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Two dependent probabilistic chip-collecting games

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ABSTRACT

Alice and Bob take turns to collect chips in the following manner. In each turn, Alice tosses a fair coin, which decides whether she collects a or b chips, where a and b are positive integers. If Alice collects a chips, then Bob collects b chips, and vice versa. We consider two variants of game play that have different rules in determining the winner. Namely, the winner of Game 1 is the first player to collect at least n chips, while the winner of Game 2 is the first player to collect a positive number of chips congruent to 0 modulo n. We fully determine the formula for the winning probabilities of each player in Game 1, and determine the best and worst case scenarios in terms of winning probabilities in Game 2.

1. Introduction

In a paper from 2018, Wong and Xu [11] investigated a two player probabilistic game that is played according to the following rules: Alice and Bob take turns to toss a coin, which decides independently whether they collect a or b chips in that turn, and the first player who accumulates at least n chips is the winner. In the game investigated by Wong and Xu, it was assumed that a and b are positive integers. This game has since been extended by Leung and Thanatipanonda [6] to the cases when $(a, b) = (-1, 1)$ and $(-1, 2)$. The games studied in both papers share one important property, namely that the number of chips collected by Alice and Bob in each turn is independent. Thus, we will refer to these games as the "independent games" throughout this paper.

Let a, b, and n be positive integers. We consider two variations of the independent games, which we shall call the "dependent game" and the "modulo dependent game". In both variations, Alice tosses a fair coin in each turn, which decides whether she collects a or b chips. If Alice collects a chips, then Bob collects b chips; if Alice collects b chips, then Bob collects a chips. Let x and y be the number of chips accumulated by Alice and Bob, respectively. Since Alice collects her chips first on each turn, in the *dependent game*, Alice wins any time $x \geq n$ and Bob wins when $y \geq n$ while $x < n$. In the *modulo dependent game*, Alice wins any time $x \equiv 0 \pmod{n}$ after the first turn and Bob wins when $y \equiv 0 \pmod{n}$ and $x \not\equiv 0 \pmod{n}$ after the first turn. In other words, in the modulo dependent game, the number of chips of a player will reset if they overshoot their goal. In both variations, the game ends once a winner is decided.

Notice that if $a = b$, then Alice is always the winner in both games. Hence, without loss of generality, we let $a < b$. Further note that in the dependent game, Alice is always the winner if $n \leq a$, and the winning probability of both Alice

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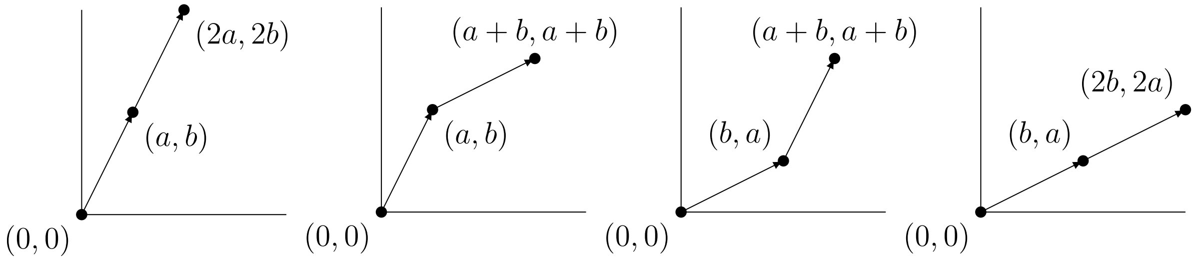

Fig. 1. All possible first two moves when $2b < n$.

and Bob is $\frac{1}{2}$ if $a < n \leq b$. As for the modulo dependent game, Alice is always the winner if $a \equiv b \equiv 0 \pmod{n}$, and the winning probability of both Alice and Bob is $\frac{1}{2}$ if exactly one of $a \equiv 0 \pmod{n}$ and $b \equiv 0 \pmod{n}$ holds. Therefore, to make our games nontrivial, we let $a < b < n$ throughout the rest of this paper.

Our games can be thought of as a random walk on a square grid, where the number of chips accumulated by Alice and Bob is recorded by the position (x, y) on the grid, and each move is represented by either $(+a, +b)$ or $(+b, +a)$. Such walks are often referred to as *generalized knight moves* in the literature. For example, Chia and Ong [3] studied generalized knight tours on rectangular grids and Watkins and Hoenigman [10] considered knight tours on a torus. The interested reader is directed to [2,4,5,7] for other examples of random walks and knight tours on various surfaces.

There are two key differences between our random walks and what is studied in the papers mentioned in the previous paragraph. First, traditional random walks allow movement in all directions: $(\pm a, \pm b)$ and $(\pm b, \pm a)$. Due to the nature of our chip-collecting games, we have to restrict our generalized knight moves to $(+a, +b)$ and $(+b, +a)$. Another key difference is our introduction of absorbing points on our surfaces. An absorbing point in a random walk is a point on the surface that cannot be left once entered. Random walks with absorbing points are commonly studied in the literature [1,9]. In this article we refer to the set of absorbing points as *winning regions*. For our games, all random walks start from the position $(0, 0)$. We define $\{(x, y) : x \geq n\}$ and $\{(0, k) : k \in [0, n-1]\}$ as *Alice's winning region* for the dependent game and for the modulo dependent game, respectively. We similarly define $\{(x, y) : x < n \leq y\}$ and $\{(k, 0) : k \in [1, n-1]\}$ as *Bob's winning region* for the dependent game and for the modulo dependent game, respectively. Thus, Alice wins if the random walk, after leaving its starting position $(0, 0)$, lands in Alice's winning region before landing in Bob's winning region, and vice versa (see Fig. 1).

The main focus of this paper is to study the winning probabilities of Alice and Bob in the dependent game and the modulo dependent game. For each of these games, we determine the best and worst case scenarios for each player in terms of their winning probabilities. In fact, for the dependent game, we completely determine the winning probabilities of the two players for each $a < b < n$.

In addition to studying the dependent game and the modulo dependent game, we also revisit the independent game presented by Leung and Thanatipanonda. In their paper, the authors presented a theorem that they proved using a computer algebra system, and they pondered the existence of a combinatorial proof. In Section 4, we present such a proof.

2. Dependent games

To study the winning probabilities of Alice and Bob in the dependent game, we shall first understand the positions on the square grid after m turns. Observe that the position after m turns is given by $(m-i)(a, b) + i(b, a)$ for some $0 \leq i \leq m$. Hence, we define

$$\mathbf{p}_{m,i} = (x_{m,i}, y_{m,i}) = ((m-i)a + ib, (m-i)b + ia).$$

Note that $\mathbf{p}_{m,i}$ lies on the straight line $x + y = m(a+b)$ for all $0 \leq i \leq m$. Furthermore, observe that the number of paths on the square grids to reach position $\mathbf{p}_{m,i}$ is precisely $\binom{m}{i}$, since each path is determined by the order of $m - i$ moves $(+a, +b)$ and i moves $(+b, +a)$. As a result, the probability that the position of the random walk is $\mathbf{p}_{m,i}$ after m turns is $\frac{1}{2^m}\binom{m}{i}$.

We now use these observations to find the winning probability of Bob in the dependent game, as presented in the following theorem. The winning probability of Alice can subsequently be found by subtracting the winning probability of Bob from 1.

Theorem 2.1. *Given positive integers $a < b < n$, the winning probability of Bob in the dependent game is given by*

$$\begin{cases} \dfrac{1}{2^M}\displaystyle\sum_{i=0}^{\lceil \frac{M}{2}\rceil -1}\binom{M}{i} & \text{if } n = \dfrac{M}{2}(a+b), \\ \dfrac{1}{2^M}\displaystyle\sum_{i=0}^{j}\binom{M}{i} & \text{if } \dfrac{M}{2}(a+b) < n \le (M-j)a + (j+1)b, \text{ and} \\ \dfrac{1}{2^{M+1}}\displaystyle\sum_{i=0}^{\widetilde{j}}\binom{M+1}{i} & \text{if } (M-j)a+(j+1)b < n, \end{cases}$$

where $M = \left\lfloor \frac{2n}{a+b} \right\rfloor$, $j = \left\lfloor \frac{Mb-n}{b-a} \right\rfloor$, and $\widetilde{j} = \left\lceil \frac{n-b-Ma}{b-a} \right\rceil$.

Proof. Recall from Section 1 that Bob wins the dependent game if and only if the random walk given by moves $(+a, +b)$ and $(+b, +a)$ lands in the region $\{(x, y) : x < n \le y\}$. Hence, the location of (n, n) relative to the set of positions of the random walk is quintessential. The unique integer M such that (n, n) lies in the region $\{(x, y) : M(a+b) \le x + y < (M+1)(a+b)\}$ is $M = \left\lfloor \frac{2n}{a+b} \right\rfloor$. Therefore, to find the winning probability of Bob, we will focus on the location of (n, n) relative to the sets of positions $\{\mathbf{p}_{M,i} : 0 \le i \le M\}$ and $\{\mathbf{p}_{M+1,i} : 0 \le i \le M+1\}$.

The point (n, n) lies on the line $x + y = M(a+b)$ if and only if $n = \frac{M}{2}(a+b)$. In this case, the position $\mathbf{p}_{M,i}$ is in Bob's winning region $\{(x, y) : x < n \le y\}$ if and only if

$$(M-i)a + ib < \frac{M}{2}(a+b) \le (M-i)b + ia,$$

which is equivalent to $\left(\frac{M}{2} - i\right)(b-a) > 0$, or $i < \frac{M}{2}$. Therefore, the set of winning positions of Bob when $n = \frac{M}{2}(a+b)$ is $\left\{\mathbf{p}_{M,i} : 0 \le i < \frac{M}{2}\right\}$, and the winning probability of Bob is

$$\sum_{i=0}^{\lceil \frac{M}{2}\rceil -1} \mathbb{P}(\text{The random walk lands on the position } \mathbf{p}_{M,i}) = \sum_{i=0}^{\lceil \frac{M}{2}\rceil -1} \frac{1}{2^M}\binom{M}{i}.$$

If the point (n, n) lies in the region $\{(x, y) : M(a+b) < x + y < (M+1)(a+b)\}$, then let j be the largest integer such that $y_{M,j} \ge n$. In other words,

$$y_{M,j+1} = (M-j-1)b + (j+1)a < n \le (M-j)b + ja = y_{M,j}, \tag{1}$$

which is equivalent to $Mb - (j+1)(b-a) < n \le Mb - j(b-a)$, or $j \le \frac{Mb-n}{b-a} < j+1$. Hence, $j = \left\lfloor \frac{Mb-n}{b-a} \right\rfloor$. If the random walk lands on the position $\mathbf{p}_{M,j+1}$, then Alice is guaranteed to win if and only if the smallest possible x-coordinate of the next position, i.e., $x_{M,j+1} + a = (M-j-1)a + (j+1)b + a$, is at least n (see Fig. 2). This inequality is equivalent to $n \le (M-j)a + (j+1)b$. When this inequality holds, Alice is guaranteed to win from the position $\mathbf{p}_{M,i}$ for all $i \ge j+1$. Therefore, the set of winning positions of Bob is $\{\mathbf{p}_{M,i} : 0 \le i \le j\}$, and the probability that Bob wins is

$$\sum_{i=0}^{j} \mathbb{P}(\text{The random walk lands on the position } \mathbf{p}_{M,i}) = \sum_{i=0}^{j} \frac{1}{2^M}\binom{M}{i}.$$

When n satisfies the inequality $(M-j)a + (j+1)b < n$ instead, some positions $\mathbf{p}_{M,i}$ with $i \ge j+1$ no longer guarantee Alice to win. Hence, we will switch our focus to the set of positions $\{\mathbf{p}_{M+1,i} : 0 \le i \le M+1\}$ instead. Let $\widetilde{j}$ be the largest integer such that $x_{M+1,\widetilde{j}} < n$. In other words,

$$x_{M+1,\widetilde{j}} = (M+1-\widetilde{j})a + \widetilde{j}b < n \le (M+1-\widetilde{j}-1)a + (\widetilde{j}+1)b = x_{M+1,\widetilde{j}+1},$$

which is equivalent to $(M+1)a + \widetilde{j}(b-a) < n \le (M+1)a + (\widetilde{j}+1)(b-a)$, or $\widetilde{j} < \frac{n-(M+1)a}{b-a} \le \widetilde{j}+1$. Hence, $\widetilde{j} = \left\lceil \frac{n-(M+1)a}{b-a} \right\rceil - 1 = \left\lceil \frac{n-b-Ma}{b-a} \right\rceil$. If the random walk lands on the position $\mathbf{p}_{M+1,\widetilde{j}+1}$, then Alice wins if the largest possible y-coordinate of the previous position, i.e., $y_{M+1,\widetilde{j}+1} - a = (M-\widetilde{j})b + (\widetilde{j}+1)a - a$, is less than n. This inequality turns out to be true, as proved in the following.

Recall that we are under the assumption $x_{M+1,j+1} = (M-j)a + (j+1)b < n$. Hence, by the definition of $\widetilde{j}$, we have $x_{M+1,j+1} \le x_{M+1,\widetilde{j}}$, which implies $y_{M+1,\widetilde{j}} \le y_{M+1,j+1}$. Therefore, $y_{M+1,\widetilde{j}+1} - a = y_{M+1,\widetilde{j}} - b \le y_{M+1,j+1} - b = y_{M,j+1} < n$, where the last inequality due to inequality (1) (also see Fig. 3).

In conclusion, Alice wins from the position $\mathbf{p}_{M+1,i}$ for all $i \ge \widetilde{j}+1$. As a result, the set of winning positions of Bob is $\{\mathbf{p}_{M+1,i} : 0 \le i \le \widetilde{j}\}$, and the probability that Bob wins is

$$\sum_{i=0}^{\widetilde{j}} \mathbb{P}(\text{The random walk lands on the position } \mathbf{p}_{M+1,i}) = \sum_{i=0}^{\widetilde{j}} \frac{1}{2^{M+1}}\binom{M+1}{i}. \quad \square$$

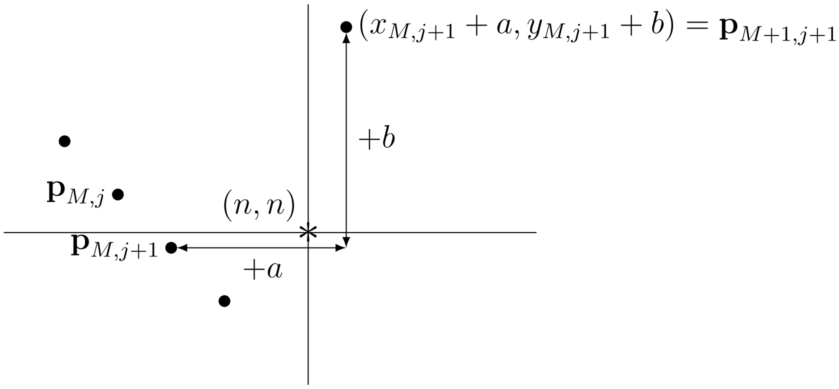

Fig. 2. $n \leq (M-j)a + (j+1)b$.

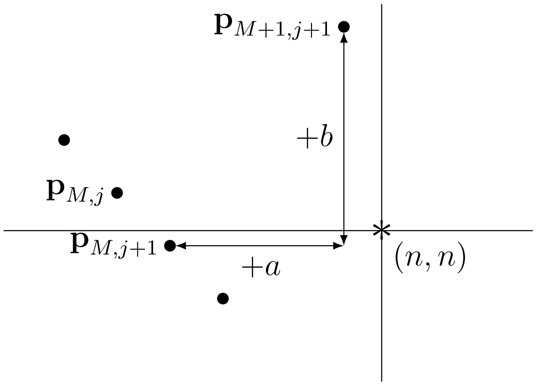

Fig. 3. $(M-j)a + (j+1)b < n$.

Since Alice always collects her chips first on each turn, it is obvious that the winning probability of Bob will never exceed $\frac{1}{2}$. Consequently, the following corollary establishes the best case scenario for Bob in terms of winning probability.

Corollary 2.2. *Let r be the smallest nonnegative integer such that $n \equiv r \pmod{a+b}$. The winning probability of Bob is $\frac{1}{2}$ if and only if $a < r \leq b$.*

Proof. We begin by writing $n = q(a+b)+r$, where q is an integer. Since $0 \leq r < a+b$,

$$M = \left\lfloor \frac{2n}{a+b} \right\rfloor = \left\lfloor \frac{2(q(a+b)+r)}{a+b} \right\rfloor = 2q + \left\lfloor \frac{2r}{a+b} \right\rfloor = \begin{cases} 2q & \text{if } 2r < a+b, \\ 2q+1 & \text{if } 2r \geq a+b. \end{cases}$$

In view of this, let

$$\varepsilon = \begin{cases} 0 & \text{if } 2r < a+b, \\ 1 & \text{if } 2r \geq a+b. \end{cases}$$

Then

$$j = \left\lfloor \frac{Mb-n}{b-a} \right\rfloor = \left\lfloor \frac{(2q+\varepsilon)b - (q(a+b)+r)}{b-a} \right\rfloor = \left\lfloor \frac{q(b-a)+\varepsilon b - r}{b-a} \right\rfloor = q + \left\lfloor \frac{\varepsilon b - r}{b-a} \right\rfloor,$$

so

$$j = \begin{cases} q - \left\lceil \dfrac{r}{b-a} \right\rceil & \text{if } 2r < a+b, \\ q + \left\lfloor \dfrac{b-r}{b-a} \right\rfloor & \text{if } 2r \geq a+b. \end{cases}$$

Similarly, we find that

$$\tilde{j} = \left\lceil \frac{(q(a+b)+r) - b - (2q+\varepsilon)a}{b-a} \right\rceil = \begin{cases} q - \left\lfloor \dfrac{b-r}{b-a} \right\rfloor & \text{if } 2r < a+b, \\ q - \left\lfloor \dfrac{b+a-r}{b-a} \right\rfloor & \text{if } 2r \geq a+b. \end{cases}$$

If $a < r \leq b$, then when $2r < a+b$, we have $M = 2q$ and $\tilde{j} = q = \frac{M}{2} = \left\lfloor \frac{M+1}{2} \right\rfloor$. Furthermore,

$$\begin{aligned}(M-j)a + (j+1)b &= (2q-j)a + (j+1)b \\ &= 2qa + j(b-a) + b \\ &\leq 2qa + (q-1)(b-a) + b \\ &= q(a+b) + a < q(a+b) + r = n,\end{aligned}$$

where the first inequality holds since $a < b$ and $j = q - \left\lceil \frac{r}{b-a} \right\rceil \leq q-1$. Therefore, by Theorem 2.1, the winning probability of Bob is

$$\frac{1}{2^{M+1}} \sum_{i=0}^{\tilde{j}} \binom{M+1}{i} = \frac{1}{2^{M+1}} \sum_{i=0}^{\left\lfloor \frac{M+1}{2} \right\rfloor} \binom{M+1}{i} = \frac{1}{2}.$$

When $2r \geq a+b$, we have $M = 2q+1$, $j = q = \left\lfloor \frac{M}{2} \right\rfloor$, and

$$(M-j)a + (j+1)b = (2q+1-q)a + (q+1)b = q(a+b) + a + b > q(a+b) + r = n.$$

Therefore, by Theorem 2.1, the winning probability of Bob is

$$\frac{1}{2^M} \sum_{i=0}^{\left\lceil \frac{M}{2} \right\rceil - 1} \binom{M}{i} \quad \text{or} \quad \frac{1}{2^M} \sum_{i=0}^{j} \binom{M}{i} = \frac{1}{2^M} \sum_{i=0}^{\left\lfloor \frac{M}{2} \right\rfloor} \binom{M}{i},$$

which are both equal to $\frac{1}{2}$.

Conversely, if the winning probability of Bob is $\frac{1}{2}$, then when $2r < a+b$, we have $M = 2q$ is an even number. This implies $\frac{1}{2^M} \sum_{i=0}^{j} \binom{M}{i}$ is never equal to $\frac{1}{2}$ regardless of the value of j. Hence, from Theorem 2.1, we deduce that $(M-j)a + (j+1)b < n$, and in order for $\frac{1}{2^{M+1}} \sum_{i=0}^{\tilde{j}} \binom{M+1}{i}$ to be $\frac{1}{2}$, we have $\tilde{j} = \left\lfloor \frac{M+1}{2} \right\rfloor = \left\lfloor \frac{2q+1}{2} \right\rfloor = q$. As a result, $\left\lfloor \frac{b-r}{b-a} \right\rfloor = 0$, which implies $0 \leq b-r < b-a$, or $a < r \leq b$.

When $2r \geq a+b$, we have $M = 2q+1$ is an odd number. This implies $\frac{1}{2^{M+1}} \sum_{i=0}^{\tilde{j}} \binom{M+1}{i}$ is never equal to $\frac{1}{2}$ regardless of the value of $\tilde{j}$. Hence, from Theorem 2.1 again, we deduce that $n \leq (M-j)a + (j+1)b$, and either $n = \frac{M}{2}(a+b)$ or $j = \left\lfloor \frac{M}{2} \right\rfloor$. If $n = \frac{M}{2}(a+b)$, then $q(a+b) + r = \frac{2q+1}{2}(a+b)$. In other words, $r = \frac{a+b}{2}$, which implies $a < r < b$. If $j = \left\lfloor \frac{M}{2} \right\rfloor = q$, then $\left\lfloor \frac{b-r}{b-a} \right\rfloor = 0$, which again implies $a < r \leq b$. □

Next, we present a corollary that gives that worst case scenario for Bob in terms of winning probability.

Corollary 2.3. *The winning probability of Bob is* 0 *if and only if* $Mb < n \leq (M+1)a$*, where* $M = \left\lfloor \frac{2n}{a+b} \right\rfloor$.

Proof. Let $j = \left\lfloor \frac{Mb-n}{b-a} \right\rfloor$ and $\tilde{j} = \left\lceil \frac{n-b-Ma}{b-a} \right\rceil$. To prove the "if" direction, we assume that $Mb < n \leq (M+1)a$. If $n = \frac{M}{2}(a+b)$, then

$$0 > Mb - n = Mb - \frac{M}{2}(a+b) = \frac{M}{2}(b-a),$$

which implies $M < 0$. As a result, $n < 0$, leading to a contradiction. Following from Theorem 2.1, the probability that Bob wins is either $\frac{1}{2^M} \sum_{i=0}^{j} \binom{M}{i}$ or $\frac{1}{2^{M+1}} \sum_{i=0}^{\tilde{j}} \binom{M+1}{i}$. Note that both sums are empty and equal to 0 since $j < 0$ and $\tilde{j} < 0$.

To prove the "only if" direction, we assume that the winning probability of Bob is 0. If $n = \frac{M}{2}(a+b)$, then in order for $\frac{1}{2^M} \sum_{i=0}^{\left\lceil \frac{M}{2} \right\rceil - 1} \binom{M}{i}$ to be 0, we must have $\left\lceil \frac{M}{2} \right\rceil - 1 < 0$. This happens if and only if $M \leq 0$, which implies $n \leq 0$, leading to a contradiction. If $\frac{M}{2}(a+b) < n \leq (M-j)a + (j+1)b$, then in order for $\frac{1}{2^M} \sum_{i=0}^{j} \binom{M}{i}$ to be 0, we must have $j < 0$. In this case,

$$\tilde{j} = \left\lceil \frac{n-b-Ma}{b-a} \right\rceil \leq \left\lceil \frac{(M-j)a + (j+1)b - b - Ma}{b-a} \right\rceil = j < 0.$$

If $(M-j)a+(j+1)b<n$, then in order for $\frac{1}{2^{M+1}}\sum_{i=0}^{\widetilde{j}}\binom{M+1}{i}$ to be 0, we must have $\widetilde{j}<0$. In this case,

$$0>\widetilde{j}=\left\lceil\frac{n-b-Ma}{b-a}\right\rceil\geq\left\lceil\frac{(M-j)a+(j+1)b-b-Ma}{b-a}\right\rceil=j.$$

Therefore, the winning probability of Bob is 0 implies that $j<0$ and $\widetilde{j}<0$. Equivalently, $\frac{Mb-n}{b-a}<0$ and $\frac{n-b-Ma}{b-a}\leq-1$. Since $b-a>0$, combining these two inequalities gives $Mb<n\leq b+Ma-(b-a)=(M+1)a$. □

3. Modulo dependent game

Now, we will focus our attention on the modulo dependent game. Recall that in this game, all positions are considered modulo n. Hence, in this section, we view the positions of the game as elements of $\mathbb{Z}_n\times\mathbb{Z}_n$, where $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}/n\mathbb{Z}$. Also recall that the sets $\{(0,k):k\in[0,n-1]\}$ and $\{(k,0):k\in[1,n-1]\}$ are Alice's and Bob's winning region, respectively. For this game, a random walk terminates when it lands on one of the winning regions for the first time after the first move. Similar to the dependent game, we observe that the position after m turns is given by $(m-i)(a,b)+i(b,a)$ in $\mathbb{Z}_n\times\mathbb{Z}_n$ for some $0\leq i\leq m$. Without loss of generality, we assume throughout this section that $\gcd(a,b,n)=1$.

Unlike the dependent game, there exist modulo dependent games with walks that never land in a winning region. For instance, when $a=1$, $b=2$, and $n=6$, the walk

$$(0,0)\to(1,2)\to(2,4)\to(4,5)\to(5,1)\to(1,2)\to(2,4)\to\cdots$$

forms an infinite loop so that a winner is never decided. However, the following lemma implies the probability that a random walk terminates after finitely many moves is 1.

Lemma 3.1. *In every modulo dependent game, the sum of the winning probabilities of Alice and Bob is* 1.

Proof. At any point in the game, the position of the game is given by $(ia+jb,ja+ib)$ for some positive integers i and j. If a winner has not yet been decided, then the probability that a winner is decided within the next $2n$ moves is at least $p=\frac{1}{2^{2n}}$. This is because any path from $(ia+jb,ja+ib)$ that consists of $n-i$ moves of $(+a,+b)$ and $n-j$ moves of $(+b,+a)$ must land on a winning region, since this path ends at $(0,0)$.

For each positive integer k, let E_k be the event that a winner is not determined on the mth turn for any $1\leq m\leq 2nk$. Then for $k\geq 2$, $\mathbb{P}(E_k|E_{k-1})\leq 1-p$. Therefore, the probability of a winner being decided after finitely many moves is

$$\begin{aligned}&1-\mathbb{P}\left(\bigcap_{k=1}^{\infty}E_k\right)\\&=1-\mathbb{P}(E_1)\cdot\prod_{k=2}^{\infty}\mathbb{P}(E_k|E_{k-1})\\&\geq 1-\mathbb{P}(E_1)\cdot\prod_{k=2}^{\infty}(1-p)\\&=1.\quad\square\end{aligned}$$

From Lemma 3.1, since the sum of the winning probabilities of Alice and Bob is 1, determining the winning probability of one player yields the winning probability of the other. The following theorem extends this idea by showing that the winning probability of each player is completely determined by the probability that a random walk terminates at the position $(0,0)$.

Theorem 3.2. *In a modulo dependent game, let q be the probability that a random walk terminates at the position* $(0,0)$. *Then the winning probability of Alice is $\frac{1}{2}(1+q)$ and the winning probability of Bob is $\frac{1}{2}(1-q)$.*

Proof. Recall that Alice's and Bob's winning regions are $\{(0,k):k\in[0,n-1]\}$ and $\{(k,0):k\in[1,n-1]\}$, respectively. Moreover, since the set of moves $\{(+a,+b),(+b,+a)\}$ and $\mathbb{Z}_n\times\mathbb{Z}_n$ are both symmetric about the diagonal $y=x$, we have

$$\mathbb{P}(\text{A random walk terminates at }(0,k))=\mathbb{P}(\text{A random walk terminates at }(k,0)).$$

Thus,

$$\begin{aligned}\mathbb{P}(\text{Alice wins})&=\sum_{k=0}^{n-1}\mathbb{P}(\text{A random walk terminates at }(0,k))\\&=q+\sum_{k=1}^{n-1}\mathbb{P}(\text{A random walk terminates at }(0,k))\end{aligned}$$

$$= q + \sum_{k=1}^{n-1} \mathbb{P}(\text{A random walk terminates at } (k, 0))$$
$$= q + \mathbb{P}(\text{Bob wins}).$$

Combining with $\mathbb{P}(\text{Alice wins}) + \mathbb{P}(\text{Bob wins}) = 1$ from Lemma 3.1 yields our desired results. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *The winning probability of Bob in every modulo dependent game is less than or equal to $\frac{1}{2}$. Further, the winning probability of Bob is $\frac{1}{2}$ if and only if no random walk terminates at the position $(0, 0)$.*

A position $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is called *reachable* if there exists a random walk that lands on (x, y) after leaving the starting position $(0, 0)$ until terminating. In view of Corollary 3.3, to identify when Bob has a fair chance of winning, we wish to determine in which modulo dependent games that $(0, 0)$ is not reachable. We will first establish some sufficient conditions on a, b, and n under which $(0, 0)$ is not reachable.

Theorem 3.4. *Let a, b, and n be such that $2a \equiv 2b \pmod{n}$ and $a + b$ is odd. Then the position $(0, 0)$ is not reachable.*

Proof. Since $a < b < n$ and $2a \equiv 2b \pmod{n}$, we deduce that $2b = n + 2a$, which implies that n is even and $b = \frac{n}{2} + a$. Moreover, since $2(a, b) = 2(b, a)$ in $\mathbb{Z}_n \times \mathbb{Z}_n$, the position after m turns is given by either $(m-1)(a, b) + (b, a)$ or $m(a, b)$, where m is a positive integer.

Note that $(m-1)(a, b) + (b, a)$ is never equal to $(0, 0)$. This is because $a + b$ is odd, so exactly one of a and b is even, which forces at least one coordinate in $(m-1)(a, b) + (b, a) = ((m-2)a + (a + b), (m-2)b + (a + b))$ to be odd. Since $\gcd(a, b, n) = 1$, the only way for $m(a, b)$ to be $(0, 0)$ is when m is a positive multiple of n.

Suppose that there is a random walk that terminates at $(0, 0)$ on the mth turn. Then $m \geq n$. Consider the $\frac{n}{2}$-th turn of this walk. Substituting $b = \frac{n}{2} + a$, the position after $\frac{n}{2}$ turns is either

$$\left(\frac{n}{2} - 1\right)\left(a, \frac{n}{2} + a\right) + \left(\frac{n}{2} + a, a\right) = \left(\frac{n}{2}(a+1), \frac{n}{2}\left(\frac{n}{2} - 1 + a\right)\right) \tag{2}$$

or

$$\frac{n}{2}\left(a, \frac{n}{2} + a\right) = \left(\frac{n}{2}a, \frac{n}{2}\left(\frac{n}{2} + a\right)\right). \tag{3}$$

Note that $\frac{n}{2}$ is odd since $b = \frac{n}{2} + a$ and a and b have opposite parity. Consequently, if a is odd, then the first coordinate in Eq. (2) and the second coordinate in Eq. (3) are multiples of n; if a is even, then the second coordinate in Eq. (2) and the first coordinate in Eq. (3) are multiples of n. This contradicts the assumption that the random walk does not terminate until the mth turn. As a result, no random walk terminates at $(0, 0)$. □

Remarkably, with the exception of $a = 1$, $b = 2$, and $n = 4$, the converse of Theorem 3.4 also holds. To prove this claim, we first expand our investigation of reachable positions to general $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$. It is important to note that not all positions in $\mathbb{Z}_n \times \mathbb{Z}_n$ are reachable. For example, for a random walk to land on (a, a), it must first land on $(0, a-b)$ or $(a-b, 0)$. Thus, the walk will have terminated before reaching (a, a), which makes (a, a) not reachable. A similar observation shows that (b, b) is also not reachable.

The following lemma allows us to search for reachable positions inductively.

Lemma 3.5. *Let $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(-a, -a), (-b, -b)\}$ such that (x, y) is a reachable position. If $x \neq 0$ and $y \neq 0$, then the position $(x + a + b, y + a + b)$ is reachable.*

Proof. If $x \not\equiv -a \pmod{n}$ and $y \not\equiv -b \pmod{n}$, then each position on the path

$$(x, y) \to (x + a, y + b) \to (x + a + b, y + a + b)$$

is reachable. Now suppose that $x \equiv -a \pmod{n}$. Since $(x, y) \neq (-a, -a)$, it follows that $y \not\equiv -a \pmod{n}$, and each position on the path

$$(x, y) \to (x + b, y + a) \to (x + a + b, y + a + b)$$

is reachable. Similarly, if $y \equiv -b \pmod{n}$, then $x \not\equiv -b \pmod{n}$ and each position on the path

$$(x, y) \to (x + b, y + a) \to (x + a + b, y + a + b)$$

is reachable. □

With this lemma, we can provide some sufficient conditions under which the position $(x, x) \notin \{(a, a), (b, b)\}$ is reachable.

Theorem 3.6. *Let a, b, and n be such that* $2a \not\equiv 2b \pmod{n}$*,* $a \not\equiv 2b \pmod{n}$*,* $2a \not\equiv b \pmod{n}$*, and* $a+b$ *is relatively prime to n. Then every position* $(x, x) \in \mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(a, a), (b, b)\}$ *is reachable.*

Proof. Let $\mathbf{p}_i = (i(a+b), i(a+b)) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Since $a+b$ is relatively prime to n, $\{\mathbf{p}_i : i \in [1, n]\} = \{(x, x) : x \in [0, n-1]\}$. Hence, it suffices to prove that $\mathbf{p}_i$ is reachable for all $1 \le i \le n$ except $\mathbf{p}_i \in \{(a, a), (b, b)\}$. Notice that $\mathbf{p}_1 = (a+b, a+b)$ is reachable via the walk

$$(0, 0) \to (a, b) \to (a+b, a+b).$$

We now proceed with a proof by induction on i.

Let $1 \le i < n$. If $\mathbf{p}_i$ is reachable and $\mathbf{p}_{i+1} \notin \{(a, a), (b, b)\}$, then

$$\mathbf{p}_i \notin \{(a, a) - (a+b, a+b), (b, b) - (a+b, a+b)\} = \{(-b, -b), (-a, -a)\}.$$

Therefore, by Lemma 3.5, $\mathbf{p}_{i+1}$ is reachable. It remains to tackle the case when $\mathbf{p}_{i+1} \in \{(a, a), (b, b)\}$. We will only show the proof for $\mathbf{p}_{i+1} = (a, a)$, since the proof for $\mathbf{p}_{i+1} = (b, b)$ is similar.

Assume that $\mathbf{p}_{i+1} = (a, a)$. Since $\mathbf{p}_i = (a - (a+b), a - (a+b)) = (-b, -b)$ is reachable, it can only be reached from positions $(-a-b, -2b)$ and $(-2b, -a-b)$. By the symmetry about the diagonal $y = x$, both $(-a-b, -2b)$ and $(-2b, -a-b)$ are reachable or are the starting position $(0, 0)$. Notice also that $\mathbf{p}_{i+2} = (b, b)$ if and only if $a + (a+b) \equiv b \pmod{n}$, which is equivalent to $2a \equiv 0 \pmod{n}$. Hence, if $\mathbf{p}_{i+2} = (b, b)$, then $2b \not\equiv 2a \equiv 0 \pmod{n}$, and $\mathbf{p}_{i+3} = (b + (a+b), b + (a+b)) = (a+2b, a+2b)$ is reachable via the walk

$$\begin{aligned}(-a-b, -2b) &\to (-a, a-2b) \to (-a+b, 2a-2b) = (-a+b, -2b) \to (b, -b) \to (2b, a-b)\\ &\to (a+2b, a) \to (2a+2b, a+b) = (2b, a+b) \to (a+2b, a+2b).\end{aligned}$$

This walk does not terminate before reaching $\mathbf{p}_{i+3}$ because $a \not\equiv 2b \pmod{n}$, $a \neq b$, $2b \not\equiv 0 \pmod{n}$, $-a \equiv a \not\equiv 2b \pmod{n}$, and $a+b$ is relatively prime to n.

Finally, if $\mathbf{p}_{i+2} \neq (b, b)$, then $\mathbf{p}_{i+2} = (a + (a+b), a + (a+b)) = (2a+b, 2a+b)$ is reachable via the walk

$$(-a-b, -2b) \to (-a, a-2b) \to (b-a, 2a-2b) \to (b, 2a-b) \to (a+b, 2a) \to (2a+b, 2a+b).$$

Once again, this walk does not terminate before reaching $\mathbf{p}_{i+2}$ because $a \not\equiv 2b \pmod{n}$, $a \neq b$, $2a \not\equiv 2b \pmod{n}$, $2a \not\equiv b \pmod{n}$, $a+b$ is relatively prime to n, and $2a \not\equiv 0 \pmod{n}$. The proof follows by induction on i. □

For the purpose of determining the winning probabilities of Alice and Bob, Theorem 3.6 provides sufficient conditions on a, b, and n that give Alice an advantage. More specifically, the theorem provides sufficient conditions under which $(0, 0)$ is a reachable position. In the following, we provide another sufficient condition.

Lemma 3.7. *Let a, b, and n be such that* $a+b$ *is not relatively prime to n. Then the position* $(0, 0)$ *is reachable.*

Proof. Let $d = \gcd(a+b, n) > 1$. If there exists $i \in [0, n-1]$ such that $(i+1)a + ib \equiv 0 \pmod{n}$, then d divides $(i+1)a + ib - i(a+b) = a$, which further implies that d divides $(a+b) - a = b$. This contradicts the assumption that $\gcd(a, b, n) = 1$. As a result, we have $(i+1)a + ib \not\equiv 0 \pmod{n}$. Similarly, $ia + (i+1)b \not\equiv 0 \pmod{n}$. In conclusion, for all $i \in [0, n-1]$, the position $((i+1)a + ib, ia + (i+1)b)$ is never in a winning region. Therefore, the position $(0, 0)$ is reachable via the walk

$$\begin{aligned}&(0, 0) \to (a, b) \to (a+b, a+b) \to (2a+b, a+2b) \to (2(a+b), 2(a+b)) \to \cdots\\ &\to ((i+1)a+ib, ia+(i+1)b) \to ((i+1)(a+b), (i+1)(a+b)) \to \cdots \to \left(\frac{n}{d}(a+b), \frac{n}{d}(a+b)\right) = (0, 0),\end{aligned}$$

which alternates the moves $(+a, +b)$ and $(+b, +a)$. □

Now, we are ready to state and prove the necessary and sufficient conditions under which the position $(0, 0)$ is not reachable.

Theorem 3.8. *The position* $(0, 0)$ *is not reachable if and only if one of the following holds:*

- $a = 1$, $b = 2$, *and* $n = 4$, *or*
- $2a \equiv 2b \pmod{n}$ *and* $a+b$ *is odd.*

Proof. If $a = 1$, $b = 2$, and $n = 4$, then we may easily see that $(0, 0)$ is not reachable by exhausting all reachable positions in this modulo dependent game. If $2a \equiv 2b \pmod{n}$ and $a+b$ is odd, then the conclusion that $(0, 0)$ is not reachable is provided by Theorem 3.4.

To prove the converse, let a, b, and n be such that the respective values of a, b, and n are not simultaneously 1, 2, and 4. If $2a \equiv 2b \pmod{n}$ and $a+b$ is even, then $2b = n + 2a$, so we deduce that n is also even. This implies that $a+b$ is not relatively prime to n, thus by Lemma 3.7, the position $(0, 0)$ is reachable. For the rest of the proof, we assume that

$2a \not\equiv 2b \pmod{n}$. Due to Lemma 3.7, we may further assume that $a+b$ is relatively prime to n, or else the position $(0, 0)$ is guaranteed to be reachable.

If $a \not\equiv 2b \pmod{n}$ and $2a \not\equiv b \pmod{n}$, then the position $(0, 0)$ is reachable by Theorem 3.6. Otherwise, we have either $a \equiv 2b \pmod{n}$ or $2a \equiv b \pmod{n}$. We will only show the proof for $2a \equiv b \pmod{n}$, since the proof for $a \equiv 2b \pmod{n}$ is similar.

Since $0 < a < b < n$, $2a \equiv b \pmod{n}$ implies $b = 2a$, so the two moves in the modulo dependent game are $(+a, +2a)$ and $(+2a, +a)$. Furthermore, since $a+b$ is relatively prime to n, we have $\gcd(3a, n) = 1$. If n is odd, then $(0, 0)$ is reachable via the walk

$$(0, 0) \to (a, 2a) \to (2a, 4a) \to \cdots \to (ia, 2ia) \to \cdots \to (na, 2na) = (0, 0).$$

This walk does not terminate before reaching $(0, 0)$ because a is relatively prime to n, $1 \le i < n$, and n is odd. If n is even, then $n \ge 6$; otherwise, $a < b < n$ and $\gcd(3a, n) = 1$ together imply $a = 1$, $b = 2$, and $n = 4$. With $n \ge 6$ established, the position $(0, 0)$ is reachable via the walk that starts from $(0, 0)$, takes the move $(+a, +2a)$ to $(a, 2a)$, followed by $n-3$ moves of $(+2a, +a)$ to $((2n-5)a, (n-1)a) = (-5a, -a)$, followed by the move $(+a, +2a)$ twice to $(-3a, 3a)$, followed by the move $(+2a, +a)$ twice to $(a, 5a)$, followed by $n-3$ moves of $(+a, +2a)$ to $((n-2)a, (2n-1)a)$, and finishes with the move $(+2a, +a)$ to $(0, 0)$. Symbolically, the walk is

$$
\begin{aligned}
&(0, 0) \xrightarrow{(+a,+2a)} (a, 2a)\\
&\underbrace{\to (3a, 3a) \to (5a, 4a) \to \cdots \to ((2i+1)a, (i+2)a) \to \cdots \to ((2n-5)a, (n-1)a)}_{n-3 \text{ times } (+2a,+a)}\\
&= (-5a, -a) \xrightarrow{(+a,+2a)} (-4a, a) \xrightarrow{(+a,+2a)} (-3a, 3a) \xrightarrow{(+2a,+a)} (-a, 4a) \xrightarrow{(+2a,+a)} (a, 5a)\\
&\underbrace{\to (2a, 7a) \to (3a, 9a) \to \cdots \to ((j+1)a, (2j+5)a) \to \cdots \to ((n-2)a, (2n-1)a)}_{n-3 \text{ times } (+a,+2a)}\\
&\xrightarrow{(+2a,+a)} (na, 2na) = (0, 0).
\end{aligned}
$$

This walk does not terminate before reaching $(0, 0)$ because a is relatively prime to n, $2i+1$ is odd, $3 \le i+2 \le n-1$, $n \ge 6$, $2 \le j+1 \le n-2$, and $2j+5$ is odd. This completes the proof of the converse. □

Combining Corollary 3.3 and Theorem 3.8, we have the following.

Corollary 3.9. *In a modulo dependent game, Alice and Bob have equal winning probabilities if and only if one of the following holds:*

- $a = 1$, $b = 2$, *and* $n = 4$, *or*
- $2a \equiv 2b \pmod{n}$ *and* $a + b$ *is odd.*

While the above corollary gives the best case scenario for Bob in terms of his winning probability, the following theorem and its corollary give his worst case scenario.

Theorem 3.10. *The position* $(0, 0)$ *is the only reachable position in the union of Alice's and Bob's winning regions, i.e., a random walk only terminates at the position* $(0, 0)$, *if and only if* $n \mid (b^2 - a^2)$.

Proof. Suppose $n \mid (b^2 - a^2)$. Then there exist integers k and k' such that $n = kk'$, $k \mid (b-a)$, and $k' \mid (b+a)$. Let ℓ be an integer such that $b = \ell k + a$. The position after m turns is given by

$$
\begin{aligned}
\mathbf{p}_{m,i} = (x_{m,i}, y_{m,i}) &= (m-i)(a, \ell k + a) + i(\ell k + a, a)\\
&= (ma + i\ell k, m\ell k + ma - i\ell k).
\end{aligned}
$$

Since $\gcd(a, b, n) = 1$, $b = \ell k + a$ implies that $\gcd(k, a) = 1$. As a result, if $x_{m,i}$ is a multiple of n, then $ma = t_1 n - i\ell k = k(t_1 k' - i\ell)$ for some integer t_1, which implies that m is a multiple of k; if $y_{m,i}$ is a multiple of n instead, then $ma = t_2 n - m\ell k + i\ell k = k(t_2 k' - m\ell + i\ell)$ for some integer t_2, which again implies that m is a multiple of k. Note that $x_{m,i} + y_{m,i} = m\ell k + 2ma = m(\ell k + 2a) = m(a+b)$. If either $x_{m,i}$ or $y_{m,i}$ is a multiple of n, then $x_{m,i} + y_{m,i}$ is also a multiple of n since $k \mid m$ and $k' \mid (a+b)$. Therefore, $x_{m,i} \equiv 0 \pmod{n}$ if and only if $y_{m,i} \equiv 0 \pmod{n}$. In other words, a random walk will only terminate at the position $(0, 0)$.

Suppose $n \nmid (b^2 - a^2)$ and a random walk only terminates at the position $(0, 0)$. Let m and i be positive integers such that $m \ge i$, $m \equiv b - a \pmod{n}$, and $i \equiv -a \pmod{n}$. Then $m - i \equiv b \pmod{n}$, and

$$((m-i)a + ib, (m-i)b + ia) = (0, b^2 - a^2) \ne (0, 0).$$

Consider the sequence of positions

$$(0, 0), (b, a), (2b, 2a), \ldots, (ib, ia), (a + ib, b + ia), (2a + ib, 2b + ia), \ldots, ((m-i)a + ib, (m-i)b + ia).$$

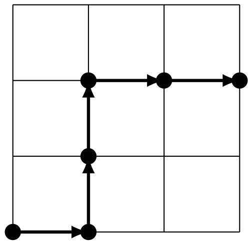

Fig. 4. The graphical representation of a player's moves in the independent game.

In this sequence, let $(\xi a + \zeta b, \xi b + \zeta a)$ be the last position that equals $(0, 0)$. Then the subsequence of positions from $(\xi a+\zeta b, \xi b+\zeta a)$ to $((m-i)a+ib, (m-i)b+ia)$ forms a random walk that starts from $(0, 0)$ and terminates at $(0, b^2-a^2)$, contradicting that $(0, 0)$ is the only position at which a random walk may terminate. □

Combining Theorem 3.2 and Theorem 3.10, we have the following.

Corollary 3.11. *In a modulo dependent game, the winning probability of Bob is* 0 *if and only if* $n \mid (b^2 - a^2)$.

4. Back to independent games

In this section, we come back to study an independent game established by Leung and Thanatipanonda. In this game, Alice and Bob independently toss a fair coin to decide whether they collect -1 or 1 chip in that turn, and the first player who accumulates at least n chips is the winner. Their paper defined the notation $q(n, k)$ to represent the probability that a player does not win on their kth move, i.e., they never accumulate at least n chips on or before their kth move. Leung and Thanatipanonda used Maple to find that

$$q(1, 2m) = \frac{\binom{2m}{m}}{2^{2m}}, \tag{4}$$

but they are interested in a combinatorial proof of this result. We note that Rényi provided a proof in 1970 [8]. However, Rényi's proof is by establishing a recurrence relation for $q(n, k)$, which is not combinatorial. In this section, we are going to provide an outline of a combinatorial proof by using André's reflection method, and further provide a new and detailed combinatorial proof by using a different path transformation.

As a player may collect -1 or 1 chip in each turn, there are 2^{2m} scenarios after a player has made $2m$ moves, which explains the denominator in the formula given by Eq. (4). To explain the numerator combinatorially, we establish a new graphical representation of the game, which is different from our standard notation in the rest of the paper. The player starts from the position $(0, 0)$, and the moves -1 and $+1$ are represented by the rightward step $(+1, +0)$ and the upward step $(+0, +1)$, respectively. For example, if the player's first 5 moves are $-1, +1, +1, -1, -1$ in that sequence, then the graphical representation is given by the lattice path as shown in Fig. 4.

In this graphical representation, the position (x, y) records the following information: x and y represent the cumulative number of -1 and $+1$ moves of the player, respectively; $x + y$ represents the total number of moves of the player; and $y - x$ represents the number of chips the player accumulates. Since we are interested in the value of $q(1, 2m)$, the player never accumulates at least 1 chip on or before their $2m$-th move. Hence, if we consider the graphical representation of the player's first $2m$ moves, the lattice path will never go above the straight line $y = x$, and it will end on the line segment $\{(x, y) : x + y = 2m \text{ and } m \le x \le 2m\}$. Define X as the set of all such lattice paths (see Fig. 5). It remains to prove that the cardinality of X is $\binom{2m}{m}$.

During the reviewing process of this paper, we were made aware by a referee of a neat proof. This proof involves recurrent application of André's reflection method, a technique commonly used in proving Bertrand's ballot theorem and counting the number of Dyck paths. In particular, for all integers $x' \ge y'$, André's reflection method establishes a bijection between the set of lattice paths from $(0, 0)$ to (x', y') that touches the line $y = x + 1$ and the set of lattice paths from $(-1, 1)$ to (x', y'). Hence, the number of lattice paths from $(0, 0)$ to (x', y') that never go above the straight line $y = x$ is $\binom{x'+y'}{x'} - \binom{x'+y'}{x'+1}$. The set of integers

$$\left\{ \binom{x'+y'}{x'} - \binom{x'+y'}{x'+1} : x' + y' = 2m,\ m \le x' \le 2m \right\}$$

forms a telescoping sum and yields $|X| = \binom{2m}{m}$. Nevertheless, we would like to present a different bijective proof, which involves a new path transformation technique.

Let Y be the set of lattice paths of length $2m$ starting from $(0, 0)$ and ending at (m, m), which travels only in the upward or rightward directions (see Fig. 6). It is well known that the cardinality of Y is $\binom{2m}{m}$. In fact, we may bject Y to a set of

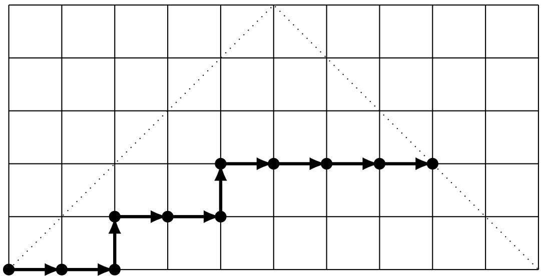

Fig. 5. A path of length 10 in X.

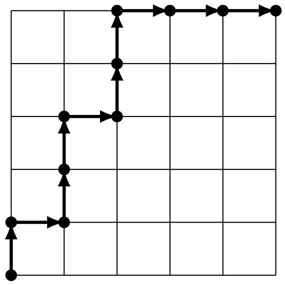

Fig. 6. A path of length 10 in Y.

"commands", where each command is a sequence of length $2m$ that contains m "upward" and m "rightward" steps. We are going to finish our combinatorial proof of the formula in Eq. (4) by constructing a bijection from Y and X.

Let $(0,0)=(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{2m},y_{2m})=(m,m)$ be a lattice path in Y. We will perform a sequence of transformations on this path according to the following algorithm.

> If the lattice path $(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{2m},y_{2m})$ never goes above the straight line $y=x$, then we have completed the transformation. Otherwise, let i be the smallest positive integer such that $y_i-x_i=\max\{y_j-x_j : 1\le j\le 2m\}$. Note then that the step $(x_{i-1},y_{i-1})\to(x_i,y_i)$ is an upward step $(0,1)$. Update $(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{2m},y_{2m})$ as the new lattice path $(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{i-1},y_{i-1})\to(x_i+1,y_i-1)\to(x_{i+1}+1,y_{i+1}-1)\to\cdots\to(x_{2m}+1,y_{2m}-1)$, and reiterate this process. (5)

Fig. 7 illustrates the above algorithm on the lattice path in Fig. 6.

The resultant lattice path of this algorithm has $2m$ steps, only travels rightward or upward, and never goes above the line $y=x$. Hence, it is a lattice path in X. In other words, this algorithm produces a well-defined function from Y to X. To see that this function is bijective, we will define a function from X to Y, and verify that these two functions are inverses to each other.

Given a lattice path $(0,0)=(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{2m},y_{2m})$ in X, let i be the unique integer obtained in algorithm (5) above. Define the vertex (x_i,y_i) as the *pinnacle* of the lattice path. The idea is to imagine that when a strong wind blows from bottom left to top right with slope 1, (x_i,y_i) is the vertex that is the most susceptible. A vertex (x_j,y_j) is a *pinnacle-elect* if $(x_{j-1},y_{j-1})\to(x_j,y_j)$ is a rightward step $(+1,+0)$, and the vertex (x_j-1,y_j+1) is the pinnacle of the new lattice path $(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{j-1},y_{j-1})\to(x_j-1,y_j+1)\to(x_{j+1}-1,y_{j+1}+1)\to\cdots\to(x_{2m}-1,y_{2m}+1)$.

Lemma 4.1. *Let $P\in X\setminus Y$ be the lattice path $(0,0)=(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{2m},y_{2m})$. Then there exists a unique integer $1\le j\le 2m$ such that (x_j,y_j) is a pinnacle-elect.*

Proof. Let (x_i,y_i) be the pinnacle of the lattice path in $X\setminus Y$, and let i' be the largest integer such that $y_{i'}-x_{i'}=y_i-x_i>0$. Note then that $i'<2m$. This is because $(x_{2m},y_{2m})\in\{(x,y): x+y=2m \text{ and } m\le x\le 2m\}$, which implies that $x_{2m}\ge y_{2m}$. Thus, we may let $j=i'+1$. We will prove that (x_j,y_j) is the unique pinnacle-elect.

First, note that $(x_{j-1},y_{j-1})\to(x_j,y_j)$ is a rightward step $(+1,+0)$. Otherwise, if $(x_{j-1},y_{j-1})\to(x_j,y_j)$ is an upward step $(+0,+1)$, then $y_j-x_j=(y_{j-1}+1)-x_{j-1}=y_{i'}-x_{i'}+1>y_i-x_i$, which contradicts that (x_i,y_i) is the pinnacle.

Next, consider the new lattice path P_1 given by $(x_0,y_0)\to(x_1,y_1)\to(x_2,y_2)\to\cdots\to(x_{j-1},y_{j-1})\to(x_j-1,y_j+1)\to(x_{j+1}-1,y_{j+1}+1)\to\cdots\to(x_{2m}-1,y_{2m}+1)$. In this lattice path P_1,

$$(y_j+1)-(x_j-1)=(y_{j-1}+1)-((x_{j-1}+1)-1)=y_{j-1}-x_{j-1}+1=y_{i'}-x_{i'}+1,$$

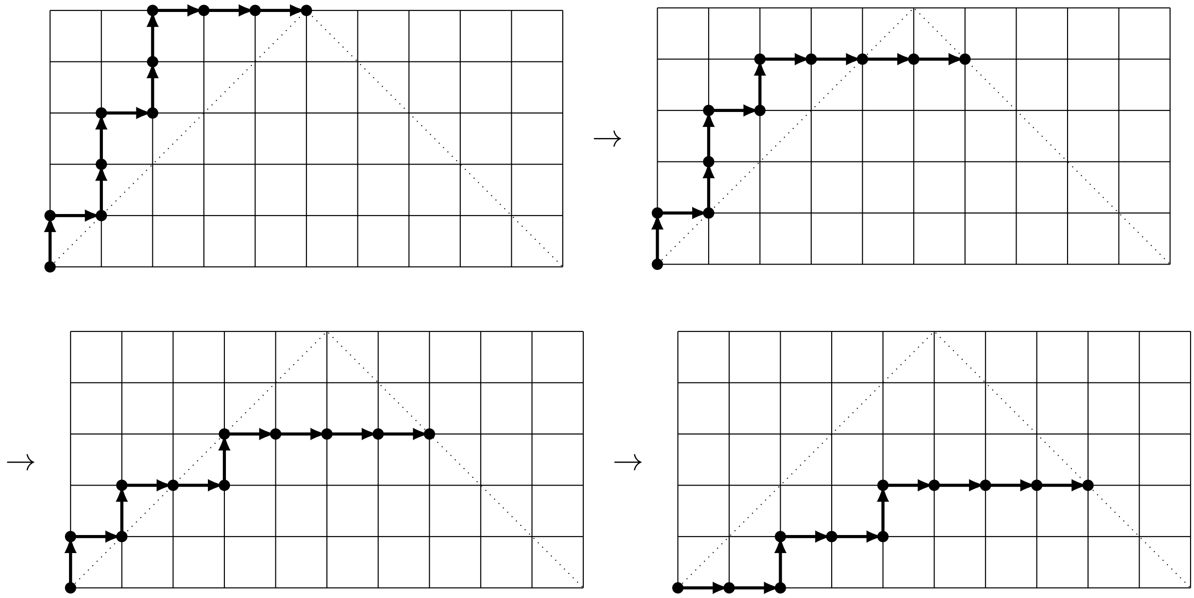

Fig. 7. The sequence of transformations on the lattice path in Fig. 6.

which is greater than $y_\ell - x_\ell$ for all $0 \le \ell \le j-1$, and is greater than or equal to $(y_\ell + 1) - (x_\ell - 1) = y_\ell - x_\ell + 2$ for all $j + 1 = i' + 2 \le \ell \le 2m$, since $y_{i'} - x_{i'} > y_\ell - x_\ell$. This shows that $(x_j - 1, y_j + 1)$ is the pinnacle of the lattice path P_1, and hence, (x_j, y_j) is a pinnacle-elect of P.

Finally, if there are two pinnacle-elects (x_j, y_j) and $(x_{j'}, y_{j'})$ in the lattice path P for some $j < j'$, then $(x_j, y_j) = (x_{j-1} + 1, y_{j-1})$ and $(x_{j'}, y_{j'}) = (x_{j'-1} + 1, y_{j'-1})$. Now, at the pinnacle of the new lattice path P_2 given by $(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_{j'-1}, y_{j'-1}) \to (x_{j'} - 1, y_{j'} + 1) \to (x_{j'+1} - 1, y_{j'+1} + 1) \to \cdots \to (x_{2m} - 1, y_{2m} + 1)$, we have $(y_{j'} + 1) - (x_{j'} - 1) > y_\ell - x_\ell$ for all $0 \le \ell \le j' - 1$. In particular, $(y_{j'} + 1) - (x_{j'} - 1) > y_{j-1} - x_{j-1}$. This implies that at the pinnacle of the lattice path P_1, we have $(y_j + 1) - (x_j - 1) = (y_{j-1} + 1) - (x_{j-1} + 1 - 1) = y_{j-1} - x_{j-1} + 1 < (y_{j'} + 1) - (x_{j'} - 1) + 1 = (y_{j'-1} + 1) - (x_{j'-1} - 1)$, which contradicts that $(x_j - 1, y_j + 1)$ is the pinnacle of the lattice path P_1. This establishes the uniqueness of the pinnacle-elect in the lattice path P. □

Now, we are ready to define an algorithm to transform a lattice path $(0, 0) = (x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_{2m}, y_{2m})$ in X.

> If $(x_{2m}, y_{2m}) = (m, m)$, then we have completed the transformation. Otherwise, note that $x_{2m} - y_{2m} > 0$, and let (x_j, y_j) be the pinnacle-elect. Update $(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_{2m}, y_{2m})$ as the new lattice path $(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_{j-1}, y_{j-1}) \to (x_j - 1, y_j + 1) \to (x_{j+1} - 1, y_{j+1} + 1) \to \cdots \to (x_{2m} - 1, y_{2m} + 1)$, and reiterate this process. (6)

It is easy to see that this algorithm produces a well-defined function from X to Y, and is the desired inverse of the function from Y to X defined above. Therefore, the cardinality of X is the same as the cardinality of Y, which is $\binom{2m}{m}$.

5. Final remarks

In Section 1, we discussed how the introduction of designated terminating positions makes our investigation different from the usual study of generalized knight tours. For instance, Watkins and Hoenigman showed that without these terminating positions, every position in the torus $\mathbb{Z}_n \times \mathbb{Z}_n$ is reachable when $(a, b) = (1, 2)$ and $n \ge 6$ is even [10]. However, our following conjecture suggests a very different result when terminating positions are introduced.

Conjecture 5.1. *Every position in $\mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(a, a), (b, b)\}$ is reachable if and only if $a \not\equiv 2b \pmod n$, $2a \not\equiv b \pmod n$, and $a^2 - b^2$ is relatively prime to n.*

While our main focus in this paper was to study two specific variations of a game presented by Wong and Xu, it is worth noting that many other variations could be considered. For example, future directions may include studying a three-player game that follows rules similar to those in Wong and Xu's paper, or rules similar to those established in this paper. Another interesting variation would be a modulo independent game that follows the winning rules of the modulo dependent game, but where Alice and Bob each flip coins independently. One may also consider a dependent game with two different winning criteria or a modulo dependent game with two different moduli. We briefly considered these variations, and provide here some preliminary observations of the last variation.

Let a, b, n_1, and n_2 be positive integers such that $\gcd(a, b, n_1, n_2) = 1$ and $a < b < \min\{n_1, n_2\}$, and consider the variation of the modulo dependent game where Alice is the winner any time $x \equiv 0 \pmod{n_1}$ after the first turn and Bob the winner if $y \equiv 0 \pmod{n_2}$ and $x \not\equiv 0 \pmod{n_1}$ after the first turn. The following is an easy application of Theorem 3.10.

Theorem 5.2. *Let $n_1 \mid n_2$. If $n_1 \mid (b^2 - a^2)$, then the positions $(0, y)$, where $n_1 \mid y$, are the only reachable position in the union of Alice's and Bob's winning regions, i.e., a random walk only terminates at the positions $(0, y)$, where $n_1 \mid y$. Thus, the winning probability of Bob is 0.*

Computational data suggests the following conjecture, which is the converse of Theorem 5.2.

Conjecture 5.3. *Let $n_1 \mid n_2$. If the positions $(0, y)$ are the only reachable position in the union of Alice's and Bob's winning regions, then $n_1 \mid (b^2 - a^2)$.*

The condition that $n_1 \mid n_2$ is not necessary for the winning probability of Bob to be 0. For example, if $(n_1, n_2, a, b) = (6, 9, 2, 4)$, then the winning probability of Bob is 0. Computationally, the only necessary condition that we can identify is $n_1 \mid (b^2 - a^2)$. As for the sufficient condition, it is still elusive. For instance, we find that if $(n_1, n_2, a, b) \in \{(10, 15, 4, 6), (12, 9, 4, 8), (20, 28, 5, 15)\}$, then the winning probability of Bob is 0, but the winning probability of Bob is nonzero when $(n_1, n_2, a, b) = (6, 9, 1, 5)$.

CRediT authorship contribution statement

Joshua Harrington: Conceptualization, Investigation, Writing - original draft, Writing - review & editing, Supervision, Funding acquisition. **Kedar Karhadkar:** Investigation, Writing - original draft. **Madeline Kohutka:** Investigation. **Tessa Stevens:** Investigation, Writing - original draft. **Tony W.H. Wong:** Conceptualization, Investigation, Writing - original draft, Writing - review & editing, Supervision.

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