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On the Lipschitz dimension of Cheeger–Kleiner

by

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Abstract. In a 2013 paper, Cheeger and Kleiner introduced a new type of dimension for metric spaces, the "Lipschitz dimension". We study the dimension-theoretic properties of Lipschitz dimension, including its behavior under Gromov–Hausdorff convergence, its (non-)invariance under various classes of mappings, and its relationship to the Nagata dimension and Cheeger's "analytic dimension". We compute the Lipschitz dimension of various natural spaces, including Carnot groups, snowflakes of Euclidean spaces, metric trees, and Sierpiński carpets. As corollaries, we obtain a short proof of a quasi-isometric non-embedding result for Carnot groups and a necessary condition for the existence of non-degenerate Lipschitz maps between certain spaces.

Contents

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1. Introduction. In a 2013 paper [CK13], Cheeger and Kleiner introduced a new type of dimension for metric spaces, the Lipschitz dimension, and proved some deep results about spaces of Lipschitz dimension ≤ 1. In this paper, we study the dimension-theoretic properties of Lipschitz dimension. We begin our introduction with a discussion of the analogies with topological dimension that lead to the definition of Lipschitz dimension, and then describe the structure and results of the present paper.

1.1. Topological dimension and Lipschitz dimension. We will be concerned with a metric analog of a well-studied concept in topology: the *topological dimension* $\dim_T(X)$ of a space X. In the setting of compact metric spaces, the topological dimension $\dim_T(X)$ admits many equivalent definitions. The "small inductive definition" defines the empty set to have $\dim_T(\emptyset) = -1$, and then declares that $\dim_T(X) \leq n$ if X has a neighborhood basis of open sets U with $\dim_T(\partial U) \leq n-1$. The "Lebesgue covering definition" declares $\dim_T(X)$ to be the minimal n such that every locally finite open cover of X admits a locally finite refinement of multiplicity at most $n+1$, meaning that every point is contained in at most $n+1$ sets of the refinement.

These two definitions are known to be equivalent for compact (in fact, for separable) metric spaces (see [Nag83, Sections I.4 and II.5]), and so we refer to them simply as "topological dimension", denoted by $\dim_T$.

There is yet another way (among many others unmentioned here) to view the topological dimension of a compact metric space X, this time through studying continuous maps from X to Euclidean space. A continuous map is called *light* if $f^{-1}(p)$ is totally disconnected for each p in the image of f. We then observe, for a compact metric space X, that

$$\dim_T(X) = \min\{n \geq 0 : \exists f\colon X \to \mathbb{R}^n \text{ light}\}, \tag{1.1}$$

where $\mathbb{R}^0$ denotes the one-point space. (This follows from [Nag83, Theorems III.6 and III.10].) Thus, the topological dimension of compact metric spaces can be seen through examining light maps to Euclidean space.

In [CK13], Cheeger and Kleiner were inspired by this fact to give a quantitative analog of topological dimension. They replace continuous maps by Lipschitz maps, give a quantitative analog of the notion of lightness, and then use the analog of (1.1) to define a new notion of dimension.

As a preliminary, we need the following discrete notion:

DEFINITION 1.1. For $r > 0$, a finite sequence $(x_1, \dots, x_k)$ in a metric space X is an *r-path* if $d(x_i, x_{i+1}) \le r$ for all $i \in \{1, \dots, k\}$.

We say that two points in X are in the same *r-component* of X if there is an r-path in X containing both of them. This defines an equivalence relation on X.

Cheeger and Kleiner then used this notion to define a quantitative analog of lightness for Lipschitz maps:

DEFINITION 1.2 (Cheeger–Kleiner [CK13]). A map $f\colon X \to Y$ between metric spaces is *Lipschitz light* if there is a constant $C > 0$ such that

- f is Lipschitz with constant C, and
- for every $r > 0$ and every subset $W \subset Y$ with $\mathrm{diam}(W) \le r$, the r-components of $f^{-1}(W)$ have diameter at most Cr.

(An astute reader may note that Definition 1.2 is not precisely the one given in [CK13, Definition 1.14], though it is the one used in [CK13, Section 11]. We address this small discrepancy in Subsection 1.3 below.)

By analogy with (1.1), Cheeger and Kleiner define the following notion of dimension, which is the main subject of the present paper.

DEFINITION 1.3 (Cheeger–Kleiner [CK13]). A metric space X has *Lipschitz dimension* $\le n$ if there is a Lipschitz light map $f\colon X \to \mathbb{R}^n$.

We let the Lipschitz dimension of X be the minimal n such that X has Lipschitz dimension $\le n$, and denote this by $\dim_L(X)$. If X admits no Lipschitz light map into any Euclidean space, we write $\dim_L(X) = \infty$.

Again, $\mathbb{R}^0$ is considered here to be the one-point metric space.

The two main theorems of [CK13] concern the structure of spaces of Lipschitz dimension ≤ 1. First of all, Cheeger and Kleiner characterize spaces of Lipschitz dimension ≤ 1 as inverse limits of systems of metric graphs satisfying certain axioms [CK13, Theorems 1.10 and 1.11]. Second, they show that each metric space of Lipschitz dimension ≤ 1 admits a bi-Lipschitz embedding into the Banach space L_1. (By contrast, [Laa00] and [CK15] construct spaces of Lipschitz dimension 1 with no bi-Lipschitz embedding into Hilbert space, or even the Banach space ℓ_1.) This provides motivation to study Lipschitz dimension from the perspective of metric embedding theory.

We are now ready to elaborate on the goals and results of the present paper.

1.2. Purpose and results of the present paper. The purpose of this paper is to study the dimension-theoretic properties of Lipschitz dimension. We explain the structure of our paper and the ideas of our main results here, referring the reader to the appropriate sections for the official statements of theorems.

After giving basic notation in Section 2, we first describe the relationship (or lack thereof) between Lipschitz dimension and other well-studied notions of metric dimension: the Nagata, Assouad, and Hausdorff dimensions. In particular, we show that Lipschitz dimension bounds Nagata dimension from above (Corollary 3.5), and that the two agree for 0-dimensional spaces but not in general (Proposition 3.6).

We then address, in Section 4, the behavior of Lipschitz dimension under completions, products, and unions.

In Section 5, the technical core of the paper, we characterize Lipschitz light maps on doubling metric spaces via their behavior under Gromov–Hausdorff convergence (Theorem 5.19), obtaining bounds on Lipschitz dimension of tangent spaces as a consequence.

We then use this and other techniques to compute the Lipschitz dimension of a number of natural examples in Section 6, including metric trees, snowflakes of Euclidean spaces, and Carnot groups. The most concrete results of this section are that

- products of n metric trees (as well as rank-n Euclidean buildings) have Lipschitz dimension n (Corollary 6.3),
- snowflakes of $\mathbb{R}^n$ have Lipschitz dimension n (Corollary 6.5), and
- non-abelian Carnot groups have infinite Lipschitz dimension (Theorem 6.8).

As a corollary of this last fact, we obtain a short proof of a quasi-isometric non-embedding result (Corollary 6.10) in the spirit of Pauls [Pau01].

Also in Section 6, in Theorem 6.16, we introduce a "self-covering" property for Euclidean subsets and use this to compute the Lipschitz dimension of some classical fractals, like the Sierpiński carpets and gasket. The results of Section 6 rely on Gromov–Hausdorff convergence arguments; in the case of trees and buildings, they rely on constructions of Lipschitz light maps provided by Lang and Schlichenmaier [LS05].

In Section 7, we consider the "Lipschitz differentiability spaces" first described by Cheeger. These are metric measure spaces X that carry a type of measurable cotangent bundle allowing for the almost-everywhere differentiation, in an appropriate sense, of Lipschitz functions from X to $\mathbb{R}$. Our main result in this section is Theorem 7.6, which states that the dimension of Cheeger's cotangent bundle is bounded above by the Lipschitz dimension, complementing earlier results of Schioppa [Sch16, Corollary 5.99] and the author [Dav15, Corollary 8.5] concerning Assouad dimension.

Lastly, in Section 8, we study the invariance and non-invariance properties of Lipschitz dimension under various categories of mappings: Lipschitz light, quasisymmetric, snowflake, and David–Semmes regular mappings. We provide a construction in Corollary 8.3 that shows that, while Lipschitz light mappings cannot decrease Lipschitz dimension, they can increase it arbitrarily, and in fact that every compact doubling metric space is the image under a Lipschitz light map of a space with Lipschitz dimension 0.

In our study of David–Semmes regular mappings in Subsection 8.3, we also obtain in Corollary 8.10 a necessary condition for the existence of non-degenerate Lipschitz maps between certain spaces.

Throughout the paper, we include a number of questions that we consider worth studying.

1.3. Remarks on the definition of Lipschitz light mappings. Before proceeding further, we remark briefly on a discrepancy between our definition of Lipschitz light in Definition 1.2 and [CK13, Definition 1.14].

In [CK13, Definition 1.14], a Lipschitz map $f\colon X \to Y$ between metric spaces is called Lipschitz light if there is $C > 0$ such that, for every bounded subset $W \subset Y$, the $\mathrm{diam}(W)$-components of $f^{-1}(W)$ have diameter at most $C\,\mathrm{diam}(W)$.

Our Definition 1.2 and [CK13, Definition 1.14] are equivalent if $Y = \mathbb{R}^n$ ($n \geq 1$), but are not equivalent in general, as Remarks 1.4 and 1.5 now show.

Remark 1.4. It is clear that if a mapping satisfies Definition 1.2, then it satisfies [CK13, Definition 1.14]. If $n \geq 1$ and $Y = \mathbb{R}^n$, it is not hard to show that the converse holds as well. Indeed, if $f\colon X \to \mathbb{R}^n$ satisfies [CK13, Definition 1.14] and $W \subseteq \mathbb{R}^n$ has $\mathrm{diam}(W) \leq r$, then one may find a point $x \in \mathbb{R}^n$ such that $W' = W \cup \{x\}$ has $\mathrm{diam}(W') = r$. Any r-component of $f^{-1}(W)$ lies in an r-component (i.e., a $\mathrm{diam}(W')$-component) of $f^{-1}(W')$, and hence has diameter at most Cr.

Remark 1.5. In general, a mapping may satisfy [CK13, Definition 1.14] and not Definition 1.2, as the following example shows. Let $X = [0,1] \times (2\mathbb{Z})$, $Y = [0,1]$, and $f\colon X \to Y$ simply be the projection to the first factor. Then f satisfies [CK13, Definition 1.14]: Any $W \subseteq Y$ has $\mathrm{diam}(W) \leq 1$, so any $\mathrm{diam}(W)$-component of $f^{-1}(W)$ is simply an isometric copy of W contained in some $[0,1] \times \{2n\}$.

However, this mapping fails Definition 1.2 in the case $W = Y$ and $r = 2$, since $f^{-1}(W)$ has 2-paths of arbitrarily large diameter.

For the remainder of this paper, we use Definition 1.2 above as our definition of Lipschitz light maps, as it is better adapted to general metric space targets. We point out that, for the purposes of computing Lipschitz dimension on spaces with positive Lipschitz dimension, it does not matter which definition one takes (by Remark 1.4), and that Definition 1.2 is in any case the one used in Section 11 of [CK13].

2. Notation and definitions

2.1. Basic metric space notions. We write (X, d) for a metric space. Often, if the metric d is understood from the context, we denote it simply by X, and also we often use the same symbol d to denote the metric on different spaces. A *pointed* metric space is simply a pair (X, x) consisting of a metric space X and a point $x \in X$.

We denote open and closed balls in a metric space X by

$$B(x,r) = \{y \in X : d(y,x) < r\} \quad \text{and} \quad \overline{B}(x,r) = \{y \in X : d(y,x) \leq r\}.$$

If we wish to emphasize the ambient space X in which the ball is taken, we may write $B_X(x,r)$. If $\lambda > 0$ and $B = B(x,r)$, it is convenient to write λB for $B(x, \lambda r)$.

The *diameter* of a set E in a metric space X is

$$\operatorname{diam}(E) = \sup\{d(x,y) : x, y \in E\}.$$

The distance between two sets E, F in a metric space X is

$$\operatorname{dist}(E,F) = \inf\{d(x,y) : x \in E,\ y \in F\}.$$

If one of these sets happens to be a single point, say $E = \{p\}$, then we write $\operatorname{dist}(p, F)$ rather than $\operatorname{dist}(\{p\}, F)$.

If E is a subset of a metric space X and $r > 0$, then the open and closed r-neighborhoods of E in X are

$$N_r(E) = \{y \in X : \operatorname{dist}(y,E) < r\} \quad \text{and} \quad \overline{N}_r(E) = \{y \in X : \operatorname{dist}(y,E) \leq r\}.$$

For $\epsilon > 0$, an *ϵ-separated set* in X is a subset in which all mutual distances are at least ϵ. An *ϵ-net* S in X is a maximal ϵ-separated set (which always exists by Zorn's lemma); in that case we have $X = N_\epsilon(S)$.

A metric space is *proper* if all closed balls in the space are compact. A metric space is *doubling* if there is a constant N such that every ball in X can be covered by N balls of half the radius. This is a finite dimensionality condition; in fact, it is equivalent to the finiteness of the Assouad dimension defined in Definition 3.2. Every complete, doubling metric space is automatically proper.

It is often useful to study the Cartesian product $X \times Y$ of two metric spaces (X, d_X) and (Y, d_Y). To fix a convention, unless otherwise noted, we take the metric on $X \times Y$ to be the ℓ_∞ combination of the metrics on the factors:

$$d((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}.$$

Of course, this choice of product metric d is bounded above and below by constant multiples of any of the other natural ℓ_p combinations of the two metrics.

In Section 7, we will need the notion of a *metric measure space*, which for us is a complete metric space X equipped with a finite Radon measure μ. A metric measure space is *doubling* if the measure μ is doubling, meaning that it is non-zero and there is a constant $C \geq 0$ such that

$$\mu(2B) \leq C\mu(B)$$

for all balls B in X. In particular, this implies that X is a doubling metric space in the sense defined above [Hei01, Section 10.13].

2.2. Mappings. A function $f : X \to Y$ between two metric spaces is called *Lipschitz* (or L-Lipschitz) if there is $L > 0$ such that

$$d(f(x), f(x')) \leq Ld(x,x') \quad \text{for all } x, x' \in X.$$

It is called *bi-Lipschitz* (or L-bi-Lipschitz) if

$$L^{-1}d(x,x') \leq d(f(x), f(x')) \leq Ld(x,x') \quad \text{for all } x, x' \in X.$$

A 1-bi-Lipschitz map is called an *isometric embedding.*

A more general class than the bi-Lipschitz mappings is the class of *quasisymmetric* mappings. An embedding $f \colon X \to Y$ is called quasisymmetric if there is a homeomorphism $\eta \colon [0,\infty) \to [0,\infty)$ such that

$$d(x,a) \leq td(x,b) \quad \text{implies} \quad d(f(x), f(a)) \leq \eta(t) d(f(x), f(b))$$

for all triples a, b, x of points in X and all $t \geq 0$. Quasisymmetric maps may wildly distort distances (in particular, they may not be Lipschitz), but in some sense they preserve "shape". See [Hei01] for an introduction to quasisymmetric mappings.

Other than the bi-Lipschitz mappings, another interesting subclass of quasisymmetric mappings are the snowflake mappings. A mapping $f \colon X \to Y$ is called a *snowflake mapping* (or an *α-snowflake mapping*) if there are constants $\alpha \in (0,1]$ and $C > 0$ such that

$$C^{-1}d(x,y)^\alpha \leq d(f(x), f(y)) \leq Cd(x,y)^\alpha \quad \text{for all } x, y \in X.$$

A metric space Z is called an *α-snowflake* if it is the image of another metric space X under an α-snowflake mapping. Of course, this is equivalent to saying that Z is bi-Lipschitz equivalent to the metric space (X, d^α).

The terminology "snowflake" arises from the fact that the standard von Koch snowflake curve in $\mathbb{R}^2$, with the induced Euclidean metric, can be viewed as an α-snowflake image of $[0,1]$, where α^{-1} is the Hausdorff dimension of the snowflake.

A few other classes of mappings will be introduced in the paper as needed.

3. Relationship to other dimensions. In this section, we discuss the relationship between Lipschitz dimension and a number of other well-known notions of dimension for metric spaces: the topological dimension (defined in the introduction) and the Hausdorff, Assouad, and Nagata dimensions.

3.1. Other dimensions for metric spaces. We first briefly recall the definitions of the latter three dimensions. For more information about the Hausdorff and Assouad dimensions, we refer the reader to [Hei01, Sections 8.3 and 10.13], and for the Nagata dimension to [LS05].

The *n-dimensional Hausdorff measure* of a set E in a metric space X is

$$\mathcal{H}^n(E) = \lim_{\delta\to 0} \inf_{\{B_i\}} \sum_i \operatorname{diam}(B_i)^n,$$

the infimum being over covers of E by closed balls B_i of diameter at most δ.

Definition 3.1. The *Hausdorff dimension* of X is

$$\dim_H(X) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\} \in [0,\infty]$$

Definition 3.2. The *Assouad dimension* $\dim_A(X)$ of a metric space X is the infimum of all $\beta > 0$ such that there is a constant C for which every set of diameter d can be covered by at most $C\epsilon^{-\beta}$ sets of diameter at most ϵd.

Equivalently, $\dim_A(X)$ can be defined as the infimum over all $\gamma > 0$ such that there is a constant C for which every ball of radius r contains at most $C\epsilon^{-\gamma}$ ϵr-separated points.

Lastly, we define the Nagata dimension $\dim_N(X)$ of a metric space X. Call a family $\{B_i\}$ of subsets of X *D-bounded* if each B_i has diameter $\leq D$. For $s > 0$, the *s-multiplicity* of the family $\{B_i\}$ is the minimal n such that every subset of X with diameter $\leq s$ meets at most n members of the family.

Definition 3.3. The *Nagata dimension* of X, which we denote $\dim_N(X)$, is the minimal integer n with the following property: there exists $c > 0$ such that, for all $s > 0$, X has a cs-bounded covering with s-multiplicity $n+1$.

The Nagata dimension is clearly a quantitative analog of the Lebesgue covering definition of topological dimension, introduced at the start of Subsection 1.1.

Each of the five notions of dimension defined above (topological, Lipschitz, Hausdorff, Assouad, and Nagata) is easily seen to be invariant under bi-Lipschitz homeomorphisms.

We have the following relationships between the above dimensions, for all separable metric spaces X:

$$\dim_T(X) \le \dim_N(X) \le \dim_A(X) \tag{3.1}$$

(see [LS05, Theorem 2.2] and [LDR15, Theorem 1.1]), and

$$\dim_T(X) \le \dim_H(X) \le \dim_A(X) \tag{3.2}$$

(see [Hei01, Theorem 8.13 and Exercise 10.6]). Each of the above inequalities may be strict; we refer the reader to the references above for examples.

We now explain where $\dim_L(X)$ does (and does not) fit into the lists (3.1) and (3.2).

3.2. Nagata dimension and topological dimension. By [LS05, Theorem 2.2], $\dim_N(X) \ge \dim_T(X)$ for every metric space X. We show that Lipschitz dimension provides an upper bound for Nagata dimension, and hence also for topological dimension.

LEMMA 3.4. *If $f : X \to Y$ is Lipschitz light, then* $\dim_N(X) \le \dim_N(Y)$.

Proof. Without loss of generality, we may assume that f is 1-Lipschitz and that $\dim_N(Y) = n < \infty$. Fix s and consider a cs-bounded cover $\{B_i\}$ of Y with s-multiplicity at most $n+1$. We may also assume without loss of generality that $c \ge 1$.

For each i, let $\{U^i_j\}$ denote the cs-components of $f^{-1}(B_i)$. Then, because f is Lipschitz light with some constant $C \ge 1$, we have

$$\operatorname{diam}(U^i_j) \le Ccs$$

for all i, j.

We claim that $\{U^i_j\}_{i,j}$ forms a cover of X with s-multiplicity at most $n+1$. Consider any set $E \subset X$ with $\operatorname{diam}(E) \le s$. First of all, note that for each fixed i, E can meet U^i_j for at most one value of j. Indeed, if E met both U^i_j and U^i_k, then there would be $x \in U^i_j$ and $y \in U^i_k$ with $d(x,y) \le s \le cs$, in which case U^i_j and U^i_k would be the same cs-component, i.e., we would have $j = k$.

So we must show that E meets some U^i_j for at most $n+1$ values of i. This is the same as saying that $f(E)$ meets B_i for at most $n+1$ values of i. This is in fact the case, because $\operatorname{diam}(f(E)) \le s$, as f is 1-Lipschitz, and because $\{B_i\}$ has s-multiplicity at most $n+1$. ∎

COROLLARY 3.5. *For any metric space X,* $\dim_N(X) \le \dim_L(X)$.

Proof. This follows immediately from the previous lemma and the fact that the Nagata dimension of $\mathbb{R}^n$ is n. ∎

On the other hand, Nagata dimension provides no non-trivial upper bound for Lipschitz dimension. This will follow from Theorem 6.8 below and

[LDR15, Theorem 4.2], which together say that non-abelian Carnot groups have infinite Lipschitz dimension and finite Nagata dimension. (See Section 6.3 for the definition of a Carnot group.)

Nagata dimension and Lipschitz dimension do agree for 0-dimensional spaces:

PROPOSITION 3.6. *A metric space X has Lipschitz dimension 0 if and only if it has Nagata dimension 0.*

Proof. By Corollary 3.5, we always have

$$\dim_L(X) \geq \dim_N(X),$$

and so if $\dim_L(X) = 0$ then $\dim_N(X) = 0$.

Conversely, suppose the Nagata dimension of X is zero. That means that, for every $s > 0$, there is a cs-bounded cover of X with s-multiplicity at most 1.

Let $f: X \to \mathbb{R}^0$ be the constant map. We claim that f is Lipschitz light. This just means that for every $s > 0$, the s-components of X have diameter at most cs. Consider the cover of X given by the Nagata dimension in the previous paragraph. Any s-component of X must be contained in a single set in the cover, so it has diameter at most cs. Hence f is Lipschitz light. ∎

QUESTION 3.7. *Is there a compact metric space with Nagata dimension 1 and Lipschitz dimension greater than 1?*

This question is interesting in light of the results in [CK13] described in the introduction.

3.3. Hausdorff dimension and Assouad dimension. There is in general no relationship between the Lipschitz dimension and the Hausdorff or the Assouad dimension of a space. The following two propositions indicate this.

Building on a construction of Laakso [Laa00], Cheeger and Kleiner [CK15] give a very flexible construction of metric spaces with Lipschitz dimension 1, including examples with arbitrary Hausdorff and Assouad dimensions.

PROPOSITION 3.8 ([CK15]). *For every $\alpha \geq 1$, there is a compact metric space of Lipschitz dimension 1 and Hausdorff and Assouad dimensions equal to α.*

This shows that the Hausdorff and Assouad dimensions can be larger than Lipschitz dimension by any desired amount.

The reverse situation can also happen: the Lipschitz dimension may be any amount larger than the Hausdorff and/or Assouad dimensions. Indeed, as we will note in Subsection 6.3.1, Carnot groups have finite Hausdorff and Assouad dimensions, but have infinite Lipschitz dimension by Theorem 6.8 below.

4. Monotonicity, completions, products, and unions. In this section, after proving some basic facts concerning Lipschitz light maps, we study the behavior of Lipschitz dimension under products and unions.

4.1. Monotonicity and completions. For future reference, we make the following simple observations:

LEMMA 4.1. *Let X, Y, Z be metric spaces and $f\colon X \to Y$ and $g\colon Y \to Z$ be Lipschitz light. Then*

(i) *$g \circ f$ is Lipschitz light,*
(ii) $\dim_L(X) \le \dim_L(Y)$,
(iii) *the extension $\overline{f}\colon \overline{X} \to \overline{Y}$ of f to the completions of X and Y is Lipschitz light, and*
(iv) $\dim_L(\overline{X}) = \dim_L(X)$.

Proof. Let $C = \max\{C_f, C_g, 1\}$, where C_f and C_g are the Lipschitz light constants of f and g, respectively.

For item (i), let $B \subseteq Z$ be a ball of radius r and consider an r-path $P \subseteq (g \circ f)^{-1}(B)$. Then $f(P) \subseteq Y$ is a Cr-path with $g(f(P))$ contained in a ball of radius $r \le Cr$, and hence $\operatorname{diam}(f(P)) \le C^2 r$ because g is Lipschitz light with constant C. Thus, P is a $C^2 r$-path with $\operatorname{diam}(f(P)) \le C^2 r$, and hence $\operatorname{diam}(P) \le C^3 r$ because f is Lipschitz light with constant C. It follows that each r-component of $(g \circ f)^{-1}(B)$ has diameter at most $C^3 r$, and hence this composition is Lipschitz light.

Item (ii) follows from (i) by taking $Z = \mathbb{R}^n$, where $n = \dim_L(Y)$.

For item (iii), consider a ball $B = B(y, r) \subseteq \overline{Y}$ and an r-path $P = (x_1, \ldots, x_n) \subseteq \overline{f}^{-1}(B) \subseteq \overline{X}$. Fix $\epsilon > 0$. For each $i \in \{1, \ldots, n\}$, choose a point $x_i' \in X$ with $d(x_i', x_i) < \epsilon$, and set $P' = (x_1', \ldots, x_n')$. Also choose a point $y' \in Y$ with $d(y, y') < \epsilon$.

Then P' is an $(r + 2\epsilon)$-path in X with $f(P') \subseteq B(y', r + 2C\epsilon) \subseteq Y$. Since f is Lipschitz light on X,

$$\operatorname{diam}(P) \le \operatorname{diam}(P') + 2\epsilon \le C(r + 2C\epsilon) + 2\epsilon.$$

Sending ϵ to 0 shows that $\operatorname{diam}(P) \le Cr$ and hence f is Lipschitz light.

Item (iv) follows from (iii) by taking $Y = \mathbb{R}^n$, where $n = \dim_L(X)$. ■

4.2. Products and unions

PROPOSITION 4.2. *Let X and Y be metric spaces with*

$$\dim_L(X) \le m \quad \textit{and} \quad \dim_L(Y) \le n.$$

Then

$$\dim_L(X \times Y) \le m + n.$$

Proof. Let $Z = X \times Y$ and write π_X and π_Y for the projections to the two factors.

Let $f : X \to \mathbb{R}^m$ and $g : Y \to \mathbb{R}^n$ be Lipschitz light mappings. Let

$$F = (f, g) : Z \to \mathbb{R}^{m+n}.$$

Fix $W \subseteq \mathbb{R}^{m+n}$ with $\mathrm{diam}(W) \le r$. We write $\pi_{\mathbb{R}^m} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ for the projection onto the first m coordinates and $\pi_{\mathbb{R}^n} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ for the projection onto the last n coordinates.

Let A be an r-component of $F^{-1}(W) \subseteq Z$. Note that

$$f(\pi_X(A)) = \pi_{\mathbb{R}^m}(F(A)) \subseteq \pi_{\mathbb{R}^m}(W),$$
$$g(\pi_Y(A)) = \pi_{\mathbb{R}^n}(F(A)) \subseteq \pi_{\mathbb{R}^n}(W).$$

If P is any r-path in A, then $\pi_X(P)$ and $\pi_Y(P)$ are r-paths in $f^{-1}(\pi_{\mathbb{R}^m}(W))$ and $g^{-1}(\pi_{\mathbb{R}^n}(W))$, respectively.

It follows that $\pi_X(P)$ and $\pi_Y(P)$ have diameters controlled by Cr, where C is the maximum of the Lipschitz light constants of f and g. Thus, P has diameter controlled by Cr. As P was an arbitrary r-path in A, $\mathrm{diam}(A) \le Cr$. This proves that F is Lipschitz light, and hence $\dim_L(Z) \le m + n$. ■

In Proposition 4.2, equality is of course sometimes attained (e.g., for $\mathbb{R} \times \mathbb{R}$) but it may be strict in some cases. An example following [LS05, Theorem 2.6] shows this: Let $X = \mathbb{Z}$ and $Y = [0, 1]$. Then it is easy to see that $\dim_L(X) = \dim_L(Y) = 1$. However, $\dim_L(X \times Y) = 1$ as well. Indeed, the map $f : X \times Y \to \mathbb{R}$ defined by $f(n, t) = 2n + t$ is a bi-Lipschitz embedding: It is clearly Lipschitz. If $n = m$, then $|f(n, t) - f(m, s)| = |t - s| = d((n, t), (m, s))$. Otherwise,

$$|f(n, t) - f(m, s)| \ge 2|n - m| - |t - s| \ge |n - m| = d((n, t), (m, s)).$$

Next we study unions. While we are able to show that a finite union of spaces with finite Lipschitz dimension has finite Lipschitz dimension, we do not appear to obtain the sharp bound.

Proposition 4.3. *Let Z be a metric space that can be written as a union $Z = X \cup Y$. Then*

$$\dim_L(Z) \le \dim_L(X) + \dim_L(Y).$$

Proof. Write $m = \dim_L(X)$ and $n = \dim_L(Y)$. Of course, if either is infinite, then there is nothing to prove.

Let $f : X \to \mathbb{R}^m$ and $g : Y \to \mathbb{R}^n$ be Lipschitz light. We may assume that both are Lipschitz light with constant $C \ge 1$. By McShane's extension theorem [Hei01, Theorem 6.2], we may extend both mappings to Lipschitz mappings defined on all of Z, though of course they will not necessarily be Lipschitz light on the entire domain Z.

Let $F : Z \to \mathbb{R}^{n+m}$ be defined by $F(z) = (f(z), g(z))$. We claim that F is Lipschitz light.

Let $W \subseteq \mathbb{R}^{n+m}$ be a set of diameter at most $r > 0$, and let

$$P = (x_1, \dots, x_k)$$

be an r-path in $F^{-1}(W) \subseteq Z$. Without loss of generality, assume that $x_1 \in X$.

We make the following immediate observation:

(4.1) If $A \geq 1$, then any Ar-path Q contained in $P \cap X$ or $P \cap Y$ has diameter at most CAr.

Indeed, for such a path Q, either f or g maps it into a set of diameter at most $r \leq Ar$, and both these maps are Lipschitz light on their respective original domains X and Y.

We now define a subpath $P' \subseteq P$ as follows.

Let $i_1 = 1$. For each $j > 1$, inductively set i_j to be the smallest index greater than i_{j-1} such that $x_{i_j} \in X$. Continue this until there is no such index i_j. We obtain a subpath

$$P' = (x_{i_1}, \dots, x_{i_\ell}) \subseteq P.$$

Observe that if $i_j > i_{j-1}+1$, then the entire subpath of P from index $i_{j-1}+1$ to $i_j - 1$ is contained in Y, since it is disjoint from X. Thus, by (4.1), the diameter of this subpath is at most Cr. The same holds for the subpath between index i_ℓ and the last index k, if it so happens that $i_\ell < k$.

Thus,

$$d(x_{i_{j-1}}, x_{i_j}) \leq (C+2)r \quad \text{for each } j,$$

i.e., P' is a $(C+2)r$-path, and moreover

$$P \subseteq \overline{N}_{Cr}(P').$$

Since P' is a $(C+2)r$-path that is entirely contained in X, it follows again from (4.1) that

$$\operatorname{diam}(P') \leq C(C+2)r.$$

Hence,

$$\operatorname{diam}(P) \leq \operatorname{diam}(P') + 2(C+2)r \leq (C+2)^2 r.$$

Thus, F is Lipschitz light and so $\dim_L(Z) \leq n+m$. ∎

Of course, Proposition 4.3 implies that any finite union of spaces with finite Lipschitz dimension has finite Lipschitz dimension.

If true, the natural bound in Proposition 4.3 would be to replace the sum by the maximum:

Question 4.4. *If $Z = X \cup Y$, is it true that*

$$\dim_L(Z) \leq \max(\dim_L(X), \dim_L(Y))?$$

5. Gromov–Hausdorff limits and weak tangents

5.1. Convergence of metric spaces. We will use the notion of convergence of “mapping packages”, a version of Gromov–Hausdorff convergence, that is described in [DS97, Chapter 8]. This version applies only to sequences of complete metric spaces that are doubling with uniform constants, and mappings defined on such spaces. Expositions of this material are also given in [Kei04] and [Dav16, Section 2.1].

Definition 5.1. We say that a sequence $\{F_j\}$ of non-empty, closed subsets of some Euclidean space $\mathbb{R}^N$ *converges* to a non-empty closed set $F \subseteq \mathbb{R}^N$ if

$$\lim_{j\to\infty} \sup_{x\in F_j\cap B(0,R)} \operatorname{dist}(x,F) = 0 \quad \text{and} \quad \lim_{j\to\infty} \sup_{y\in F\cap B(0,R)} \operatorname{dist}(y,F_j) = 0$$

for all $R > 0$.

We now move on to defining convergence of mappings.

Definition 5.2. Suppose $\{F_j\}$ is a sequence of closed sets converging to a closed set F in $\mathbb{R}^N$ as in the previous definition. Let Y be a metric space and $f_j\colon F_j \to Y$, $f\colon F \to Y$ be mappings. We say that $\{f_j\}$ *converges* to f if for each sequence $\{x_j\}$ in $\mathbb{R}^N$ such that $x_j \in F_j$ for all j and $x_j \to x \in F$,

$$\lim_{j\to\infty} f_j(x_j) = f(x).$$

We have the following compactness statements for these notions of convergence:

Lemma 5.3 ([DS97, Lemmas 8.2 and 8.6]). *Let $\{F_j\}$ be a sequence of non-empty, closed subsets of $\mathbb{R}^n$ that all intersect a fixed ball $B(0,r)$. Let $f_j\colon F_j \to \mathbb{R}^m$ be L-Lipschitz mappings such that, for each bounded set $B\subseteq\mathbb{R}^n$,*

$$\sup_j \sup_{x\in B\cap F_j} |f_j(x)| < \infty.$$

Then there is a subsequence along which $\{F_j\}$ converges to a non-empty, closed subset F of $\mathbb{R}^n$ (in the sense of Definition 5.1) and $\{f_j\}$ converges to an L-Lipschitz mapping $f\colon F \to \mathbb{R}^m$ (in the sense of Definition 5.2).

Now we begin to define convergence for general metric spaces and mappings.

Definition 5.4. A sequence $\{(X_j, d_j, p_j)\}$ of pointed metric spaces *converges* to a pointed metric space (X, d, p) if the following holds. There exists $\alpha \in (0,1]$, $N \in \mathbb{N}$, and L-bi-Lipschitz embeddings $e_j : (X_j, d_j^\alpha) \to \mathbb{R}^N$, $e : (X, d^\alpha) \to \mathbb{R}^N$ with $e_j(p_j) = e(p) = 0$ for all j. Furthermore, we require that $e_j(X_j)$ converge to $e(X)$ in the sense of Definition 5.1, and that the real-valued functions $d_j(e_j^{-1}(x), e_j^{-1}(y))$ defined on $e_j(X_j) \times e_j(X_j)$ converge to $d(e^{-1}(x), e^{-1}(y))$ on $e(X) \times e(X)$ in the sense of Definition 5.2.

In the case where the metric spaces $\{(X_j, d_j)\}$ and (X, d) are uniformly doubling, the embeddings e_j and e can always be found, by Assouad's embedding theorem (see [Hei01, Theorem 12.2]).

Definition 5.5. A *mapping package* consists of a pair of pointed metric spaces (M, d_M, p) and (N, d_N, q) as well as a mapping $g : M \to N$ such that $g(p) = q$. It is written $((M, d_M, p), (N, d_N, q), g)$.

We slightly abuse notation and call a mapping package "doubling" if the underlying spaces are both doubling, and "uniformly doubling" if all underlying spaces are doubling with the same doubling constant.

Definition 5.6. A sequence $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ of mapping packages is said to *converge* to another mapping package $((X, d, p), (Y, \rho, q), h)$ if the following conditions hold. The sequences $\{(X_j, d_j, p_j)\}$ and $\{(Y_j, \rho_j, q_j\}$ converge to (X, d, p) and (Y, ρ, q), respectively, in the sense of Definition 5.4. Furthermore, the maps $g_j \circ h_j \circ f_j^{-1}$ converge to $g \circ h \circ f^{-1}$ in the sense of Definition 5.2, where f_j, g_j, f, g are the embeddings of Definition 5.4.

We take this opportunity to remark that the limit of a sequence of mapping packages is unique up to isometry; see [DS97, Lemma 8.20]. That is, two limits of the same sequence of mapping packages are isometrically equivalent by an isometry that preserves base points and intertwines the mappings.

We often use the $\to$ notation to indicate convergence of a sequence of pointed metric spaces or mapping packages, e.g., $(X_j, d_j, p_j) \to (X, d, p)$.

The following proposition is a special case of [DS97, Lemma 8.22].

Proposition 5.7. *Let $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ be a sequence of mapping packages in which all the metric spaces are complete and uniformly doubling, and in which the maps h_j are equicontinuous and uniformly bounded on bounded sets and satisfy $h_j(p_j) = q_j$. Then there exists a mapping package $((X, d, p), (Y, \rho, q), h)$ that is the limit of a subsequence of $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$.*

Here, the assumption that the $\{h_j\}$ are *equicontinuous and uniformly bounded on bounded sets* means that for each $R > 0$ and $\epsilon > 0$, there is $\delta > 0$ such that

$$\rho_j(h_j(x), h_j(y)) < \epsilon \quad \text{for all } x, y \in B_{X_j}(p_j, R) \text{ with } d_j(x, y) < \delta$$

and

$$\sup_j \sup_{x \in B_{X_j}(p_j, R)} \rho_j(h_j(x), q_j) < \infty.$$

In particular, this assumption is satisfied when the h_j are Lipschitz or snowflake maps with constants controlled independent of j, which is how we will always use this result.

We will now describe some consequences of the convergence of a sequence of mapping packages, which are Lemmas 8.11 and 8.19 of [DS97].

PROPOSITION 5.8. *Suppose a sequence $\{(X_k, d_k, p_k)\}$ of pointed metric spaces converges to the pointed metric space (X, d, p), in the sense of Definition* 5.4. *Then there exist (not necessarily continuous) mappings $\phi_k\colon X \to X_k$ and $\psi_k\colon X_k \to X$ such that:*

(i) *For all k, $\phi_k(p) = p_k$ and $\psi_k(p_k) = p$.*
(ii) *For all $R > 0$,*

$$\lim_{k\to\infty} \sup\{d_X(\psi_k(\phi_k(x)), x) : x \in B_X(p, R)\} = 0,$$
$$\lim_{k\to\infty} \sup\{d_{X_k}(\phi_k(\psi_k(x)), x) : x \in B_{X_k}(p_k, R)\} = 0.$$

(iii) *For all $R > 0$,*

$$\lim_{k\to\infty} \sup\{|d_{X_k}(\phi_k(x), \phi_k(y)) - d_X(x, y)| : x, y \in B_X(p, R)\} = 0,$$
$$\lim_{k\to\infty} \sup\{|d_X(\psi_k(x), \psi_k(y)) - d_{X_k}(x, y)| : x, y \in B_{X_k}(p_k, R)\} = 0.$$

PROPOSITION 5.9. *Suppose a sequence of mapping packages*

$$\{((X_k, d_k, p_k), (Y_k, \rho_k, q_k), h_k)\}$$

converges to a mapping package

$$((X, d, p), (Y, \rho, q), h),$$

where the mappings h_k are uniformly Lipschitz and satisfy $h_k(p_k) = q_k$. Then there exist (not necessarily continuous) mappings $\phi_k\colon X \to X_k$ and $\psi_k\colon X_k \to X$ satisfying exactly the conditions of Proposition 5.8, *and mappings $\sigma_k\colon Y \to Y_k$ and $\tau_k\colon Y_k \to Y$ satisfying the analogous properties of Proposition* 5.8, *such that in addition, for all $x \in X$,*

$$\lim_{k\to\infty} \tau_k\big(h_k(\phi_k(x))\big) = h(x) \tag{5.1}$$

and this convergence is uniform on bounded subsets of X.

The following lemma is needed to ensure that subsets of our spaces converge. It is a simple consequence of [DS97, Lemma 8.31].

LEMMA 5.10. *Suppose that $\{(X_k, d_k, p_k)\}$ is a sequence of pointed metric spaces that converges to the pointed metric space $\{(X, d, p)\}$ in the sense of Definition* 5.4. *Let $\{F_k\}$ be a sequence of non-empty closed sets with*

$$p_k \in F_k \subset X_k \quad \textit{for each } k.$$

Then we can pass to a subsequence to get convergence of (F_k, d_k, p_k) to (F, d, p), where F is a non-empty closed subset of X.

Lastly, we record the following basic facts about the preservation of mapping properties under limits.

LEMMA 5.11. *Let $\{((X_k, d_k, p_k), (Y_k, \rho_k, q_k), f_k)\}$ be a sequence of mapping packages converging to $((X, d, p), (Y, \rho, q), f)$.*

(i) *If all the f_k are L-Lipschitz, then so is f.*
(ii) *If all the f_k are L-bi-Lipschitz, then so is f.*
(iii) *If all the f_k are α-snowflake maps with constant C, then so is f.*
(iv) *If all the f_k are surjective α-snowflake maps with constant C, then f is a surjective α-snowflake map.*

Proof. The first two statements in the lemma are given in [DS97, Lemma 8.20], and the third is easy to verify by the same means. The fourth follows by passing to a subsequence along which the packages

$$\{((Y_k, \rho_k, q_k), (X_k, d_k^\alpha, p_k), (f_k)^{-1})\}$$

converge as well, which we can do by (iii) and Proposition 5.7. ■

5.2. Tangents and weak tangents. We can now define the notions of tangent and weak tangent to a space or mapping package.

DEFINITION 5.12. If (X, d) is a metric space, a *weak tangent* of X is any limit of pointed metric spaces of the form $(X, \lambda_k d, x_k)$, where $\lambda_k > 0$ and $x_k \in X$.

If $f\colon (X, d) \to (Y, \rho)$ is a mapping, a *weak tangent* of f is any limit of mapping packages of the form

$$((X, \lambda_k d, x_k), (Y, \lambda \rho_k, f(x_k)), f),$$

where $\lambda_k > 0$ and $x_k \in X$ are arbitrary.

We denote the collection of weak tangents of X or f by $\mathrm{WTan}(X)$ or $\mathrm{WTan}(f)$, respectively.

As a special case of the notion of weak tangent, one may force the base points x_k to be fixed and the sequence of scales to tend to infinity, corresponding to the notion of "blowing up" the space at a given point. This is the notion of a tangent.

DEFINITION 5.13. If (X, d) is a metric space and $x \in X$, a *tangent* of X at x is any limit of pointed metric spaces of the form $(X, \lambda_k d, x)$, where $\lambda_k \to \infty$.

If $f : (X, d) \to (Y, \rho)$ is a mapping and $x \in X$, a *tangent* of f is any limit of mapping packages of the form

$$((X, \lambda_k d, x), (Y, \lambda_k \rho, f(x)), f),$$

where $\lambda_k \to \infty$.

We denote the collection of tangents of X at x by $\mathrm{Tan}(X,x)$, and the collections of tangents of f at x by $\mathrm{Tan}(f,x)$.

Of course, tangents are always weak tangents. Proposition 5.7 guarantees that a doubling metric space has at least one tangent at each of its points, and that a Lipschitz mapping has a tangent at each point of its domain.

If F is a non-empty, closed subset of $\mathbb{R}^n$, one may wish to pass to a tangent or weak tangent of F inside $\mathbb{R}^n$, rather than viewing it simply as an abstract metric space. We have the following simple consequence of the results above.

LEMMA 5.14. *Let $F \subseteq \mathbb{R}^n$ be a non-empty, closed set, and let $f\colon F \to \mathbb{R}^m$ be a Lipschitz mapping. Let $\{x_j\}$ be a sequence of points in F and $\{\lambda_j\}$ be a sequence of positive numbers. Then:*

(i) *We may pass to a subsequence along which the sets*

$$\lambda_j(F - x_j) \tag{5.2}$$

and the mappings

$$z \mapsto \lambda_j(f(\lambda_j^{-1}z + x_j) - f(x_j)) \tag{5.3}$$

converge to a set $\hat{F} \subseteq \mathbb{R}^n$ and a mapping $\hat{f}\colon \hat{F} \to \mathbb{R}^m$, in the sense of Definitions 5.1 *and* 5.2. *Furthermore, $(\hat{F}, 0)$ is in* $\mathrm{WTan}(F)$ *and the mapping package*

$$((\hat{F},0),(\mathbb{R}^m,0),\hat{f})$$

is in $\mathrm{WTan}(f)$.

(ii) *Conversely, if $((Z,p),(\mathbb{R}^m,0),h) \in \mathrm{WTan}(f)$ arises from the choice of points $\{x_j\}$ and scales $\{\lambda_j\}$, then, after passing to a subsequence, the sets in* (5.2) *converge to a set $\hat{F}$ isometric to (Z,p) and the mappings in* (5.3) *converge to a mapping on $\hat{F}$ that agrees with h, up to composition with an isometry.*

Proof. This is an immediate consequence of Lemma 5.3 and Proposition 5.7, and the fact that limits of isometric spaces and mappings are themselves isometric. (See [DS97, Lemma 8.12].) ∎

5.3. Gromov–Hausdorff limits of Lipschitz light mappings. In this subsection, we study the Gromov–Hausdorff convergence properties of Lipschitz light mappings, culminating in a characterization result, Theorem 5.19, and corresponding consequences for Lipschitz dimension.

We begin by establishing the persistence of the Lipschitz light property during Gromov–Hausdorff convergence.

PROPOSITION 5.15. *Let $\{((X_k,d_k,p_k),(Y_k,\rho_k,q_k),f_k)\}$ be a sequence of complete, uniformly doubling mapping packages converging to* $\{((X,d,p),$

$(Y,\rho,q),f)\}$. *Assume that each* f_n *is Lipschitz light with constant* C, *independent of* k. *Then* f *is Lipschitz light with constant* C.

Proof. To begin, the map f is C-Lipschitz by Lemma 5.11.

Consider a sequence of mapping packages as in the proposition. We have "almost-isometries" $\phi_k\colon X\to X_k$, $\psi_k\colon X_k\to X$, $\sigma_k\colon Y\to Y_k$, and $\tau_k\colon Y_k\to Y$ as in Proposition 5.9.

Fix $r>0$ and $W\subseteq Y$ with $\operatorname{diam}(W)\leq r$. Let U be an r-component of $f^{-1}(W)\subseteq X$, and P an arbitrary r-path in U.

We can choose $R>0$ large enough so that $P\subseteq B(p,R/2)$ and $P_k:=\phi_k(P)\subseteq B(p_k,R/2)$ for each $k\in\mathbb{N}$. Let $\epsilon\in(0,r)$ be arbitrary. We may then choose $k\in\mathbb{N}$ sufficiently large so that all distortions of $\phi_k,\psi_k,\sigma_k,\tau_k$ are less than ϵ. In other words,

$$\begin{aligned}
&\sup\{d(\psi_k(\phi_k(x)),x) : x\in B_X(p,R)\}<\epsilon,\\
&\sup\{d_k(\phi_k(\psi_k(x)),x) : x\in B(p_k,R)\}<\epsilon,\\
&\sup\{|d_k(\phi_k(x),\phi_k(y))-d(x,y)| : x,y\in B(p,R)\}<\epsilon,\\
&\sup\{|d(\psi_k(x),\psi_k(y))-d_k(x,y)| : x,y\in B(p_k,R)\}<\epsilon,
\end{aligned}$$

with analogous properties for τ_k and σ_k. We may furthermore ensure that

$$|\tau_k(f_k(\phi_k(x)))-f(x)|<\epsilon \quad \text{for all } x\in B(p,R).$$

Then $P_k=\phi_k(P)$ is an $(r+2\epsilon)$-path in X_k. Moreover,

$$\operatorname{diam}(f_k(P_k))\leq \operatorname{diam}(\tau_k(f_k(P_k)))+2\epsilon\leq \operatorname{diam}(f(P))+4\epsilon\leq r+4\epsilon.$$

Since f_k is Lipschitz light with constant C and $f_k(P_k)$ is a set of diameter at most $r+4\epsilon$, it follows that

$$\operatorname{diam}(P_k)\leq C(r+4\epsilon).$$

Lastly, we have

$$\operatorname{diam}(P)\leq \operatorname{diam}(P_k)+2\epsilon\leq C(r+4\epsilon)+2\epsilon.$$

Letting ϵ tend to 0, we get

$$\operatorname{diam}(P)\leq Cr,$$

which proves that f is Lipschitz light with constant C. ■

COROLLARY 5.16. *If* X *and* Y *are complete and doubling, and* $f\colon X\to Y$ *is Lipschitz light with constant* C, *then so is each* $\hat{f}\in\operatorname{WTan}(f)$.

Proof. We need only observe that the mapping

$$f:(X,\lambda d)\to(Y,\lambda\rho)$$

is also Lipschitz light with constant C, for each $\lambda>0$, and then apply Proposition 5.15. ■

As an immediate consequence, we show that Lipschitz dimension cannot increase when passing to weak tangents.

COROLLARY 5.17. *If X is doubling,* $\dim_L(X) \leq n$ *and* $(Z,z) \in \mathrm{WTan}(X)$, *then* $\dim_L(Z) \leq n$.

Proof. If $f\colon X \to \mathbb{R}^n$ is Lipschitz light, then by Proposition 5.7 there is a mapping $\hat{f} \in \mathrm{WTan}(f)$ from Z to $\mathbb{R}^n$. It is Lipschitz light by Corollary 5.16. ∎

In fact, we can also characterize Lipschitz light mappings among all Lipschitz mappings by examining their weak tangents.

Before we do so, we need the following lemma.

LEMMA 5.18. *Let* $\{(X_n, d_n, p_n)\} \to (X, d, p)$ *be a converging sequence of complete, uniformly doubling pointed metric spaces. Suppose that for each n, there is a δ_n-path* $P_n \subseteq \overline{B}(p_n, 1) \subseteq X_n$ *containing p_n, and* $\delta_n \to 0$. *Then, after passing to a subsequence,*

$$\{(P_n, d_n, p_n)\} \to (P, d, p),$$

where $P \subseteq X$ is compact and connected.

Proof. The existence of a subsequence under which P_n converges to a compact set $P \subseteq X$ is ensured by Lemma 5.10. Assume that P is not connected. It follows that P can be written as $A \cup B$, where $\epsilon := \mathrm{dist}(A, B) > 0$.

Fix mappings $\phi_n\colon P \to P_n$ and $\psi_n : P_n \to P$ as in Proposition 5.8. By choosing n large, we may ensure that

$$\delta_n < \epsilon/10, \quad |d_n(\phi_n(x), \phi_n(y)) - d(x,y)| < \epsilon/10, \quad |d_n(\psi_n(\phi_n(x)), x)| < \epsilon/10$$

for all $x, y \in P$.

Fix $a \in A \subseteq P$ and $b \in B \subseteq P$. Then $\phi_n(a)$ and $\phi_n(b)$ are in P_n, so there is a δ_n-path

$$(\phi_n(a) = x_n^1, \dots, \phi_n(b) = x_n^m) \subseteq P_n$$

between them. Then

$$(a, \psi_n(x_n^1), \psi_n(x_n^2), \dots, \psi_n(x_n^m), b) \subseteq P$$

is a $\frac{3\epsilon}{10}$-path from a to b in P. But this is impossible, since $\mathrm{dist}(A, B) = \epsilon$. ∎

THEOREM 5.19. *Let $f\colon X \to Y$ be a Lipschitz mapping between complete, doubling metric spaces. Then the following are equivalent:*

(i) *f is Lipschitz light.*
(ii) *Each weak tangent of f is Lipschitz light.*
(iii) *Each weak tangent of f is light.*

Proof. We have already shown in Corollary 5.16 that (i) implies (ii). Since Lipschitz light maps are automatically light, (ii) immediately implies (iii). It remains to show that (iii) implies (i).

Suppose that every weak tangent of f is light, but that f is not Lipschitz light. That means that, for each $n \in \mathbb{N}$, we have

- a positive number r_n,
- a subset $W_n \subseteq Y$ with $\operatorname{diam}(W_n) \le r_n$, and
- an r_n-path $P_n \subseteq f^{-1}(W_n)$ with $\operatorname{diam}(P_n) \ge n r_n$.

Let x_n be the initial point of P_n. We consider the following sequence of mapping packages:

$$\left\{\left(\left(X, \frac{1}{\operatorname{diam}(P_n)} d, x_n\right), \left(Y, \frac{1}{\operatorname{diam}(P_n)}\rho, f(x_n)\right), f\right)\right\}.$$

By Proposition 5.7, a subsequence of this sequence converges to a mapping package

$$\{((\hat{X}, \hat{d}, \hat{x}), (\hat{Y}, \hat{\rho}, \hat{f}(\hat{x})), f)\}$$

in $\operatorname{WTan}(f)$.

In the space $\left(X, \frac{1}{\operatorname{diam}(P_n)} d, x_n\right)$, P_n is a $\frac{1}{n}$-path of diameter exactly 1. By passing to a further subsequence, we may assume that P_n converges to a connected subset $\hat{P} \subset \hat{X}$, as in Lemma 5.18. Furthermore, $\operatorname{diam}(\hat{P}) = 1$, since $\operatorname{diam}(\hat{P_n}) = 1$ for each n.

On the other hand, $f(P_n) \subseteq W_n$ has diameter at most $\frac{1}{n}$ in $\left(Y, \frac{1}{\operatorname{diam}(P_n)}\rho\right)$, and therefore $\hat{f}(\hat{P})$ is a single point in $\hat{Y}$.

Thus, $\hat{f}$ collapses the non-trivial connected set $\hat{P}$ to a point. It follows that $\hat{f}$ is not light, contradicting our assumption (iii). ∎

6. Lipschitz dimensions of various spaces. In this section, we use a variety of techniques to compute or bound the Lipschitz dimension of a number of spaces. The main concrete results of interest are Corollary 6.3 concerning trees and buildings, Corollary 6.5 on snowflakes of Euclidean spaces, Theorem 6.8 concerning Carnot groups, and Theorem 6.16 covering certain fractals in Euclidean space.

6.1. Trees and Euclidean buildings. In this section, we study two classes of non-positively curved spaces: metric trees and Euclidean buildings. A *metric tree* is a geodesic metric space such that all geodesic triangles are degenerate. In other words, a metric tree is a space T such that any two points $x, y \in T$ can be joined by a curve γ_{xy} of length $d(x,y)$, and if $x, y, z \in T$ then $\gamma_{xz} \subseteq \gamma_{xy} \cup \gamma_{yz}$. In particular, as in [LS05], no compactness or local finiteness is assumed, so a metric tree may have arbitrarily large Hausdorff dimension, for example.

The definition of Euclidean building would take us rather far afield here, so we refer those who are interested to [LPS00, Section 6] or [KL97] for details. The definition of Euclidean building will not directly enter our arguments in this section; we just use two results from [LS05] about trees and buildings.

In [LS05, Lemma 3.1], Lang and Schlichenmaier study mappings that satisfy certain technical conditions (those in Lemma 6.1 below), and show that such mappings cannot decrease Nagata dimension. They then construct such mappings from trees and buildings into $\mathbb{R}^n$, in order to bound the Nagata dimensions of these spaces.

We show that the mappings studied by Lang and Schlichenmaier are in fact Lipschitz light.

LEMMA 6.1. *Suppose $f\colon X \to Y$ is 1-Lipschitz and $h\colon X \times [0,\infty) \to X$ are mappings with the following three properties, for some $\lambda, \mu > 0$:*

(i) *Whenever $C \subset Y$ is non-empty and bounded, there exists $y \in Y$ with*
$$f^{-1}(C) \subset \overline{N}_{\lambda \operatorname{diam}(C)}(f^{-1}(y)).$$
(ii) *For all $x \in X$ and $t \geq 0$, $d(h(x,t),x) \leq t$.*
(iii) *If $f(x) = f(x')$ and $t \geq \mu d(x,x')$, then $h(x,t) = h(x',t)$.*

Then f is Lipschitz light.

Proof. Consider any $C \subset Y$ with $\operatorname{diam}(C) \leq r$. Consider also any r-path $(x_1,\ldots,x_n)$ in $f^{-1}(C)$. By (i), there is a corresponding $(1+2\lambda)r$-path $(z_1,\ldots,z_n) \subset f^{-1}(y)$ for some $y \in Y$, with $d(x_i,z_i) \leq \lambda r$ for each i.

By (iii), we see that
$$h(z_i, \mu(1+2\lambda)r) = h(z_{i+1}, \mu(1+2\lambda)r)$$
for each $i \in \{1,\ldots,n-1\}$. So
$$h(z_i, \mu(1+2\lambda)r) = h(z_j, \mu(1+2\lambda)r)$$
for each $i,j \in \{1,\ldots,n\}$. Thus, there is a point $p \in X$ with
$$h(z_i, \mu(1+2\lambda)r) = p \quad \text{for each } i \in \{1,\ldots,n\}.$$

It follows from (ii) that
$$d(z_i,p) = d\big(z_i, h(z_i,\mu(1+2\lambda)r)\big) \leq \mu(1+2\lambda)r \quad \text{for each } i \in \{1,\ldots,n\},$$
and so
$$\operatorname{diam}(\{z_1,\ldots,z_n\}) \leq 2\mu(1+2\lambda)r.$$
Therefore,
$$\operatorname{diam}(\{x_1,\ldots,x_n\}) \leq 2\mu(1+2\lambda)r + 2\lambda r = 2(\mu + 2\mu\lambda + \lambda)r$$
and so f is Lipschitz light. ∎

It follows immediately from Lemma 4.1(ii) that if X and Y are as in Lemma 6.1, then $\dim_L(X) \leq \dim_L(Y)$. We observe that Lang and Schlichenmaier in fact prove the following en route to Theorems 3.2 and 3.3 of [LS05].

Theorem 6.2 (Lang–Schlichenmaier [LS05, Theorems 3.2 and 3.3]). *Let T be a metric tree and let X be a Euclidean building of rank n. Then:*

(i) *There are maps $f_T : T \to \mathbb{R}$ and $h_T : T \times \mathbb{R} \to T$ satisfying the assumptions of Lemma* 6.1.
(ii) *There are maps $f_X : X \to \mathbb{R}^n$ and $h_X : X \times \mathbb{R} \to X$ satisfying the assumptions of Lemma* 6.1.

As a consequence, we have:

Corollary 6.3. *Let X be a product of n (non-trivial) metric trees or a Euclidean building of rank n. Then the Lipschitz dimension of X is n.*

Proof. By Lemma 6.1 and Theorem 6.2, a Euclidean building of rank n has Lipschitz dimension at most n, and a metric tree has Lipschitz dimension at most 1.

Since a Euclidean building of rank n contains an isometric copy of $\mathbb{R}^n$, its dimension must be n.

By Proposition 4.2 and the above, a product of n metric trees has Lipschitz dimension at most n. If each tree is non-trivial, the product contains an embedded copy of a cube in $\mathbb{R}^n$, and hence has Lipschitz dimension equal to n. ∎

6.2. Snowflakes of Euclidean spaces. Unlike Nagata dimension (see [LS05, Theorem 1.2]), Lipschitz dimension is not a quasisymmetric or even snowflake invariant, as we will discuss in Section 8. However, we have the following result:

Theorem 6.4. *For every $\epsilon \in (0,1]$, the Lipschitz dimension of the ϵ-snowflake $X := (\mathbb{R}, |\cdot|^\epsilon)$ is* 1.

Proof. The case $\epsilon = 1$ is immediate, so we assume $\epsilon < 1$.

By Assouad's theorem [Hei01, Theorem 12.2], there is a bi-Lipschitz embedding of X into some Euclidean space. Let n be the minimial dimension of a Euclidean space $\mathbb{R}^n$ into which X bi-Lipschitz embeds. Note that $n \geq 2$ since $\dim_H(X) > 1$.

Let $Y \subseteq \mathbb{R}^n$ be the image of X under such a bi-Lipschitz embedding. Since Lipschitz dimension is clearly a bi-Lipschitz invariant, it suffices to show that Y has Lipschitz dimension 1.

Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first coordinate. We claim that $\pi|_Y$ is Lipschitz light.

Suppose not. Then, by Theorem 5.19, there is a weak tangent

$$((Z,z),(\mathbb{R},0),\hat{\pi}) \in \mathrm{WTan}(\pi|_Y)$$

such that $\hat{\pi}$ is not light.

By Lemma 5.14, the weak tangent package above may be viewed as a limit of rescalings inside $\mathbb{R}^n$. In other words, there is an isometry ι from Z

onto a set $\hat{Y} \subseteq \mathbb{R}^n$ with $\iota(z) = 0$. Moreover, since rescalings and translations like those in Lemma 5.14 do not affect the linear map π, we have $\hat{\pi} = \pi \circ \iota$.

Thus, we have a weak tangent

$$((\hat{Y} \subseteq \mathbb{R}^n, 0), (\mathbb{R}, 0), \pi) \in \mathrm{WTan}(\pi|_Y)$$

such that the linear projection π is constant on a connected subset E of $\hat{Y}$.

Since Y is bi-Lipschitz equivalent to $X = (\mathbb{R}, |\cdot|^\epsilon)$, the space $\hat{Y}$ is also bi-Lipschitz equivalent to $(\mathbb{R}, |\cdot|^\epsilon)$ by Lemma 5.11. Therefore, any compact, connected subset F of $E \subseteq \hat{Y}$ is bi-Lipschitz equivalent to $([-1,1], |\cdot|^\epsilon)$.

Since F is contained in $\pi^{-1}(p)$ (for some $p \in \mathbb{R}$), which is isometric to $\mathbb{R}^{n-1}$, there is a bi-Lipschitz embedding

$$h\colon ([-1,1], |\cdot|^\epsilon) \to \mathbb{R}^{n-1}.$$

The mappings

$$t \mapsto \lambda^\epsilon h(t/\lambda)\colon ([-\lambda, \lambda], |\cdot|^\epsilon) \to \mathbb{R}^{n-1}$$

are then uniformly bi-Lipschitz, and so subconverge to a bi-Lipschitz embedding of $X = (\mathbb{R}, |\cdot|^\epsilon)$ into $\mathbb{R}^{n-1}$ as $\lambda \to \infty$.

This contradicts our choice of n as the minimal integer such that X admits a bi-Lipschitz embedding into $\mathbb{R}^n$. Thus, $\pi|_Y$ must in fact have been Lipschitz light, which implies $\dim_L(X) = \dim_L(Y) \leq 1$. Of course, $\dim_L(X) \geq 1$ by Corollary 3.5 and (3.1). ∎

Corollary 6.5. *The Cartesian product of n snowflakes of $\mathbb{R}$ has Lipschitz dimension n. In particular, each snowflake of $\mathbb{R}^n$ has Lipschitz dimension n.*

Proof. By Theorem 6.4 and Proposition 4.2, the Cartesian product of n snowflakes of $\mathbb{R}$ has Lipschitz dimension at most n. It has Lipschitz dimension at least its topological dimension (by Corollary 3.5 and (3.1)), which is also n.

For the second statement, we simply observe that $(\mathbb{R}^n, |\cdot|^\epsilon)$ is bi-Lipschitz equivalent to the Cartesian product of n copies of $(\mathbb{R}, |\cdot|^\epsilon)$. ∎

We can also prove a result about the Lipschitz dimensions of more general snowflakes in Euclidean space.

Theorem 6.6. *Let $E \subseteq \mathbb{R}^n$ be a closed set that is an α-snowflake for some $\alpha \in (0,1)$. Let k be an integer with*

$$k > n - \frac{1}{\alpha}.$$

Then

$$\dim_L(E) \leq k.$$

Proof. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be an arbitrary choice of linear projection, which of course is 1-Lipschitz. We claim that $\pi|_E$ is Lipschitz light.

Suppose not. Then, exactly as in Theorem 6.4, we find a set $\hat{E} \subseteq \mathbb{R}^n$ such that

$$((\hat{E}, 0), (\mathbb{R}^k, 0), \pi) \in \mathrm{WTan}(\pi|_E),$$

and such that π is constant on a non-trivial connected subset $C \subseteq \hat{E}$. Thus, C is contained in some $\pi^{-1}(p)$, which is an $(n-k)$-plane in $\mathbb{R}^n$.

On the other hand, $\hat{E}$ is also an α-snowflake. Indeed, if E is the image of a metric space (Z, d) under an α-snowflake map with constant C, then λE is the image of $(Z, \lambda^{1/\alpha} d)$ under an α-snowflake map with constant C. It then follows from Proposition 5.7 and Lemma 5.11 that $\hat{E}$ is the image of a metric space $\hat{Z}$ under an α-snowflake map.

Since $\hat{E}$ is an α-snowflake, each connected subset of $\hat{E}$ has Hausdorff dimension at least $1/\alpha$. (A connected set always has Hausdorff dimension at least 1, and α-snowflake maps multiply Hausdorff dimension by $1/\alpha$.)

Thus, using our assumption, we get

$$\dim_H(C) = 1/\alpha > n - k = \dim_H(\pi^{-1}(p)),$$

which contradicts our observation that $C \subseteq \pi^{-1}(p)$.

Therefore, $\pi|_E$ must have been Lipschitz light, forcing $\dim_L(E) \leq k$. ∎

6.3. Carnot groups. The so-called Carnot groups are central objects of study in the modern theory of analysis on metric spaces and non-smooth calculus. We begin this subsection with a very brief introduction to Carnot groups, referring the reader elsewhere for more background. We then show that non-abelian Carnot groups have infinite Lipschitz dimension, and proceed by discussing some consequences of this fact. We thank Bruce Kleiner for pointing out to us a number of years ago that Carnot groups should have infinite Lipschitz dimension.

6.3.1. *Background on Carnot groups.* We give a very brief background summary on Carnot groups. For more, we refer the reader to [Mon02, CD+07, LD17]. Very little of the Lie group structure of Carnot groups is directly used in our arguments below, but it is necessary to set the stage.

A *Carnot group* is a simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s,$$

where the first layer V_1 generates the rest via $V_{i+1} = [V_1, V_i]$ for all $1 \leq i \leq s$, and we set $V_{s+1} = \{0\}$.

Given an inner product $\langle \cdot, \cdot \rangle$ on the horizontal layer V_1, the associated sub-Riemannian Carnot–Carathéodory metric d on $\mathbb{G}$ is defined by

$$d(x, y) = \inf \Big\{ \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle^{1/2} \, dt : \gamma \text{ a horizontal curve joining } x \text{ to } y \Big\},$$

where an absolutely continuous curve $\gamma\colon [0,1] \to \mathbb{G}$ is called *horizontal* if $\gamma'(t) \in V_1$ for a.e. $t \in [0,1]$.

Like the standard Euclidean metric, which is just the special case in which the stratification has a single layer, the Carnot–Carathéodory distance d is invariant under left-translations, the maps L_x defined by $L_x(p) = x \cdot p$. It also admits a family of dilations: For each $\lambda > 0$, there is a homeomorphism $\delta_\lambda\colon \mathbb{G} \to \mathbb{G}$ such that

$$d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y) \quad \text{for each } x, y \in \mathbb{G}.$$

Together, these facts imply that every element of $\mathrm{WTan}(\mathbb{G})$ is pointedly isometric to $(\mathbb{G}, 0)$ itself.

The simplest non-abelian Carnot group is the (first) Heisenberg group $\mathbb{H}$. The underlying manifold of $\mathbb{H}$ is $\mathbb{R}^3$, and its Lie algebra $\mathfrak{h}$ can be written

$$\mathfrak{h} = V_1 \oplus V_2,$$

where $\dim(V_1) = 2$, $\dim(V_2) = 1$, $[V_1, V_1] = V_2$, and $[V_1, V_2] = 0$. In exponential coordinates, $\mathbb{H}$ can be viewed as $\mathbb{C} \times \mathbb{R}$ with the group law

$$(z,t) \times (z',t') = \left(z + z', t + t' - \tfrac{1}{2}\operatorname{Im}(z\overline{z'})\right).$$

On the Heisenberg group, the Korányi metric

$$d_K(p,q) = \|q^{-1}p\|,$$

where

$$\|(z,t)\| = (|z|^4 + 16t^2)^{1/4},$$

yields a bi-Lipschitz equivalent distance to d (see [CD$^+$07, p. 18]). If we define the standard projection $\pi\colon \mathbb{H} \to \mathbb{C} \cong \mathbb{R}^2$ by $\pi(z,t) = z$, we see that π is Lipschitz and that $\pi^{-1}(y)$ is a snowflake of $\mathbb{R}$ for each $y \in \mathbb{R}$.

The main result about Carnot groups that we will use is the celebrated Pansu differentiation theorem:

Theorem 6.7 (Pansu [Pan89]). *Let $f\colon \mathbb{G}_1 \to \mathbb{G}_2$ be a Lipschitz map between Carnot groups. Then for almost every $x \in \mathbb{G}_1$, the sequence of maps*

$$\delta_\lambda \circ (L_{f(x)^{-1}} \circ f \circ L_x) \circ \delta_{\lambda^{-1}}$$

converges uniformly on compact sets, as $\lambda \to \infty$, to a Lie group homomorphism $Df(x)\colon \mathbb{G}_1 \to \mathbb{G}_2$ that commutes with dilations.

We will use Theorem 6.7 in the case where $\mathbb{G}_1$ is non-abelian and $\mathbb{G}_2$ is a Euclidean space $\mathbb{R}^n$. In that case, $Df(x)$ must collapse the (connected) commutator subgroup of $\mathbb{G}_1$. Also, in the setting of Theorem 6.7,

$$((\mathbb{G}_1, 0), (\mathbb{G}_2, 0), Df(x)) \in \mathrm{Tan}(f,x) \subseteq \mathrm{WTan}(f).$$

Carnot groups are doubling metric spaces. (See [LD17, p. 116] and note that Ahlfors regular spaces are always doubling.) Therefore, they have finite Hausdorff and Assouad dimensions. In addition, their Nagata dimension is equal to their topological dimension [LDR15, Theorem 4.2], and hence also finite, though generally smaller.

Nonetheless, in the next section, we show that non-abelian Carnot groups have infinite Lipschitz dimension.

6.3.2. *Lipschitz dimension of Carnot groups*

THEOREM 6.8. *If* $\mathbb{G}$ *is a non-abelian Carnot group, then* $\dim_L(\mathbb{G}) = \infty$.

Proof. Suppose there is a Lipschitz light map $f\colon \mathbb{G} \to \mathbb{R}^n$. Then by Theorem 6.7, and the remark following it, there is

$$((\mathbb{G}, 0), (\mathbb{R}^n, 0), Df(x)) \in \mathrm{WTan}(f)$$

such that $Df(x)$ is a group homomorphism that commutes with dilations. In particular, $Df(x)$ must collapse the (connected) commutator subgroup of $\mathbb{G}$ to a point.

However, the mapping $Df(x)$ is Lipschitz light by Corollary 5.16, so cannot collapse a connected set to a point. This is a contradiction. ■

We note that the same result holds for positive-measure subsets of Carnot groups:

COROLLARY 6.9. *Let* $\mathbb{G}$ *be a non-abelian Carnot group and let* $K \subseteq \mathbb{G}$ *be compact with positive measure. Then* $\dim_L(K) = \infty$.

Proof. By [LD11, Proposition 3.1], there is a point $x \in K$ and a tangent $(\hat{K}, \hat{x}) \in \mathrm{Tan}(K, x)$ such that $\hat{K}$ is isometric to $\mathbb{G}$. It follows from Corollary 5.17 that

$$\dim_L(K) \geq \dim_L(\hat{K}) = \dim_L(\mathbb{G}) = \infty. \ ■$$

6.3.3. *Quasi-isometric non-embedding for Carnot groups.* As a corollary of Theorem 6.8, we prove a "coarse" non-embedding result for Carnot groups, Corollary 6.10. Our theorem overlaps with a result of Pauls [Pau01], but our approach is somewhat different.

We recall a notion from coarse geometry: A *quasi-isometric embedding* of a space X into a space Y is a (not necessarily continuous) map $g\colon X \to Y$ with constants $C \geq 1$ and $\epsilon > 0$ such that

$$C^{-1}d(x, x') - \epsilon \leq d(g(x), g(x')) \leq Cd(x, x') + \epsilon$$

for all $x, x' \in X$. Quasi-isometric embeddings are coarse generalizations of bi-Lipschitz embeddings.

Our methods give a short proof of the following result.

COROLLARY 6.10. *If* $\mathbb{G}$ *is a non-abelian Carnot group, then* $\mathbb{G}$ *does not admit a quasi-isometric embedding into any space of finite Lipschitz dimension. In particular,* $\mathbb{G}$ *does not admit a quasi-isometric embedding into any finite product of trees or finite-rank Euclidean building.*

The statement about trees and buildings in Corollary 6.10 already follows from a general result of Pauls [Pau01, Theorem C]. Both approaches rely at heart on Pansu's theorem. One advantage of our short approach is that it does not require proving a "metric differentiation" form of Pansu's theorem (see [Pau01, Theorem 4.7]) but rather relies directly on the original result of Pansu. On the other hand, Pauls' result allows for quite general targets, including infinite-dimensional spaces, which our result does not address.

Proof of Corollary 6.10. Suppose $\mathbb{G}$ admits a quasi-isometric embedding $g\colon \mathbb{G} \to Y$, where Y has finite Lipschitz dimension. There are constants $C \geq 1$ and $\epsilon > 0$ such that

$$C^{-1}d(x,x') - \epsilon \leq d(g(x), g(x')) \leq Cd(x,x') + \epsilon$$

for all $x, x' \in \mathbb{G}$.

Let N be a $2C\epsilon$-net in $\mathbb{G}$ containing 0. On the one hand, $g|_N$ is easily seen to be a bi-Lipschitz embedding of N into Y, and therefore N has finite Lipschitz dimension.

On the other hand, the pointed spaces $(\delta_{1/k}(N), 0)$ converge to the pointed space $(\mathbb{G}, 0) \in \mathrm{WTan}(N)$ as $k \in \mathbb{N}$ tends to infinity. It follows from Corollary 5.17 that

$$\dim_L(\mathbb{G}) \leq \dim_L(N) < \infty,$$

which contradicts Theorem 6.8. Therefore, there can be no such quasi-isometric embedding g.

The statement about trees and buildings now follows from Corollary 6.3. ∎

6.3.4. *Carnot groups as counterexamples.* We close this discussion of Carnot groups by observing that they, in particular the first Heisenberg group $\mathbb{H}$, provide counterexamples to two natural hopes for Lipschitz dimension.

First of all, in contrast to Proposition 4.3, we observe that the finiteness of Lipschitz dimension is not stable under countable unions, even locally finite ones. Indeed, consider a 1-net N in the Heisenberg group $\mathbb{H}$, with $0 \in N$. Exactly as in the proof of Corollary 6.10, we must have $\dim_L(N) = \infty$, even though N is countable.

Next, we observe that the Heisenberg group also serves as a counterexample to any "Hurewicz-type" theorem for Lipschitz dimension. Recall first the classical Hurewicz theorem for topological dimension, which we state in the compact case: If $f\colon X \to Y$ is a continuous map between compact metric spaces, then

$$\dim_T(X) \leq \dim_T(Y) + \sup\{\dim_T(f^{-1}(y)) : y \in Y\}$$

(see, for example, [Nag83, Theorem III.6]).

No such result holds with $\dim_L$ replacing $\dim_T$: Let X denote the closed unit ball in the Heisenberg group $\mathbb{H}$, let $\pi\colon X \to Y := \mathbb{R}^2$ denote the restriction to X of the standard projection from $\mathbb{H}$ to $\mathbb{R}^2$. Then $\dim_T(Y) = 2$. Furthermore, for each $y \in \mathbb{R}^2$, $\pi^{-1}(y)$ is contained in a space that is bi-Lipschitz equivalent to a $\frac{1}{2}$-snowflake of $\mathbb{R}$, and hence has Lipschitz dimension ≤ 1 by Theorem 6.4. However, X itself has infinite Lipschitz dimension.

6.4. Subsets of $\mathbb{R}$. In this subsection, we characterize the Lipschitz dimension of subsets of $\mathbb{R}$ by a simple metric property.

Definition 6.11. A set E in a metric space X is called *porous*, with constant $c > 0$, if for every $x \in E$ and $r > 0$, there is a point y with

$$B(y, cr) \subseteq B(x, r) \setminus E.$$

Proposition 6.12. *Let $E \subseteq \mathbb{R}$. Then the following are equivalent:*

(i) *E has Lipschitz dimension* 0.
(ii) *Every weak tangent of E is totally disconnected.*
(iii) *E is porous.*

Proof. First, assume E has Lipschitz dimension 0. Then every weak tangent of E has Lipschitz dimension 0, by Corollary 5.17, hence topological dimension 0, hence is totally disconnected. Thus, (i) implies (ii).

Suppose E satisfied condition (ii) but E was not porous. Then we could find balls $B(x_i, r_i) \subseteq \mathbb{R}$, with $x_i \in E$, such that

$$N_{1/i}(\overline{B}(x_i, r_i) \cap E) \supseteq \overline{B}(x_i, r_i).$$

Passing to a weak tangent of E along the sequence of scales $\lambda_i = 1/r_i$ and the sequence of points x_i, we obtain a weak tangent $(\hat{E}, 0) \in \mathrm{WTan}(E)$ such that $\hat{E}$ contains an isometric copy of $[-1, 1]$. This contradicts (ii), proving that (ii) implies (iii).

Finally, suppose $E \subseteq \mathbb{R}$ is porous. Then no weak tangent of E contains a non-trivial interval. Indeed, suppose $\{(\lambda_k E, x_k)\}$ converged to a pointed metric space $(\hat{E}, 0)$ containing a non-trivial interval. Then, along a subsequence, the sets $\{\lambda_k(E - x_k)\}$ would converge in $\mathbb{R}$ to a set F that is isometric to $\hat{E}$, and so contains a non-trivial interval. It would follow that, for arbitrarily large λ, there is an interval I_λ such that $E \cap I_\lambda$ is $\frac{1}{\lambda}$-dense in I_λ, which contradicts the porosity of E.

Thus, every weak tangent mapping of the constant map $\kappa\colon E \to \mathbb{R}^0$ is light, and hence $\kappa : E \to \mathbb{R}^0$ is Lipschitz light. Therefore (iii) implies (i). ∎

For sets in $\mathbb{R}^n$, we do not know whether having Lipschitz dimension $\leq n - 1$ is equivalent to porosity.

One direction is clear: If a set in $\mathbb{R}^n$ has Lipschitz dimension $\leq n - 1$, it must be porous. If it is not, then by an argument similar to that in Proposition 6.12, it has a weak tangent containing an isometric copy of a ball in $\mathbb{R}^n$, contradicting Corollary 5.17.

QUESTION 6.13. *Is it true that a set $E \subseteq \mathbb{R}^n$ has Lipschitz dimension $\leq n-1$ if and only if it is porous?*

REMARK 6.14. The answer to Question 6.13 is "yes" if one replaces Lipschitz dimension by Nagata dimension. One direction (Nagata dimension $\leq n-1$ implies porosity) follows from the same argument as in the remark above Question 6.13, since Nagata dimension can also only drop under weak tangents [LDR15, Proposition 2.18]. For the other direction, it is well-known that porous subsets of $\mathbb{R}^n$ have Assouad dimension $< n$ and hence Nagata dimension $\leq n-1$ by [LDR15, Theorem 1.1].

6.5. Self-covering sets and classical fractals. In this subsection, our goal is to show that some classical fractals in the plane have Lipschitz dimension 1. As concrete examples, our results apply to the standard Sierpiński carpets S_p, indexed by odd integers $p \geq 3$. Recall that for such p, the "first generation" $S_p^1 \subseteq \mathbb{R}^2$ is formed by dividing the unit square into axis-parallel subsquares of side length $\frac{1}{p}$ in the usual way and removing the central square. The nth generation S_p^n is formed by doing the same procedure on each of the squares of side length $p^{-(n-1)}$ remaining in S_p^{n-1}. The Sierpiński carpet S_p is defined as $\bigcap_{n=1}^{\infty} S_p^n$.

Our results will also apply to the standard Sierpiński gasket, similarly formed by starting with an equilateral triangle in the plane, dividing it into four congruent equilateral triangles, removing the central triangle, and then iterating this construction on the remaining three triangles of half the size. See, for example, [DS97, pp. 7–8] for pictures and descriptions of these constructions.

We frame our result for a certain class of sets that includes the above examples, which we now describe. For a compact set K, a *rescaled translate* of K is a set of the form $sK + v$ for some $s > 0$ and $v \in \mathbb{R}^n$.

DEFINITION 6.15. We call a compact set $K \subseteq \mathbb{R}^n$ *self-covering* if there are constants $N \in \mathbb{N}$ and $C > 0$ such that the following holds: For each $x \in K$ and $r > 0$, there are rescaled translates $K_1, \ldots, K_M$ of K such that

- $M \leq N$,
- $\operatorname{diam}(K_i) \leq Cr$, and
- $\overline{B}(x,r) \cap K \subseteq \cup K_i$.

In other words, a set K is self-covering if every ball of radius r in K can be covered by a controlled number of rescaled copies of K of size at most Cr. Note that we allow the copies of K covering $B(x,r) \cap K$ to overlap arbitrarily and to contain points outside of K, but we do not allow rotations.

It is easy to see that the Sierpiński carpets S_p and the Sierpiński gasket are self-covering. On the other hand, the self-covering property is somewhat different from standard notions of self-similarity; for example, the set $[0,1] \cup [2,3]$ in $\mathbb{R}$ is self-covering. For non-examples, we point out that the set $\{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ in $\mathbb{R}$ and the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ are examples of non-self-covering sets.

Of course, the whole unit cube in $\mathbb{R}^n$ is also an example of a self-covering set, so we cannot expect to interestingly bound the Lipschitz dimension based only on the self-covering property. Our additional assumption is that the self-covering set does not contain any non-trivial line segments in some fixed direction.

THEOREM 6.16. *Let $K \subseteq \mathbb{R}^n$ be compact and self-covering, according to Definition* 6.15. *Assume that there exists $v \in \mathbb{S}^{n-1}$ such that K contains no non-trivial line segment in direction v. Then the Lipschitz dimension of K is at most $n-1$.*

Proof. Let K and v be as in the theorem. Assume without loss of generality that $\operatorname{diam}(K) = 1$ and $0 \in K$.

Let π denote the orthogonal projection from $\mathbb{R}^n$ onto the orthogonal complement V of v; of course, V is isometric to $\mathbb{R}^{n-1}$.

We claim that $\pi|_K$ is Lipschitz light, which will suffice to prove the theorem. The spirit of this argument is similar to some above that use Gromov–Hausdorff convergence. However, in this setting we need to be a bit careful *not* to identify isometric sets, as we want to avoid rotation.

Suppose that $\pi|_K$ is not Lipschitz light. Then for each $j \in \mathbb{N}$ there is a set $W_j \subseteq V$ of diameter at most r_j such that $\pi^{-1}(W_j)$ contains an r_j-path P_j with $R_j := \operatorname{diam}(P_j) \geq j r_j$.

Let $B_j = \overline{B}(x_j, R_j) \cap K$ be a ball in K of radius R_j containing P_j, where x_j is the initial point of P_j. By Definition 6.15, there are rescaled translates $K_j^1, \dots, K_j^{N_j}$ of diameter at most CR_j covering B_j, with $N_j \leq N$. Note that we may freely assume that each K_j^i actually intersects B_j, and therefore is contained in $\overline{B}(x_j, (C+1)R_j)$.

By passing to a subsequence, we may further assume that there is M in $\{1, \dots, N\}$ such that $N_j = M$ for all $j \in \mathbb{N}$.

For each $i \in \{1, \dots, M\}$, consider the sequence of sets

$$\frac{1}{R_j}(K_j^i - x_j).$$

This is a sequence of rescaled translates of K, all contained in $\overline{B}(0, C+1)$. Thus, we may pass to a subsequence (which we continue to label by j) such that for each i, this sequence converges in the Hausdorff sense (equivalently, in the sense of Definition 5.1) to a set K_∞^i that is a rescaled translate of K.

Indeed, each set $\frac{1}{R_j}(K_j^i - x_j)$ is simply $s_j^i K_j^i + v_j^i$ for some $s_j^i \in [0, 2(C+1)]$ and $v_j^i \in \overline{B}(0, C+1)$, so we may simply pass to a subsequence along which $\{s_j^i\}_{j=1}^\infty$ and $\{v_j^i\}_{j=1}^\infty$ both converge.

In particular, our assumption on K implies that no K_∞^i can contain a non-trivial line segment in direction v.

Let $K_\infty = \bigcup_{i=1}^M K_\infty^i$.

By passing to a further subsequence, we may also assume that the sets

$$\frac{1}{R_j}(P_j - x_j)$$

converge to a compact subset $P_\infty \subseteq \mathbb{R}^n$. We also claim that $P_\infty \subseteq K_\infty$: If $y \in P_\infty$, then $y = \lim y_j$ for some $y_j \in \frac{1}{R_j}(P_j - y_j)$. Each such y_j is in some $\frac{1}{R_j}(K_j^i - x_j)$. Therefore there is some $i_0 \in \{1, \dots, M\}$ such that $y_j \in \frac{1}{R_j}(K_j^{i_0} - x_j)$ for infinitely many j, from which it follows that $y \in K_\infty^i \subseteq K_\infty$.

By Lemma 5.18, P_∞ is a connected set, and it has diameter 1. Moreover,

$$\operatorname{diam}(\pi(P_\infty)) = \lim_{j\to\infty} \operatorname{diam}\left(\pi\left(\frac{1}{R_j}(P_j - x_j)\right)\right) \leq \lim_{j\to\infty} \frac{1}{j} = 0.$$

Thus, $P_\infty \subseteq K_\infty$ is a non-trivial line segment in direction v.

To conclude the proof, we now argue that in fact some K_∞^i must contain a non-trivial subsegment of P_∞. Indeed, if not, then $K_\infty^i \cap P_\infty$ has empty interior in P_∞ for each j. However, by the Baire Category Theorem, P_∞ cannot be the union of a finite collection of subsets with empty interior. Thus, some $K_\infty^i \cap P_\infty$ must contain a non-trivial subsegment of P_∞. Since K_∞^i is a rescaled translate of K, this is a contradiction. ∎

Corollary 6.17. *For each odd $p \in \mathbb{N}$, the Sierpiński carpets S_p have Lipschitz dimension* 1. *The same holds for the Sierpiński gasket G.*

Proof. The Sierpiński carpets S_p and the Sierpiński gasket G are easily seen to satisfy Definition 6.15. Moreover, S_p contains no non-trivial line segments in directions of irrational slope (see [DCT11, Corollary 4.5] or [CN14, Theorem 3.4]), and the gasket G clearly contains no non-trivial vertical line segments. Thus, these fractals all satisfy the conditions of Theorem 6.16 and so have Lipschitz dimension at most 1. Since each contains some non-trivial line segments, their Lipschitz dimensions must be equal to 1. ∎

7. Cheeger's analytic dimension. In this section, we describe Cheeger's version of Rademacher's theorem on certain non-smooth metric measure spaces, which equips them with a type of "measurable cotangent bundle", and we show that Lipschitz dimension bounds the dimension of this cotangent bundle.

7.1. Cheeger's differentiation theory and Lipschitz quotient mappings. Recall that Rademacher's theorem says that a Lipschitz mapping from $\mathbb{R}^n$ to $\mathbb{R}$ is differentiable almost everywhere, with respect to Lebesgue measure. In [Che99], Cheeger gave a far-reaching generalization of this result to a large class of non-smooth metric measure spaces. In order to do so, he defined a very general notion of differentiable structure on a metric measure space. (The name "Lipschitz differentiability space" used below for this notion was coined by Bate [Bat15].)

Definition 7.1 ([Che99]). A metric measure space (X, μ) is called a *Lipschitz differentiability space* if it satisfies the following condition: There are countably many Borel sets ("charts") U_i of positive measure covering X, positive integers n_i (the "dimensions of the charts"), and Lipschitz maps $\phi_i \colon X \to \mathbb{R}^{n_i}$ with respect to which any Lipschitz function $f \colon X \to \mathbb{R}$ is differentiable almost everywhere, in the sense that for each i and for μ-almost every $x \in U_i$, there exists a unique $df(x) \in \mathbb{R}^{n_i}$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - df(x) \cdot (\phi_i(y) - \phi_i(x))|}{d(x, y)} = 0. \tag{7.1}$$

Here $df(x) \cdot (\phi_i(y) - \phi_i(x))$ denotes the standard scalar product in $\mathbb{R}^{n_i}$.

Although the choice of charts (U_i, ϕ_i) is by no means unique, the numbers n_i are uniquely determined, in the sense that if (U, ϕ) and (V, ψ) are charts and $\mu(U \cap V) > 0$, then their associated dimensions must be the same. Thus, the numbers n_i reflect something about the geometry of the space (X, d, μ), which motivates the following, chart-independent, definition:

Definition 7.2. If (X, d, μ) is a Lipschitz differentiability space, we call the supremum of the numbers n_i from Definition 7.1 the *analytic dimension* of X, and denote it $\dim_C(X)$.

For a nice introduction to Cheeger's theory, we refer the reader to [KM18] and for more recent developments in the subject to [Bat15, Sch16, EB19]. For more specific results on the interaction between analytic dimension and metric geometry, which is an active area of research, we refer the reader to [Dav15, BL17, DK19, KS17].

The main theorem of [Che99] is that all the so-called "Poincaré inequality (PI) spaces" are Lipschitz differentiability spaces. Examples of these include Euclidean spaces and Carnot groups [Hei01], as well as a selection of more exotic examples appearing in [BP99, Laa00, CK15, KS17]. The full spectrum of possibilities does not seem to be well-understood yet.

A key tool in the study of Lipschitz differentiability spaces has been the following notion.

Definition 7.3 ([BJ$^+$99]). Let X and Y be metric spaces and $f\colon X\to Y$ a mapping. We say that f is a *Lipschitz quotient mapping* if there are constants $C,c>0$ such that

$$B(f(x),cr)\subseteq f(B(x,r))\subseteq B(f(x),Cr) \tag{7.2}$$

for all $x\in X$ and $r>0$.

Note that the second inclusion in (7.2) simply says that the mapping is C-Lipschitz. Lipschitz quotient mappings were first defined and studied in [BJ$^+$99, JL$^+$00], where the following path-lifting property was observed. (For a proof in the generality below, see [DK19, Lemma 3.3].)

Lemma 7.4. *Let X be a proper metric space and let $f\colon X\to Y$ be a Lipschitz quotient map. Let $\gamma\colon[0,T]\to Y$ be a 1-Lipschitz curve with $\gamma(0)=f(x)$. Then there is a Lipschitz curve $\tilde{\gamma}\colon[0,T]\to X$ with $\tilde{\gamma}(0)=x$ such that $f\circ\tilde{\gamma}=\gamma$.*

Lipschitz quotient maps enter the study of Lipschitz differentiability spaces through the following proposition. Independent proofs of this fact were given in [Sch16, Theorem 5.56] and (in the doubling case) [Dav15, Corollary 5.1]. A stronger statement appears in [CKS16, Theorem 1.11] but is not needed here.

Proposition 7.5. *Let (X,d,μ) be a complete, metrically doubling Lipschitz differentiability space with a chart $(U,\phi\colon X\to\mathbb{R}^k)$. Then for almost every $x\in X$ and every mapping package*

$$((\hat{X},\hat{x}),(\mathbb{R}^k,0),\hat{\phi})\in\operatorname{Tan}(\phi,x),$$

the mapping $\hat{\phi}$ is a Lipschitz quotient map of $\hat{X}$ onto $\mathbb{R}^k$.

Without the assumption that (X,d) is metrically doubling, one can still interpret $\operatorname{Tan}(\phi,x)$ with a bit of care and this proposition still holds (see [DK19, Remark 2.11]). However, we will not need this case below.

7.2. Lipschitz dimension bounds analytic dimension. It was proven in [Sch16, Corollary 5.99] and [Dav15, Corollary 8.5] that Assouad dimension is an upper bound for the analytic dimension of Lipschitz differentiability spaces. In fact, a stronger result is now known to hold: Hausdorff dimension is an upper bound for analytic dimension. This follows from [DP$^+$17, Theorem 1.1]; see also the approaches in [KM18] and [GP16].

On the other hand, it is a very interesting open question whether Nagata dimension bounds analytic dimension: see [KS17, Question 1.2].

We show here that Lipschitz dimension is an upper bound for analytic dimension. Note that by the results in Subsection 3.3, this neither implies nor is implied by the above-mentioned results concerning Assouad and Hausdorff dimensions.

THEOREM 7.6. *Let* (X, d, μ) *be a complete Lipschitz differentiability space. Then* $\dim_C(X) \le \dim_L(X)$.

Proof. Without loss of generality, we may assume $n := \dim_L(X) < \infty$, otherwise the theorem is trivial. Let $f\colon X \to \mathbb{R}^n$ be a Lipschitz light map.

Let k denote the analytic dimension of (X, d, μ), so that there is a chart

$$(U, \phi\colon X \to \mathbb{R}^k)$$

in X. Our goal is to show that $k \le n$, so assume to the contrary that $k > n$.

Lipschitz differentiability spaces satisfy a property called "pointwise doubling", which in particular implies that they can be covered up to measure zero by compact, metrically doubling subsets. (See [BS13, Corollary 2.6] and [Bat15, Lemma 8.3].) We can therefore find a compact, metrically doubling subset $A \subset U$ with $\mu(A) > 0$. Moreover, by [BS13, Corollary 2.7], the space (A, d, μ) is a complete Lipschitz differentiability space consisting of one chart $(A, \phi\colon A \to \mathbb{R}^k)$.

We now work entirely on A and forget about the rest of X. Of course, f restricts to a Lipschitz light map $f|_A\colon A \to \mathbb{R}^n$, which we continue to call f.

Choose a point $x \in A$ at which each of the n $\mathbb{R}$-valued component functions f_i of f are differentiable. We may also choose x such that the conclusion of Proposition 7.5 holds at x. By rescaling and passing to a suitable subsequence, we find that

$$\begin{aligned}
&(\hat{A}, \hat{x}) \in \mathrm{Tan}(A, x),\\
&((\hat{A}, \hat{x}), (\mathbb{R}^n, 0), \hat{f}) \in \mathrm{Tan}(f, x),\\
&((\hat{A}, \hat{x}), (\mathbb{R}^k, 0), \hat{\phi}) \in \mathrm{Tan}(\phi, x).
\end{aligned}$$

From the definition of differentiability in (7.1), there exists a linear map $L\colon \mathbb{R}^k \to \mathbb{R}^n$, formed from the df_i, such that

$$\hat{f} = L \circ \hat{\phi}.$$

Since we have assumed that $k > n$, L has a non-trivial kernel. In other words, there is a line $\ell \in \mathbb{R}^k$ such that $L(\ell) = \{0\}$.

By Proposition 7.5, $\hat{\phi}$ is a Lipschitz quotient map. By Lemma 7.4, there must therefore be a non-trivial curve $\gamma \subseteq \hat{A}$ such that $\hat{\phi}(\gamma) \subseteq L$. It follows that

$$\hat{f}(\gamma) = L \circ \hat{\phi}(\gamma) = \{0\},$$

i.e., that $\hat{f}$ is constant on γ.

On the other hand, $\hat{f}$ is a light mapping, by Proposition 5.15. It can therefore not collapse the non-trivial curve γ to a point. This is a contradiction, and therefore we must have $k \le n$. ∎

8. Mapping properties. In this section, we discuss the invariance and non-invariance properties of Lipschitz dimension under various classes of mappings. We show that Lipschitz light mappings cannot decrease Lipschitz dimension but can arbitrarily increase it, and we point to examples that show that Lipschitz dimension is in general not a quasisymmetric or snowflake invariant. We conclude by studying David–Semmes regular mappings and using them to prove Corollary 8.10, which provides a necessary condition for certain metric spaces to admit non-degenerate Lipschitz maps between them.

8.1. Lipschitz light mappings. In Lemma 4.1, we already made the easy observation that if $f\colon X \to Y$ is Lipschitz light, then $\dim_L(X) \leq \dim_L(Y)$. In other words, Lipschitz light mappings cannot decrease Lipschitz dimension.

Here, we observe that this inequality may be strict (even if f is surjective). The preliminary lemma we need is the following:

LEMMA 8.1. *Let X be a metric space of Nagata dimension* 0*, let Y be a metric space, and let $f\colon X \to Y$ be Lipschitz. Then f is Lipschitz light.*

Proof. We showed in Proposition 3.6 that X must also have Lipschitz dimension 0, i.e., that X admits a Lipschitz light map to the one-point metric space $\mathbb{R}^0$. It follows that there is a constant $C > 0$ such that every r-path P in X has diameter at most Cr. Hence, if $W \subseteq Y$ has $\mathrm{diam}(W) \leq r$, then every r-component of $f^{-1}(W)$ has diameter at most Cr, making f Lipschitz light. ∎

The following fact is probably well-known, but we include a proof. It is an analog of the well-known topological fact that every compact metric space is a continuous image of the Cantor set.

PROPOSITION 8.2. *Let Y be a compact, doubling metric space. Then there is a compact metric space X of Nagata dimension* 0 *and a Lipschitz map from X onto Y.*

Proof. Let Y be a compact, doubling metric space. Assume without loss of generality that $\mathrm{diam}(Y) = 1$.

For each $k \in \mathbb{N}$, let $N_k \subseteq Y$ be a sequence of nested 2^{-k}-nets in Y, i.e., $N_1 \subseteq N_2 \subseteq \cdots$. Given a point y in some N_k, let

$$N_{k+1}(y) := \{z \in N_{k+1} : d(y, z) \leq 2^{-k}\}.$$

Since Y is bounded and doubling, there is an $M \in \mathbb{N}$ such that $|N_1| \leq M$ and $|N_{k+1}(y)| \leq M$ for each $k \in \mathbb{N}$ and $y \in N_k$.

We form X as an abstract Cantor set, as follows. Let X denote the set of infinite words on the alphabet $\mathcal{A} = \{1, \ldots, M\}$, i.e.

$$X = \{(a_1, a_2, \ldots) : a_i \in \mathcal{A} \text{ for each } i \in \mathbb{N}\}.$$

Define a metric d on X by

$$d((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = 2^{-\min\{i:\, a_i \neq b_i\}}.$$

It is standard that d defines a compact metric (indeed, an "ultra-metric") on X.

Given $s > 0$, we may choose k with $2^{-(k+1)} \leq s < 2^{-k}$. Given a word w of length k on the alphabet $\mathcal{A}$, the set B_w of all elements of X beginning with w has diameter $2^{-(k+1)} \leq s$. Moreover, if $W \subseteq X$ has diameter $\leq s < 2^{-k}$, then W is contained in some B_w with $|w| = k$, from which it follows that the disjoint cover$\{B_w : |w| = k\}$ of X has s-multiplicity at most 1. Therefore, X has Nagata dimension $\dim_N(X) = 0$.

We now define a Lipschitz map from X onto Y. For each $k \in \mathbb{N}$ and $y \in N_k$, choose an arbitrary surjective map

$$\phi_{k,y} \colon \mathcal{A} \to N_{k+1}(y),$$

which we can do since $|\mathcal{A}| = M \geq |N_{k+1}(y)|$.

Similarly, choose an arbitrary surjective map

$$\phi_1 \colon \mathcal{A} \to N_1.$$

Note that, for each sequence $(a_1, a_2, \dots) \in X$, the sequence

$$y_1 := \phi_1(a_1), \quad y_2 := \phi_{1,y_1}(a_2), \quad y_3 := \phi_{2,y_2}(a_3)$$

is Cauchy in Y as $d(y_i, y_{i+1}) \leq 2^{-i}$. We therefore define a map $f \colon X \to Y$ by

$$f((a_i)) = \lim_{i\to\infty} y_i,$$

where y_i is defined as above.

We now show that f is Lipschitz. Let $a = (a_i)$ and $b = (b_i)$ be distinct elements of X. Let $k \in \mathbb{N} \cup \{0\}$ be the length of the maximal shared initial segment between a and b, so that $d(a, b) = 2^{-(k+1)}$. Then the first k terms of the sequences (y_i) defining $f(a)$ and $f(b)$ agree. Therefore,

$$d(f(a), f(b)) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-(k-1)} = 4d(a, b).$$

Hence, f is Lipschitz.

Lastly, we show that f is surjective. Note that, for each $k \in \mathbb{N}$ and $y \in N_k$, the point y itself is an element of $\phi_{k,y}(\mathcal{A})$ because $N_k \subseteq N_{k+1}$. Thus, there is a choice of (a_i) making y_i equal to y for all i sufficiently large. This implies that $y = f((a_i))$, meaning that $N_k \subseteq f(X)$ for each $k \in \mathbb{N}$. It follows that $f(X)$ is dense in Y, and hence $f(X) = Y$ since X is compact and f is continuous. ∎

We immediately have the following corollary of Proposition 8.2 and Proposition 3.6.

Corollary 8.3. *Every compact, doubling metric space is the image under a Lipschitz light map of a metric space with Lipschitz dimension* 0.

In particular, using Corollary 6.9, we may find compact doubling spaces of infinite Lipschitz dimension, and thus by Corollary 8.3 we see that Lipschitz light mappings may arbitarily increase the Lipschitz dimension of a metric space.

8.2. Quasisymmetric and snowflake non-invariance. As noted earlier, it is apparent that Lipschitz dimension, like Hausdorff, Assouad, and Nagata dimensions, is invariant under bi-Lipschitz deformations.

Nagata dimension is in addition a quasisymmetric invariant [LS05, Theorem 1.2]. However, Lipschitz dimension is not.

Corollary 8.4. *Lipschitz dimension is not a quasisymmetric, or even snowflake, invariant.*

Proof. Let $\mathbb{G}$ be a non-abelian Carnot group. Every snowflake of $\mathbb{G}$ admits a bi-Lipschitz embedding into some $\mathbb{R}^n$ by Assouad's embedding theorem [Hei01, Theorem 12.2]. Therefore, each snowflake of $\mathbb{G}$ has finite Lipschitz dimension. On the other hand, $\dim_L(\mathbb{G}) = \infty$ by Theorem 6.8. ∎

More specifically, Corollary 8.4 shows that snowflake mappings can arbitrarily decrease Lipschitz dimension.

Question 8.5. *Can a snowflake map increase the Lipschitz dimension of a compact metric space?*

Recall that in Corollary 6.5, we showed that snowflakes of $\mathbb{R}^n$ have Lipschitz dimension n. However, for general quasisymmetric deformations of Euclidean space, Lipschitz dimension is not an invariant. This follows from a construction of Semmes [Sem96].

Corollary 8.6. *There exists $n \in \mathbb{N}$ and an Ahlfors n-regular quasisymmetric deformation of $\mathbb{R}^n$ with infinite Lipschitz dimension.*

We recall that a metric space X is *Ahlfors n-regular* if there is a constant $C > 0$ such that

$$C^{-1}r^n \leq \mathcal{H}^n(\overline{B}(x,r)) \leq Cr^n \quad \text{for all } r \leq \operatorname{diam}(X). \tag{8.1}$$

Proof of Corollary 8.6. By [Sem96, Theorem 1.15], there is an Ahlfors n-regular quasisymmetric deformation of $\mathbb{R}^n$ that contains a bi-Lipschitz embedded copy of the Heisenberg group. Thus, it has infinite Lipschitz dimension by Theorem 6.8. ∎

The example provided by the proof of Corollary 8.6 must have $n > 4$, since n arises in [Sem96, Theorem 1.15] as the dimension of a Euclidean space containing a snowflake embedding of the Heisenberg group, which must have Hausdorff dimension greater than 4. An interesting question would be to explore the Lipschitz dimension of quasisymmetric deformations of low-dimensional Euclidean spaces:

Question 8.7. *Does every quasi-arc (quasisymmetric image of* $[0,1] \subseteq \mathbb{R}$*) have Lipschitz dimension* 1*? Does every quasi-plane (quasisymmetric image of* $\mathbb{R}^2$*) have finite Lipschitz dimension?*

We note that a positive answer to Question 3.7 would imply a positive answer to the first part of Question 8.7.

8.3. David–Semmes regularity and non-degenerate Lipschitz maps. Another well-studied class of mappings are the so-called David–Semmes regular mappings.

Definition 8.8 ([DS97, Definition 12.1]). A Lipschitz map $f\colon X \to Y$ is *David–Semmes regular* if there is a constant $C > 0$ such that, if $B = B(y,r) \subseteq Y$, then $f^{-1}(B)$ can be covered by at most C balls of radius Cr in X.

David–Semmes regular mappings are finite-to-one in a controlled, quantitative manner.

Lipschitz light mappings need not be David–Semmes regular: David–Semmes regular mappings are always bounded-to-one, in particular discrete, whereas Lipschitz light mappings need not be. However, we do have the following direction.

Lemma 8.9. *David–Semmes regular mappings are Lipschitz light.*

Proof. Let $f\colon X \to Y$ be David–Semmes regular. We may assume without loss of generality that the constant C from Definition 8.8 is at least 1.

Let W be a set of diameter at most r in Y. Then $f^{-1}(W) \subseteq X$ can be covered by a collection $\mathcal{B}$ of at most C closed balls, each of radius Cr.

Let $P = (x_1, \dots, x_k)$ be any r-path in $f^{-1}(W) \subseteq \bigcup_{B \in \mathcal{B}} B \subseteq X$. Without loss of generality, assume that $\operatorname{diam}(P) = d(x_1, x_k)$. Let $(B_1, \dots, B_m)$ be a list of balls in $\mathcal{B}$ such that

$$x_1 \in B_1 \quad \text{and} \quad x_k \in B_m, \tag{8.2}$$

and

$$2B_i \cap 2B_{i+1} \neq \emptyset \quad \text{for each } i \in \{1, \dots, m-1\}, \tag{8.3}$$

and moreover such that m is the minimal length of such a "chain of balls" satisfying (8.2) and (8.3). Note that simply choosing one ball from $\mathcal{B}$ containing each x_i yields a path satisfying (8.2) and (8.3), so such chains exist.

The fact that m is minimal implies that $B_i \neq B_j$ if $1 \leq i < j \leq m$. Indeed, if $B_i = B_j$ for such i, j, then excising all the balls between indices $i+1$ and $j-1$ yields a shorter list satisfying (8.2) and (8.3).

Since there are only at most C distinct balls in $\mathcal{B}$, we have $m \leq C$, and therefore $\operatorname{diam}(P) = d(x_1, x_k) \leq 4C^2 r$. As P was an arbitary r-path in

$f^{-1}(W)$, it follows that the r-components of $f^{-1}(W)$ have diameter at most $4C^2r$, which proves that f is Lipschitz light. ∎

Combined with a result of David–Semmes, Lemma 8.9 yields one way to control Lipschitz dimension of weak tangents of certain metric spaces by Hausdorff dimension. Recall the definition of Ahlfors regularity from (8.1).

COROLLARY 8.10. *Let X be an Ahlfors n-regular metric space, Y a complete, doubling metric space, and Z a compact subset of X. Suppose that there is a Lipschitz map $g\colon Z \to Y$ such that $\mathcal{H}^n(g(X)) > 0$. Then there are weak tangents $(\hat{X}, \hat{x}) \in \mathrm{WTan}(X)$ and $(\hat{Y}, \hat{y}) \in \mathrm{WTan}(Y)$ such that $\dim_L(\hat{X}) \leq \dim_L(\hat{Y})$. In particular, if $\dim_L(Y) \leq m$, then X has a weak tangent with Lipschitz dimension at most m.*

Proof. By [DS97, Proposition 12.8], there is a weak tangent

$$((\hat{X}, \hat{x}), (\hat{Y}, \hat{y}), \hat{g}) \in \mathrm{WTan}(g)$$

such that $\hat{g}$ is David–Semmes regular. The map $\hat{g}$ is then Lipschitz light by Lemma 8.9, and so the result follows from the observation at the beginning of Subsection 8.1.

The "In particular..." statement follows from Corollary 5.17. ∎

Corollary 8.10 can be viewed as a statement about which Ahlfors n-regular spaces can admit "non-degenerate" Lipschitz maps into other spaces, where a "non-degenerate" Lipschitz map is one whose image has positive $\mathcal{H}^n$-measure.

In particular, if X is an Ahlfors n-regular metric space such that every weak tangent of X has Lipschitz dimension greater than m, then X cannot admit a non-degenerate Lipschitz map into a space of Lipschitz dimension at most m.

Specializing further, if X is an Ahlfors n-regular metric space such that every weak tangent of X has infinite Lipschitz dimension (as we have seen is the case of non-abelian Carnot groups), then X cannot admit a non-degenerate Lipschitz map into any metric space of finite Lipschitz dimension. This is closely related to the notion of "strong unrectifiability" studied in [AK00, Section 7] and [DK20, Section 4.1], among other places.

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