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A short proof for stronger version of DS decomposition in set function optimization

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Abstract
Using a short proof, we show that every set function f can be decomposed into the difference of two monotone increasing and strictly submodular functions g and h, i.e., $f = g - h$, and every set function f can also be decomposed into the difference of two monotone increasing and strictly supermodular functions g and h, i.e., $f = g - h$.

Keywords Set function · DS decomposition · Monotone nondecreasing · Submodular · Supermodular

1 Introduction

Consider a real function f over subsets of a finite set X. f is said to be *submodular* if for any two subsets A and B of X,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

f is said to be *monotone nondecreasing* if for any two subsets A and B of X

$$A \subset B \Rightarrow f(A) \leq f(B).$$

In study of wireless sensor networks, social networks, cloud computing, data science, and machine learning, a lot of problems can be formulated into set function

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optimizations (Weili et al. 2019). The following theorem plays an important role in the set function optimization.

Theorem 1 (1st DS Decomposition) *Every set function $f : 2^X \to R$ can be decomposed into the difference of two monotone nondecreasing submodular functions g and h, i.e., $f = g - h$.*

This theorem is proved by Iyer and Bilmes (Iyer and Bilmes 2012). They first proved that every set function $f : 2^X \to R$ can be decomposed into the difference of two submodular functions g and h, i.e., $f = g - h$. (This weak version of DS deposition is first proposed by Narasimhan and Bilmes (Narasimhan and Bilmes 2005) with a nonrigorous proof.) Then they prove Theorem 1 by using a fact that every submodular function can be decomposed into the sum of a polymatroid function and a modular function, where a polymatroid function is a monotone nondecreasing submodular function with zero function value at empty set, and a set function $f : 2^X \to R$ is modular if for any two subsets A and B of X,

$$f(A) + f(B) = f(A \cup B) + f(A \cap B).$$

Note that a set function $f : 2^X \to R$ is *supermodular* if for any two subsets A and B of X,

$$f(A) + f(B) \leq f(A \cup B) + f(A \cap B).$$

Li, Du, and Pardalos (Xiang Li 2020) showed a variation of Theorem 1 as follows.

Theorem 2 (2nd DS Decomposition) *Every set function $f : 2^X \to R$ can be decomposed into the difference of two monotone nondecreasing supermodular functions g and h, i.e., $f = g - h$.*

In this paper, we will give a short proof for Theorems 1 and 2. Meanwhile, we will make them stronger by replacing monotone nondecreasing, submodular, and supermodular properties by monotone increasing, strictly submodular, and strictly supermodular properties, respectively, which are defined as follows.

A set function $f : 2^X \to R$ is *monotone increasing* if for any two subsets A and B of X

$$A \subset B \Rightarrow f(A) < f(B).$$

A set function $f : 2^X \to R$ is *strictly submodular* if for any two subsets A and B of X,

$$f(A) + f(B) > f(A \cup B) + f(A \cap B).$$

A set function $f : 2^X \to R$ is *strictly supermodular* if for any two subsets A and B of X,

$$f(A) + f(B) < f(A \cup B) + f(A \cap B).$$

2 Main results

Following is well-known fact about submodularity and monotone nondecreasing property (Ding-Zhu and Ko 2012).

Lemma 3 *For any subset A and element x, denote $\Delta_x f(A) = f(A \cup \{x\}) - f(A)$. Then following holds.*

(a) *A set function $f : 2^X \to R$ is submodular if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in X \setminus B$, $\Delta_x f(A) \geq \Delta_x f(B)$.*

(b) *A set function $f : 2^X \to R$ is monotone nondecreasing if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in B \setminus A$, $\Delta_x f(A) \leq \Delta_x f(B)$.*

(c) *A set function $f : 2^X \to R$ is supermodular if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in X \setminus B$, $\Delta_x f(A) \leq \Delta_x f(B)$.*

Similarly, it is easy to establish following.

Lemma 4 *For any subset A and element x, denote $\Delta_x f(A) = f(A \cup \{x\}) - f(A)$. Then following holds.*

(a) *A set function $f : 2^X \to R$ is strictly submodular if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in X \setminus B$, $\Delta_x f(A) > \Delta_x f(B)$.*

(b) *A set function $f : 2^X \to R$ is monotone increasing if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in B \setminus A$, $\Delta_x f(A) > \Delta_x f(B)$.*

(c) *A set function $f : 2^X \to R$ is strictly supermodular if and only if for any two subsets A and B with $A \subseteq B$, and for any $x \in X \setminus B$, $\Delta_x f(A) < \Delta_x f(B)$.*

Next, we present short proofs for stronger version of Theorems 1 and 2. They are obtained by modifying the original proof for the weak DS decomposition theorem in Iyer and Bilmes [2012].

Theorem 5 *Every set function $f : 2^X \to R$ can be decomposed into the difference of two monotone increasing and strictly submodular functions g and h, i.e., $f = g - h$.*

Proof Define $\zeta(f) = \min_{A \subset B \subseteq X, x \in X \setminus A}\{\Delta_x f(A) - \Delta_x f(B)\}$. By Lemma 4, $\zeta(f) > 0$ if and only if f is monotone increasing and strictly submodular. Consider set function $p(A) = \sqrt{|A|}$. Then $\zeta(p) > 0$ since

$$\begin{aligned}
\min_{A \subset B \subseteq X, x \in B \setminus A}\{\Delta_x p(A) - \Delta_x p(B)\} &= \min_{A \subset B \subseteq X, x \in B \setminus A}\{\Delta_x p(A)\} \\
&= \min_{A \subset X}\{\sqrt{|A|+1} - \sqrt{|A|}\} \\
&\geq \sqrt{|X|+1} - \sqrt{|X|} \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
\min_{A \subset B \subset X, x \in X \setminus B}\{\Delta_x p(A) - \Delta_x p(B)\} = &\min_{A \subset B \subset X}\{\Big(\sqrt{|A|+1} \\
&-\sqrt{|A|}\Big) - \Big(\sqrt{|B|+1} - \sqrt{|B|}\Big)\} \\
= &\min_{A \subset X}\{\Big(\sqrt{|A|+1} - \sqrt{|A|}\Big) \\
&- \Big(\sqrt{|A|+2} - \sqrt{|A|+1}\Big)\}
\end{aligned}$$

$$\geq 2\sqrt{|X|+1} - \sqrt{|X|} - \sqrt{|X|+2}$$
$$> 0. \quad (\text{note:}\sqrt{n}\text{is a strictly concave function.})$$

Now, we consider two cases.

Case 1 $\zeta(f) \geq 0$. Define set functions $g = f + p$ and $h = p$. Then $f = g - h$. Moreover, $\zeta(g) = \zeta(f) + \zeta(p) > 0$ and $\zeta(h) = \zeta(p) > 0$.

Case 2 $\zeta(f) < 0$. Define set functions $h = 2 \cdot \frac{-\zeta(f)}{\zeta(p)} \cdot p$ and $g = f + h$. Then $\zeta(h) = 2 \cdot \frac{-\zeta(f)}{\zeta(p)} \cdot \zeta(p) = -2\zeta(f) > 0$ and $\zeta(g) = \zeta(f) + \zeta(h) = -\zeta(f) > 0$. □

Corollary 6 *For any set function $f : 2^X \to R$, there exist two monotone increasing and strictly submodular set functions u and l such that $l \leq f \leq u$.*

Proof By Theorem 5, there exist two monotone increasing and strictly submodular functions g and h such that $f = g - h$. Define $u(A) = g(A) - h(\emptyset)$ and $l(A) = g(A) - h(X)$. Then u and l are monotone increasing and strictly submodular such that $l \leq f \leq u$. □

Theorem 7 *Every set function $f : 2^X \to R$ can be decomposed into the difference of two monotone increasing strictly supermodular functions g and h, i.e., $f = g - h$.*

Proof Define $\eta(f) = \min_{A \subset B \subset X, x \in X \setminus B}\{\Delta_x f(B) - \Delta_x f(A)\}$. By Lemma 4, $\eta(f) > 0$ if and only if f is strictly submodular. Define $\tau(f) = \min_{A \subset B \subseteq X}\{f(B) - f(A)\}$. Then $\tau(f) > 0$ if and only if f is monotone increasing.

Consider set function $q(A) = |A|^2$. Then

$$\begin{aligned}
\eta(q) &= \min_{A \subset B \subset X, x \in X \setminus B}\{\Delta_x q(B) - \Delta_x q(A)\} \\
&= \min_{A \subset B \subset X}\{((|B|+1)^2 - |B|^2) - ((|A|+1)^2 - |A|^2)\} \\
&= \min_{A \subset B \subset X}\{2(|B| - |A|)\} \\
&\geq 2 > 0
\end{aligned}$$

and

$$\begin{aligned}
\tau(q) &= \min_{A \subset B \subseteq X}\{q(B) - q(A)\} \\
&= \min_{A \subset B \subseteq X}(|B|^2 - |A|^2) \\
&\geq \min_{A \subset X}\{(|A|+1)^2 - |A|^2\} \\
&\geq 1 > 0.
\end{aligned}$$

Now, we consider two cases.

Case 1. $\min\{\eta(f), \tau(f)\} \geq 0$. Define set functions $g = f + q$ and $h = q$. Then $f = g - h$. Moreover, $\eta(g) = \eta(f) + \eta(q) > 0$, $\tau(g) = \tau(f) + \tau(q) > 0$, $\eta(h) = \eta(q) > 0$, and $\tau(h) = \tau(q) > 0$.

Case 2. $\min\{\eta(f), \tau(f)\} < 0$. Define set functions $h = 2 \cdot \frac{-\min\{\eta(f),\tau(f)\}}{\min\{\eta(q),\tau(q)\}} \cdot q$ and $g = f + h$. Then

$$\eta(h) = 2 \cdot \frac{-\min\{\eta(f), \tau(f)\}}{\min\{\eta(q), \tau(q)\}} \cdot \eta(q) \geq -2 \cdot \min\{\eta(f), \tau(f)\} > 0$$

and

$$\tau(h) = 2 \cdot \frac{-\min\{\eta(f), \tau(f)\}}{\min\{\eta(q), \tau(q)\}} \cdot \tau(q) \geq -2 \cdot \min\{\eta(f), \tau(f)\} > 0.$$

Therefore,

$$\eta(g) = \eta(f) + \eta(h) \geq \eta(f) - 2 \cdot \min\{\eta(f), \tau(f)\} \geq -\min\{\eta(f), \tau(f)\} > 0$$

and

$$\tau(g) = \tau(f) + \tau(h) \geq \tau(f) - 2 \cdot \min\{\eta(f), \tau(f)\} \geq -\min\{\eta(f), \tau(f)\} > 0.$$

□

3 Discussion

Although the proof of DS decomposition is constructive, it does not give an efficient method to find a DS decomposition. In fact, it is still open whether such an efficient method exists or not. It is very likely to be NP-hard for finding a DS decomposition for a given set function.

The first DS decomposition has applications in design of algorithms for a lot of problem (Iyer and Bilmes 2012; Narasimhan and Bilmes 2005) while the second DS decomposition is found to have application for active friending (Yang and Hung 2013; Shuyang et al. 2020; Yuan and Weili 2017). For a given set function, the DS decomposition is not unique. Different DS decompositions may induce algorithms with different performances. This gives researchers a lot of opportunity to make their contributions. In this paper, we provide a stronger version of DS decomposition. However, it is unknown if this stronger version could improve certain type of algorithms. Hopefully, Corollary 6 can be found to improve some switch methods (Wei et al 2015; Zhang and Dinh 2013; Guo and Weili 2019).

It is worth mentioning that this work may be extended to functions on lattice points (Shuyang et al 2020; Lei et al 2019), which is our potential work in the future.

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