Closest separable state when measured by a quasi-relative entropy

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Abstract

It is well known that for pure states the relative entropy of entanglement is equal to the reduced entropy, and the closest separable state is explicitly known as well. The same holds for Renyi relative entropy per recent results. We ask the same question for a quasi-relative entropy of entanglement, which is an entanglement measure defined as the minimum distance to the set of separable state, when the distance is measured by the quasi-relative entropy. First, we consider a maximally entangled state, and show that the closest separable state is the same for any quasi-relative entropy as for the relative entropy of entanglement. Then, we show that this also holds for a certain class of functions and any pure state. And at last, we consider any pure state on two qubit systems and a large class of operator convex function. For these, we find the closest separable state, which may not be the same one as for the relative entropy of entanglement.

1 Introduction

Entanglement describes strong quantum "connections" (stronger than any classical ones) that parts of a system can posses even when separated by long distance and having no connecting matter in between. It found applications in quantum optics [8, 27], nuclear magnetic resonance [36, 37], condensed mater physics [19, 29], quantum cryptography [16, 30, 43], and quantum algorithms and protocols [4, 5]. Understanding the behavior of entanglement in a system, its creation and loss (decoherence), is one of the fundamental goals of quantum science. For an overview on entanglement see [26, 40].

The amount of entanglement in a quantum state can be captured using entanglement measures. Any such measure, $E(\rho)$ of a quantum state ρ, should fulfill at least the following criteria [45]:

(E1) $E(\rho) \geq 0$, and $E(\rho) = 0$ if and only if ρ is separable;

(E2) Invariance under local unitary operations: $E(\rho) = E(U_A \otimes U_B \rho U_A^* \otimes U_B^*)$.

(E3) E cannot increase under local operations and classical communication (LOCC), i.e. $E(\Lambda(\rho)) \leq E(\rho)$ for any LOCC map Λ.

It's interesting to note that these conditions are not set in stone for all entanglement measures. They can be relaxed, strengthen or new ones added depending on specific circumstances one would like to consider. For example, even the simples condition (E1) is not satisfied for a distillable entanglement, as it is zero on the bound entangled states [24], which are entangled. However, the measure is nice enough for us to overlook that lapse, explaining that there are various kinds of entanglement, all of which cannot be captured by one single measure, yet. New conditions on entanglement measures sometimes take the form of normalization on the maximally entangled states, convexity, additivity, continuity, and reduction to the marginal state on pure states [25, 46].

A large class of entanglement measures is defined using a 'distance' between two quantum states [45, 46]. For a distance measure $D(\rho\|\sigma)$ between two quantum states ρ and σ, which is not required to be a metric, entanglement measure is defined as

$$E(\rho) := \min_{\sigma \in \mathcal{S}} D(\rho\|\sigma) \ ,$$

where $\mathcal{S}$ is the set of separable states.

This construction gives entanglement measures such as: relative entropy of entanglement [45], Bures measure of entanglement [45], several Rényi relative entropies of entanglement [44, 47, 54], Tsallis relative entropy of entanglement [47], Groverian measure of entanglement [7, 41, 42], and possibly others.

In [45, 46] a set of conditions on the distance was given in order to induce an entanglement measure that satisfies conditions (E1)-(E3). These conditions are:

(D1) $D(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$.

(D2) Invariance under unitary operations: $D(\rho\|\sigma) = D(U\rho U^*\|U\sigma U^*)$.

(D3) Data processing inequality (or monotonicity under completely positive trace-preserving (CPTP) maps): $D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \leq D(\rho\|\sigma)$, for any CPTP map $\mathcal{N}$.

In [46] it was shown that a relative entropy was a good candidate for a distance, since it induces an entanglement measure that satisfies (E1)-(E3). The relative entropy is defined as

$$S(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma) \ .$$

It was also shown in [46] that a relative entropy of entanglement reduces to the marginal entropy on a maximally entangled state, and later in [45] it was generalized to all pure states: for a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$, the closest separable state when measured by the relative entropy is $\sigma^* = \sum_j p_j |jj\rangle \langle jj|$ and the entropy of entanglement is $E_r(|\Psi\rangle \langle \Psi|) = -\sum_j p_j \log p_j$. Sometimes this property of the relative entropy is elevated to a condition on an entanglement measure:

(E4) Entanglement of a pure state $\rho = |\Psi\rangle \langle \Psi|$ is the marginal entropy, i.e. $E(|\Psi\rangle) = S(\rho_A)$, where $\rho_A = \mathrm{Tr}_B |\Psi\rangle \langle \Psi|$, and $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the quantum entropy.

Similar results were found for the Renyi entropy of entanglement, defined by the minimal distance to the space of separable states when the distance is taken to be the Renyi relative entropy

$$S_\alpha^R(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}) \ .$$

An explicit expression was found for the Renyi entropy of entanglement in [12, 54]. In [12], it was shown that for a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ and $\alpha < 2$, the closest separable state is

$$\sigma = \frac{1}{P_\alpha} \sum_j p_j^{1/\alpha} |jj\rangle \langle jj| \ ,$$

where $P_\alpha = \sum_j p_j^{1/\alpha}$. The Renyi entropy of entanglement is then given by

$$E_\alpha^R(\Psi) = \frac{\alpha}{\alpha - 1} \log \sum_j p_j^{1/\alpha} \ .$$

We consider a large class of distances, called quasi-relative entropy, defined as

$$S_f(\rho\|\sigma) = \mathrm{Tr}(f(\Delta_{\sigma,\rho})\rho) \ ,$$

where the relative modular operator, $\Delta_{A,B}(X) = L_A R_B^{-1}(X) = AXB^{-1}$. This class includes the regular relative entropy, the exponential Renyi relative entropy, and scaled Tsallis relative entropy.

Since the quasi-relative entropy satisfies conditions (D1)-(D3), the entanglement measure it induces, satisfy (E1)-(E3). The question now is what about (E4)? Or, what is the closest separable state to a pure entangled state when measured by the quasi-relative entropy?

This optimization question was asked in [21], where the authors developed a method of solving this problem for any convex function on the set of quantum states. While authors considered the case of quasi-relative entropy as an example, they focused more on developing a general method rather than giving an explicit solution, as we aim in this manuscript. The method we are using is similar, if not the same, to the method in [21], but it could be simplified and traced back to the original relative entropy of entanglement paper [45].

In Theorem 3.1 we consider a maximally entangled (Bell) state $|\Psi^+\rangle = \sum \frac{1}{\sqrt{d}} \sum_j |jj\rangle$, and show that the closest state is $\sigma^* = \sum \frac{1}{d} \sum_j |jj\rangle \langle jj|$, which means that the quasi-relative entropy of entanglement is equal to

$$E_f(|\Psi^+\rangle) = f(1/d) \ .$$

In Theorem 3.2 we consider any pure state, and a class of functions f that define quasi-relative entropy. In these conditions, we show that the closest separable state to a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ is the state $\sigma^* = \sum_j p_j |jj\rangle \langle jj|$. The quasi-relative entropy of entanglement is then

$$E_f(|\Psi\rangle) = \sum_j p_j f(p_j) \ .$$

In Theorem 3.3 we take any qubit state $|\Psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle$ and a large class of functions. We show that the closest separable state is in a form $\sigma^* = q |00\rangle \langle 00| + (1-q) |11\rangle \langle 11|$. We explicitly define these q_j's in the proof of the theorem. In this case, the quasi-relative entropy becomes

$$E_f(|\Psi\rangle \langle \Psi|) = \sum_j p_j f(q_j) \ .$$

2 Preliminaries (for more details see [6, 11, 48, 49])

2.1 Operator Monotone Functions

Denote the set of operator monotone decreasing functions f as $\mathcal{Q}_{(0,\infty)}$.

2.1 Example. From [6, Exercise V.4.8] The following functions belong to $\mathcal{Q}_{(0,\infty)}$:

- $f(x) = -\log x$,

- $f(x) = -x^p$ for $p \in [0,1]$,

- $f(x) = x^p$ for $p \in [-1,0]$.

According to [15, Chapter II, Theorem I] every function $f \in \mathcal{Q}_{(0,\infty)}$, has a canonical integral representation

$$f(x) = -a_f x - b_f + \int_0^\infty \left(\frac{1}{t+x} - \frac{t}{t^2+1} \right) \mathrm{d}\mu_f(t) , \tag{2.1}$$

where $a_f := -\lim_{y\uparrow\infty} \frac{f(iy)}{iy} \geq 0$, $b_f := -\operatorname{Re} f(i) \in \mathbb{R}$ and μ_f is a positive measure on $(0,\infty)$ such that $\int_0^\infty \frac{1}{t^2+1} \mathrm{d}\mu_f(t) < \infty$, and

$$\mu_f(x_1) - \mu_f(x_0) = -\lim_{y\downarrow 0} \frac{1}{\pi} \int_{x_0}^{x_1} \operatorname{Im} f(-x+iy) \mathrm{d}x . \tag{2.2}$$

Conversely, every such function belongs in $\mathcal{Q}_{(0,\infty)}$.

We consider functions $f \in \mathcal{Q}_{(0,\infty)}$ such that $f(1) = 0$. The last condition is equivalent to

$$0 = f(1) = -a_f - b_f + \int_0^\infty \left(\frac{1}{t+1} - \frac{t}{t^2+1} \right) \mathrm{d}\mu_f(t) ,$$

in other words,

$$a_f + b_f = \int_0^\infty \left(\frac{1}{t+1} - \frac{t}{t^2+1} \right) \mathrm{d}\mu_f(t) . \tag{2.3}$$

Therefore, the operator monotone decreasing function f such that $f(1) = 0$ has the following integral representation

$$f(x) = a_f(1-x) + \int_0^\infty \left(\frac{1}{t+x} - \frac{1}{t+1} \right) \mathrm{d}\mu_f(t) . \tag{2.4}$$

2.2 Example. Consider the power function $f(x) = -x^p$ for $p \in (0,1)$. It is operator monotone decreasing. Then

$$a_f = -\lim_{y\uparrow\infty} f(iy)/(iy) = 0 , \quad \text{and} \quad b_f = \cos(p\pi/2) .$$

For $x > 0$, $\lim_{y\downarrow 0} \operatorname{Im} f(-x+iy) = -x^p \sin(p\pi)$ so that

$$\mathrm{d}\mu_f(x) = \pi^{-1} \sin(p\pi) x^p \mathrm{d}x .$$

This yields the representation

$$-x^p = -\cos(p\pi/2) + \frac{\sin(p\pi)}{\pi} \int_0^\infty t^p \left(\frac{1}{t+x} - \frac{t}{t^2+1} \right) \mathrm{d}t . \tag{2.5}$$

2.3 Example. Let $f(x) = -\log(x)$. It is operator monotone decreasing. Then

$$b_f = \text{Re}\left(\log(i)\right) = 0 \ ,$$

and

$$a_f = \lim_{y\uparrow\infty} \log(iy)/(iy) = \lim_{y\uparrow\infty} (\log y + i\pi/2)/(iy) = 0 \ .$$

It is clear from (2.2) that

$$\mathrm{d}\mu_f(x) = \frac{1}{\pi} \lim_{y\downarrow 0} \text{Im} \ \log(-x + iy)\mathrm{d}x = \mathrm{d}x \ .$$

Then the integral representation (2.1) gives the following formula for the logarithmic function

$$-\log x = \int_0^\infty \left(\frac{1}{t+x} - \frac{t}{t^2+1} \right) \mathrm{d}t \ , \tag{2.6}$$

which is also obvious from the direct computation of the integral.

2.2   Quasi-relative entropy

Quantum quasi-relative entropy was introduced by Petz [38, 39] as a quantum generalization of a classical Csiszár's f-divergence [13]. It is defined in the context of von Neumann algebras, but we consider only the Hilbert space setup. Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and ρ and σ be two states (given by density operators).

2.4 Definition. For an operator monotone decreasing function f, such that $f(1) = 0$, and strictly positive states ρ and σ acting on a finite-dimensional Hilbert space $\mathcal{H}$, *the quasi-relative entropy* (or sometimes referred to as *the f-divergence*) is defined as

$$S_f(\rho||\sigma) = \text{Tr}(f(\Delta_{\sigma,\rho})\rho) \ ,$$

where the relative modular operator, introduced by Araki [1],

$$\Delta_{A,B}(X) = L_A R_B^{-1}(X) = AXB^{-1}$$

is a product of left and right multiplication operators, $L_A(X) = AX$ and $R_B(X) = XB$. Throughout this paper we consider finite-dimensional setup, so the operators are invertible. (In general, A^{-1} is stands for the generalized inverse of A.)

There is a straightforward way to calculate the quasi-relative entropy from the spectral decomposition of states. Let ρ and σ have the following spectral decomposition

$$\rho = \sum_j \lambda_j \left|\phi_j\right\rangle \left\langle\phi_j\right|, \quad \sigma = \sum_k \mu_k \left|\psi_k\right\rangle \left\langle\psi_k\right| \ , \tag{2.7}$$

where the eigenvalues are ordered:

$$\lambda_n \leq \cdots \leq \lambda_1, \quad \mu_n \leq \cdots \leq \mu_1 \ .$$

the set $\{|\phi_k\rangle \langle \psi_j|\}_{j,k}$ forms an orthonormal basis of $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators, with respect to the Hilbert-Schmidt inner product defined as $\langle A, B \rangle = \text{Tr}(A^*B)$. By [51], the modular operator can be written as

$$\Delta_{\sigma,\rho} = \sum_{j,k} \frac{\mu_k}{\lambda_j} P_{j,k} , \qquad (2.8)$$

where $P_{j,k} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by

$$P_{j,k}(X) = |\psi_k\rangle \langle \phi_j| \langle \psi_k| X |\phi_j\rangle .$$

The quasi-relative entropy is calculated as follows

$$S_f(\rho||\sigma) = \sum_{j,k} \lambda_j f \left(\frac{\mu_k}{\lambda_j} \right) |\langle \psi_k| |\phi_j\rangle|^2 . \qquad (2.9)$$

2.5 Example. For $f(x) = -\log x$, the quasi-relative entropy becomes the Umegaki relative entropy

$$S_{-\log}(\rho||\sigma) = S(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma) .$$

2.6 Example. For $p \in (-1, 2)$ and $p \neq 0, 1$ let us take the function

$$f_p(x) := \frac{1}{p(1-p)}(1 - x^p) ,$$

which is operator convex. The quasi-relative entropy for this function is calculated to be

$$S_{f_p}(\rho||\sigma) = \frac{1}{p(1-p)} \left(1 - \text{Tr}(\sigma^p \rho^{1-p}) \right) .$$

2.7 Example. For $p \in (-1, 1)$ take $q = 1 - p \in (0, 2)$, the function

$$f_q(x) = \frac{1}{1-q}(1 - x^{1-q})$$

is operator convex. The quasi-relative entropy for this function is known as Tsallis q-entropy

$$S_q(\rho||\sigma) = \frac{1}{1-q} \left(1 - \text{Tr}(\rho^q \sigma^{1-q}) \right) .$$

3 Quasi-relative entropy of entanglement

Quasi-relative entropy of entanglement is defined as follows

$$E_f(\rho) = \min_{\sigma \in \mathcal{S}} S_f(\rho||\sigma) ,$$

where $\mathcal{S}$ is the set of separable states.

Note that the quasi-relative entropy satisfies all conditions (D1)-(D3). Condition (D1) was explicitly proved in [48], or could easily be derived from Pinsker inequality [23]. Condition (D2) is clear from the explicit form of the quasi-relative entropy (2.9).

Throughout the paper consider a bipartite system AB with finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let $\{|j\rangle_A\}_j$ and $\{|j\rangle_B\}_j$ denote some bases for each Hilbert space (we will drop the subscript A and B since these are the only two systems we consider). For a probability distribution $\{p_j\}_j$ (i.e. $0 \le p_j \le 1$ and $\sum_j p_j = 1$) and a basis $\{|j\rangle\}_j$ of each finite-dimensional Hilbert space, denote a separable state

$$\sigma(\{p_j\}) = \sum_j p_j |jj\rangle \langle jj|_{AB} \ . \tag{3.1}$$

3.1 Theorem. *Let f be an operator monotone decreasing function, such that $f(1) = 0$. For a maximally entangled pure state $|\Psi^+\rangle = \sum_j \frac{1}{\sqrt{d}} |jj\rangle$, the quasi-relative entropy of entanglement is reached for a state $\sigma(\{\frac{1}{d}\}) = \sum_j \frac{1}{d} |jj\rangle \langle jj|$, and becomes*

$$E_f(|\Psi^+\rangle \langle \Psi^+|) = f(1/d) \ .$$

Proof. For any state $\sigma = \sum_k \mu_k |\psi_k\rangle \langle \psi_k|$, the quasi-relative entropy is

$$S_f(|\Psi^+\rangle \langle \Psi^+| \, \| \sigma) = \sum_k f(\mu_k) |\langle \psi_k| |\Psi^+\rangle|^2 = \langle \Psi^+| f(\sigma) |\Psi^+\rangle \ .$$

Since f is convex and monotonically decreasing, we have

$$\min_{\sigma \in \mathcal{S}} S_f(|\Psi^+\rangle \langle \Psi^+| \, \| \sigma) = \min_{\sigma \in \mathcal{S}} \langle \Psi^+| f(\sigma) |\Psi^+\rangle \tag{3.2}$$

$$\ge \min_{\sigma \in \mathcal{S}} f\left(\langle \Psi^+| \sigma |\Psi^+\rangle\right) \tag{3.3}$$

$$\ge f\left(\max_{\sigma \in \mathcal{S}} \langle \Psi^+| \sigma |\Psi^+\rangle\right) = f(1/d) \ . \tag{3.4}$$

The last equality is due to [26], for example. All inequalities are reached for $\sigma(\frac{1}{d})$, and $S_f(|\Psi^+\rangle \langle \Psi^+| \, \| \sigma(\{\frac{1}{d}\})) = f(1/d)$. $\qquad \square$

The proof of the following theorem is inspired by [45].

3.2 Theorem. *Let f be an operator monotone decreasing function, such that $f(1) = 0$ and $a_f = 0$ in (2.1). Let $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ be a pure state. If there exists a constant $H_f > 0$ such that for any $p \in [0,1]$*

$$\int_0^\infty p(t+p)^{-2} d\mu_f(t) = H_f \ , \tag{3.5}$$

where $\mu_f(t)$ is the measure in the integral representation (2.1), then the quasi-relative entropy of entanglement is reached at a state $\sigma(\{p_j\})$ (3.1), and becomes

$$E_f(|\Psi\rangle \langle \Psi|) = \sum_j p_j f(p_j) \ .$$

Proof. Let $\rho = |\Psi\rangle \langle \Psi|$ and $\sigma^* := \sigma(\{p_j\}) = \sum_j p_j |jj\rangle \langle jj|$. Consider a function

$$F(x, \sigma) := S_f(\rho \| (1-x)\sigma^* + x\sigma) \ .$$

For any state σ, adopting notation $\Delta(x) := \Delta_{(1-x)\sigma^*+x\sigma,\rho}$ and $\Delta^* := \Delta_{\sigma^*,\rho}$, we have

$$\frac{\partial F}{\partial x}(0,\sigma) = \lim_{x\to 0}\frac{1}{x}\mathrm{Tr}\{[f(\Delta(x)) - f(\Delta^*)]\rho\} \tag{3.6}$$

$$= \lim_{x\to 0}\frac{1}{x}\int_0^\infty d\mu_f(t)\mathrm{Tr}\left[\{(t\mathbb{1} + \Delta(x))^{-1} - (t\mathbb{1} + \Delta^*)^{-1}\}\rho\right] \tag{3.7}$$

$$= \lim_{x\to 0}\frac{1}{x}\int_0^\infty d\mu_f(t)\mathrm{Tr}\left[\{(t\mathbb{1} + \Delta(x))^{-1}(\Delta^* - \Delta(x))(t\mathbb{1} + \Delta^*)^{-1}\}\rho\right] \tag{3.8}$$

$$= \int_0^\infty d\mu_f(t)\mathrm{Tr}\left[\{(t\mathbb{1} + \Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{1} + \Delta^*)^{-1}\}(\rho)\right] \tag{3.9}$$

$$\tag{3.10}$$

The formula $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ holds for any invertible operators A and B.

For ρ and σ^* defined above we have

$$\Delta^* = \sum_j p_j P_j\,, \qquad P_j(X) = \langle jj| X |\Psi\rangle\ |jj\rangle\langle\Psi|\ .$$

Then

$$(t\mathbb{1} + \Delta^*)^{-1}(\rho) = \sum_j (t + p_j)^{-1}P_j(I) = \sum_j (t + p_j)^{-1}\langle jj| |\Psi\rangle |jj\rangle\langle\Psi|\ .$$

And therefore,

$$\{(t\mathbb{1}+\Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{1}+\Delta^*)^{-1}\}(\rho) = \sum_{jk}(t+p_k)^{-1}(t+p_j)^{-1}\langle jj||\Psi\rangle |kk\rangle\langle\Psi|\langle kk|(\sigma^* - \sigma)|jj\rangle\langle\Psi||\Psi\rangle\ .$$

Denote

$$g_t(p,q) := \sqrt{pq}\,(t+p)^{-1}(t+q)^{-1}\ .$$

Then taking the trace above, we obtain

$$\mathrm{Tr}\left[\{(t\mathbb{1}+\Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{1}+\Delta^*)^{-1}\}(\rho)\right]$$
$$= \sum_{jk}(t+p_k)^{-1}(t+p_j)^{-1}\langle jj||\Psi\rangle\langle\Psi||kk\rangle\langle kk|(\sigma^* - \sigma)|jj\rangle \tag{3.11}$$

$$= \sum_{jk}g_t(p_j,p_k)\langle kk|(\sigma^* - \sigma)|jj\rangle \tag{3.12}$$

$$= \sum_j p_j^2(t+p_j)^{-2} - \sum_{jk}g_t(p_j,p_k)\langle kk|\sigma|jj\rangle\ . \tag{3.13}$$

Therefore,

$$\frac{\partial F}{\partial x}(0,\sigma) = \int_0^\infty d\mu_f(t)\sum_j p_j^2(t+p_j)^{-2} - \sum_{jk}g_t(p_j,p_k)\langle kk|\sigma|jj\rangle \tag{3.14}$$

$$= \sum_j \int_0^\infty p_j^2(t+p_j)^{-2}d\mu_f(t) - \int_0^\infty \mathrm{Tr}\left(\sigma\sum_{jk}g_t(p_j,p_k)|jj\rangle\langle kk|\right)d\mu_f(t) \tag{3.15}$$

$$\tag{3.16}$$

Define
$$G_f(p,q) := \int_0^\infty g_t(p,p)d\mu_f(t) = \int_0^\infty \sqrt{pq}(t+p)^{-1}(t+q)^{-1}d\mu_f(t) \ ,$$

and
$$H_f(p) := G_f(p,p) = \int_0^\infty p(t+p)^{-2}d\mu_f(t) \ .$$

Since $(t+p)(t+q) \geq (\sqrt{pq}+t)^2$, we have

$$0 \leq G_f(p,q) = \int_0^\infty \sqrt{pq}(t+p)^{-1}(t+q)^{-1}d\mu_f(t) \leq \int_0^\infty \sqrt{pq}(t+\sqrt{pq})^{-2}d\mu_f(t) = H_f(\sqrt{pq}) \ .$$

Take $\sigma = |\alpha\beta\rangle\langle\alpha\beta|$, where $|\alpha\rangle = \sum_j a_j|j\rangle$ and $|\beta\rangle = \sum_j b_j|j\rangle$. Then

$$\frac{\partial F}{\partial x}(0,|\alpha\beta\rangle\langle\alpha\beta|) = \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j,p_k)\langle kk|\sigma|jj\rangle \tag{3.17}$$

$$= \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j,p_k)\langle kk||ab\rangle\langle ab||jj\rangle \tag{3.18}$$

$$= \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j,p_k)a_k b_k \overline{a_j}\overline{b_j} \tag{3.19}$$

We assumed that the function f is such that $H_f(p) = H_f$ is independent of p, then

$$\frac{\partial F}{\partial x}(0,|\alpha\beta\rangle\langle\alpha\beta|) = H_f - \sum_{jk} G_f(p_j,p_k)a_k b_k \overline{a_j}\overline{b_j} \ .$$

And therefore,

$$\left|\frac{\partial F}{\partial x}(0,|\alpha\beta\rangle\langle\alpha\beta|) - H_f\right| \leq \sum_{jk} G_f(p_j,p_k)|a_j||b_j||a_k||b_k| \leq \sum_{jk} H_f(\sqrt{p_j p_k})|a_j||b_j||a_k||b_k| \leq H_f \ .$$

Since every $\sigma \in \mathcal{S}$ can be written as a convex combination $\sigma = \sum_j q_j|\alpha_j\beta_j\rangle\langle\alpha_j\beta_j|$, we obtain a non-negative partial derivate $\frac{\partial F}{\partial x}(0,\sigma) \geq 0$. Moreover, quasi-relative entropy of entanglement for a pure state is calculated to be

$$E_f(|\Psi\rangle\langle\Psi|) = S_f(|\Psi\rangle\langle\Psi|\,\|\sigma^*) = \text{Tr}(f(\Delta^*)|\Psi\rangle\langle\Psi|) = \sum_j p_j f(p_j) \ .$$

$\square$

3.3 Theorem. *Let $|\Psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$. Let f be an operator monotone decreasing function f, such that $f(1) = 0$ and $a_f = 0$ in (2.1). Let f be such that for any $p \in [0,1]$ the function*

$$H_f(p) = \int_0^\infty p(t+p)^{-2}d\mu_f(t) \ ,$$

is either monotonically increasing or decreasing in p (here $\mu_f(t)$ is the measure in the integral representation (2.1)). Then, the closest separable state to $|\Psi\rangle$, when measured by the quasi-relative entropy S_f, is in a form $\sigma(\{q,1-q\})$ (3.1) for some $0 < q < 1$. In this case, the quasi-relative entropy becomes,

$$E_f(|\Psi\rangle\langle\Psi|) = \sum_j p_j f(q_j) \ .$$

Proof. Let us denote $\sigma^* = \sigma(\{q, 1-q\}) = q\,|00\rangle\,\langle 00| + (1-q)\,|11\rangle\,\langle 11|$. From a proof of Theorem 3.2, we have

$$\Delta^* = \sum_j q_j P_j , \qquad P_j(X) = \langle jj|\,X\,|\Psi\rangle\,|jj\rangle\,\langle\Psi| .$$

Then denoting $\rho = |\Psi\rangle\,\langle\Psi|$,

$$(t\mathbb{1} + \Delta^*)^{-1}(\rho) = \sum_j (t+q_j)^{-1} P_j(I) = \sum_j (t+q_j)^{-1}\,\langle jj|\,|\Psi\rangle\,|jj\rangle\,\langle\Psi| .$$

And therefore, for any state σ, we have

$$\{(t\mathbb{1}+\Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{1}+\Delta^*)^{-1}\}(\rho) = \sum_{jk}(t+q_k)^{-1}(t+q_j)^{-1}\,\langle jj|\,|\Psi\rangle\,|kk\rangle\,\langle\Psi|\,\langle kk|\,(\sigma^*-\sigma)\,|jj\rangle\,\langle\Psi|\,|\Psi\rangle .$$

And taking the trace, we obtain

$$\mathrm{Tr}\left[\{(t\mathbb{1}+\Delta^*)^{-1}(\sigma^*-\sigma)(t\mathbb{1}+\Delta^*)^{-1}\}(\rho)\right]$$

$$= \sum_{jk}(t+q_k)^{-1}(t+q_j)^{-1}\,\langle jj|\,|\Psi\rangle\,\langle\Psi|\,|kk\rangle\,\langle kk|\,(\sigma^*-\sigma)\,|jj\rangle \tag{3.20}$$

$$= \sum_{jk} g_t(p_j, p_k)\,\langle kk|\,(\sigma^*-\sigma)\,|jj\rangle \tag{3.21}$$

$$= \sum_j p_j q_j (t+q_j)^{-2} - \sum_{jk} g_t(p_j, p_k)\,\langle kk|\,\sigma\,|jj\rangle . \tag{3.22}$$

Therefore,

$$\frac{\partial F}{\partial x}(0,\sigma) = \int_0^\infty d\mu_f(t)\sum_j p_j q_j(t+q_j)^{-2} - \sum_{jk} g_t(p_j,p_k)\,\langle kk|\,\sigma\,|jj\rangle \tag{3.23}$$

$$= \sum_j \int_0^\infty p_j q_j(t+q_j)^{-2} d\mu_f(t) - \int_0^\infty \mathrm{Tr}\left(\sigma \sum_{jk}\sqrt{\frac{p_j p_k}{q_j q_k}}\,g_t(p_j,p_k)\,|jj\rangle\,\langle kk|\right) d\mu_f(t) \tag{3.24}$$

$$= \sum_j p_j H_f(q_j) - \sum_{jk}\sqrt{\frac{p_j p_k}{q_j q_k}}\,G_f(q_j,q_k)\,\langle kk|\,\sigma\,|jj\rangle . \tag{3.25}$$

Therefore, for $\sigma = |\alpha\beta\rangle\,\langle\alpha\beta|$ with $|\alpha\rangle = \sum_j a_j\,|j\rangle$ and $|\beta\rangle = \sum_j b_j\,|j\rangle$, we obtain

$$\frac{\partial F}{\partial x}(0,|\alpha\beta\rangle\,\langle\alpha\beta|) = \sum_j p_j H_f(q_j) - \sum_{jk}\sqrt{\frac{p_j p_k}{q_j q_k}}\,G_f(q_j,q_k)a_k b_k \overline{a_j}\,\overline{b_j} .$$

For $|\Psi\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$, take $\sigma = |00\rangle\,\langle 00|$,

$$\frac{\partial g}{\partial x}(0,|00\rangle\,\langle 00|) = (1-p)H_f(1-q) - p\frac{1-q}{q}H_f(q) . \tag{3.26}$$

And for $\sigma = |11\rangle\,\langle 11|$,

$$\frac{\partial F}{\partial x}(0,|11\rangle\,\langle 11|) = pH_f(q) - (1-p)\frac{q}{1-q}H_f(1-q) = -\frac{q}{1-q}\frac{\partial F}{\partial x}(0,|00\rangle\,\langle 00|) . \tag{3.27}$$

Since we need both (3.26) and (3.27) to be non-negative, choose q such that $\frac{\partial F}{\partial x}(0, |00\rangle \langle 00|) = 0$. In other words, choose q such that

$$\frac{1-p}{1-q} H_f(1-q) - \frac{p}{q} H_f(q) = 0 .$$

This expression can be written as

$$p = \frac{q H_f(1-q)}{q H_f(1-q) + (1-q) H_f(q)} = \frac{\int_0^\infty (t+1-q)^{-2} d\mu_f(t)}{\int_0^\infty [(t+1-q)^{-2} + (t+q)^{-2}] d\mu_f(t)} . \tag{3.28}$$

Let us note a few cases here:

- If $H_f(p)$ is a constant, then $p = q$ for all $p \in [0, 1]$, which is in line with Theorem 3.2.

- If $p = 1/2$, then since the function $(t + x)^{-2}$ is monotone decreasing in x, and μ_f is a positive measure, we have $q = 1/2$. And vice versa. This is in line with Theorem 3.1.

- If $p > 1/2$, then it implies that

$$\int_0^\infty (t+q)^{-2} d\mu_f(t) < \int_0^\infty (t+1-q)^{-2} d\mu_f(t) .$$

  Since the function $(t + x)^{-2}$ is monotone decreasing in x, and μ_f is a positive measure, we have that $q > 1 - q$, or $q > 1/2$. And vice versa, i.e. if $q > 1/2$, then $p > 1/2$.

- If $H_f(q)$ is an increasing function and $p > 1/2$, then we also have that $p < q$. So,

$$\frac{1}{2} < p < q \leq 1 .$$

  And similarly,

$$0 < q < p < \frac{1}{2} .$$

- Similarly, if $H_f(q)$ is a decreasing function, we have

$$0 < p < q < \frac{1}{2}, \qquad \frac{1}{2} < q < p \leq 1 .$$

Consider the difference

$$\left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle \alpha\beta|) - \sum_j p_j H_f(q_j) \right| = \left| \sum_{jk} \sqrt{\frac{p_j p_k}{q_j q_k}} G_f(q_j, q_k) a_k b_k \overline{a_j} \overline{b_j} \right| \tag{3.29}$$

$$\leq \sum_{jk} \sqrt{\frac{p_j p_k}{q_j q_k}} G_f(q_j, q_k) |a_k b_k a_j b_j| \tag{3.30}$$

$$= 2 \sum_{k<j} \sqrt{\frac{p_j p_k}{q_j q_k}} G_f(q_j, q_k) |a_k b_k a_j b_j| + \sum_j \frac{p_j}{q_j} H_f(q_j) |a_j|^2 |b_j|^2 \tag{3.31}$$

$$\tag{3.32}$$

The last equality is due to the fact that G_f is symmetric in its arguments. Therefore, using (3.28), we obtain

$$\left|\frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - pH_f(q) - (1-p)H_f(1-q)\right| \tag{3.33}$$

$$= \left|\frac{\partial g}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \frac{p}{q}H_f(q)\right| \tag{3.34}$$

$$= 2\sqrt{\frac{p(1-p)}{q(1-q)}}G_f(q, 1-q)|a_0 a_1 b_0 b_1| + \frac{p}{q}H_f(q)|a_0|^2|b_0|^2 + \frac{1-p}{1-q}H_f(1-q)|a_1|^2|b_1|^2 \tag{3.35}$$

$$= 2\sqrt{\frac{p(1-p)}{q(1-q)}}G_f(q, 1-q)|a_0 a_1 b_0 b_1| + \frac{p}{q}H_f(q)\left(|a_0|^2|b_0|^2 + |a_1|^2|b_1|^2\right) \tag{3.36}$$

$$\tag{3.37}$$

Let us denote $A := |a_0||b_0|$ and $B := |a_1||b_1|$. Then $(A+B)^2 \leq \sum_j |a_j|^2 \sum_j |b_j|^2 \leq 1$. Therefore, $B \leq 1 - A$. And thus

$$\left|\frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \frac{p}{q}H_f(q)\right| \tag{3.38}$$

$$= 2\sqrt{\frac{p(1-p)}{q(1-q)}}G_f(q, 1-q)|a_0 a_1 b_0 b_1| + \frac{p}{q}H_f(q)\left(|a_0|^2|b_0|^2 + |a_1|^2|b_1|^2\right) \tag{3.39}$$

$$\leq \frac{p}{q}H_f(q) + 2A(1-A)\left[\sqrt{\frac{p(1-p)}{q(1-q)}}G_f(q, 1-q) - \frac{p}{q}H_f(q)\right]. \tag{3.40}$$

Note that from (3.28) the following holds

$$\sqrt{\frac{p(1-p)}{q(1-q)}}G_f(q, 1-q) - \frac{p}{q}H_f(q) = \sqrt{\frac{p(1-p)}{q(1-q)}}\left[G_f(q, 1-q) - \sqrt{\frac{p(1-q)}{q(1-p)}}H_f(q)\right] \tag{3.41}$$

$$= \sqrt{\frac{p(1-p)}{q(1-q)}}\left[G_f(q, 1-q) - \sqrt{H_f(q)H_f(1-q)}\right] \tag{3.42}$$

$$\leq 0. \tag{3.43}$$

The last inequality is due to the Cauchy-Swartz inequality:

$$G_f(p, q)^2 = pq\left(\int_0^\infty (t+p)^{-1}(t+q)^{-1}d\mu_f(t)\right)^2 \tag{3.44}$$

$$\leq pq\int_0^\infty (t+p)^{-2}d\mu_f(t)\int_0^\infty (t+q)^{-2}d\mu_f(t) = H_f(p)H_f(q). \tag{3.45}$$

Therefore, going back we obtain

$$\left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle \alpha\beta|) - \frac{p}{q} H_f(q) \right| \tag{3.46}$$

$$\leq \frac{p}{q} H_f(q) + 2A(1-A) \left[\sqrt{\frac{p(1-p)}{q(1-q)}} G_f(q, 1-q) - \frac{p}{q} H_f(q) \right] \tag{3.47}$$

$$\leq \frac{p}{q} H_f(q) . \tag{3.48}$$

This means that $\frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle \alpha\beta|) \geq 0$. Since every $\sigma \in \mathcal{S}$ can be written as a convex combination $\sigma = \sum_j q_j |\alpha_j\beta_j\rangle \langle \alpha_j\beta_j|$, we obtain a non-negative partial derivate $\frac{\partial F}{\partial x}(0, \sigma) \geq 0$. Therefore, the chosen σ^* is the closest separable state to $|\Psi\rangle$. $\qquad\square$

As a consequence we obtain partial cases of Theorems discussed above:

- If $H_f(p)$ is a constant, then $p = q$ for all $p \in [0, 1]$. And the closest separable state to a state $|\Psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle$ is the state $\sigma^* = p |00\rangle \langle 00| + (1-p) |11\rangle \langle 11|$, which is a case of a two-qubit system in Theorem 3.2.

- If $p = 1/2$, then $q = 1/2$. Therefore, the closest separable state to a maximally entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the state $\sigma^* = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|)$. This is a case of a two-qubit system in Theorem 3.1.

4   Examples

4.1 Example. For $f(x) = -\log(x)$, the quasi-relative entropy becomes regular Umegaki relative entropy

$$S_{-log}(\rho\|\sigma) = S(\rho\|\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma) .$$

For this function, $H_f(p) = 1$ for all $p \in [0, 1]$. As it was shown in [45] the relative entropy of entanglement reduces to the marginal entropy on pure states, coinciding with Theorem 3.2,

$$E(|\Psi\rangle \langle \Psi|) = S(\rho_A) = -\sum_j p_j \log p_j ,$$

where $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| = \sum_j p_j |j\rangle \langle j|$. This holds for any pure state $\rho = |\Psi\rangle \langle \Psi|$.

4.2 Example. The power function $f(x) = 1 - x^{1-\alpha}$ for $\alpha \in (0, 1)$ defines the quasi-relative entropy as

$$S_\alpha(\rho\|\sigma) := 1 - \text{Tr}(\rho^\alpha \sigma^{1-\alpha}) .$$

By Theorem 3.1, for a maximally entangled state $|\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_j |jj\rangle$, the minimum of the α-relative entropy over separable states is

$$\min_{\sigma \in \mathcal{S}} S_\alpha \left(|\Psi^+\rangle \langle \Psi^+| \|\sigma \right) = 1 - d^{\alpha-1} .$$

4.3 Example. Renyi entropy of entanglement is defined as

$$E_\alpha^R(\rho) = \min_{\sigma \in \mathcal{S}} S_\alpha^R(\rho \| \sigma) \ ,$$

for $\alpha \geq 0$, $\alpha \neq 1$, where the Renyi relative entropy is defined as

$$S_\alpha^R(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \operatorname{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha - 1} \log \left(1 - S_\alpha(\rho \| \sigma)\right) \ .$$

Consider the case when $\alpha < 1$. Then

$$E_\alpha^R(\rho) = \frac{1}{\alpha - 1} \log[1 - \min_{\sigma \in \mathcal{S}} S_\alpha(\rho \| \sigma)] \ .$$

And therefore, for a maximally entangled state, we have

$$E_\alpha^R(|\Psi^+\rangle \langle \Psi^+|) = \frac{1}{\alpha - 1} \log d^{\alpha - 1} = \log d.$$

This means that for a maximally entangled state, the Renyi entropy of entanglement reduces to the marginal Renyi entropy, since

$$S_\alpha^R(\rho_A) = \frac{1}{1 - \alpha} \log \operatorname{Tr} \rho_A^\alpha = \log d = E_\alpha^R(|\Psi^+\rangle \langle \Psi^+|) \ ,$$

where $\rho_A = \operatorname{Tr}_B |\Psi^+\rangle \langle \Psi^+| = \frac{1}{d} \sum_j |jj\rangle \langle jj|$. This is in line with the explicit expression found for the Renyi entropy of entanglement in [12, 54]. Note that in [12] it was shown that for a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ and $\alpha < 2$, the closest separable state is

$$\sigma = \frac{1}{P_\alpha} \sum_j p_j^{1/\alpha} |jj\rangle \langle jj| \ ,$$

where $P_\alpha = \sum_j p_j^{1/\alpha}$. The Renyi entropy of entanglement is then given by

$$E_\alpha^R(\Psi) = \frac{\alpha}{\alpha - 1} \log \sum_j p_j^{1/\alpha} \ .$$

This would be a generalization of Theorem 3.3 in Renyi relative entropy case, although more work is needed to explicitly show that $H_f(p)$ is monotonic in this case, and to explicitly find coefficients in the separable state to apply the Theorem for a general pure state.

4.4 Example. Tsallis entropy of entanglement is defined as

$$E_\alpha^T(\rho) = \min_{\sigma \in \mathcal{S}} S_\alpha^T(\rho \| \sigma) \ ,$$

for $\alpha \geq 0$, $\alpha \neq 1$, where Tsallis relative entropy is defined as

$$S_\alpha^T(\rho \| \sigma) = \frac{1}{1 - \alpha} \left(1 - \operatorname{Tr}(\rho^\alpha \sigma^{1-\alpha})\right) = \frac{1}{1 - \alpha} S_\alpha(\rho \| \sigma) \ .$$

By Theorem 3.1, for $\alpha < 1$, and a maximally entangled state, we have

$$E_\alpha^T(|\Psi^+\rangle \langle \Psi^+|) = \frac{1}{1-\alpha} \min_{\sigma \in \mathcal{S}} S_\alpha(|\Psi^+\rangle \langle \Psi^+| \|\sigma) = \frac{1}{1-\alpha} \left[1 - d^{\alpha-1}\right] \ .$$

Interestingly enough, Tsallis entropy of entanglement does not reduce to the marginal Tsallis entropy on a maximally entangled state, since

$$S_\alpha^T(\rho_A) = \frac{1}{\alpha-1}(1 - \mathrm{Tr}\rho_A^\alpha) = \frac{1}{\alpha-1}\left[1 - d^{1-\alpha}\right] \ ,$$

where $\rho_A = \mathrm{Tr}_B |\Psi^+\rangle \langle \Psi^+| = \frac{1}{d}\sum_j |jj\rangle \langle jj|$.

5 Conclusion

We asked a question: what is the closest separable state to a pure state when the distance between two states is measured by the quasi-relative entropy? We answered this question in three cases: when a state is maximally entangled; when a state is on a bipartite qubit systems with a large class of functions defining quasi-relative entropy; and for any pure state and a certain class of functions. It would be interesting to generalize a qubit case (Theorem 3.3) to any finite dimension and/or remove the assumptions on the functions f, which would result in a complete answer of the initial question.

There are multiple directions one could take to further develop this problem. Let us a list just a few possibilities, besides the direct improvement of the presented results.

- It is important to answer the same question for a variety of other entanglement measures, not obtained from the quasi-relative entropy of entanglement. These measures can be induced, for example, by sandwiched relative entropy [35, 52], $\alpha - z$-Renyi relative entropy [2, 10, 53], geometric Renyi divergence [18, 31], to name a few.

- A more general question of explicitly finding a closest separable state for a mixed bipartite entangled stats is still open. The way to find an answer for two-qubit systems has been discussed in [33] for some restricted cases. For a review on the problem, see [28]. In [9, 14, 17, 22, 55] authors provide an algorithm for the calculation of the relative entropy of entanglement and for the separability problem. The existence of a similar algorithm could be explored for a quasi-relative entropy.

- An inverse problem has be considered for a while: given a separable state, which entangled states are the closest to it? For two qubit states a closed formula for entangled states has been found in [33], which was generalized to any dimensions and any number of parties in [20]. A natural question arises to generalize this result for quasi-relative entropy.

- As mentioned in Introduction, there are a number of entanglement measure for a mixed state. For a review see, for example, [25]. It is interesting to note that if an entanglement of one state is less than an entanglement of another state when measured by, say, relative entropy, it is not necessarily

the case that this order will be preserved using other measures. In fact it was demonstrated in [50] that all asymptotic entanglement measures satisfying (E1)-(E3) are either identical or have a different ordering on the set of all quantum states. So the following problem arises: for a fixed entanglement value of one kind of entanglement measure, find a state with a maximum or minimum value of entanglement measured by another. Some numerical results have been done, for example, in [3, 32, 34]. Adding quasi-relative entropy into the mix, poses new frontiers in this problem.

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