On Aharoni's rainbow generalization of the Caccetta-Häggkvist conjecture

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Abstract

For a digraph G and $v \in V(G)$, let $\delta^+(v)$ be the number of out-neighbors of v in G. The Caccetta-Häggkvist conjecture states that for all $k \geq 1$, if G is a digraph with $n = |V(G)|$ such that $\delta^+(v) \geq k$ for all $v \in V(G)$, then G contains a directed cycle of length at most $\lceil n/k \rceil$. In [2], Aharoni proposes a generalization of this conjecture, that a simple edge-colored graph on n vertices with n color classes, each of size k, has a rainbow cycle of length at most $\lceil n/k \rceil$. In this paper, we prove this conjecture if each color class has size $\Omega(k \log k)$.

1 Introduction and preliminaries

A graph or digraph is *simple* if there are no loops or parallel edges. For a simple digraph G and a vertex $v \in V(G)$, let $\delta^+(v)$ denote the number of out-neighbors of v in G. A famous conjecture in graph theory is the following, due to Caccetta and Häggkvist [1]:

Conjecture 1.1 (Caccetta-Häggkvist) *Suppose n, k are positive integers, and let G be a simple digraph on n vertices with $\delta^+(v) \geq k$ for all $v \in V(G)$; then G contains a directed cycle of length at most $\lceil n/k \rceil$.*

For a graph G and a function $c : E(G) \to \{1, \ldots, |V(G)|\}$, a *rainbow cycle (with respect to c)* is a cycle C in G such that for all $e, f \in E(C)$ with $e \neq f$, we have $c(e) \neq c(f)$. We will refer to c as a *coloring* of the edges of G.* We say that c has *color classes of size at least k* for $k \in \mathbb{N}$ if $|c^{-1}(i)| \geq k$ for all $i \in \{1, \ldots, |V(G)|\}$.

In [2], Aharoni proposes a generalization of Conjecture 1.1:

Conjecture 1.2 ([2]) *Let n, k be positive integers, and let G be a simple graph on n vertices. Let $c : E(G) \to \{1, \ldots, n\}$ be a coloring of the edges of G with color classes of size at least k; then G has a rainbow cycle of length at most $\lceil n/k \rceil$.*

In a recent paper, Devos et al. [4] prove that Conjecture 1.2 is true for $k = 2$:

Theorem 1.3 ([4]) *Let G be a simple graph on n vertices, and let c be a coloring of the edges of G with color classes of size at least 2; then there exists a rainbow cycle of length at most $\lceil n/2 \rceil$.*

We also make use of the following results due to Bollobás and Szemerédi [3] and Shen [5], respectively. The first deals with the girth of a simple graph, while the second is an approximate result for Conjecture 1.1. In this paper, log denotes the logarithm with base 2.

Theorem 1.4 ([3]) *For all $n \geq 4$ and $k \geq 2$, if G is a simple graph on n vertices with $n + k$ edges, then G contains a cycle of length at most*

$$\frac{2(n+k)}{3k}(\log k + \log \log k + 4).$$

Theorem 1.5 ([5]) *Let G be a simple digraph with $\delta^+(v) \geq k$ for all $v \in V(G)$. Then G contains a directed cycle of length at most $\lceil n/k \rceil + 73$.*

2 Main result

Our main result is the following:

Theorem 2.1 *Let $k > 1$ be an integer, and let G be a graph. Let c be a coloring of the edges of G with color classes of size at least $301k \log k$. Then G contains a rainbow cycle of length at most $\lceil n/k \rceil$.*

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*Note that c is not required to be a proper edge-coloring.

Proof. We proceed by induction on the number of vertices. Let $f(k) = 7k \log k$, and let G be a graph on n vertices. Let c be a coloring of the edges of G with color classes of size at least $43f(k)$. Suppose for a contradiction that there is no rainbow cycle of length at most $\lceil n/k \rceil$. Note that G has at least $43f(k)n$ edges, and therefore, $n > 43f(k)$.

For $v \in V(G)$ and $i \in \{1, \ldots, n\}$, we say that i *is dominant at* v if v is incident with at least $7f(k)$ edges e such that $c(e) = i$. We call a vertex $v \in V(G)$ *color-dominating* if there exists $i \in \{1, \ldots, n\}$ such that i is dominant at v. We call a color $i \in \{1, \ldots, n\}$ *vertex-dominating* if there exists a vertex $v \in V(G)$ such that i is dominant at v. Let us say that $H \subseteq V(G)$ is *nice* if

- for every vertex-dominating color $i \in \{1, \ldots, n\}$, there is a vertex $v \in V(G) \setminus H$ such that i is dominant at v; and

- there are at most $|H|$ colors $i \in \{1, \ldots, n\}$ such that i is not vertex-dominating and for all $e \in c^{-1}(i)$, at least one end of e is in H.

(Claim 1)   *If there is a nice set $H \subseteq V(G)$ with $6f(k) \leq |H| < n$, then there is a nice set $H' \subseteq V(G)$ with $|H'| = \lceil 6f(k) \rceil$.*

We remove vertices from H one-by-one such that the remaining set is nice. Suppose that we have removed $j \geq 0$ vertices from H, leaving a nice set H_j with $|H_j| > \lceil 6f(k) \rceil$. Let C_j be the set of colors $i \in \{1, \ldots, n\}$ which are not vertex-dominating and also do not have an edge e with $c(e) = i$ such that both ends of e are in $V(G) \setminus H_j$. From the definition of a nice set, we know $|C_j| \leq |H_j|$. If $|C_j| < |H_j|$, then removing any vertex from H_j gives a smaller nice set. So, we may assume that $|C_j| = |H_j|$. If there is a color i in C_j and an edge $e = uv \in c^{-1}(i)$ with $v \in H_j$ and $u \in G \setminus H_j$, then $H_j \setminus \{v\}$ is nice. If there is no such $i \in C_j$, then for every color $i \in C_j$, all edges in $c^{-1}(i)$ have both their ends in H_j. Now applying induction to the subgraph of G with vertex set H_j and edge set $c^{-1}(C_j)$ gives a rainbow cycle of length at most $\lceil n/k \rceil$ in G, a contradiction. This proves Claim 1.

(Claim 2)   *There is a nice set $H' \subseteq V(G)$ with $|H'| = \lceil 6f(k) \rceil$.*

For each vertex-dominating color i, we pick a vertex v_i such that i is dominant at v_i, and let S be the set of these vertices v_i. Let $H = V(G) \setminus S$. Note that H is nice; thus by Claim 1, we may assume that either $|H| < 6f(k)$ or $|H| = n$.

We first consider the case when $|H| = n$. Since $43f(k) \geq 2$, Theorem 1.3 guarantees the existence of a rainbow cycle K of length at most $n/2 + 1$ in G. Let $H' = V(G) \setminus V(K)$. Then H' is nice, and $n > |H'| \geq n/2 - 1 \geq 6f(k)$; so by Claim 1, G contains a nice set of size $\lceil 6f(k) \rceil$.

Now we may assume that $|H| < 6f(k)$. We construct a digraph G' with $V(G') = S$, and for all i, j with $v_i, v_j \in S$, there is an arc $v_i \to v_j$ if $v_i v_j \in E(G)$ and $c(v_i v_j) = i$. Every vertex v_i is incident with at least $7f(k)$ edges e with $c(e) = i$, and since $|H| < 6f(k)$, there are at least $f(k)$ edges $e = v_i u$ with $c(e) = i$ and $u \in S$. Therefore, $\delta^+(G') \geq f(k)$.

Now, we claim $n/f(k) + 74 \leq n/k$, which is equivalent to $74kf(k) \leq n(f(k) - k)$. Since $k \geq 2$, we have $\log(k) \geq 117/301$, and thus $74kf(k) \leq 43f(k)(f(k) - k) \leq n(f(k) - k)$, as claimed.

Then, by applying Theorem 1.5 to G' we obtain a directed cycle K of length at most $\lceil n/f(k) \rceil + 73 \leq \lceil n/k \rceil$ in G'. The edges of G that correspond to arcs of K form a rainbow cycle of length at most $\lceil n/k \rceil$ in G, a contradiction. This proves Claim 2. ∎

(Claim 3) *Let $H \subseteq V(G)$ be a nice set with $|H| = \lceil 6f(k) \rceil$. Then there exists $H' \subseteq H$ such that $|H'| \geq \lceil 2f(k) \rceil$ and such that for at least $n - \lceil f(k) \rceil + 1$ colors i, at least one edge $e \in c^{-1}(i)$ has both ends in $V(G) \setminus H'$.*

Let C be the set of colors i which are not vertex-dominating and for which no edge of $c^{-1}(i)$ has both ends in $V(G) \setminus H$. Since H is nice, it follows that $|C| \leq |H| = \lceil 6f(k) \rceil$. Let $D \subseteq C$ be the set of colors $i \in C$ such that there is a vertex $v \in H$ which is incident with all edges in $c^{-1}(i)$ that have one end in H and the other in $V(G) \setminus H$. We claim that $|D| \leq \lceil f(k) \rceil - 1$. Indeed, for each color $i \in D$, there are at least $\lceil 36f(k) \rceil$ edges in $c^{-1}(i)$ with both ends in H since i is not vertex-dominating. If $|D| > \lceil f(k) \rceil - 1$, then we obtain more than $(f(k)-1)(36f(k))$ edges with both ends in H. Now, since $k \geq 2$, we have $f(k) \geq 72/23$, and it follows that:

$$(f(k) - 1)(36f(k)) \geq \frac{49f(k)^2}{2} \geq \frac{(6f(k)+1)^2}{2} \geq \frac{|H|^2}{2}$$

which gives a contradiction. Thus, $|D| \leq \lceil f(k) \rceil - 1$.

Next, we claim there exists $H' \subseteq H$ such that $|H'| = \lceil 2f(k) \rceil$ and such that for all $i \in \{1, \ldots, n\} \setminus D$, there is an edge $e \in c^{-1}(i)$ with both ends in $V(G) \setminus H'$. To see this, we construct a graph J with vertex set H and the following set of edges. For each $i \in C \setminus D$, we choose two vertices $v_1^i, v_2^i \in H$, each incident with an edge in $c^{-1}(i)$ whose other end is in $V(G) \setminus H$; we know from the definition of D that this is possible. Now, the graph J has $|H|$ vertices and at most $|H|$ edges, and so J has a stable set $H' \subseteq V(J)$ of size at least $|V(J)|/3 \geq 2f(k)$; and so $|H'| \geq \lceil 2f(k) \rceil$.

Now, for every color $i \in C \setminus D$, $V(G) \setminus H'$ contains at least one of v_1^i, v_2^i, and therefore, there is an edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H'$. Moreover, for every $i \in \{1, \ldots, n\} \setminus C$, either i dominates a vertex v in $V(G) \setminus H \subseteq V(G) \setminus H'$ (and so, since $|H'| < 7f(k)$, there is an edge in $c^{-1}(i)$ incident with v whose other end is not in H'); or there is an edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H \subseteq V(G) \setminus H'$. Thus, for at least $n - |D| \geq n - \lceil f(k) \rceil + 1$ colors i, at least one edge in $c^{-1}(i)$ has both ends in $V(G) \setminus H'$. This proves Claim 3.

By combining Claim 2 and Claim 3, we conclude that there exists $H' \subseteq V(G)$ with $|H'| \geq \lceil 2f(k) \rceil$, and such that for at least $n - \lceil f(k) \rceil + 1$ colors i, at least one edge in $c^{-1}(i)$ has both ends in $V(G) \setminus H'$. Let H'' be a subgraph of G with vertex set $V(G) \setminus H'$, obtained by taking exactly one edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H'$ for all $i \in \{1, \ldots, n\}$ which have such an edge. It follows that $|E(H'')| \geq |V(H'')| + \lceil f(k) \rceil$.

Now, we claim that $\frac{2(n+f(k))}{3(f(k))}(\log\log(f(k)) + \log(f(k)) + 4) \leq \frac{n}{k}$. Using $f(k) < n/43$, it suffices to show:

$$\frac{88(\log\log(f(k)) + \log(f(k)) + 4)}{129} \leq 7\log(k)$$

Let $g(k) = 7\log(k) - \frac{88}{129}(\log\log(f(k)) + \log(f(k)) + 4)$. We have that $g(2) > 0$, and for $k \geq 2$ we have:

$$f(k)g'(k)\ln(2) = 49\log(k) - \frac{88}{129}f'(k)\left(\frac{1}{\log(f(k))\ln(2)} + 1\right) > 0$$

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since for $k \geq 2$ we have:

$$f'(k) \left(\frac{1}{\log(f(k)) \ln(2)} + 1 \right) < (7 + 7 \log(k))(3) \leq 49 \log(k)$$

So $g'(k) > 0$ for $k \geq 2$, and it follows that $g(k) \geq 0$ for $k \geq 2$, as desired.

Then, Theorem 1.4 gives a rainbow cycle of length at most $\frac{2(n+f(k))}{3(f(k))}(\log \log(f(k)) + \log(f(k)) + 4) \leq \lceil \frac{n}{k} \rceil$, a contradiction. This proves Theorem 2.1. ∎

We have an immediate corollary which gives us a result for the case of $\Omega(n \log n)$ color classes each of size k:

Corollary 2.2 *Let k be a positive integer and let G be a simple graph on n vertices. Let $c : E(G) \to \{1, \ldots, t\}$ with $t \geq 303n \log n$, and with $|c^{-1}(i)| \geq k$ for all $i \in \{1, \ldots, t\}$. Then G contains a rainbow cycle in G of length at most $\lceil n/k \rceil$.*

Proof. Note that $t \geq 303n \log n \geq 303n \log k$. Since $303n \log k \geq n\lceil 301 \log k \rceil$, we can partition $\{1, \ldots, t\}$ into n parts, each of size at least $\lceil 301 \log k \rceil$; that is, there is a function $f : \{1, \ldots, t\} \to \{1, \ldots, n\}$ such that $|f^{-1}(i)| \geq \lceil 301 \log k \rceil$ for all $i \in \{1, \ldots, n\}$. By Theorem 2.1, applied to G and $f \circ c$, we obtain a rainbow cycle of length at most $\lceil n/k \rceil$ in G with respect to $f \circ c$, which is also rainbow with respect to c. This proves Corollary 2.2. ∎

Acknowledgments

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