THE FACTORIZATION PROPERTY OF $\ell^\infty(X_k)$

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ABSTRACT. In this paper we consider the following problem: Let X_k, be a Banach space with a normalized basis $(e_{(k,j)})_j$, whose biorthogonals are denoted by $(e^*_{(k,j)})_j$, for $k \in \mathbb{N}$, let $Z = \ell^\infty(X_k : k \in \mathbb{N})$ be their ℓ^∞-sum, and let $T : Z \to Z$ be a bounded linear operator with a large diagonal, *i.e.*,

$$\inf_{k,j} \left|e^*_{(k,j)}(T(e_{(k,j)})\right| > 0.$$

Under which condition does the identity on Z factor through T? The purpose of this paper is to formulate general conditions for which the answer is positive.

Contents

1. Introduction

Throughout this paper, we assume X_k is for each $k \in \mathbb{N}$ a Banach space which has a normalized basis $(e_{(k,j)})_j$ and let $(e^*_{(k,j)})_j \subset X_k^*$ be the coordinate functionals. Let Z be the space

$$Z = \ell^\infty(X_k : k \in \mathbb{N}) = \big\{(x_k) : x_k \in X_k,\, k \in \mathbb{N},\, \|(x_k)\| = \sup_{k\in\mathbb{N}} \|x_k\|_{X_k} < \infty\big\}. \tag{1.1}$$

We say a bounded linear operator $T\colon Z \to Z$ has *large diagonal*, if

$$\inf_{k,j} \left|e^*_{(k,j)}\big(Te_{(k,j)}\big)\right| > 0.$$

The main focus of this work is the following problem concerning operators on Z.

Problem 1.1. *Does the identity operator I_Z on Z factor through every bounded linear operator $T\colon Z \to Z$ with a large diagonal, i.e., do there exist bounded linear operators $A, B\colon Z \to Z$ such that $I_Z = ATB$?*

If **??** has a positive answer, we say that Z has the *factorization property* (with respect to the array $(e_{(k,j)})$).

In our previous work [**?**, Theorem 7.6] we solved the factorization problem for unconditional sums of Banach spaces with bases (e.g. ℓ^p or c_0 sums). In that case, an appropriate linear ordering of the array $(e_{(k,j)})$ is a basis of the unconditional sum. Since Z is a non-separable Banach space, the array $(e_{(k,j)})$ *cannot* be reordered into a basis of Z. In particular, we lose

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the norm convergence of the series expansion of vectors in Z which are not in the c_0-sum of the X_k. Consequently, the arguments given in [**?**] are not applicable to the space Z.

Historically, the first factorization problem of that type appeared in the 1967 paper [**?**] by Lindenstrauss, in which he proved that the space ℓ^∞ is prime. Later in 1982, Capon [**?**] actually showed that whenever X has a symmetric basis, $\ell^\infty(X)$ has the factorization property. Bourgain proved in his 1983 work [**?**] that H^∞ is primary, by solving a factorization problem of ℓ^∞-sums of finite dimensional spaces (Bourgain's localization method). The first applications of Bourgain's localization method appear shortly thereafter in works by the third named author [**?**] and by Blower [**?**]. The cases $X_k = L^p$, $1 < p < \infty$, $k \in \mathbb{N}$ and $X_k = H^1$, $k \in \mathbb{N}$, were treated by Wark [**?**] in 2007 and the third named author [**?**] in 2012, respectively. The ℓ^∞-sum of mixed-norm Hardy and BMO spaces and the ℓ^∞-sum of non-separable Banach spaces with a subsymmetric weak* Schauder bases were recently treated by the first named author in [**?**, **?**].

In our previous paper [**?**], we developed an approach to factorization problems based on two player games; the type of games we are referring to were first considered by Maurey, Milman, Tomczak-Jaegermann in [**?**] and further developed by Odell-Schlumprecht [**?**] and Rosendal [**?**] who coined the term *infinite asymptotic games* (see also [**?**, **?**, **?**]). Thereby, we were able to unify the proofs of several known factorization results as well as provide new ones. We exploited those infinite asymptotic games to define the concept of *strategically reproducible bases* in Banach spaces.

In the present paper, we develop a two player game approach to solve the factorization **??** on Z if the array $(e_{(k,j)})_{k,j}$ is *uniformly asymptotically curved*; that is, if for every bounded array $(x_{(k,j)})_{k,j}$ for which the k-th row, $(x_{(k,j)})_j$, is a block basis of $(e_{(k,j)})_j$, for every $k \in \mathbb{N}$, we have

$$\lim_n \sup_k \Big\| \frac{1}{n} \sum_{j=1}^{n} x_{(k,j)} \Big\|_{X_k} = 0. \tag{1.2}$$

Our first main **??** isolates conditions on the array $(e_{(k,j)})$ which guarantee that **??** has a positive solution. Moreover, if we drop the restriction that the array $(e_{(k,j)})_{k,j}$ is uniformly asymptotically curved, then we were able to successfully treat the following

Problem 1.2. *Does for every $T\colon Z \to Z$ with large diagonal with respect to $(e_{(k,j)})$ exist an infinite $\Gamma \subset \mathbb{N}$ such that the identity on $Z_\Gamma := \ell^\infty(X_k : k \in \Gamma)$ factor through T.*

In the special case that $X_k = X$, $k \in \mathbb{N}$, our solution to **??** implies a positive solution to **??**.

2. Preliminaries

In this section, we introduce the necessary notation and concepts.

2.1. **Review of strategically reproducible bases.** Let X denote a Banach space and $S \subset X$. We define $[S]$ as the norm-closure of $\operatorname{span} S$, where $\operatorname{span} S$ denotes the linear span of S. Given sequences (x_i) in X and $(\widetilde{x}_i)$ in possibly another Banach space $\widetilde{X}$, we say that

(x_i) and $(\widetilde{x}_i)$ are *impartially C-equivalent* if for any finite choice of scalars $(a_i) \in c_{00}$ we have

$$\frac{1}{\sqrt{C}}\Big\|\sum_{i=1}^{\infty} a_i\widetilde{x}_i\Big\| \leq \Big\|\sum_{i=1}^{\infty} a_i x_i\Big\| \leq \sqrt{C}\Big\|\sum_{i=1}^{\infty} a_i\widetilde{x}_i\Big\|.$$

For a Banach space X we denote by $\mathrm{cof}(X)$ the set of cofinite-dimensional subspaces of X, while $\mathrm{cof}_{w^*}(X^*)$ denotes the set of cofinite-dimensionl w^*-closed subspaces of X^*.

Let $C > 0$. Given an operator $T\colon X \to X$, we say that *the identity C-factors through T* if there are bounded linear operators $A, B : X \to X$ with $\|A\|\|B\| \leq C$ and $I = ATB$; moreover, we say that *the identity almost C-factors through T* if it $(C+\varepsilon)$-factors through T for all $\varepsilon > 0$. If (e_j) is a basis for X and (e_j^*) denotes its biorthogonal sequence and an operator T on X satisfies $\inf_j \left|e_j^*(Te_j)\right| > 0$, then we say that T has *large diagonal (with respect to (e_j))*. An operator T on X satisfying $e_m^*(Te_j) = 0$ whenever $j \neq m$, is called a *diagonal operator*.

We recall some definitions from [**?**].

Definition 2.1. Let X be a Banach space with a normalized Schauder basis (e_j) and its biorthogonals $(e_j^*) \subset X^*$.

(i) We say that (e_j) has the *factorization property* if whenever $T\colon X \to X$ is a bounded linear operator with $\inf_j |e_j^*(Te_j)| > 0$ then the identity of X factors through T. More precisely, for a map $K\colon (0,\infty) \to \mathbb{R}^+$ we say that (e_j) has the *$K(\cdot)$-factorization property in X* if for every $\delta > 0$ and bounded linear operator $T\colon X \to X$, with $\inf_j |e_j^*(Te_j)| \geq \delta$ the identity I_X on X almost $K(\delta)$-factors through T, *i.e.*, for every $\varepsilon > 0$ there are operators $A, B\colon X \to X$, with $\|A\|\|B\| \leq K(\delta) + \varepsilon$ and $I_X = BTA$.

(ii) We say that the basis (e_j) has the *uniform diagonal factorization property in X* if for every $\delta > 0$ there exists $K(\delta) > 0$ so that for every bounded diagonal operator $T\colon X \to X$ with $\inf_j \left|e_j^*(Te_j)\right| \geq \delta$ the identity almost $K(\delta)$-factors through T. If we wish to be more specific we shall say that (e_j) has the *$K(\delta)$-diagonal factorization property*.

Remark 2.2. First, we remark that if (e_j) is unconditional, then it satisfies **??** (**??**). Secondly, recall that by [**?**, Remark 3.11] we have $1/\delta \leq K(\delta) \leq K(1)/\delta$.

Also recall the following definition of *strategic reproducibility* [**?**] of a Banach space X with a basis (e_j).

Definition 2.3. Let X be a Banach space with a normalized Schauder basis (e_j) and fix positive constants $C \geq 1$, and $\eta > 0$.

Consider the following two-player game between Player I and Player II:

Pregame: Before the first turn Player I is allowed to choose at the beginning of the game a partition of $\mathbb{N} = N_1 \cup N_2$.

Turn n, Step 1: Player I chooses $\eta_n > 0$, $W_n \in \mathrm{cof}(X)$, and $G_n \in \mathrm{cof}_{w^*}(X^*)$,

Turn n, Step 2: Player II chooses $i_n \in \{1,2\}$, a finite subset E_n of N_{i_n} and sequences of non-negative real numbers $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ satisfying

$$1 - \eta < \sum_{i\in E_n} \lambda_i^{(n)}\mu_i^{(n)} < 1 + \eta.$$

Turn n, Step 3: Player I chooses $(\varepsilon_i^{(n)})_{i\in E_n}$ in $\{-1,1\}^{E_n}$.

We say that Player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_i))}(C,\eta)$ if he can force the following properties on the result:

For all $j\in\mathbb{N}$ we set

$$x_j=\sum_{i\in E_j}\varepsilon_i^{(j)}\lambda_i^{(j)}e_i \text{ and } x_j^*=\sum_{i\in E_j}\varepsilon_i^{(j)}\mu_i^{(j)}e_i^*$$

and demand:

(i) the sequences (x_j) and (e_j) are impartially $(C+\eta)$-equivalent,
(ii) the sequences (x_j^*) and (e_j^*) are impartially $(C+\eta)$-equivalent,
(iii) for all $j\in\mathbb{N}$ we have $\mathrm{dist}(x_j,W_j)<\eta_j$, and
(iv) for all $j\in\mathbb{N}$ we have $\mathrm{dist}(x_j^*,G_j)<\eta_j$.

We say that (e_j) is *C-strategically reproducible in X* if for every $\eta>0$ Player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_j))}(C,\eta)$.

It was shown in [**?**, Remark 3.5] that in the case that (e_j) is shrinking and unconditional, then **??** is equivalent to a considerably simpler formulation.

?? was used in [**?**] to prove the following factorization result:

Theorem 2.4 ([**?**, Theorem 3.12])**.** *Let X be a Banach space with a normalized Schauder basis (e_j) that has a basis constant λ. Assume also that*

(i) the basis (e_j) has the $K(\delta)$-diagonal factorization property and
(ii) the basis (e_j) is C-strategically reproducible in X.

Then (e_j) has the $\lambda C^2K(\delta)$-factorization property.

2.2. **Dyadic Hardy spaces and** BMO**.** We now turn to defining the dyadic Hardy spaces, BMO and VMO.

For a more in depth discussion of the biparameter Hardy spaces, we refer to [**?**]; see also [**?**]. Let $\mathcal{D}$ denote the collection of dyadic intervals given by

$$\mathcal{D}=\{[k2^{-n},(k+1)2^{-n}) \,:\, n,k\in\mathbb{N}_0, 0\le k\le 2^n-1\}.$$

For $I\in\mathcal{D}$ we let $|I|$ denote the length of the dyadic interval I. Let h_I be the L^∞-normalized Haar function supported on $I\in\mathcal{D}$; that is, for $I=[a,b)$ and $c=(a+b)/2$, we have $h_I(x)=1$ if $a\le x<c$, $h_I(x)=-1$ if $c\le x<b$, and $h_I(x)=0$ otherwise. For $1\le p<\infty$, the *Hardy space* H^p is the completion of

$$\mathrm{span}\{h_I \,:\, I\in\mathcal{D}\}$$

under the square function norm

$$\Big\|\sum_{I\in\mathcal{D}}a_Ih_I\Big\|_{H^p}=\bigg(\int_0^1\Big(\sum_{I\in\mathcal{D}}a_I^2h_I^2(x)\Big)^{p/2}dx\bigg)^{1/p}. \tag{2.1}$$

The Haar system $(h_I)_{I\in\mathcal{D}}$ is a 1-unconditional basis of H^p, and thus gives rise to a canonical lattice structure. Finally, we define VMO as the norm closure of $(h_I)_{I\in\mathcal{D}}$ inside BMO, the dual of H^1, where we canonically identify h_I with the linear functional $f\mapsto\int h_I(x)f(x)dx$.

Next, let X denote any Banach space. We will now define the vector-valued Banach spaces $H^p[X]$, $1\le p<\infty$, BMO$[X]$ and VMO$[X]$. Put $\mathcal{D}_n=\{I\in\mathcal{D} : |I|=2^{-n}\}$,

$\mathcal{D}^n = \{I \in \mathcal{D} : |I| \geq 2^{-n}\}$, $n \geq 0$ and let (r_I) denote a sequence of independent Rademacher functions. We define

$$H^p[X] = \Big\{ f \in L^1(X) : \|f\|_{H^p[X]} < \infty \Big\}, \tag{2.2}$$

where for every $f = \sum_{I\in\mathcal{D}} f_I h_I \in L^1(X)$, $f_I \in X$, $I \in \mathcal{D}$, the norm is given by

$$\|f\|_{H^p[X]} = \int_0^1 \Big\| \sum_{I\in\mathcal{D}} r_I(t) f_I h_I \Big\|_{L^p(X)} dt.$$

Of special interest for us is the case $p = 1$. Müller and Schechtman observed that Davis' inequality holds for Banach spaces with the UMD property [**?**, Theorem 6], *i.e.*, there exists a constant $C > 0$ depending only on the UMD-constant of the Banach space X such that

$$C^{-1}\|f\|_{H^1[X]} \leq \int_0^1 \sup_n \| \mathbb{E}_n(f)\|_X dt \leq C\|f\|_{H^1[X]},$$

where $\mathbb{E}_n$ denotes the conditional expectation with respect to $\mathcal{D}_n$. For a detailed presentation of UMD spaces, we refer to Pisier's recent monograph [**?**].

We now define BMO$[X]$:

$$\mathrm{BMO}[X] = \Big\{ f \in L^1(X) : \|f\|_{\mathrm{BMO}[X]} < \infty \Big\},$$

where for every $f = \sum_{I\in\mathcal{D}} f_I h_I \in L^1(X)$, $f_I \in X$, $I \in \mathcal{D}$, the norm is given by

$$\|f\|^2_{\mathrm{BMO}[X]} = \sup_{I\in\mathcal{D}} \frac{1}{|I|} \int_I \Big\| \sum_{J\subset I} f_J h_J(x) \Big\|^2_X dx.$$

Taking Davis' inequality into account, we observe that for UMD spaces X Bourgain [**?**, Theorem 12] proved that the dual of $H^1[X]$ is BMO$[X^*]$. Finally, we define VMO$[X]$ as the norm closure of $\mathrm{span}\{x_I h_I : x_I \in X,\ I \in \mathcal{D}\}$ in BMO$[X]$.

Moreover, let $\mathcal{R} = \{I \times J : I, J \in \mathcal{D}\}$ be the collection of dyadic rectangles contained in the unit square, and set

$$h_{I,J}(x, y) = h_I(x)h_J(y), \qquad I \times J \in \mathcal{R},\ x, y \in [0, 1).$$

For $1 \leq p, q < \infty$, the *mixed-norm Hardy space* $H^p(H^q)$ is the completion of

$$\mathrm{span}\{h_{I,J} \,:\, I \times J \in \mathcal{R}\}$$

under the square function norm

$$\|f\|_{H^p(H^q)} = \bigg(\int_0^1 \Big(\int_0^1 \big(\sum_{I,J} a^2_{I,J} h^2_{I,J}(x, y)\big)^{q/2} dy \Big)^{p/q} dx \bigg)^{1/p}, \tag{2.3}$$

where $f = \sum_{I,J} a_{I,J} h_{I,J}$. The system $(h_{I,J})_{I\times J\in\mathcal{R}}$ is a 1-unconditional basis of $H^p(H^q)$, called the *bi-parameter Haar system*. Note that in view of the Khinchin-Kahane inequality, the norms in the spaces $H^p(H^q)$ and $H^p[H^q]$ are equivalent for $1 \leq p, q < \infty$; to be precise, the identity operator $J\colon H^p(H^q) \to H^p[H^q]$ satisfies

$$\|J\| \cdot \|J^{-1}\| \leq C(p, q).$$

We refer to [**?**, Theorem 4, p.20].

First note that H^p, $1 < p < \infty$ is a UMD space; the UMD constant depends only on p. We will now recall that the dual of $\mathrm{VMO}(H^p)$, $1 < p < \infty$, is $H^1(H^{p'})$, where $p' = \frac{p}{p-1}$.

Theorem 2.5. *Let $1 < p < \infty$, define $p' = \frac{p}{p-1}$ and $J\colon H^1(H^{p'}) \to (\mathrm{VMO}(H^p))^*$ by $f \mapsto (g \mapsto \langle f, g\rangle)$. Then J is an isomorphism with $\|J\| \cdot \|J^{-1}\| \le C(p)$; hence, $\big(\mathrm{VMO}(H^p)\big)^* = H^1(H^{p'})$.*

Proof. Define $J\colon H^1(H^{p'}) \to (\mathrm{VMO}(H^p))^*$ by $f \mapsto (g \mapsto \langle f, g\rangle)$. First, we observe that by Bourgain's vector-valued version of Fefferman's inequality [**?**], we know that $\|J\| \le C(p)$. Secondly, let $f \in H^1(H^{p'})$ be given as the finite linear combination $f = \sum_{I\in\mathcal{D}} f_I h_I$, $f_I \in H^{p'}$. Define the family of functions $f_t = \sum_{I\in\mathcal{D}} r_I(t) f_I h_I$, where the (r_I) are independent Rademacher functions. Since

$$\int_0^1 \|f_t\|_{L^1(H^{p'})} dt \ge c(p)\|f\|_{H^1(H^{p'})},$$

we find a $t_0 \in [0,1]$ such that

$$\|f_{t_0}\|_{L^1(H^{p'})} \ge c(p)\|f\|_{H^1(H^{p'})}.$$

Next, we choose $g \in L^\infty(H^p)$ with $\|g\|_{L^\infty(H^p)} = 1$ such that $\langle f_{t_0}, g\rangle = \|f_{t_0}\|_{L^1(H^{p'})}$. Since, f is a finite linear combination of h_I's, so is g, and we write $g = \sum_{I\in\mathcal{D}} g_I h_I$. Now, we define $h = \sum_{I\in\mathcal{D}} r_I(t_0) g_I h_I$ and note

$$\langle f, h\rangle = \langle f_{t_0}, g\rangle = \|f_{t_0}\|_{L^1(H^{p'})} \ge c(p)\|f\|_{H^1(H^{p'})}.$$

Taking into account that H^p, $1 < p < \infty$ is a UMD space, we observe

$$\|h\|_{\mathrm{BMO}(H^p)} \le C(p)\|g\|_{\mathrm{BMO}(H^p)} \le 4C(p)\|g\|_{L^\infty(H^p)} = 4C(p).$$

We summarize the above calculation

$$\|Jf\|_{(\mathrm{VMO}(H^p))^*} \ge c(p)\|f\|_{H^1(H^{p'})}, \qquad f \in H^1(H^{p'}).$$

Finally, let $L\colon \mathrm{VMO}(H^p) \to \mathbb{R}$ denote any bounded linear functional. Define the conditional expectation $\mathbb{E}_n$ by

$$\mathbb{E}_n\Big(\sum_{I,J\in\mathcal{D}} a_{I,J} h_{I,J}\Big) = \sum_{I,J\in\mathcal{D}^{n-1}} a_{I,J} h_{I,J}$$

and note that $\mathbb{E}_n$ is a contraction on $\mathrm{BMO}(H^p)$. Next, we now define $h = \sum_{I,J\in\mathcal{D}} \frac{L(h_{I,J})}{|I\times J|} h_{I,J}$ and calculate

$$\begin{aligned}
\|h\|_{H^1(H^{p'})} &= \sup_n \|\,\mathbb{E}_n(h)\|_{H^1(H^{p'})} \le C(p) \sup_n \sup\big\{\langle \mathbb{E}_n(h), g\rangle : \|g\|_{\mathrm{BMO}(H^p)} \le 1\big\} \\
&= C(p) \sup_n \sup\Big\{ \sum_{I,J\in\mathcal{D}^n} g_{I,J} L(h_{I,J}) : \|g\|_{\mathrm{BMO}(H^p)} \le 1\Big\} \\
&= C(p) \sup_n \sup\Big\{ L\big(\mathbb{E}_n(g)\big) : \|g\|_{\mathrm{BMO}(H^p)} \le 1\Big\} \le C(p)\|L\|.
\end{aligned}$$

It follows that $L(f) = (Jh)(f)$, $f \in \mathrm{VMO}(H^p)$. □

Remark 2.6. Later, in **??**, we will show that $H^1(H^{p'})$ does not contain c_0. This observation allows us to give another proof of **??**, which we will discuss below. Since $H^1(H^{p'})$ has a 1-unconditional basis and it does not contain c_0, we obtain by James characterization [**?**, Lemma 1] (see also [**?**, Theorem 1.c.10]) that the biparameter Haar basis of $H^1(H^{p'})$ is boundedly complete. By 1-unconditionality $H^1(H^{p'})$ is isometrically isomorphic to the dual of the $(H^1(H^{p'}))^*$-norm-closed linear span of $\{h_{I,J} : I, J \in \mathcal{D}\}$ in $(H^1(H^{p'}))^*$ [**?**, Theorem 1.b.4]. Hence, $H^1(H^{p'})$ is isomorphic to the dual of $\mathrm{VMO}(H^p) = [h_{I,J} : I, J \in \mathcal{D}] \subset \mathrm{BMO}(H^p)$ and the isomorphism constant between them depends just on the isomorphism constant between $H^1(H^p)^*$ and $\mathrm{BMO}(H^p)$.

Proposition 2.7. *Let X denote a Banach space with a normalized shrinking basis (e_j) and assume that $(e_j^*/\|e_j^*\|)$ is C-strategically reproducible in X^*. Then (e_j) is C-strategically reproducible in X.*

Proof. For the sake of simplicity, we assume that (e_j) is bimonotone (and thus, $\|e_j^*\| = \|e_j\| = 1$); the statement still holds without that assumption and can be proved by slightly modifying the argument given below.

We are now describing a winning strategy for Player II, assuming he has a winning strategy in X^*. Assume that in Turn n Step 1 Player I picks $W_n \in \mathrm{cof}(X)$ and $G_n \in \mathrm{cof}_{w^*}(X^*)$. Using his winning strategy in X^* for the spaces $\widetilde{G}_n = \overline{W_n}^{w^*} \in \mathrm{cof}_{w^*}(X^{**})$ and $\widetilde{W}_n = G_n \in \mathrm{cof}(X^*)$, Player II completes Step 2 of Turn n. Obviously, (**??**), (**??**) and (**??**) are satisfied, while (**??**) follows from the fact that $\mathrm{dist}(x_n, \overline{W_n}^{w^*}) = \mathrm{dist}(x_n, W_n)$, which is a consequence of the Hahn-Banach theorem. □

Remark 2.8. Using **??**, we are able to deduce the following two assertions. By [**?**, Theorem 5.2] the Haar basis (h_I) is strategically reproducible in H^1, and hence is also strategically reproducible in VMO. Moreover, the biparameter Haar system $(h_{I,J})$ in $\mathrm{VMO}(H^p)$ is C_p-strategically reproducible for a constant $C_p > 0$, which satisfies $\sup_{p_0 \le p \le p_1} C_p \le C_{p_0,p_1} < \infty$ whenever $1 < p_0 \le p_1 < \infty$ [**?**, Theorem 5.3].

Definition 2.9. Let X be a Banach space with a basis (e_n). We say that X is *asymptotically curved* (with respect to (e_j)) if for every bounded block basis (x_n)

$$\lim_{n\to\infty} \frac{1}{n} \Big\| \sum_{j=1}^{n} x_n \Big\| = 0.$$

As already defined in the introduction we call the sequence of Banach spaces (X_k) *uniformly asymptotically curved* with respect to the array $(e_{(k,j)})$, if for every bounded array $(x_{(k,j)})_{k,j}$, for which $(x_{(k,j)})_j$ is for every $k \in \mathbb{N}$ a block basis of $(e_{(k,j)})_j$, we have

$$\lim_n \sup_k \Big\| \frac{1}{n} \sum_{j=1}^{n} x_{(k,j)} \Big\|_{X_k} = 0.$$

The following special case of Proposition 3 in [**?**] is well known. It can also be easily shown directly.

Lemma 2.10. *Let X denote a Banach space with a Schauder basis (e_j), and let $1 \le r \le \infty$, $1 \le s \le \infty$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Assume that each block sequence (x_j^*) of the coordinate functionals (e_j^*) of (e_j) satisfies the lower r-estimate*

$$\Big\|\sum_{j=1}^n x_j^*\Big\|_{X^*} \ge c\Big(\sum_{j=1}^n \|x_j^*\|_{X^*}^r\Big)^{1/r}, \qquad n \in \mathbb{N},$$

for some constant $c > 0$ independent of n. Then each block sequence (x_j) of (e_j) satisfies the upper s-estimate

$$\Big\|\sum_{j=1}^n x_j\Big\|_X \le \frac{1}{c}\Big(\sum_{j=1}^n \|x_j\|_X^s\Big)^{1/s}, \qquad n \in \mathbb{N}.$$

The following Lemma is proved easily.

Lemma 2.11. *Let $1 < s < \infty$. Assume that the array $(e_{k,j})_{k,j}$ is such that $(e_{k,j})_j$ satisfies an upper s-estimate for each k, where the constant C is independent of k. Then the array $(e_{k,j})_{k,j}$ is uniformly asymptotically curved.*

Proposition 2.12. *Let $1 \le p, q < \infty$. Then every block sequence of the biparameter Haar system in $H^p(H^q)$ satisfies the lower $\max(2,p,q)$-estimate with constant 1 and the upper $\min(2,p,q)$-estimate also with constant 1.*

Proof. Before we begin with the actual proof, we define the biparameter square function $\mathbb{S}$ by

$$\mathbb{S}\Big(\sum_{I,J\in\mathcal{D}} a_{I,J}h_{I,J}\Big) = \Big(\sum_{I,J\in\mathcal{D}} a_{I,J}^2 h_{I,J}^2\Big)^{1/2}.$$

Let (f_i) denote a block sequence of the biparameter Haar system. Note that

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} = \bigg(\int\Big(\int\Big(\sum_{j=1}^n \big(\mathbb{S}f_j\big)^2\Big)^{q/2}dy\Big)^{p/q}dx\bigg)^{1/p}. \tag{2.4}$$

First, we will show that $H^p(H^q)$ satisfies the upper $\min(2,p,q)$-estimate with constant 1.

Case $p \ge 2$, $q \ge 2$: Since $q/2 \ge 1$, we reinterpret (**??**) and use Minkowski's inequality to obtain

$$\begin{aligned}
\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} &= \bigg(\int\Big\|\sum_{j=1}^n \big(\mathbb{S}f_j\big)^2\Big\|_{L^{q/2}(y)}^{p/2}dx\bigg)^{1/p} \le \bigg(\int\Big(\sum_{j=1}^n\big\|\big(\mathbb{S}f_j\big)^2\big\|_{L^{q/2}(y)}\Big)^{p/2}dx\bigg)^{1/p}\\
&= \bigg(\int\Big(\sum_{j=1}^n\Big(\int\big(\mathbb{S}f_j\big)^q dy\Big)^{2/q}\Big)^{p/2}dx\bigg)^{1/p} = \Big\|\sum_{j=1}^n\Big(\int\big(\mathbb{S}f_j\big)^q dy\Big)^{2/q}\Big\|_{L^{p/2}(x)}^{1/2}\\
&\le \bigg(\sum_{j=1}^n\Big\|\Big(\int\big(\mathbb{S}f_j\big)^q dy\Big)^{2/q}\Big\|_{L^{p/2}(x)}\bigg)^{1/2} = \Big(\sum_{j=1}^n\|f_j\|_{H^p(H^q)}^2\Big)^{1/2}.
\end{aligned}$$

Case $p \le 2$, $q \ge 2$: The first step is the same as in the previous case, *i.e.*, we have

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \bigg(\int\Big(\sum_{j=1}^n\Big(\int\big(\mathbb{S}f_j\big)^q dy\Big)^{2/q}\Big)^{p/2}dx\bigg)^{1/p}.$$

Since $p/2 \le 1$, we obtain

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \Big(\int\Big(\sum_{j=1}^n\Big(\int(\mathbb{S}f_j)^q dy\Big)^{2/q}\Big)^{p/2}dx\Big)^{1/p}$$
$$\le \Big(\sum_{j=1}^n\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p} = \Big(\sum_{j=1}^n\|f_j\|^p_{H^p(H^q)}\Big)^{1/p}.$$

Case $p \ge 2$, $q \le 2$: Since $q/2 \le 1$, (**??**) yields

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \Big(\int\Big(\sum_{j=1}^n\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p} = \Big\|\sum_{j=1}^n\int(\mathbb{S}f_j)^q dy\Big\|^{1/q}_{L^{p/q}(x)}$$
$$\le \Big(\sum_{j=1}^n\Big(\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{q/p}\Big)^{1/q} = \Big(\sum_{j=1}^n\|f_j\|^q_{H^p(H^q)}\Big)^{1/q}.$$

Case $p \le 2$, $q \le 2$: The first step is similar to the previous case, *i.e.*,

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \Big(\int\Big(\sum_{j=1}^n\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p}. \tag{2.5}$$

If $p \ge q$, we use Minkowski's inequality in $L^{p/q}$ and obtain

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \Big\|\sum_{j=1}^n\Big\|\int(\mathbb{S}f_j)^q dy\Big\|_{L^{p/q}(x)}\Big\|^{1/q} = \Big(\sum_{j=1}^n\|f_j\|^q_{H^p(H^q)}\Big)^{1/q}.$$

If $p \le q$, (**??**) yields

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \le \Big(\sum_{j=1}^n\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p} = \Big(\sum_{j=1}^n\|f_j\|^p_{H^p(H^q)}\Big)^{1/p}.$$

Secondly, we will prove that $H^p(H^q)$ satisfies the lower $\max(2,p,q)$-estimate with constant 1.

Case $p \ge q \ge 2$: Since $q/2 \ge 1$, we obtain from (**??**)

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \ge \Big(\int\Big(\sum_{j=1}^n\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p}. \tag{2.6}$$

Using $p/q \ge 1$ yields

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \ge \Big(\sum_{j=1}^n\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p} = \Big(\sum_{j=1}^n\|f_j\|^p_{H^p(H^q)}\Big)^{1/p}.$$

Case $q \ge p \ge 2$: Using (**??**) and Minkowski's inequality yields

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \ge \Big(\int\Big(\sum_{j=1}^n\int(\mathbb{S}f_j)^q dy\Big)^{p/q}dx\Big)^{1/p} = \Big(\int\Big\|\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}\Big\|_{\ell^{q/p}(j)}dx\Big)^{1/p}$$

$$\geq \Big(\Big\|\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q} dx\Big\|_{\ell^{q/p}(j)}\Big)^{1/p} = \Big(\sum_{j=1}^n \|f_j\|^q_{H^p(H^q)}\Big)^{1/q}.$$

Case $p \geq 2 \geq q$: By (**??**) and Minkowski's inequality, and since $2/q \geq 1$, $p/2 \geq 1$ we obtain

$$\begin{aligned}\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} &= \Big(\int\Big(\int\Big\|(\mathbb{S}f_j)^q\Big\|_{\ell^{2/q}(j)} dy\Big)^{p/q} dx\Big)^{1/p} \geq \Big(\int\Big\|\int(\mathbb{S}f_j)^q dy\Big\|^{p/q}_{\ell^{2/q}(j)} dx\Big)^{1/p}\\ &= \Big(\int\Big(\sum_{j=1}^n\Big(\int(\mathbb{S}f_j)^q dy\Big)^{2/q}\Big)^{p/2} dx\Big)^{1/p}\\ &\geq \Big(\sum_{j=1}^n\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q} dx\Big)^{1/p} = \Big(\sum_{j=1}^n \|f_j\|^p_{H^p(H^q)}\Big)^{1/p}.\end{aligned}$$

Case $p, q \leq 2$: The first step is the same as in the previous case, *i.e.*,

$$\begin{aligned}\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} &\geq \Big(\int\Big(\sum_{j=1}^n\Big(\int(\mathbb{S}f_j)^q dy\Big)^{2/q}\Big)^{p/2} dx\Big)^{1/p}\\ &= \Big(\int\Big\|\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q}\Big\|_{\ell^{2/p}(j)} dx\Big)^{1/p}\end{aligned}$$

Here, $2/p \geq 1$, hence, by Minkowski's inequality, we obtain

$$\Big\|\sum_{j=1}^n f_j\Big\|_{H^p(H^q)} \geq \Big(\Big\|\int\Big(\int(\mathbb{S}f_j)^q dy\Big)^{p/q} dx\Big\|_{\ell^{2/p}(j)}\Big)^{1/p} = \Big(\sum_{j=1}^n \|f_j\|^2_{H^p(H^q)}\Big)^{1/2}.$$

□

Remark 2.13. Let $1 < r, s < \infty$, then the identity operator provides an isomorphism between $H^r(H^s)$ and $L^r(L^s)$ (see Capon [**?**]); hence, by **??**, each block sequence with respect to the biparameter Haar system in $L^r(L^s)$ satisfies an upper $\min(r,s)$-estimate with constant $C = C_{r,s}$. Moreover, $\sup_{p_0\leq r,s\leq p_1} C_{r,s} \leq C_{p_0,p_1} < \infty$ whenever $1 < p_0 \leq p_1 < \infty$.

3. Simultaneous strategical reproducibility and statement of the main results

In order to state our main results, we will now state precisely the necessary definitions. Recall that we defined Z as the space

$$Z = \ell^\infty(X_k : k \in \mathbb{N}) = \big\{(x_k) : x_k \in X_k,\ k\in\mathbb{N},\ \|(x_k)\| = \sup_{k\in\mathbb{N}} \|x_k\|_{X_k} < \infty\big\}. \tag{3.1}$$

We also put

$$Y = c_0(X_k : k \in \mathbb{N}) = \big\{(x_k) : x_k \in X_k,\ k\in\mathbb{N},\ \lim_{k\to\infty} \|x_k\|_{X_k} = 0\big\}.$$

If for some space X we have $X_k = X$, for all $k \in \mathbb{N}$, we will write $\ell^\infty(X)$ and $c_0(X)$ instead of $\ell^\infty(X_k : k \in \mathbb{N})$ and $c_0(X_k : k \in \mathbb{N})$.

For $\overline{x} = (x_k)$ in Z (or Y) we call the set $\mathrm{supp}(\overline{x}) = \{k \in \mathbb{N} : x_k \neq 0\}$ *the support of* $\overline{x}$ *in* Z *(or* Y*).*

For $N \subset \mathbb{N}$, and $\overline{x} = (x_k) \in Z$ we let $P_N(x) \in Z$ be the projection of x on the coordinates in N,*i.e.*,

$$P_N(\overline{x}) = (y_k), \text{ with } y_k = \begin{cases} x_k & \text{if } k \in N, \text{ and} \\ 0 & \text{if } k \notin N, \end{cases} \tag{3.2}$$

and we put $Z_N := P_N(Z)$ which is isometrically isomorphic to $\ell^\infty(X_k : k \in N)$ and will be identified with that space. For $\overline{x} = (x_k) \in Z$ and $k \in \mathbb{N}$ we put $P_k(\overline{x}) = x_k$. In particular we identify X_k with its image under the canonical embedding into Z. We also identify X_k^* in the canonical way as a subspace of Z^* ($x^* \in X_k^*$ is acting on the k-component of $\bar{z} = (z_k) \in Z$). Note that X_k^* is thus a w^*-closed subspace of Z^*. For $k, j \in \mathbb{N}$ we also consider $e_{(k,j)}$ to be an element of Z, and $e^*_{(k,j)}$ to be an element of Z^* in the obvious way.

Convention 3.1. We fix from now on a bijective map $\nu(\cdot,\cdot)\colon \mathbb{N}^2 \to \mathbb{N}$, $(k,j) \mapsto \nu(k,j)$ with the property that for any $i, j, k \in \mathbb{N}$ we have $\nu(k,i) < \nu(k,j)$ if and only if $i < j$. We denote the inverse map by $(\kappa, \iota) : \mathbb{N} \to \mathbb{N}^2$, $n \mapsto (\kappa(n), \iota(n))$. We order the array $(e_{(k,j)} : k, j \in \mathbb{N})$ into a sequence (e_n), by putting $e_n = e_{(\kappa(n),\iota(n))}$ and $e_n^* = e^*_{(\kappa(n),\iota(n))}$. More generally, whenever $(x_{(k,j)})_j$ is a sequence in X_k, $k \in \mathbb{N}$, then we order the array $(x_{(k,j)})$ into the sequence (x_n) defined by $x_n = x_{(\kappa(n),\iota(n))}$; we do the same for $(x^*_{(k,j)})$.

Let $\mathcal{P}$ denote the product topology on Z, *i.e.*, the coarsest topology such that all the P_k, $k \in \mathbb{N}$, are continuous. Let $\overline{z}^{(j)} \in Z$, $j \in \mathbb{N}$, and $\overline{z} \in Z$. Then $(\overline{z}^{(j)})$ converges to $\overline{z}$ with respect to $\mathcal{P}$, if and only if

$$\lim_{j\to\infty} P_k \overline{z}^{(j)} = P_k \overline{z}, \text{ for all } k \in \mathbb{N}.$$

Whenever a sequence converges in Z, we implicitly refer to convergence with respect to the product topology $\mathcal{P}$. Whenever a sequence converges in some X_k, we refer to the norm topology in X_k.

Remark 3.2. For each $k \in \mathbb{N}$, assume $(e_{(k,j)})_j$ has basis constant $\lambda \geq 1$. Let $\sum_{n=1}^\infty a_n e_n \in Y$, where the series converges in the relative topology $\mathcal{P}|_Y$. Then the series $\sum_{n=1}^\infty a_n e_n$ converges in the norm topology of Y.

We now consider the following "simultaneous version" of the game described in [**?**].

Definition 3.3. Let $C \geq 1$. We say that the array $(e_{(k,j)})$ is *C-simultaneously strategically reproducible in Z* if for every $k \in \mathbb{N}$ $(e_{(k,j)})_j$ is C-strategically reproducible in X_k.

Remark 3.4. Note that we can also describe simultaneous strategic reproducibility in terms of the following two-player game: The array $(e_{(k,j)})$ is C- simultaneously strategically reproducible if and only if for every $\eta > 0$, Player II has a winning strategy for the game $\mathrm{Rep}_{(Z,(e_{(k,j)}))}(C, \eta)$ between Player I and Player II:

Assume the space Z, P_N, $(e_n : n \in \mathbb{N})$ and $(e_n^* : n \in \mathbb{N})$, are defined as in (**??**), (**??**) and in **??**.

Pregame: Before the first turn Player I is allowed to choose a partition of $\mathbb{N} = N_1 \cup N_2$. For $k \in \mathbb{N}$, and $r = 1, 2$ let $N_r^{(k)} = \{\nu(k,j) : j \in \mathbb{N}\} \cap N_r$.

Turn n, Step 1: Player I chooses $\eta_n > 0$, $G_n \in \mathrm{cof}_{w^*}(X^*_{\kappa(n)})$, and $W_n \in \mathrm{cof}(X_{\kappa(n)})$.

Turn n, Step 2: Player II chooses $i_n \in \{1,2\}$, a finite subset E_n of $N_{i_n}^{(\kappa(n))}$ and sequences of non-negative real numbers $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ satisfying

$$1-\eta < \sum_{i\in E_n} \lambda_i^{(n)}\mu_i^{(n)} < 1+\eta.$$

Turn n, Step 3: Player I chooses $(\varepsilon_i^{(n)})_{i\in E_n}$ in $\{-1,1\}^{E_n}$.

We say that Player II has a winning strategy in the game $\mathrm{Rep}_{Z,(e_{(k,j)})}(C,\eta)$ if he can force the following properties on the result:

For all $k,j\in\mathbb{N}$ we set $n=\nu(k,j)$ and put

$$x_n = x_{(k,j)} = \sum_{i\in E_n} \varepsilon_i^{(n)}\lambda_i^{(n)} e_{(k,i)} \qquad \text{and} \qquad x_n^* = x^*_{(k,j)} = \sum_{i\in E_n} \varepsilon_i^{(n)}\mu_i^{(n)} e^*_{(k,i)}$$

and demand:

(i) the sequences $(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially $(C+\eta)$-equivalent for each $k\in\mathbb{N}$;
(ii) the sequences $(x^*_{(k,j)})_j$ and $(e^*_{(k,j)})_j$ are impartially $(C+\eta)$-equivalent for each $k\in\mathbb{N}$;
(iii) for all $n\in\mathbb{N}$ we have $\mathrm{dist}(x_n^*, G_n) < \eta_n$;
(iv) for all $n\in\mathbb{N}$ we have $\mathrm{dist}(x_n, W_n) < \eta_n$.

Completely analogous to **??**, we define the corresponding notions in Z, below.

Definition 3.5. Let $T\colon Z\to Z$ be an operator.

(i) We call T a *diagonal operator on Z*, if $e_m^*(Te_n)=0$, whenever $m\neq n$.
(ii) We say that T has *a large diagonal* if $\inf_n \left|e_n^*\big(Te_n\big)\right| > 0$.

We are now in the position to state our two main results.

Theorem 3.6. *Assume that there are $C,\lambda\geq 1$, and a map $K\colon(0,\infty)\to(0,\infty)$ so that*

(i) the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k\in\mathbb{N}$;
(ii) $(e_{(k,j)})_j$ has the K-diagonal factorization property in X_k, for each $k\in\mathbb{N}$;
(iii) the array $(e_{(k,j)})_{k,j}$ is C-simultaneously strategically reproducible in Z.

Let $T\colon Z\to Z$ be bounded and linear, with

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}(Te_{(k,j)})\right| > 0.$$

Then for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there is an infinite $\Gamma\subset\mathbb{N}$ so that $\Gamma\cap\Omega_l$ is infinite for all $l\in\mathbb{N}$ and the identity on Z_Γ $\lambda CK(\delta)$-factors through T.

Remark 3.7. Note, that we did not simply state in **??** that there is an infinite Γ so that the identity on Z_Γ factors through T. However, we show more: additionally we require that the intersection of Γ with any prespecified sequence of infinite sets $\Omega_l\subset N$, $l\in\mathbb{N}$, also has to stay infinite. This means that if an infinite number of elements of the spaces X_n belong to a certain category, then also the spaces in an infinite subset of Γ will belong to that category. In **??** we will provide an application of that additional condition on Γ.

Corollary 3.8. *Assume that X is a Banach space with a normalized basis which is C-strategically reproducible and has the K-diagonal factorization property for some $K : (0,\infty) \to (0,\infty)$. Define $e_{(k,j)}$ to be the j-th basis element of the k-th component in $\ell^\infty(X)$, for $k, j \in \mathbb{N}$.*

Then the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in $Z = \ell^\infty(X)$, and the identity on Z factors through every operator $T\colon Z \to Z$ with large diagonal.

The following result describes a situation where it is not necessary, like in **??**, to pass to an infinite subset Γ of $\mathbb{N}$.

Theorem 3.9. *Assume the array $(e_{(k,j)})$ is uniformly asymptotically curved (see (**??**)), and furthermore, that there are $C, \lambda \geq 1$, and a map $K\colon (0,\infty) \to (0,\infty)$ so that*

(i) the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k \in \mathbb{N}$;
(ii) $(e_{(k,j)})_j$ has the K-diagonal factorization property in X_k, for each $k \in \mathbb{N}$;
(iii) the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in Z.

Let $T\colon Z \to Z$ be bounded and linear, with

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}(Te_{(k,j)})\right| > 0.$$

Then the identity on Z $\lambda CK(\delta)$-factors through T.

4. Factorization through diagonal operators

The main purpose of this section is to prove the pivotal **??**.

Lemma 4.1. *Let $S\colon Z \to Z$ be a bounded operator, and let (Ω_k) denote a sequence of infinite subsets of $\mathbb{N}$. For any $\rho > 0$ there is an infinite $\Gamma \subset \mathbb{N}$ so that*

$$\Gamma \cap \Omega_k \text{ is infinite for all } k \in \mathbb{N},$$

and

$$\left\|P_\Gamma \circ S|_{Z_\Gamma}\right\| \leq 2\left\|S|_Y\right\| + \rho.$$

Proof. Let $l \in \mathbb{N}$. We first observe that for a fixed $x^* \in X_l^*$, and for any infinite set $\Lambda \subset \mathbb{N}$, such that $\Lambda \cap \Omega_k$ is infinite for all $k \in \mathbb{N}$, there is an infinite $\Lambda' \subset \Lambda$ so that $\Lambda' \cap \Omega_k$ is still infinite, for all $k \in \mathbb{N}$, and $\|x^* \circ P_l \circ S \circ P_{\Lambda'}\| \leq \rho$. Indeed, if that were not true we could choose for any $n \in \mathbb{N}$ a partition of Λ into n infinite subsets $\Lambda_1, \Lambda_2, \dots \Lambda_n$ such that $\Lambda_j \cap \Omega_k$ is infinite, for all $1 \leq j \leq n$, $k \in \mathbb{N}$, and find $\overline{x}_1, \overline{x}_2, \dots \overline{x}_n$ in Z with $\|\overline{x}_j\| \leq 1$, $\mathrm{supp}(\overline{x}_j) \subset \Lambda_j$ and $x^*\big(P_l \circ S(\overline{x}_j)\big) \geq \rho$. But then we would have for $\overline{x} = \sum_{j=1}^n \overline{x}_j$ that $\|\overline{x}\| \leq 1$ and $x^*\big(P_l \circ S(\overline{x})\big) \geq n\rho$, which is impossible assuming that n is chosen large enough.

We now choose a sequence $(y_j^* : j \in \mathbb{N})$ in X_l^* with $\|y_j^*\| = 1$ which norms the elements of X_l. Using our previous observation we find infinite sets $\Lambda_1 \supset \Lambda_2 \supset \dots$, so that for all $j, k \in \mathbb{N}$ theset $\Lambda_j \cap \Omega_k$ is infinite and

$$\left\|y_j^* \circ P_l \circ S \circ P_{\Lambda_j}\right\| \leq \rho.$$

Then choose a diagonal set $\Gamma' = \{\lambda_j : j \in \mathbb{N}\}$ of the sets Γ_j, $j \in \mathbb{N}$, which has furthermore the property that $\Gamma' \cap \Omega_k$ is infinite for all $k \in \mathbb{N}$.

Now, we can choose for a given $\varepsilon > 0$ and an $\overline{x} = (x_k) \in Z$ a number $j \in \mathbb{N}$ for which

$$y_j^*\big(P_l \circ S \circ P_{\Gamma'}(\overline{x})\big) \geq (1-\varepsilon)\left\|P_l \circ S \circ P_{\Gamma'}(\overline{x})\right\|$$

and thus,

$$\begin{aligned}(1-\varepsilon)\|P_l\circ S\circ P_{\Gamma'}(\overline{x})\|&\\ &\leq \big|y_j^*\circ P_l\circ S\circ P_{\Gamma'\setminus\Gamma_j}(\overline{x})\big|+\big|y_j^*\circ P_l\circ S\circ P_{\Gamma_j}\circ P_{\Gamma'}(\overline{x})\big|\\ &\leq \|P_l\circ S|_Y\|+\|y_j^*\circ P_l\circ S\circ P_{\Gamma_j}\|\leq \|S|_Y\|+\rho.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we deduce that

$$\big\|P_l\circ S|_{Z_{\Gamma'}}\|\leq \|S|_Y\big\|+\rho.$$

Let $(k_j)\subset\mathbb{N}$ be a sequence in which every $k\in\mathbb{N}$ appears infinitely often. Starting by letting $\Gamma_0=\mathbb{N}$ and $\gamma_1\in\Omega_{k_1}$, we can apply our observation and recursively choose infinite sets $\Gamma_0\supset\Gamma_1\supset\Gamma_2\supset\ldots$, and $\gamma_1<\gamma_2<\ldots$ so that

$$\Gamma_j\cap\Omega_k \text{ is infinite for all } j,k\in\mathbb{N},\gamma_j\in\Gamma_{j-1}\cap\Omega_{k_j} \text{ and } \big\|P_{\gamma_j}\circ S|_{Z_{\Gamma_j}}\|\leq\|S|_Y\big\|+\rho.$$

Finally, letting $\Gamma=\{\gamma_j:j\in\mathbb{N}\}$, we deduce that

$$\begin{aligned}\|P_\Gamma S|_{Z_\Gamma}\|&=\sup_{j\in\mathbb{N}}\|P_{\gamma_j}S|_{Z_\Gamma}\|\\ &\leq\sup_{j\in\mathbb{N}}\Big(\big\|P_{\gamma_j}S|_{Z_{\{\gamma_1,\gamma_2,\ldots\gamma_j\}}}\big\|+\big\|P_{\gamma_j}S|_{Z_{\Gamma_j}}\big\|\Big)\leq 2\big\|S|_Y\big\|+\rho,\end{aligned}$$

which proves our claim. □

Lemma 4.2. *We assume that the array $(e_{(k,j)})_{k,j}$ is uniformly asymptotically curved. Let $z^*\in Z^*$ and $\eta>0$. Then there exists $(m_k)\subset\mathbb{N}$ so that for every $w=(w_k)\in Z$, $\|w\|\leq 1$ with $w_k\in[e_{(k,j)}:j\geq m_k]$, $k\in\mathbb{N}$, it follows that $|z^*(w)|\leq\eta$. In other words, letting $W_k=[e_{(k,j)}:j\geq m_k]$ it follows that*

$$\|z^*|_{\ell^\infty(W_k:k\in\mathbb{N})}\|\leq\eta.$$

Proof. Assume that our claim is not true for some $z^*\in Z^*$ and $\eta>0$. Then we can choose inductively for every $n\in\mathbb{N}$, sequences $(m_k^{(n)})_{k\in\mathbb{N}}\subset\mathbb{N}$, and $\overline{z}_n=(w_k^{(n)})_k\in B_Z$, so that

$$m_k^{(n-1)}<m_k^{(n)} \text{ for all } k\in\mathbb{N} \text{ (with } m_k^{(0)}=0), \tag{4.1}$$

$$w_k^{(n)}\in[e_{(k,j)}:m_k^{(n-1)}<j\leq m_k^{(n)}], \text{ for all } k\in\mathbb{N}, \text{ and} \tag{4.2}$$

$$z^*((w_k^{(n)})_k)>\eta/2. \tag{4.3}$$

Then for $n\in\mathbb{N}$ define

$$\overline{u}_n=\Big(\frac{1}{n}\sum_{m=1}^n w_k^{(m)}:k\in\mathbb{N}\Big)\in Z,$$

It follows from our assumption on the spaces X_k that $\lim_n\|\overline{u}_n\|_Z=0$, but on the other hand we have

$$z^*\big(\overline{u}_n\big)=\frac{1}{n}\sum_{m=1}^n z^*\big((w_k^{(n)})_k\big)>\eta/2$$

which for large enough n leads to a contradiction. □

Let (e_j) denote a basic sequence in a Banach space X. We say that a sequence (x_j) in X is a *perturbation of a block basic sequence of* (e_j) if there exists a block basis sequence $(\widetilde{x}_j)$ of (e_j) such that $\sum_{j=1}^{\infty} \|x_j - \widetilde{x}_j\|_X < \infty$.

Notation 4.3. *Let $\lambda \geq 1$ and assume the basis constant of $(e_{(k,j)})_j$, is not larger than λ, for all $k \in \mathbb{N}$. Assume that for each $k \in \mathbb{N}$, $(x_{(k,j)})_j$ is a sequence in X_k and that $(x^*_{(k,j)})_j$ is a perturbation of a block basic sequence of $(e^*_{(k,j)})_j$ in X_k^*. Moreover, assume that*

(i) $(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(ii) $(x^*_{(k,j)})_j$ and $(e^*_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(iii) $1 - \eta < x^*_{(k,i)}(x_{(k,i)}) < 1 + \eta$, for all $k, i \in \mathbb{N}$.

Then for each $k, j \in \mathbb{N}$, we define

$$A_k \colon X_k \to X_k, \qquad A_k e_{(k,j)} = x_{(k,j)},$$
$$B_k \colon X_k \to X_k, \qquad B_k x = \sum_{j=1}^{\infty} x^*_{(k,j)}(x) e_{(k,j)}.$$

and their respective vector operator version

$$A \colon Z \to Z, \qquad A\big((z_k)_k\big) = \big(A_k z_k\big)_k,$$
$$B \colon Z \to Z, \qquad B\big((z_k)_k\big) = \big(B_k z_k\big)_k.$$

Remark 4.4. In view of our hypothesis, the operators A_k, B_k, $k \in \mathbb{N}$ and consequently A, B in **??** are well defined and satisfy $\|A\| \leq \sqrt{C}$ and $\|B\| \leq \lambda\sqrt{C}$. Indeed, for $k \in \mathbb{N}$ we have

$$\begin{aligned}\sup_{x \in X_k, \|x\| \leq 1} \Big\| \sum_j x^*_j(x) e_{k,j} \Big\| &= \sup_{\substack{x \in X_k \|x\| \leq 1 \\ x^* \in X_k^* \|x^*\| \leq 1}} \sum_j x^*_j(x) x^*(e_{k,j}) = \sup_{\substack{x \in X_k, \|x\| \leq 1 \\ x^* \in X_k^* \|x^*\| \leq 1}} \Big(\sum_j x^*(e_{k,j}) x^*_j \Big)(x) \\ &= \sup_{x^* \in X_k^*, \|x^*\| \leq 1} \Big\| \sum_j x^*(e_{k,j}) x^*_j \Big\| \leq \lambda \sup_{x^* \in \operatorname{span}(e^*_{k,j}), \|x^*\| \leq 1} \Big\| \sum_j x^*(e_{k,j}) x^*_j \Big\| \\ &\leq \lambda\sqrt{C} \sup_{x^* \in \operatorname{span}(e^*_{k,j}), \|x^*\| \leq 1} \Big\| \sum_j x^*(e_j) e^*_{k,j} \Big\| = \lambda\sqrt{C}.\end{aligned}$$

Moreover, for each $k, i \in \mathbb{N}$ we have

$$BTA(e_{(k,i)}) = \Big(\sum_{j=1}^{\infty} x^*_{(l,j)}(Tx_{(k,i)}) e_{(l,j)} : l \in \mathbb{N} \Big) \in Z \tag{4.4}$$

it follows (infinite sums are meant to converge with respect to the topology $\mathcal{P}$, introduced in Convention **??**) for an $x = \sum_{n=1}^{\infty} a_n e_n \in Z$ that

$$\begin{aligned} BTA\Big(\sum_{n=1}^{\infty} a_n e_n\Big) &= \sum_{m=1}^{\infty} x_m^*\Big(TA\sum_{n=1}^{\infty} a_n e_n\Big)e_m \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{m-1} a_n x_m^*(Tx_n)e_m \\ &\quad + \sum_{m=1}^{\infty} a_m x_m^*(Tx_m)e_m + \sum_{m=1}^{\infty} x_m^*\big(T\sum_{n=m+1}^{\infty} a_n x_n\big)e_m. \end{aligned} \tag{4.5}$$

(Note that T might not be continuous with respect to $\mathcal{P}$, we only used the linearity of T). If $x = \sum_{n=1}^{\infty} a_n e_n \in Y$ (which implies that this series is norm convergent) then

$$BTA\Big(\sum_{n=1}^{\infty} a_n e_n\Big) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_n x_m^*\big(Tx_n\big)e_m, \tag{4.6}$$

where the convergence of the double sum is meant with respect to the topology $\mathcal{P}$.

We now formulate and prove a rather technical proposition which presents the heart of the proof of **??** and **??**.

Proposition 4.5. *Assume that for some $\lambda \geq 1$ the basis constant of $(e_{(k,j)})_j$, is not larger than λ. Let $T\colon Z \to Z$ be a bounded linear operator, for each $k\in\mathbb{N}$ let $(x_{(k,j)})_j$ be a sequence in X_k and let $(x^*_{(k,j)})_j$ be a perturbation of a block basis of $(e^*_{(k,j)})_j$ in X_k^*.*

Let $0 < \eta \leq 1$, $C \geq 1$ and $(\eta_n) \subset (0,1]$ so that $\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\eta_n < \eta/2$. Consider the following conditions:

(i) $(x_{(k,j)})_j$ and $(e_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(ii) $(x^*_{(k,j)})_j$ and $(e^*_{(k,j)})_j$ are impartially C-equivalent, for all $k \in \mathbb{N}$;
(iii) $\sum_{n=1}^{m-1} |x_m^*(Tx_n)| < \eta_m$, for all $m \in \mathbb{N}$;
(iv) $\sum_{n=m+1}^{\infty} |x_m^*(Tx_n)| < \eta_m$, for all $m \in \mathbb{N}$;

In order to formulate the last condition, we assume that for each $n \in \mathbb{N}$, we are given a sequence $(W_k^{(n)})_k$, where $W_k^{(n)}$ is a cofinite-dimensional subspace of X_k, with $W_k^{(n+1)} \subset W_k^{(n)}$ for $k, n \in \mathbb{N}$.

(v) For each $n \in \mathbb{N}$, assume that

$$\|T^* x_n^*|_{\ell_\infty(W_k^{(n+1)}:k\in\mathbb{N})}\| < \eta_n \qquad \text{and} \qquad \operatorname{dist}\big(x_n, W_{\kappa(n)}^{(n)}\big) < \eta_n,$$

for all $n \in \mathbb{N}$.

Let $D\colon Z \to Z$ denote the diagonal operator given by

$$De_{(k,j)} = x^*_{(k,j)}(Tx_{(k,j)})e_{(k,j)}, \qquad \text{for all } k, j \in \mathbb{N}.$$

*Then the following assertions (**??**) and (**??**) hold true.*

*(a) If (**??**)–(**??**) is satisfied, then $BTA - D\colon Y \to Y$ is well defined and*

$$\|BTA - D : Y \to Y\| \leq 2\lambda\eta.$$

D is a bounded operator from Z to Z and thus also $D - BTA$. Moreover, for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there exists an infinite set $\Gamma \subset \mathbb{N}$ such that

$$\Gamma \cap \Omega_l \text{ is infinite, for all } l \in \mathbb{N}, \text{ and } \|D_\Gamma - P_\Gamma BTA : Z_\Gamma \to Z_\Gamma\| < 5\lambda\eta,$$

where $D_\Gamma = P_\Gamma \circ D$.

If we additionally assume that $K \geq 1$ is such that the identity K-factors through $P_\Gamma D|_{Z_\Gamma}$ and $\eta < 1/(5\lambda K)$, then the identity on Z_Γ $\left(\frac{\lambda KC}{1-5\lambda K\eta}\right)$-factors through T.

(b) If alternatively, (??)–(??) and (??) are satisfied, then

$$\|BTA - D\colon Z \to Z\| < 2\lambda\sqrt{C}(3 + \|T\|)\eta.$$

If we additionally assume that $K \geq 1$ is such that the identity on Z K-factors through D and $\eta < 1/(2\lambda\sqrt{C}(3+\|T\|)K)$ then the identity on Z $\left(\frac{\lambda KC}{1-2\lambda\sqrt{C}(3+\|T\|)K\eta}\right)$-factors through T.

Proof. Naturally, the proof splits into two parts.

Proof of (??). Let $y = \sum_{n=1}^\infty a_n e_n \in Y$ be finitely supported and observe that by (??)

$$(BTA - D)y = \sum_{m=1}^\infty \sum_{n:n\neq m} a_n x_m^*(Tx_n)e_m. \tag{4.7}$$

Hence, for each $k \in \mathbb{N}$ we obtain

$$P_k(BTA - D)y = \sum_{m:\kappa(m)=k} \sum_{n:n\neq m} a_n x_m^*(Tx_n)e_m,$$

and thus

$$\begin{aligned} \|P_k(BTA - D)y\|_{X_k} &\leq \sum_{m:\kappa(m)=k} \sum_{n\neq m}^\infty |a_n||x_m^*(Tx_n)| \\ &\leq 2\lambda\|y\|_Y \sum_{m:\kappa(m)=k} \sum_{n\neq m}^\infty |x_m^*(Tx_n)|. \end{aligned} \tag{4.8}$$

By (??) and (??) we have $\sum_{m,n:m\neq n} |x_m^*(Tx_n)| < \infty$, and hence

$$\lim_{k\to\infty} \sum_{m:\kappa(m)=k} \sum_{n\neq m}^\infty |x_m^*(Tx_n)| = 0.$$

Together with estimate (??) we obtain $(BTA - D)y \in Y$.

Using (??) and (??), (??) yields

$$\|(BTA - D)y\|_Y \leq \sum_{m=1}^\infty \sum_{n=1}^{m-1} |a_n||x_m^*(Tx_n)| + \sum_{m=1}^\infty \sum_{n=m+1}^N |a_n||x_m^*(Tx_n)| \leq 2\lambda\eta\|y\|_Y.$$

We conclude

$$\|BTA - D\colon Y \to Y\| \leq 2\lambda\eta.$$

Next, we observe that since D is a diagonal operator, which means that D is of the form

$$D\colon Z \to Z, \quad (x_k) \mapsto (D_k x_k),$$

where the operators $D_k\colon X_k \to X_k$, $k \in \mathbb{N}$, are uniformly bounded. Thus, D is also a bounded operator on Z with norm $\sup_k \|D_k\colon X_k \to X_k\|$ and the operator $P_\Gamma BTA|_{Z_\Gamma} - D$ is a well defined and a bounded operator on all of Z. Our conclusion follows therefore from **??** for some infinite set $\Gamma \subset \mathbb{N}$.

For the additional part, assume that $\hat{B}\colon Z_\Gamma \to Z_\Gamma$ and $\hat{A}\colon Z_\Gamma \to Z_\Gamma$ are such that $\|\hat{B}\|\|\hat{A}\| \le K$ and $I = \hat{B}D_\Gamma\hat{A}$. It follows that $\|I - \hat{B}P_\Gamma BTA|_{Z_\Gamma}\hat{A}\| = \|\hat{B}(D_\Gamma - P_\Gamma BTA|_{Z_\Gamma})\hat{A}\| < 5\lambda K\eta < 1$. Hence, the map $Q = \hat{B}P_\Gamma BTA|_{Z_\Gamma}\hat{A}$ is invertible with $\|Q^{-1}\| \le 1/(1-5\lambda K\eta)$. In conclusion, if we set $\tilde{B} = Q^{-1}\hat{B}B$, $\tilde{A} = A\hat{A}$ then $\tilde{B}T\tilde{A} = I$ and $\|\tilde{B}\|\|\tilde{A}\| \le \lambda KC/(1-5\lambda K\eta)$.

Proof of (**??**). Note that since A is a bounded operator (see **??**), it follows that

$$A(z) = \Big(\sum_{j=1}^{\infty} a_{\nu(k,j)} x_{(k,j)} : k \in \mathbb{N}\Big) = \sum_{n=1}^{\infty} a_n x_n,$$

whenever $z = \sum_{n=1}^{\infty} a_n e_n \in Z$, where the series converges in the product topology $\mathcal{P}$. So, in particular the sum $\sum_{n=1} a_n x_n$ is well defined in Z if the sum $\sum_{n=1} a_n e_n$ is well defined. Let us assume that $z = \sum_{n=1}^{\infty} a_n e_n \in S_Z$ and thus $\|A(z)\| \le \sqrt{C}$.

By (**??**) and the definition of D we obtain

$$(BTA - D)\Big(\sum_{n=1}^{\infty} a_n e_n\Big) = \sum_{m=1}^{\infty}\sum_{n=1}^{m-1} a_n x_m^*(Tx_n)e_m + \sum_{m=1}^{\infty} x_m^*\big(T\sum_{n=m+1}^{\infty} a_n x_n\big)e_m.$$

The norm of the first sum is dominated by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{m-1} |a_n||x_m^*(Tx_n)| < 2\lambda\sum_{m=1}^{\infty}\eta_m \le 2\lambda\eta. \tag{4.9}$$

To estimate the norm of the second sum, first choose $y_n \in W_{\kappa(n)}^{(n)}$ according to (**??**) such that $\|x_n - y_n\| < \eta_n$. Let $m \in \mathbb{N}$ be fixed. We claim that $y = \sum_{n=m+1}^{\infty} a_n y_n$ is a well defined element in $\ell^\infty\big(W_k^{(m+1)} : k \in \mathbb{N}\big)$. Indeed, it is well defined since $\sum_{n=m+1}^{\infty} a_n x_n \in Z$ and $\sum_{n=1}^{\infty}\eta_n < \eta < \infty$. Moreover, by the properties of our enumeration (see **??**) and since $W_k^{(n+1)} \subset W_k^{(n)}$, $k, n \in \mathbb{N}$, we have that

$$y = \sum_{n=m+1}^{\infty} a_n y_n = \sum_{k=1}^{\infty}\sum_{\substack{n>m\\ \kappa(n)=k}} a_n y_n \in \ell^\infty\big(W_k^{(n_k)} : k\in\mathbb{N}\big),$$

where $n_k = \min\{n > m : \kappa(n) = k\} \ge m+1$; thus we proved $y \in \ell^\infty\big(W_k^{(m+1)} : k\in\mathbb{N}\big)$. By a standard perturbation argument, we obtain

$$\begin{aligned}\sum_{m=1}^{\infty}\big|x_m^*\big(T\sum_{n=m+1}^{\infty} a_n x_n\big)\big| &\le \sum_{m=1}^{\infty}\big|T^*x_m^*\big(\sum_{n=m+1}^{\infty} a_n y_n\big)\big| + \sum_{m=1}^{\infty}\big|T^*x_m^*\big(\sum_{n=m+1}^{\infty} a_n(x_n - y_n)\big)\big| \\ &\le \sum_{m=1}^{\infty}\eta_m\Big\|\sum_{n=m+1}^{\infty} a_n y_n\Big\| + \sum_{m=1}^{\infty}\|T^*x_m^*\|\sum_{n=m+1}^{\infty}|a_n|\eta_n \quad \text{(by (vi))}\end{aligned}$$

$$\leq \sum_{m=1}^{\infty} \eta_m(2\lambda\sqrt{C} + 2\lambda\eta) + \sum_{m=1}^{\infty} \|T\|\sqrt{C}2\lambda \sum_{n=m+1}^{\infty} \eta_n$$
$$\leq \eta 2\lambda\sqrt{C}(2 + \|T\|)$$

which together with (**??**) establishes the first part of (b).

For the additional part, assume that $\hat{B}\colon Z \to Z$ and $\hat{A}\colon Z \to Z$ are such that $\|\hat{B}\|\|\hat{A}\| \leq K$ and $I = \hat{B}D\hat{A}$. It follows that $\|I - \hat{B}BTA\hat{A}\| = \|\hat{B}(D - BTA)\hat{A}\| < 2\lambda\sqrt{C}(3+\|T\|)K\eta < 1$. Hence, the map $Q = \hat{B}BTA\hat{A}$ is invertible with $\|Q^{-1}\| \leq 1/(1 - 2\lambda\sqrt{C}(3 + \|T\|)K\eta)$. In conclusion, if we set $\tilde{B} = Q^{-1}\hat{B}B$, $\tilde{A} = A\hat{A}$ then $\tilde{B}T\tilde{A} = I$ and

$$\|\tilde{B}\|\|\tilde{A}\| \leq \lambda KC/(1 - 2\lambda\sqrt{C}(3 + \|T\|)K\eta).$$

□

We finally prepare the work of the next section by isolating the following technical **??**.

Lemma 4.6. *Assume that $\lambda \geq 1$ and $K\colon (0,\infty) \to (0,\infty)$ is continuous, so that for each k the basis constant of $(e_{(k,j)})_j$, is not larger than λ and which has the $K(\delta)$-factorization property in X_k.*

Then for every bounded diagonal operator $D\colon Z \to Z$ (i.e. $D(e_{k,j})$ is a multiple of $e_{k,j}$, for $k, j \in \mathbb{N}$), for which

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}(De_{(k,j)})\right| > 0,$$

the identity almost $K(\delta)$-factors through D.

Proof. Let $D\colon Z \to Z$ be a bounded diagonal operator with $\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}\big(Te_{(k,j)}\big)\right| > 0$. For $k \in \mathbb{N}$ let $D_k = D|_{X_k}$. Then D_k is a diagonal operator from X_k to X_k. Next, let $\eta > 0$. Then there are for each $k \in \mathbb{N}$ bounded operators $A_k\colon X_k \to X_k$ and $B_k\colon X_k \to X_k$, so that $\|A_k\| \cdot \|B_k\| \leq K(\delta) + \eta$ and $I_k = B_k \circ D_k \circ A_k$, where I_k is the identity on X_k. We can assume that $\|A_k\| = 1$ and that $\|B_k\| \leq K(\delta) + \eta$. Putting

$$A\colon Z \to Z, \quad (z_k) \mapsto \big(A_k z_k\big), \qquad B\colon Z \to Z, \quad (z_k) \mapsto \big(B_k z_k\big),$$

it follows that $\|A\| = 1$ and $\|B\| \leq K(\delta) + \eta$, and $I_Z = B \circ D \circ A$. □

5. Proof of ?? and ??

The proof of both theorems **??** and **??** is organized as depicted in the flowchart **??**. According to the flowchart **??**, their respective proofs deviate at two distinct points. In the proof below we indicate that by arranging the text side by side in separate columns. We use a corresponding color scheme in the flowchart and in the proof.

Let $T\colon Z \to Z$ be a bounded linear operator with $\inf_{k,j\in\mathbb{N}} \left|e^*_{(k,j)}(Te_{(k,j)})\right| =: \delta > 0$. By **??**, Player II has a winning strategy in the game $\mathrm{Rep}_{(Z,(e_{(k,j)}))}(C,\eta)$. Let us fix $\eta > 0$ to be determined later.

We will now describe a strategy for Player I in a game $\mathrm{Rep}_{(X,(e_n))}(C,\eta)$, and assume Player II answers by following his winning strategy.

Pregame. At the beginning, Player I chooses $N_1 = \{n \in \mathbb{N} : e^*_n(Te_n) \geq \delta\}$ and $N_2 = \{n \in \mathbb{N} : e^*_n(Te_n) \leq -\delta\}$. For $k \in \mathbb{N}$, and $i = 1, 2$ let $N_i^{(k)} = \{\nu(k,j) : j \in \mathbb{N}\} \cap N_i$.

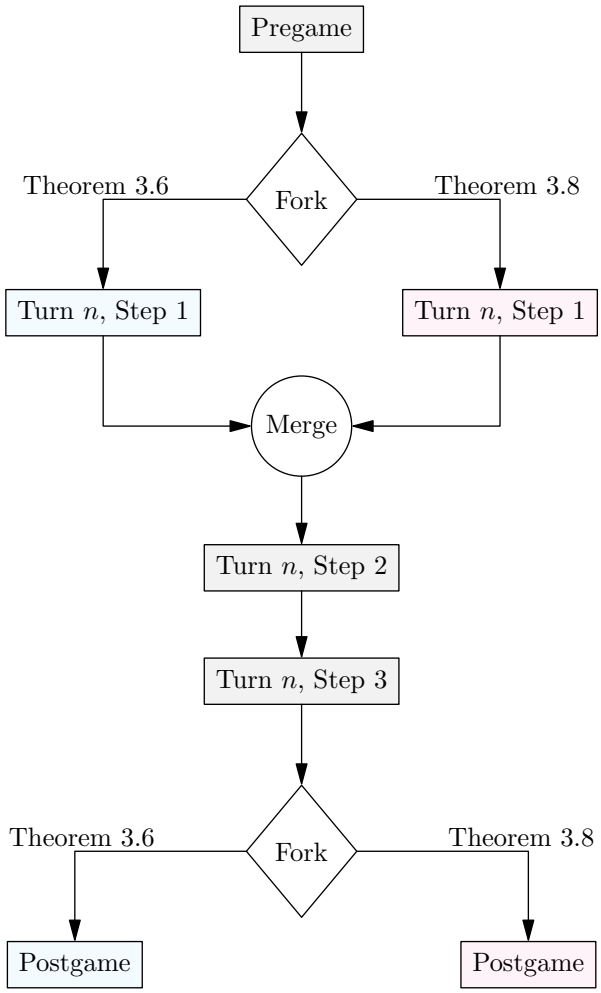

FIGURE 1. Flowchart of the proof of **??** and **??**.

Turn n, Step 1. During the n-th turn, Player I proceeds as follows. In the first step of the n-th turn he chooses

$$\eta_n < \frac{\eta}{2^{n+2}n(1+\|T\|)\sqrt{C+\eta}}. \tag{5.1}$$

Let $l_n = \max(\bigcup_{m=1}^{n-1} E_m)$ if $n > 1$ (where the finite sets E_1, E_2,...E_{n-1} were chosen in previous turns) and put $l_1 = 1$. Define

$$A_n = \big\{x_m, Tx_m : m < n\big\} \cup \{e_{(\kappa(n),i)}, Te_{(\kappa(n),i)} : i \le l_n\}$$

and put

$$G_n = A_n^{\perp} \cap X^*_{\kappa(n)} \tag{5.2}$$

as subspace of $X^*_{\kappa(n)}$ and Z^*.

Choice of W_n in **??**.

Now, let

$$B_n = \{x_m^*, T^*x_m^* : m < n\} \cup \{e^*_{(\kappa(n),i)}, T^*e^*_{(\kappa(n),i)} : i \le l_n\}$$

and define

$$W_n = (B_n)_\perp \cap X_{\kappa(n)} \tag{5.3}$$

as a subspace of $X_{\kappa(n)}$ and Z.

Choice of W_n in **??**.

In order to choose the required cofinite-dimensional subspace W_n of $X_{\kappa(n)}$ Player I also chooses at the n-th step a sequence $(V_l^{(n)})_l$, where $V_l^{(n)}$ is a finite codimensional subspace of X_l of the form

$$V_l^{(n)} = [e_{(l,j)} : j \ge m(n,l)], \tag{5.4}$$

with $V_l^{(n)} \subset V_l^{(m)}$ for $l \in \mathbb{N}$, $0 \le m < n$ and we put $V_l^{(0)} = X_l$ for $l \in \mathbb{N}$. By using **??** finitely many times Player I can find for each l a subspace $V_l^{(n)}$ of $V_l^{(n-1)}$ so that for all $j < n$

$$\|T^*x_j^*|_{\ell^\infty(V_l^{(n)}:l\in\mathbb{N})}\| \le \eta_n \tag{5.5}$$

and he chooses

$$W_n = V^{(n)}_{\kappa(n)} = [e_{(\kappa(n),j)} : j \ge m_n], \tag{5.6}$$

where $m_n = m(n, \kappa(n))$.

Turn n, Step 2. Player II, following a winning strategy, chooses $i_n \in \{1,2\}$, picks a finite set $E_n \subset N_{i_n}^{(\kappa(n))}$ and sequences of non-negative scalars $(\lambda_i^{(n)})_{i\in E_n}$, $(\mu_i^{(n)})_{i\in E_n}$ with

$$1 - \eta < \sum_{i\in E_n} \lambda_i^{(n)}\mu_i^{(n)} < 1 + \eta.$$

Turn n, Step 3. Then Player I picks signs $(\varepsilon_i^{(n)})_{i\in E_n} \in \{-1,+1\}^{E_n}$ so that whenever

$$x_n = \sum_{i\in E_n} \lambda_i^{(n)}\varepsilon_i^{(n)} e_{(\kappa(n),i)} \qquad \text{and} \qquad x_n^* = \sum_{i\in E_n} \mu_i^{(n)}\varepsilon_i^{(n)} e^*_{(\kappa(n),i)},$$

we have

$$|x_n^*(Tx_n)| > (1-\eta)\delta. \tag{5.7}$$

That it is possible to choose such signs $(\varepsilon_i^{(n)})_{i\in E_n} \in \{-1,+1\}^{E_n}$. Indeed, $r = (r_j)_{j\in E_n}$ be a Rademacher sequence, meaning that r_j, $j \in E_n$, are independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$, with $\mathbb{P}(r_j = 1) = \mathbb{P}(r_j = -1) = \frac{1}{2}$.

$$\mathbb{E}\Big(\Big(\sum_{i\in E_n} r_i\mu_i^{(n)} e^*_{(\kappa(n),i)}\Big)\Big(T\Big(\sum_{j\in E_n} r_j\lambda_j^{(n)} e_{(\kappa(n),j)}\Big)\Big)\Big) = \mathbb{E}\Big(\sum_{i,j\in E_n} r_ir_j\mu_i^{(n)}\lambda_j^{(n)} e_j^*(Te_{(\kappa(n),i)})\Big)$$

$$= \sum_{i\in E_n} \mu_i^{(n)}\lambda_i^{(n)} e^*_{(\kappa(n),i)}(Te_{(\kappa(n),i)}) \begin{cases} > \delta(1-\eta) & \text{if } i_n = 1, \\ < -\delta(1-\eta) & \text{if } i_n = 2. \end{cases}$$

The latter inequality follows from the large diagonal of the operator.

Postgame. After the game is completed the conditions (??) to (??) of ?? are satisfied.

Now, let $m, n \in \mathbb{N}$, $m < n$. By the winning strategy of Player II (see (??) of ??) and (??), we obtain

$$|x_n^*(Tx_m)| \leq \|Tx_m\| \cdot \operatorname{dist}(x_n^*, G_n) < \|T\|\sqrt{C+\eta} \cdot \eta_n. \tag{5.8}$$

We now estimate $|x_n^*(Tx_m)|$ if $m > n$.

Postgame for ??.

By the winning strategy of Player II (see ?? (??)) together with (??) yields for all $n < m$ that

$$\begin{aligned} |x_n^*(Tx_m)| &= |T^*x_n^*(x_m)| \\ &\leq \|T^*x_n^*\| \cdot \operatorname{dist}(x_m, W_m) \\ &< \|T^*\|\sqrt{C+\eta} \cdot \eta_m. \end{aligned}$$

This inequality, (??) and (??) imply that the conditions of Proposition ?? are satisfied for $\eta'_m = m\eta_m$, $m \in \mathbb{N}$.
By ?? (??), there is an infinite set $\Gamma \subset \mathbb{N}$ so that the diagonal operator $D_\Gamma \colon Z_\Gamma \to Z_\Gamma$ given by

$$D_\Gamma e_{(k,i)} = x^*_{(k,i)}(Tx_{(k,i)})e_{(k,i)}$$

is bounded. By ??, the identity on Z_Γ $(K\delta(1-\eta)+\xi)$ factors through D_Γ for any $\xi > 0$. Hence, if η is sufficiently small then by the additional assertion in ?? (??), the identity $\left(\frac{(\lambda K(\delta(1-\eta))+\xi)(C+\eta)}{1-5\lambda(K\delta(1-\eta))+\xi)\eta}\right)$-factors through T.

Postgame for ??.

Note that by (??), condition (??) of ?? is satisfied if we replace η_n by $\eta/2^{n+2}$. Secondly, for $n_0 < n$ if $w_k \in V_k^{(n)} \subset V_k^{(n_0+1)}$, for $k \in \mathbb{N}$ with $\|(w_k)\|_Z \leq 1$ and $z = (w_k)$, then by (??)

$$\left|\left(T^*_{n_0}x^*_{n_0}\right)(z)\right| \leq \eta_{n_0} < \eta/2^{n_0+2}.$$

Thus, condition (??) of ?? is satisfied, as well. By ?? (??), the diagonal operator $D : Z \to Z$ given by

$$De_{(k,i)} = x^*_{(k,i)}(Tx_{(k,i)})e_{(k,i)}$$

is bounded. By ??, the identity on Z $(K(\delta(1-\eta))+\xi)$-factors through D for any $\xi > 0$. Hence, if η is sufficiently small, then by the additional assertion of ?? (??), the identity $\left(\frac{(\lambda K(\delta(1-\eta))+\xi)(C+\eta)}{1-2\lambda\sqrt{C+\eta}(3+\|T\|)K(\delta(1-\eta))+\xi)\eta}\right)$-factors through T.

Recall that the function $K \colon (0,\infty) \to \mathbb{R}$ is continuous (see [?, Remark 3.11]). As we could have picked η and ξ arbitrarily close to zero we deduce that the identity on Z_Γ (for ??), or the identity on Z, (for ??), respectivly, almost $\lambda K(\delta)C$-factors through T. □

6. Heterogenous ℓ^∞-sums of classical Banach spaces

Given $1 < p_0 < p_1 < \infty$, we define the sets of Banach spaces

$$\mathcal{W} = \{L^p, H^p, \mathrm{VMO}, \mathrm{VMO}(H^r), H^p(H^q), L^r(L^s) : 1 \leq p, q < \infty, p_0 \leq r, s \leq p_1\} \tag{6.1}$$

$$\mathcal{X} = \{L^r, H^p, \mathrm{VMO}, \mathrm{VMO}(H^r), H^p(H^q), L^r(L^s) : p_0 \leq p, q < \infty, p_0 \leq r, s \leq p_1\}. \tag{6.2}$$

In this section, we use the following notation: Assume that we are given a sequence of Banach spaces (V_k) in either $\mathcal{W}$ or $\mathcal{X}$. Then for each $k, j \in \mathbb{N}$, $e_{k,j}$ denotes the j-th Haar

function whenever $V_k \in \{L^r, H^p, \mathrm{VMO}\}$, and $e_{k,j}$ denotes the j-th biparameter Haar function if $V_k \in \{\mathrm{VMO}(H^r), H^p(H^q), L^r(L^s)\}$. For details on the enumeration of the biparameter Haar system we refer to [**?**]; see also [**?**].

Corollary 6.1. *Let $(W_k)_{k=1}^\infty$ denote a sequence of Banach spaces in $\mathcal{W}$ and let $T : \ell^\infty(W_k : k\in\mathbb{N}) \to \ell^\infty(W_k : k\in\mathbb{N})$ be bounded and linear, with*

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{k,j}(Te_{k,j})\right| > 0.$$

Then for each sequence of infinite subsets (Ω_l) of $\mathbb{N}$, there is an infinite $\Gamma \subset \mathbb{N}$ so that $\Gamma\cap\Omega_l$ is infinite for all $l \in \mathbb{N}$ and the identity on $\ell^\infty(W_k : k \in \Gamma)$ $\frac{C}{\delta}$-factors through T, where C depends only on p_0, p_1.

Proof. First, we note that by [**?**, Proof of Theorem 5.1, Theorem 5.2, Theorem 5.3, Theorem 6.1, Remark 6.6] the one- or two-parameter Haar system in each of the spaces L^p, H^p, $H^p(H^q)$, $1 \le p, q < \infty$ is C-strategically reproducible for some universal constant, and $L^r(L^s)$, $p_0 \le r, s \le p_1$ is C_{p_0,p_1}-strategically reproducible. By **??**, the Haar system is strategically reproducible in VMO, and the biparameter Haar system is C_{p_0,p_1}-strategically reproducible in $\mathrm{VMO}(H^p)$. In each of the spaces in $\mathcal{W}$, the Haar system has the $\frac{C}{\delta}$-diagonal factorization property (for the L^1 case we refer to [**?**, Proposition 6.2], for the other cases, this follows by unconditionality) for some universal constant C. The assertion follows from **??**. □

Before we can proceed to our next application, we need the following observation.

Lemma 6.2. *Let $(X_k)_{k=1}^\infty$ denote a sequence of Banach spaces in $\mathcal{X}$. Then the array $(e_{k,j})$ is uniformly asymptotically curved.*

Proof. For each $k \in \mathbb{N}$, let $(f_{(k,j)})_j$ be a normalized block basis of (h_I). Let $k \in \mathbb{N}$ be fixed for now. Since the $(f_{(k,j)})_j$ have disjoint Haar spectra, observe that for each $n \in \mathbb{N}$

$$\Big\|\sum_{j=1}^n f_{(k,j)}\Big\|_{X_k} \le C\int_0^1 \Big\|\sum_{j=1}^n r_j(t) f_{(k,j)}\Big\|_{X_k} dt \le C\Big(\sum_{j=1}^n \|f_{(k,j)}\|_{X_k}^r\Big)^{1/r},$$

where $r_1, \dots, r_K$ are independent Rademacher functions and $r = p_0$ if $X_n \in \{L^p, H^p\}$ and $r = 2$ if $X_n = \mathrm{VMO}$ [**?**, Proposition 5.1.1, p. 268]. The constant C depends on p_0 and p_1 if $X_n = L^p$, and $C = 1$ if $X_n \in \{H^p, \mathrm{VMO}\}$. In either case, we obtain

$$\sup_k \Big\|\sum_{j=1}^n f_{(k,j)}\Big\|_{X_k} \le Cn^{1/r}.$$

Moving on to the biparameter spaces, let r' be such that $\frac{1}{r} + \frac{1}{r'} = 1$. By **??**, $H^1(H^{r'})$ satisfies a lower r'-estimate with constant 1; hence by **??** and **??**, the array formed by the biparameter Haar system in the predual of $H^1(H^{r'})$, which is $\mathrm{VMO}(H^r)$, is uniformly asymptotically curved. For the spaces $H^p(H^q)$ and $L^r(L^s)$ we refer to **??**, **??** and **??**. □

?? and **??** combined yield the following factorization result.

Corollary 6.3. *Let $(X_k)_{k=1}^\infty$ denote a sequence of Banach spaces in $\mathcal{X}$, and let $T : \ell^\infty(X_k : k\in\mathbb{N}) \to \ell^\infty(X_k : k\in\mathbb{N})$ be bounded and linear, with*

$$\delta = \inf_{k,j\in\mathbb{N}} \left|e^*_{k,j}(Te_{k,j})\right| > 0.$$

Then the identity on $\ell^\infty(X_k : k\in\mathbb{N})$ $\frac{C}{\delta}$-factors through T, where C depends only on p_0, p_1.

Proof. Let $(e_{k,j})$ be defined as in **??**. First, note that $(e_{k,j})_j$ has unconditional basis constant C_{p_0,p_1} for each $k \in \mathbb{N}$; hence, $(e_{k,j})_j$ has the $\frac{C_{p_0,q_1}}{\delta}$-diagonal factorization property for all $k \in \mathbb{N}$. Secondly, for the simultaneous strategical reproducibility of the array $(e_{k,j})$, we refer to the argument presented in the proof of **??**. Finally, applying **??** and **??** concludes the proof. □

7. Final remarks and open questions

Our first question asks whether **??** can be true if we drop the assumption that the array $(e_{(k,j)})$ is uniformly asymptotically curved.

Question 1. Assume that

(i) the basis constant of $(e_{(k,j)})_j$, is at most λ in X_k, for each $k \in \mathbb{N}$;
(ii) $(e_{(k,j)})_j$ has the K-diagonal factorization property in X_k, for each $k \in \mathbb{N}$;
(iii) the array $(e_{(k,j)})_{k,j}$ is simultaneously C-strategically reproducible in Z.

Is it true that the array $(e_{(k,j)})$ has the factorization property in $Z = \ell^\infty(X_k)$?

The following example exhibits a sequence of spaces the array of which is *not* uniformly asymptotically curved, yet the array has the factorization property in Z.

Example 7.1. *Let (p_k) be a dense sequence in $[1,\infty)$. Then*

(i) the basis constant of $(h_{k,I})_I$ is 1 in L^{p_k};
(ii) $(h_{k,I})_I$ has the $\frac{1}{\delta}$-diagonal factorization property [**?**, Remark 6.6]*;*
(iii) the array $(h_{k,I})$ is simultaneously 1-strategically reproducible [**?**, Remark 6.6]*;*
(iv) the array $(h_{k,I})$ is not *uniformly asymptotically curved.*

In contrast, the array $(h_{k,I})$ has in $Z = \ell^\infty(L^{p_k} : k\in\mathbb{N})$ the factorization property.

In order to show this is indeed true, we need the following proposition.

Proposition 7.2. *Let (p_k) and (q_k) denote two dense sequences in $[1,\infty)$. Then $\ell^\infty(L^{p_k} : k\in\mathbb{N})$ is isomorphic to $\ell^\infty(L^{q_k} : k\in\mathbb{N})$.*

Proof. In the first step of this proof we fix $p \in [1,\infty)$ and a dense sequence $(q_k)_k$ in $[1,\infty)$ and show that L^p is isomorphic to a 2-complemented subspace of $\ell^\infty(L^{q_k} : k \in \mathbb{N})$. By the density of $(q_k)_k$ we may pick a subsequence $(q_{k_n})_n$ so that for each $n \in \mathbb{N}$ the identity map $I_n : L_n^{q_{k_n}} \to L_n^p$ is a 2-isomorphism. We then define $T : L^p \to \ell^\infty(L^{q_{k_n}} : n \in \mathbb{N})$ given as follows: if $f \in L^p$ can be written as $f = \sum_j \sum_{|I|=2^{-j}} a_I h_I$ then $Tf = (\sum_{j\le n} \sum_{|I|=2^{-j}} a_I h_I)_n$. Then clearly, T is a 2-isomorphic embedding of the image. We will now show that the image of T is 4-complemented in $\ell^\infty(L^{q_{k_n}} : n\in\mathbb{N})$. To see this, we define $Q\colon \ell^\infty(L^{q_{k_n}} : n \in \mathbb{N}) \to L^p$ by

$$Q\Big(\sum_{j=0}^\infty \sum_{|I|=2^{-j}} a_{n,I} h_I\Big)_n = \sum_{j=0}^\infty \sum_{|I|=2^{-j}} (\lim_{n\in\mathcal{U}} a_{n,I}) h_I,$$

where $\mathcal{U}$ is some fixed non-principal ultrafilter on $\mathbb{N}$. For each $m \in \mathbb{N}$, define $R_m \colon L^p \to L^p$ by

$$R_m\Big(\sum_{j=0}^{\infty}\sum_{|I|=2^{-j}} a_I h_I\Big) = \sum_{j=0}^{m}\sum_{|I|=2^{-j}} a_I h_I.$$

We now verify that Q is indeed well defined. Observe that for $(f_n)_n \in \ell^\infty(L^{q_{k_n}} : n \in \mathbb{N})$, where $f_n = \sum_{j=0}^{\infty}\sum_{|I|=2^{-j}} a_{n,I} h_I \in L^{q_{k_n}}$, $n \in \mathbb{N}$, we have

$$\begin{aligned}\big\|R_m Q((f_n)_n)\big\|_{L^p} &= \Big\|\sum_{j=0}^{m}\sum_{|I|=2^{-j}} (\lim_{n\in\mathcal{U}} a_{n,I}) h_I\Big\|_{L^p} = \lim_{n\in\mathcal{U}}\Big\|\sum_{j=0}^{m}\sum_{|I|=2^{-j}} a_{n,I} h_I\Big\|_{L^p}\\ &= \lim_{n\in\mathcal{U}} \|R_m f_n\|_{L^p} \le \lim_{n\in\mathcal{U}} \|R_n f_n\|_{L^p}\\ &\le 2\sup_n \|R_n f_n\|_{L^{q_{k_n}}} \le 2\|(f_n)\|_{\ell^\infty(L^{q_{k_n}}:n\in\mathbb{N})}.\end{aligned}$$

Evidently, $QTf = f$ for all $f \in L^p$, hence, the image of T is 4-complemented. Moreover, T can be extended to a 2-isomorphism $\tilde{T} : L^p \to \ell^\infty(L^{q_k} : k \in \mathbb{N})$ with 4-complemented image. This type of argument goes back to Johnson [**?**].

In the second step we show that if $(p_k)_k$, $(q_k)_k$ are as in the assumption then $\ell^\infty(L^{p_k} : k \in \mathbb{N})$ is 2-isomorphic to a 4-complemented subspace of $\ell^\infty(L^{q_k} : k \in \mathbb{N})$. Indeed, we may decompose $\mathbb{N}$ into infinite disjoint sets $(M_n)_n$ so that for all $n \in \mathbb{N}$ the sequence $(p_k)_{k\in M_n}$ is dense in $[1, \infty)$. By the first step, for each $n \in \mathbb{N}$ we can find a 2-embedding of L^{p_n} into $\ell^\infty(L^{q_k} : k \in M_n)$ with 4-complemented image. The second step then easily follows.

For the final step we fix a dense sequence $(q_k)_k$ in $[1, \infty)$ with the property that each term q_k is repeated infinitely many times. This implies that the space $X = \ell^\infty(L^{q_k} : k \in \mathbb{N})$ is isometrically isomorphic to $\ell^\infty(X)$, *i.e.*, it satisfies the *accordion property* (see [**?**, II.B.24]). To conclude, we show that for an arbitrary dense sequence $(q_k)_k$ the space $V = \ell^\infty(L^{p_k} : k \in \mathbb{N})$ is isomorphic to X. Indeed, by the second step we have that X is complemented in V and V is complemented in X. By the accordion property of X we deduce that X is isomorphic to V. □

Verification of **??**. Given an operator $T \colon Z \to Z$ with large diagonal, choose infinite sets $\Omega_k \subset \mathbb{N}$ so that $p_k = \lim_{j\in\Omega_k} p_j$, $k \in \mathbb{N}$. By **??** there exists an infinite set $\Gamma \subset \mathbb{N}$ for which $\Gamma \cap \Omega_k$ is infinite for each $k \in \mathbb{N}$ so that I_{Z_Γ} factors through T. Since $\{p_\gamma : \gamma \in \Gamma\}$ is again dense in $[1, \infty)$, we obtain by **??** that Z_Γ is isomorphic to Z. Hence, I_Z factors through T. □

An interesting special case of **??** is the following

Question 2. Assume that (p_k), $1 \le p_k < \infty$ either converges to 1 or diverges to ∞. Does the array $(h_{k,I})$ have the factorization property in $Z = \ell^\infty(L^{p_k} : k \in \mathbb{N})$?

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