}{2}(k-2q-f_2(e))$ pairs of edges with opposite directions. Hence this orientation of $[e]$ in $(k-2q)G$ exactly matches the flow value $f_2(e)$ of $e$ in $G$. Therefore, the constructed orientation $D$ of $(k-2q)G$ is indeed a modulo $k$-orientation, since it is balanced at each vertex from the corresponding behavior of modulo $k$-flow $(D_2, f_2)$ of $G$. This finishes the proof for the case $k = 2s+1$.

**Case 2:** $k = 2s$.

The basic idea of the proof is similar to that in Case 1 with some additional and necessary modifications/discussions on the orientations and modules.

**"(i)⇒(iii)":** By Lemma 2.1, let $(D', f')$ be a modulo $2s$-flow of $G$ with $f' : E(G) \to \{q, \dots, 2s-q\}$. We shall construct an orientation $D$ of $(2s-2q)G$ from $(D', f')$, in which $D$ satisfies Eq. (2).

For each directed edge $e = (u, v)$ of $D'(G)$, orient $2s - q - f'(e)$ edges in $[e]$ from $u$ to $v$, and the remaining $f'(e) - q$ edges in $[e]$ from $v$ to $u$. Note that the orientation $D'$ is well-defined by the definition of $f'$.

Notice that, $(D', 2f')$ is a modulo $4s$-flow. Hence it is balanced modulo $4s$ at each vertex, that is, the deficiency

$$\partial_v(2f') = \sum_{e\in E^+_{D'}(v)} 2f'(e) - \sum_{e\in E^-_{D'}(v)} 2f'(e) = 2\partial_v(f') \equiv 0 \pmod{4s}, \forall v \in V(G). \quad (4)$$

For each vertex $v \in V((2s-2q)G)$, we have

$$\begin{aligned}
&d^+_D(v) - d^-_D(v)\\
&\equiv \sum_{e\in E^+_{D'}(v)} [(2s-q-f'(e)) - (f'(e)-q)] + \sum_{e\in E^-_{D'}(v)} [(f'(e)-q) - (2s-q-f'(e))]\\
&\equiv 2s(|E^+_{D'}(v)| - |E^-_{D'}(v)|) + \sum_{e\in E^+_{D'}(v)} (-2f'(e)) + \sum_{e\in E^-_{D'}(v)} 2f'(e)\\
&\equiv 2sd_G(v) - 2\partial_v(f') \pmod{4s}, \qquad (5)
\end{aligned}$$

where the last line in the equation holds since $2s(|E^+_{D'}(v)| - |E^-_{D'}(v)|) \equiv 2sd_G(v) \pmod{4s}$ by parity.

Therefore, Eq. (2) follows from Eq. (4) and (5). This proves that (i) implies (iii).

**"(iii)⇒(i)":** Let $D$ be an orientation of $(k-2q)G$ satisfying Eq. (2). Fix an orientation $D'$ of $G$. For each directed edge $e = (u, v)$ of $D'(G)$, denote by $m(e)$ the number of edges of $[e]$ directed from $u$ to $v$ in $D$ of $(k-2q)G$, and define $f'(e) = 2s - q - m(e)$. Then we have $q \le f'(e) \le 2s - q$. Moreover, with a similar calculation, Eq. (2) and Eq. (5)

yield that $2\partial_v(f') \equiv 0 \pmod{4s}$. Hence $\partial_v(f') = \sum_{e\in E^+_{D'}(v)} f'(e) - \sum_{e\in E^-_{D'}(v)} f'(e) \equiv 0 \pmod{2s}$ for any $v \in V(G)$, and so $(D', f')$ is a modulo $2s$-flow with $q \le f'(e) \le 2s - q$ for every edge $e \in E(G)$. Thus $G$ admits a circular $\frac{k}{q}$-flow by Lemma 2.1. The proof is completed. ■

## 3. Circular flows and odd edge connectivity

Before proving Theorem 1.4, we remark that Theorem 1.2 using odd $k$, together with Theorem 1.3, already lead to a flow less than $2 + \frac{1}{p-1}$ in $(6p-1)$-odd-edge-connected graphs. Specifically, let $k = 10p - 1$ and $q = 5p - 3$ in Theorem 1.2. Then $k - 2q = 5$, and $5G$ is a $5(6p-1)$-odd-edge-connected graph provided that $G$ is $(6p-1)$-odd-edge-connected. Notice that, as $5(6p-1) = 6(5p-1) + 1 = 3k - 2$, we have $\phi(5G) \le 2 + \frac{1}{5p-1}$ and $5G$ admits a modulo $k$-orientation by Theorem 1.3. Hence $\phi(G) \le \frac{k}{q} = 2 + \frac{5}{5p-3}$ by Theorem 1.2. However, Theorem 1.4 provides a better flow index $2 + \frac{2}{2p-1} < 2 + \frac{5}{5p-3}$ by applying Theorem 1.2 with an appropriate orientation of $2G$. Actually, our proofs of Theorems 1.4 and 1.5 are based on similar approaches, while much more technical. To this end, we shall introduce modulo orientation with boundaries and some needed results.

**Definition 3.1.** (a) A function $\beta : V(G) \mapsto \{0, \pm1, \dots, \pm k\}$ is called a $(2k,\beta)$-*boundary* if $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{2k}$ and $\beta(v) \equiv d(v) \pmod 2$ for every $v \in V(G)$. For a vertex subset $A \subset V(G)$, define its boundary $\beta(A) \in \{0, \pm1, \dots, \pm k\}$ such that $\beta(A) \equiv \sum_{v\in A} \beta(v) \pmod{2k}$.

(b) Given a $(2k,\beta)$-boundary, an orientation $D$ of $G$ is called a $(2k,\beta)$-*orientation* if, for every vertex $v \in V(G)$, $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod{2k}$.

For example, the $(k,q)$-ETO $D$ in $(k-2q)G$ defined in Eq. (2) is a $(2k,\beta)$-orientation with $\beta(v) \equiv d^+_D(v) - d^-_D(v) \equiv kd_G(v) \pmod{2k}$ for every vertex $v \in V((k-2q)G)$. When $k$ is odd, a modulo $k$-orientation $D$ of a graph $H$ is also a $(2k,\beta)$-orientation with $\beta(v) \equiv kd_H(v) \pmod{2k}, \forall v \in V(H)$. As a preparation of the proofs of Theorems 1.4 and 1.5, by considering circular $\frac{2t+1}{t}$-flows and circular $\frac{4p}{2p-1}$-flows in Theorem 1.2, we provide orientations as follows.

**Remark 3.2.** *(a) (Preparation for Theorem 1.5)* A graph $G$ admits a circular $\frac{2t+1}{t}$-flow if and only if it has a $(2t+1,t)$-ETO, which is a $(4t+2,\beta)$-orientation of $G$ with $\beta(v) \equiv (2t+1)d_G(v) \pmod{4t+2}, \forall v \in V(G)$.

*(b) (Preparation for Theorem 1.4)* A graph $G$ admits a circular $\frac{4p}{2p-1}$-flow if and only if it has a $(4p, 2p-1)$-ETO, which is an $(8p,\beta)$-orientation of $2G$ with $\beta(v) \equiv 4pd_G(v) \pmod{8p}, \forall v \in V(G)$.

The following is the modulo $k$-orientation theorem of Lovász et al. [8]. As remarked in [8] (or see [16]), it holds for even $k$ as well, the proof follows from the line of the proof of Theorem 3.1 in that paper.

**Theorem 3.3.** (Lovász et al. [8], Wu [16]) *Let $G$ be a graph with a $(2k, \beta)$-boundary. Let $z_0$ be a vertex of $V(G)$, and let $D_{z_0}$ be a pre-orientation of $E(z_0)$ which achieves boundary $\beta(z_0)$ at $z_0$. Let $V_0 = \{v \in V(G) - z_0 : \beta(v) = 0\}$. If $V_0 \neq \emptyset$, we let $v_0$ be a vertex of $V_0$ with smallest degree. Assume that*

*(i)* $|V(G)| \geq 3$;
*(ii)* $d(z_0) \leq 2k - 2 + |\beta(z_0)|$;
*(iii)* $d(A) \geq 2k - 2 + |\beta(A)|$ *for any* $A \subset V(G) \setminus \{z_0\}$ *with* $A \neq \{v_0\}$ *and* $|V(G) \setminus A| > 1$.

*Then pre-orientation $D_{z_0}$ at $z_0$ can be extended to a $(2k, \beta)$-orientation of the entire graph $G$.*

There are several minor differences between the present version of Theorem 3.3 in this paper and the original Theorem 3.1 in [8]. First, Theorem 3.1 from [8] uses a $\tau$-function $\tau : V(G) \mapsto \{0, \pm 1, \ldots, \pm k\}$ which is calculated from the boundary function modulo $k$ and the parity of degree modulo 2. This $\tau$-function is indeed the same as the $(2k, \beta)$-boundary modulo $2k$ in Definition 3.1 and Theorem 3.3 of this paper. Second, the proof of Theorem 3.1 is stated for odd $k$ in [8], but all those arguments are essentially valid for even $k$ as long as it is properly defined in the current form of $(2k, \beta)$-boundary and $(2k, \beta)$-orientation. In fact, the proof of Theorem 3.1 in [8] applies three types of reductions: contracting a subgraph, lifting a pair of incident edges, and deleting an edge with modifying the boundaries of its end vertices. All those reductions, as well as each claim line by line in [8], work well for $(2k, \beta)$-boundary and $(2k, \beta)$-orientation here, no matter $k$ is even or odd. We refer the reader to [8] for more details.

An orientation is called *balanced orientation* modulo $2k$ if for each vertex the deficiency (outdegree minus indegree) is congruent to 0 or $k$ modulo $2k$. For circular $\frac{k}{q}$-flow problems (such as, Theorems 1.4 and 1.5), we only need to consider special balanced orientation modulo $2k$ as in Eq. (2) of Theorem 1.2 (iii). Thus, by Remark 3.2 (and Definition 3.1(b)), the $(2k, \beta)$-boundaries of $(k - 2q)G$ must have $\beta(v) \equiv 0$ or $\beta(v) \equiv k \pmod{2k}, \forall v \in V(G)$. In this case, we also have $\beta(A) \in \{0, k\}$ for any subset $A \subset V(G)$ by Definition 3.1(a). The following corollary is a special case of Theorem 3.3 where $\beta(v) \in \{0, k\}, \forall v \in V(G)$. It is easier to use in proving Theorems 1.4 and 1.5.

**Theorem 3.4.** *Let $G$ be a graph with a $(2k, \beta)$-boundary, where $\beta(v) \in \{0, k\}, \forall v \in V(G)$. Let $z_0$ be a vertex of $V(G)$ with $\beta(z_0) = 0$ and $d(z_0) \leq 2k-2$. Let $D_{z_0}$ be a pre-orientation of $E(z_0)$ in which the indegree is congruent to outdegree modulo $2k$. Assume that for any $A \subset V(G) \setminus \{z_0\}$ with $|V(G) \setminus A| > 1$, we have*

*(a) $d(A) \geq 2k - 2$, and additionally,*
*(b) $d(A) \geq 3k - 2$ if $\beta(A) = k$.*

*Then pre-orientation $D_{z_0}$ at $z_0$ can be extended to a $(2k, \beta)$-orientation of the entire graph $G$.*

The following lifting lemma of Zhang [18] shows that the odd edge connectivity is preserved under certain lifting operation.

**Lemma 3.5.** (Zhang [18]) *Let $G$ be a graph with odd edge connectivity $t$. Assume there is a vertex $v \in V(G)$ with $d(v) \neq t$ and $d(v) \neq 2$. Then there exists a pair of edges $u_1v, u_2v$ in $E(v)$ such that after lifting $u_1v, u_2v$, the resulting graph remains odd edge connectivity $t$.*

Note that the flow index of a graph does not decrease after lifting operation.

Now, we are ready to prove Theorem 1.4 restated below.

**Theorem 3.6.** *Every $(6p-1)$-odd-edge-connected graph admits a circular $(2+\frac{2}{2p-1})$-flow.*

**Proof.** Let $G$ be a counterexample of Theorem 3.6 with $|E(G)|$ minimized. We shall show the following claims to get a contradiction.

**Claim I.** *The graph $G$ is $(6p-1)$-regular and $|V(G)| \geq 4$.*

**Proof of Claim I.** By Lemma 3.5, we may assume every vertex in $G$ is of odd degree, and so minimal degree $\delta(G) \geq 6p-1$. Otherwise, we lift all the edges incident with an even vertex, yielding a smaller counterexample. If $G$ contains a vertex of degree at least $6p+1$, then Lemma 3.5 applies as well. So $G$ must be $(6p-1)$-regular. It is straightforward to verify this statement for a $(6p-1)$-regular graph $H$ with two vertices. Specifically, $H$ admits a modulo $(6p-1)$-orientation, and so $\phi(H) \leq 2 + \frac{1}{3p-1} < 2 + \frac{2}{2p-1}$. This shows $|V(G)| \geq 3$. As $G$ is $(6p-1)$-regular, we also have $|V(G)|$ is even and, therefore, $|V(G)| \geq 4$. $\square$

**Claim II.** *For any $A \subset V(G)$ with $|A|$ odd, $d_G(A) \geq 6p-1$.*

**Proof of Claim II.** In fact, by Claim I, we have $d_G(A)$ is odd for any $A \subset V(G)$ with $|A|$ odd. Hence Claim II follows since $G$ is $(6p-1)$-odd-edge-connected. $\square$

**Claim III.** *The graph $G$ is $4p$-edge-connected.*

**Proof of Claim III.** Otherwise, $G$ contains edge-cuts of size at most $4p-2$. By Claim II, we may let $W$ be a minimal vertex set of even order such that $d_G(W) < 4p$. That is, $d_G(W') \geq 4p$ for any $W' \subsetneq W$. Notice that $G/W$ admits a circular $(2+\frac{2}{2p-1})$-flow by the minimality of $G$. By Remark 3.2(b), $2(G/W) = (2G)/W$ admits a $(8p, \beta)$-orientation $D_1$ with $\beta(v) \equiv 4pd_G(v) \pmod{8p}$ for any $v \in V(2G) \setminus W$, and $\beta(w) \equiv 4pd_{G/W}(w) \pmod{8p}$ for the contracted vertex $w$. Now contract $W^c$, the complement of $W$, in $2G$ to view it as new $z_0$. Keep the pre-orientation of $z_0$ as in $D_1$. We will apply Theorem 3.4 in the new graph $H = (2G)/W^c$. Notice that in the new graph

$$d_H(z_0) = d_{2G}(W) = 2d_G(W) \leq 8p - 4 \leq 8p - 2,$$

and $\beta(z_0) = \beta(W^c) = 0$. By the minimality of $|W|$, for any $A \subsetneq W$ we have $d_H(A) = 2d_G(A) \geq 8p$, thus condition (a) of Theorem 3.4 is satisfied. By Claim I, for any $A \subsetneq W$ we have $\beta(A) = 4p$ if and only if $|A|$ is odd. By Claim II, for all $A \subsetneq W$ with $\beta(A) = 4p$ we have

$$d_H(A) = 2d_G(A) \geq 3 \cdot 4p - 2,$$

and so condition (b) of Theorem 3.4 also holds. Thus Theorem 3.4 is applied, and the pre-orientation of $z_0$ can be extended to a $(8p, \beta)$-orientation $D_2$ of $H = (2G)/W^c$ with $\beta(v) \equiv 4pd_G(v) \pmod{8p}$ for any $v \in W$. Finally, we combine the orientations $D_1$ and $D_2$ to result an $(8p, \beta)$-orientation of $2G$ with $\beta(v) \equiv 4pd_G(v) \pmod{8p}, \forall v \in V(2G)$. By Remark 3.2(b), this implies that $G$ admits a circular $(2+\frac{2}{2p-1})$-flow, a contradiction. □

**Claim IV.** *The graph $2G$ has a $(8p, \beta)$-orientation with $\beta(v) \equiv 4pd_G(v) \pmod{8p}, \forall v \in V(2G)$.*

**Proof of Claim IV.** We subdivide an edge in $2G$ to obtain a new vertex $z_0$, and denote this graph by $H$. Let the two incident edges with $z_0$ be pre-oriented as one in and one out at $z_0$. By Claim III, for any $A \subsetneq V(H) \setminus \{z_0\}$ we have $d_H(A) = 2d_G(A) \geq 8p$, and so condition (a) of Theorem 3.4 is verified. For any $A \subsetneq V(H) \setminus \{z_0\}$ with $\beta(A) = 4p$, we have that $|A|$ is odd by Claim I, and so $d_H(A) = 2d_G(A) \geq 12p - 2$ by Claim II. Thus condition (b) of Theorem 3.4 is satisfied, and so Theorem 3.4 is applied. This orientation of $H$ extended from $z_0$ provides a $(8p, \beta)$-orientation of $2G$, and hence Claim IV holds. □

By Claim IV and by Remark 3.2(b), $G$ admits a circular $(2 + \frac{2}{2p-1})$-flow, again a contradiction. This completes the proof of Theorem 3.6. ■

In [7], the relation of strongly connected modulo $(2p + 1)$-orientations and the flow index $\phi < 2 + \frac{1}{p}$ was discovered and used as the key lemma in the proof of the main theorem. It is also proved in [7] that every 8-edge-connected graph admits a strongly connected modulo 3-orientation, and, therefore, has flow index strictly less than 3. In this paper, instead of using this strongly connected orientation lemma, we introduce a very different approach. The main idea of the method is outlined as follows. *For some subset $S \subset V(G)$ with $G[S]$ dense in certain sense, if the flow index $\phi(G/S) \leq 2 + \frac{1}{p} - \epsilon'$*

*for some positive number* $\epsilon'$*, then there is another positive number* $\epsilon$ *with* $0 < \epsilon \leq \epsilon'$ *such that* $\phi(G) \leq 2 + \frac{1}{p} - \epsilon$*.*

In proving $\phi(G) \leq 2 + \frac{1}{p} - \epsilon$ for some positive number $\epsilon$, a barrier is that the extended graph of $G$ may not satisfy the required condition (b) of Theorem 3.4. Fortunately, for sufficiently small $\epsilon$ (related to the order of $G$), there exists a sufficiently large integer $t = t(\epsilon)$ such that the graph modified from $tG$ by adding certain pre-oriented edges is suitable for applying Theorem 3.4 to get a desired orientation.

Now we prove Theorem 1.5, restated as Theorem 3.7 below in a slightly different form.

**Theorem 3.7.** *For any* $(6p+3)$*-odd-edge-connected graph* $G$*, there exists a positive integer* $t = t(G)$ *such that* $\phi(G) \leq 2 + \frac{1}{p} - \frac{1}{2pt} < 2 + \frac{1}{p}$*.*

**Proof.** Suppose, by contradiction, that $G$ is a $(6p+3)$-odd-edge-connected graph satisfying $\phi(G) \geq 2 + \frac{1}{p}$ with $|V(G)| + |E(G)|$ minimized. By a proof similar to that of Claims I and II in Theorem 3.6, we apply Lemma 3.5 to obtain the following fact.

**Claim I′.** *(a)* $G$ *is* $(6p+3)$*-regular and* $|V(G)| \geq 4$*.*

*(b) For any* $A \subset V(G)$ *with* $|A|$ *odd,* $d_G(A) \geq 6p+3$*.*

Next, we continue to apply the similar approach as in the proof of Claim III. Among all vertex subset $A \subset V(G)$ with $d(A) \leq 4p$, choose one with minimum cardinality, denoted by $S$. That is, for any $S' \subsetneq S$ we have $d(S') \geq 4p+2$. Note that, here it is possible that $S = V(G)$, and in this case $G$ is $(4p+2)$-edge-connected.

Notice that $|S| \geq 2$ by Claim I′(a). By the minimality of $G$, we have $\phi(G/S) < 2 + \frac{1}{p}$, and a further property of $G/S$ is as follows.

**Claim II′.** *There exists a positive integer* $t_0$ *such that for any* $t \geq t_0$*, we have*

$$\phi(G/S) \leq 2 + \frac{2t-1}{2pt} = \frac{4pt+2t-1}{2pt}.$$

*Thus* $(2t-1)(G/S)$ *admits a modulo* $(4pt+2t-1)$*-orientation* $D_1$ *by Theorem 1.2.*

**Proof of Claim II′.** Note that the circular flow is monotonic (see [2]). That is, for rational numbers $\lambda_1 \geq \lambda_2 \geq 2$, if a graph admits a circular $\lambda_2$-flow, then it has a circular $\lambda_1$-flow as well.

Since $\phi(G/S) < 2 + \frac{1}{p}$, there exists a positive integer $t_0$ such that $\phi(G/S) < 2 + \frac{1}{p} - \frac{1}{2pt_0}$, and so $\phi(G/S) < 2 + \frac{1}{p} - \frac{1}{2pt_0} \leq 2 + \frac{1}{p} - \frac{1}{2pt}$ whenever $t \geq t_0$. □

For convenience, denote $k = k(t) = 4pt + 2t - 1$. Then by Claim II′, $(2t-1)(G/S)$ admits a modulo $k$-orientation $D_1$ for any given $t \geq t_0$. Ideally, we would like to extend this orientation $D_1$ of $(2t-1)(G/S)$ to be a modulo $k$-orientation of $(2t-1)G$. Then it follows from Theorem 1.2 that $\phi(G) \leq \frac{k}{2pt} = 2 + \frac{1}{p} - \frac{1}{2pt} < 2 + \frac{1}{p}$.

To this end, we contract $S^c$, the complement of $S$, in $(2t-1)G$ to obtain a new graph $G_1$ and view the contracted vertex as $z_0$. (In the case that $G$ is $(4p+2)$-edge-connected, $S^c = \emptyset$ and we add a new isolated vertex $z_0$ to $(2t-1)G$ to form a new graph $G_1$.) At the same time, we also keep the pre-orientation of $z_0$ as in $D_1$. Note that

$$d_{G_1}(z_0) = (2t-1)d_G(S) \leq 4p(2t-1) = 2k + 2 - 4p - 4t, \tag{6}$$

and for each $v \in S$, by Claim I′(a) we have

$$d_{G_1}(v) = (2t-1)d_G(v) = (2t-1)(6p+3) = 3k - 6p.$$

With the above calculations, the condition (b) of Theorem 3.4 (or the condition (iii) in original Theorem 3.3) is not satisfied for $G_1$ yet, and therefore, it cannot be applied here directly. However, with a new construction in the next claim, this problem is resolved.

**Claim III′.** *Construct a new graph $G_2$ from $G_1$ by adding $3p$ directed edges from $v$ to $z_0$ and $3p$ directed edges from $z_0$ to $v$ for each $v \in S$. Then for $t \geq \max\{t_0, 2p|V(G)|\}$, the graph $G_2$ with its pre-orientation at $z_0$ satisfies the conditions of Theorem 3.4, and so we can apply Theorem 3.4 to obtain a modulo $k$-orientation $D_2$ of $G_2$.*

**Proof of Claim III′.** As $k = 4pt + 2t - 1$ is odd and each vertex in $G_2$ other than $z_0$ is of odd degree, by Remark 3.2(a), seeking a modulo $k$-orientation is equivalent to find a $(2k, \beta)$-orientation with $\beta(v) = k$ for each $v \in S$ and $\beta(z_0) = 0$. We will verify the conditions of Theorem 3.4 as follows.

For any $A \subset S$ of odd order, by Claim I′(a) we have $\beta(A) \equiv \sum_{v \in A} \beta(v) \equiv k \pmod{2k}$, and it follows from Claim I′(b) that $d_{G_2}(A) \geq (6p+3)(2t-1) + 6p = 3k$ by the construction of $G_2$. For any $A \subseteq S$ of even order, we have $\beta(A) \equiv \sum_{v \in A} \beta(v) \equiv k|A| \equiv 0 \pmod{2k}$ by Claim I′(a). Moreover, $d_G(A) \geq 4p+2$ by the minimality of $S$, which implies $d_{G_2}(A) \geq (4p+2)(2t-1) + 12p = 2k + 8p > 2k - 2$. So conditions (a)(b) of Theorem 3.4 are verified for $G_2$. By Eq. (6) and since $t \geq \max\{t_0, 2p|V(G)|\}$, we have

$$d_{G_2}(z_0) = d_{G_1}(z_0) + 6p|S| \leq 2k + 2 - 4p - 4t + 6p|V(G)| \leq 2k - 2,$$

and so Theorem 3.4 is applied for $G_2$.

By Theorem 3.4, the pre-orientation at $z_0$ can be extended to a $(2k, \beta)$-orientation $D_2$ of $G_2$. This $D_2$ is a modulo $k$-orientation of $G_2$. □

We delete all the added directed edges of $z_0$ in $D_2$, and then the combination of $D_1$ and rest of $D_2$ results a modulo $k$-orientation of $(2t-1)G$. Hence by Theorem 1.2

$$\phi(G) \leq \frac{k}{2pt} = 2 + \frac{1}{p} - \frac{1}{2pt} < 2 + \frac{1}{p},$$

which completes the proof. ■

## 4. Remarks on strongly connected orientations and contractible configurations

It was proved in [7] that *the flow index* $\phi(G) < \frac{k}{q}$ *if and only if* $G$ *admits a modulo* $k$*-flow* $(D, f)$ *with* $f : E(G) \to \{q, \cdots, k-q-1\}$ *such that the orientation* $D$ *is strongly connected.* In particular, it follows that $\phi(G) < 2 + \frac{1}{p}$ if and only if $G$ admits a strongly connected modulo $(2p+1)$-orientation. Those results can be applied to show the following analogy of Theorem 1.2.

**Theorem 4.1.** *Let* $G$ *be a connected graph and* $k, q$ *be two positive integers with* $k \geq 2q+1$. *Then* $\phi(G) < \frac{k}{q}$ *if and only if* $(k-2q)G$ *has a strongly connected orientation* $D$ *such that, for every vertex* $v \in V((k-2q)G)$,

$$d_D^+(v) \equiv -q d_G(v) \pmod{k}.$$

*Consequently, for any odd* $k$,

$$\phi(G) < \frac{k}{q} \Leftrightarrow \phi((k-2q)G) < 2 + \frac{1}{q}.$$

By Theorem 1.5 we obtain the following corollary immediately.

**Corollary 4.2.** *Every* $(6p+3)$*-odd-edge-connected graph admits a strongly connected modulo* $(2p+1)$*-orientation.*

In [7], the authors obtain a $\phi < 3$ result for 8-edge-connected graphs from strongly connected modulo 3-orientations. In the current paper, we derive an inverse result that to obtain strongly connected modulo $(2p+1)$-orientation from the fact of flow index $\phi < 2 + \frac{1}{p}$. It is also shown in [7] that every 8-edge-connected graph is a contractible configuration for the graph property of $\phi < 3$, while the current proof of Theorem 1.5 does not imply this stronger property. Here we can further extend this contractible configuration property under a slightly higher edge connectivity $6p + 3$.

**Theorem 4.3.** *Let* $H$ *be a* $(6p+3)$*-edge-connected graph. Then for any supergraph* $G$ *of* $H$*, we have*

$$\phi(G) < 2 + \frac{1}{p} \ \Leftrightarrow \ \phi(G/H) < 2 + \frac{1}{p}.$$

**Proof.** The following is an outline of the proof. The main frame of the proof is similar to Theorem 3.7. We aim to extend a $(2 + \frac{1}{p} - \epsilon)$-flow of $G/H$ to $G$ using Tutte orientations from Theorem 1.2. However, the boundary $\beta$ of $H$ may be arbitrary (not just 0 and $k$). Thus, $H$ requires to be $(6p+3)$-edge-connected, and Theorem 3.3 is applied in the proof instead of its simplified version, Theorem 3.4.

**(Sketch of proof).** Clearly, $\phi(G) < 2 + \frac{1}{p}$ implies $\phi(G/H) < 2 + \frac{1}{p}$ since the flow is preserved under contraction. It suffices to justify the reverse. Assume that $\phi(G/H) < 2+\frac{1}{p}$. Similar as in Theorem 3.7, there exists a positive integer $t_0$ such that for any $t_1 \geq t_0$ we have $\phi(G/H) \leq 2+\frac{1}{p} - \frac{1}{2pt_1}$. Let $t = \max\{t_0, 2p|V(G)|\}$, and let $q = 2pt+t-1$. Then $(2t-1)(G/H)$ has a modulo $(2q+1)$-orientation $D_1$ since $\phi(G/H) \leq 2+\frac{1}{p} - \frac{1}{2pt}$. We shall apply Theorem 3.3 to show that this modulo $(2q+1)$-orientation $D_1$ of $(2t-1)(G/H)$ can be extended to a modulo $(2q+1)$-orientation of $(2t-1)G$ below. Then it follows from Theorem 1.2 that $\phi(G) \leq 2 + \frac{1}{p} - \frac{1}{2pt} < 2 + \frac{1}{p}$.

Notice that the modulo $(2q+1)$-orientation $D_1$ of $(2t-1)(G/H)$ results an orientation of $(2t-1)(G-E(H))$, where each vertex $x \in V(H)$ receives a boundary $\beta_1(x)$ (which may or may not be zero), and for any $y \in V(G) \setminus V(H)$ we have boundary $\beta_1(y) \equiv 0 \pmod{2q+1}$. In particular, $\sum_{x\in V((2t-1)H)} \beta_1(x) \equiv 0 \pmod{2q+1}$. Thus by Definition 3.1 we can transfer $\beta_1$ to be a $(4q+2, \beta_2)$-boundary of $(2t-1)H$, where the function $\beta_2 : V((2t-1)H) \to \{0, \pm 1, \dots, \pm(2q+1)\}$ satisfies $\sum_{x\in V((2t-1)H)} \beta_2(x) \equiv 0 \pmod{4q+2}$, $\beta_2(v) \equiv -\beta_1(v) \pmod{2q+1}$ and $\beta_2(v) \equiv d_{(2t-1)H}(v) \pmod 2$ for every $v \in V((2t-1)H)$. It suffices to show that $(2t-1)H$ admits a $(4q+2, \beta_2)$-orientation $D_2$. Then $D_1 \cup D_2$ results a modulo $(2q+1)$-orientation of $(2t-1)G$.

To this end, construct a new graph $H_2$ from $(2t-1)H$ by adding a new vertex $z_0$ with $3p$ directed edges from $v$ to $z_0$ and $3p$ directed edges from $z_0$ to $v$ for each $v \in V((2t-1)H)$. We shall show that the graph $H_2$ with the pre-orientation of $z_0$ is applied for Theorem 3.3. First, since $d_{H_2}(z_0) = 6p|V(H)| \leq 6p|V(G)| \leq 2t \leq q \leq 4q + |\beta_2(z_0)|$, condition (ii) of Theorem 3.3 is satisfied. Then, for any $A \subset V((2t-1)H)$ with $|A| \leq |V(H_2)| - 2$, we have

$$d_{H_2}(A) \geq (6p+3)(2t-1) + 6p = 6q + 3 \geq 4q + |\beta_2(A)|,$$

and so condition (iii) of Theorem 3.3 is verified. Hence by Theorem 3.3 the pre-orientation of $z_0$ is extended to a $(4q+2, \beta_2)$-orientation of $H_2$, resulting a desired orientation $D_2$ of $(2t-1)H$. Therefore, it provides a modulo $(2q+1)$-orientation $D_1 \cup D_2$ of $(2t-1)G$, which completes the proof. ∎

Similar contractible configuration property holds for circular $(2 + \frac{2}{2p-1})$-flows as well. The proof is similar and thus omitted.

**Theorem 4.4.** *Let $H$ be a $(6p-1)$-edge-connected graph. Then for any supergraph $G$ of $H$, we have*

$$\phi(G) \leq 2 + \frac{2}{2p-1} \;\Leftrightarrow\; \phi(G/H) \leq 2 + \frac{2}{2p-1}.$$

## 5. A summary of flow indices

In this paper, we obtain circular flow results for all $(6p-1)$- and $(6p+3)$-odd-edge-connected graphs, filling the odd edge connectivity left from Theorem 1.3 of Lovász et al. [8]. The current known status of flow indices is summarized in Table 1.

**Table 1**
Flow index and odd edge connectivity of graphs.

| Odd-Edge-Conn. | Conjectured $\phi$ | Known $\phi$ |
|---|---|---|
| 3 | $\phi \le 5$ (Tutte 1954 [14]) | $\phi \le 6$ (Seymour 1981 [10])) |
| 5 | $\phi \le 3$ (Tutte 1972 [11] [1]) | $\phi \le 4$ (Jaeger 1979 [4]) |
| 7 | $\phi < 3$ (LTWZ 2018 [7]) | $\phi \le 3$ (LTWZ 2013 [8]) |
| 9 | $\phi \le 2.5$ (Jaeger 1988 [5]) | $\phi < 3$ (LTWZ 2018 [7]) |
| 11 | * | $\phi \le \frac{8}{3}$ (**This paper**) |
| 13 | $\phi \le \frac{7}{3}$ (Jaeger [5]. False, [3]) | $\phi \le \frac{5}{2}$ (LTWZ 2013 [8]) |
| 15 | * | $\phi < \frac{5}{2}$ (**This paper**) |
| 17 | $\phi \le \frac{9}{4}$ (Jaeger [5]. False, [3]) | $\phi \le \frac{12}{5}$ (**This paper**) |
| 19 | * | $\phi \le \frac{7}{3}$ (LTWZ 2013 [8]) |
| ... | ... | ... |
| $6p-1$ | * | $\phi \le 2 + \frac{2}{2p-1}$ (**This paper**) |
| $6p+1$ | * | $\phi \le 2 + \frac{1}{p}$ (LTWZ 2013 [8]) |
| $6p+3$ | * | $\phi < 2 + \frac{1}{p}$ (**This paper**) |

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