



Decision Support

Equilibrium strategies for multiple interdiction on a common network

Harikrishnan Sreekumaran^a, Ashish R. Hota^b, Andrew L. Liu^{c,*}, Nelson A. Uhan^d,
Shreyas Sundaram^e

^a Amazon, Seattle, WA, USA

^b Department of Electrical Engineering, Indian Institute of Technology (IIT), Kharagpur, India

^c School of Industrial Engineering, Purdue University, West Lafayette, IN, USA

^d Mathematics Department, United States Naval Academy, Annapolis, MD, USA

^e School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN, USA



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ABSTRACT

In this work, we introduce *multi-interdiction games*, which model interactions among multiple interdiction with differing objectives operating on a common network. As a starting point, we focus on *shortest path multi-interdiction (SPMI) games*, where multiple interdiction try to increase the shortest path lengths of their own adversaries attempting to traverse a common network. We first establish results regarding the existence of equilibria for SPMI games under both discrete and continuous interdiction strategies. To compute such an equilibrium, we present a reformulation of the SPMI game, which leads to a generalized Nash equilibrium problem (GNEP) with non-shared constraints. While such a problem is computationally challenging in general, we show that under continuous interdiction actions, an SPMI game can be formulated as a linear complementarity problem and solved by Lemke's algorithm. In addition, we present decentralized heuristic algorithms based on best response dynamics for games under both continuous and discrete interdiction strategies. Finally, we establish theoretical lower bounds on the worst-case efficiency loss of equilibria in SPMI games, with such loss caused by the lack of coordination among noncooperative interdiction, and use the decentralized algorithms to numerically study the average-case efficiency loss.

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1. Introduction

In an interdiction problem, an agent attempts to limit the actions of an adversary operating on a system (e.g., a network) by intentionally disrupting certain components of the system. Such problems are usually modeled in the framework of leader-follower games and can be formulated as bilevel optimization problems. Interdiction models have been used in various military and homeland security applications, such as dismantling drug traffic networks (e.g. Washburn & Wood, 1995; Wood, 1993), preventing nuclear smuggling (e.g. Brown, Carlyle, Harney, Skroch, & Wood, 2009; Morton, Pan, & Saeger, 2007), planning tactical air strikes (e.g. Ghare, Montgomery, & Turner, 1971), and identifying critical infrastructure (e.g. Church, Scaparra, & Middleton, 2004; O'Hanley & Church, 2011; Perea & Puerto, 2013; Scaparra & Church, 2008; Starita & Scaparra, 2016). Interdiction models have also found applications in other areas such as controlling the spread of pan-

demics (e.g. Assimakopoulos, 1987) and defending attacks on computer communication networks (e.g. Mofya & Smith, 2006).

Traditionally, interdiction problems have been analyzed from a centralized perspective; namely, a single agent is assumed to analyze, compute and implement interdiction strategies. In many situations, however, it might be desirable and even necessary to consider an interdiction problem from a decentralized perspective. Arguably the most prominent example of such situations today is the war against the terrorist group, the Islamic State of Iraq and Syria (ISIS, also known as ISIL or Daesh). It is believed that oil smuggling is the “biggest single source of revenue” of ISIS (Solomon, Kwong, & Bernard, 2016), and hence, a sensible strategy to defeat ISIS is to disrupt their oil smuggling operation. Such a strategy has been deployed by the multiple parties involved in the war (America, 2004; Sanger & Davis, 2014). The parties involved, however, which include the US-led coalition, Russia, Turkey, Iran, among others, do not operate as a single coalition, and often do not share information (Harris, 2016). Without any coordination between the parties, one might expect that a decentralized interdiction strategy may be inefficient compared to one determined by a central decision maker. A central decision maker in the war against ISIS is of course impractical, and hence, we would like

* Corresponding author.

E-mail addresses: harikrishnan.sreekumaran@gmail.com (H. Sreekumaran), ahota@ee.iitkgp.ac.in (A.R. Hota), andrewliu@purdue.edu (A.L. Liu), uhan@usna.edu (N.A. Uhan), sundara2@purdue.edu (S. Sundaram).

to understand better the equilibrium state of such settings with multiple interdiction on a common network, and especially the efficiency loss due to the lack of cooperation among the interdiction. This is both the motivation and the focus of this paper.

In this paper, we introduce *decentralized multiple interdiction games*, in which multiple agents with differing objectives are interested in interdicting parts of a common network. We focus on a specific class of these games, which we call *shortest path multi-interdiction (SPMI) games*. We show the existence of pure-strategy Nash equilibria. In addition, we show through an example that uniqueness of such equilibria cannot be expected in general. We also propose both an equilibrium-based and decentralized algorithms to compute such an equilibria of these games. Using these algorithms, we conduct numerical studies on the efficiency loss of equilibria in the SPMI game compared to optimal solutions obtained through centralized decision making.

Decentralized network interdiction games, as will be formally defined in Section 2, appear to be new. To the best of our knowledge, there has been no previous research on such games. As a result, not much is known about the inefficiency of equilibria for these games or intervention strategies to reduce such inefficiencies. There has been a considerable amount of work, however, on interdiction problems from a centralized decision-maker's perspective. As mentioned earlier, interdiction problems have been studied in the context of various military and security applications. For extensive reviews of the existing academic literature on other network interdiction problems, we refer the readers to Smith and Lim (2008) and Smith, Prince, and Geunes (2013).

There have also been many studies on the inefficiency of equilibria in other game-theoretic settings. Most of the efforts have been focused on routing games (Pigou, 1924; Wardrop, 1952), in which selfish agents route traffic through a congested network, and congestion games (Rosenthal, 1973), a generalization of routing games. Some examples include Roughgarden and Tardos (2002), Correa, Schulz, and Stier-Moses (2004), Awerbuch, Azar, and Epstein (2005), Christodoulou and Koutsoupias (2005), Cole, Dodis, and Roughgarden (2006), Suri, Tóth, and Zhou (2007). Several researchers have also studied the inefficiency of equilibria in network formation games, in which agents form a network subject to potentially conflicting connectivity goals (Albers, Eilts, Even-Dar, Mansour, & Roditty, 2006; Anshelevich et al., 2008; Fabrikant, Luthra, Maneva, Papadimitriou, & Shenker, 2003). The inefficiency of equilibria has been studied in other games as well, such as facility location games (Vetta, 2002), scheduling games (Koutsoupias & Papadimitriou, 1999), and resource allocation games (Johari & Tsitsiklis, 2004; 2009). Almost all of the work described above study the worst-case inefficiency of a given equilibrium concept. Although a few researchers have studied the average inefficiency of equilibria, either theoretically or empirically, and have used it as a basis to design interventions to reduce the inefficiency of equilibria (Corbo & Parkes, 2005; Thompson & Leyton-Brown, 2009), research in this direction has not received much attention.

One potential reason for the lack of attention paid to multiple interdiction games may be that such games often involve nondifferentiability, as each interdiction's optimization problem usually entails a max-min type of objective function. Games involving non-differentiable functions are generally challenging, in terms of both theoretical analysis of their equilibria and computing an equilibrium. While in some cases (such as in the case of shortest path interdiction), a smooth formulation (through total unimodularity and duality) is possible, such a reformulation will yield a generalized Nash equilibrium problem (GNEP), in which both the agents' objective functions as well as their feasible action spaces depend on other agents' actions. Although the conceptual framework of GNEPs can be dated back to Debreu (1952), rigorous theoretical and algorithmic treatments of GNEPs began much later (see Facchinei &

Kanzow, 2007; Facchinei & Kanzow, 2010a, for example, for a literature review). Several techniques have been proposed to solve GNEPs, including penalty-based approaches (Facchinei & Kanzow, 2010b; Fukushima, 2011), variational-inequality-based approaches (Nabetani, Tseng, & Fukushima, 2011), Newton's method (Dreves, von Heusinger, Kanzow, & Fukushima, 2013), projection methods (Zhang, Qu, & Xiu, 2010), and relaxation approaches (Krawczyk & Uryasev, 2000; Uryasev & Rubinstein, 1994). Most of the work on GNEPs has focused on games with shared constraints due to their tractability (Facchinei, Fischer, & Piccialli, 2007; Harker, 1991). In such games, a set of identical constraints appear in each agent's feasible action set. However, as will be seen later, in a typical decentralized network interdiction game, the constraints involving multiple agents' actions that appear in each agent's action space are not identical. As a result, such games give rise to more challenging instances of GNEPs.

Based on the discussions above, the major contributions of this work are as follows.

- First, we establish the existence of pure-strategy equilibria for SPMI games with continuous interdiction. In SPMI games with discrete interdiction, the existence of a pure strategy Nash equilibrium (PNE) is more subtle. We first demonstrate that a PNE does not necessarily exist in SPMI games with discrete interdiction. However, when all agents have the same source-target pairs (i.e., multiple agents try to achieve a common goal independently), a PNE exists in these games.
- Second, we show that SPMI games can be equivalently formulated as a GNEP. In addition, for SPMI games under continuous interdiction, we show that the corresponding GNEP formulation can be written as a linear complementarity problem, which can be proven to be solved by the well-known Lemke algorithm (Lemke (1965)). This is a much stronger result than the case for general GNEPs with non-shared constraints, where they are only equivalent to quasi-variational inequalities (Harker (1991)), which do not lend to efficient algorithms to compute an equilibrium. For SPMI games with discrete interdiction (and continuous interdiction as well), we present decentralized algorithms for finding an equilibrium, based on the well-known best-response dynamics (aka the Gauss-Seidel) approach. While such an approach is only a heuristic method in general, we establish convergence for the special case when the agents have common source-target pairs. For more general cases, we obtain encouraging numerical results for the performance of the method on several classes of network structures.
- Third, we measure the efficiency loss in SPMI games due to the lack of coordination among noncooperative interdictors, compared to a centralized interdiction strategy (that is, a strategy implemented by a single interdictor with respect to all the adversaries). In particular, we establish a theoretical lower bound for the worst-case price of anarchy of SPMI games under continuous interdiction. Such an efficiency loss measure, however, may be too conservative; as a result, we use the decentralized algorithms to numerically quantify the *average-case* efficiency loss over some instances of SPMI games. These results can help central authorities design mechanisms to reduce such efficiency losses for practical instances.

The remainder of this paper is organized as follows. We begin in Section 2 with definitions and formulations of general multi-interdiction games and the specific SPMI games. In Section 3, we present the main theoretical results of the paper, including an analysis of the existence and (lack of) uniqueness of equilibria in SPMI games. In Section 4 we investigate algorithms for solving SPMI games. We describe an equilibrium-based algorithm through linear complementarity formulation, as well as decentralized algorithms for computing equilibria of SPMI games. Section 5 contains

numerical results, showing the performance of our algorithms for computing equilibria and quantifying the price of anarchy of various instances. Finally, in Section 6, we provide some concluding remarks.

2. Games with multiple interdictors on a common network

2.1. General formulation

Network interdiction problems involve interactions between two types of parties – adversaries and interdictors – with conflicting interests. An adversary operates on a network and attempts to optimize some objective, such as the flow between two nodes. An interdictor tries to limit an adversary's objective by changing elements of the network, such as the arc capacities. Such interactions have historically been viewed from a leader-follower-game perspective. The interdictor acts as the leader and chooses an action while anticipating the adversary's potential responses, while the adversary acts as the follower and makes a move after observing the interdictor's actions. From the interdictor's perspective, this captures the pessimistic viewpoint of guarding against the worst possible result given its actions.

In this work, we consider strategic interactions among multiple interdictors who operate on a common network. The interdictors may each have their own adversary or they may have a common adversary. If there are multiple adversaries, we assume there is no strategic interaction among them. We also assume that the interdictors are allies in the sense that they are not interested in deliberately impeding each other.

Formally, we have a set $\mathcal{F} = \{1, \dots, F\}$ of interdictors or agents, who operate on a network $G = (V, A)$, where V is the set of nodes and A is the set of arcs. Each agent's actions or decisions correspond to interdicting each arc of the network with varying intensity: the decision variables of agent $f \in \mathcal{F}$ are denoted by $x^f \in X^f \subset \mathbb{R}^{|A|}$, where X^f is an abstract set that constrains agent f 's decisions. For any agent $f \in \mathcal{F}$, let x^{-f} denote the collection of all the other agents' decision variables; that is, $x^{-f} = (x^1, \dots, x^{f-1}, x^{f+1}, \dots, x^F)$. The network obtained after every agent executes interdiction actions is called the *aftermath network*. The strategic interaction between the agents occurs due to the fact that the properties of each arc in the aftermath network are affected by the combined actions of all the agents.

In addition to the abstract constraint set X^f , we assume that each agent $f \in \mathcal{F}$ has a total interdiction budget of b^f . The cost of interdicting an arc is linear in the intensity of interdiction; in particular, agent f 's cost of interdicting arc (u, v) by x_{uv}^f units is $c_{uv}^f x_{uv}^f$. Without loss of generality, we assume that $b^f > 0$ and $c_{uv}^f > 0$ for each arc $(u, v) \in A$ and for each agent $f \in \mathcal{F}$. The optimization problem for each agent $f \in \mathcal{F}$ is:

$$\begin{aligned} & \underset{x^f}{\text{maximize}} && \theta^f(x^f, x^{-f}) \\ & \text{subject to} && \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f, \\ & && x^f \in X^f, \end{aligned} \quad (1)$$

where the objective function θ^f is agent f 's *obstruction function*, or measure of how much agent f 's adversary has been obstructed. Henceforth, we refer to the game in which each agent $f \in \mathcal{F}$ solves the above optimization problem (1) as a *decentralized multi-interdictor game*. As a starting point, we restrict our attention to simultaneous-move games with complete information. Simultaneous-move means that the agents must make their decisions without being aware of the other agents' decisions. A complete information game means that the number of agents, their payoffs and their feasible action spaces are common knowledge to all the agents.

The obstruction function θ^f can capture various types of interdiction problems. Typically θ^f is the (implicit) optimal value function of the adversary's network optimization problem parametrized by the agents' decisions. For example, θ^f might be the minimum flow cost or path length subject to flow conservation, arc capacity and side constraints in the aftermath network.

Suppose that a central planner, with a comprehensive view of the network and the agents' objectives, could pool the agents' interdiction resources together and determine an interdiction strategy that maximizes some global measure of how much the agents' adversaries have been obstructed. Let $\theta^c(x^1, \dots, x^F)$ represent the global obstruction function for a given interdiction strategy $(x^1, \dots, x^F)$. The central planner's problem corresponding to the multi-interdictor game (1) is then:

$$\begin{aligned} & \underset{x^1, \dots, x^F}{\text{maximize}} && \theta^c(x^1, \dots, x^F) \\ & \text{subject to} && \sum_{f \in \mathcal{F}} \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq \sum_{f \in \mathcal{F}} b^f, \\ & && x^f \in X^f \quad \forall f \in \mathcal{F}. \end{aligned} \quad (2)$$

Without loss of generality, we assume that $\theta^c(\cdot) \geq 0$ for any feasible $(x^1, \dots, x^F)$. We refer to (2) as the centralized problem, and focus primarily on when the global obstruction function is *utilitarian*; that is,

$$\theta^c(x^1, \dots, x^F) := \sum_{f \in \mathcal{F}} \theta^f(x^f, x^{-f}).$$

Note also that we assume the resources involved in the budgetary constraints may be “pooled” amongst the interdictors. Such resources may then be allocated optimally by the central planner. The case where the resources are not shareable can easily be modeled by enforcing each interdictor's resource constraints separately in the central planner's problem.

As mentioned above, the generic form of problem (1) can be used to describe various network interdiction settings. To start with models that are both theoretically and computationally tractable, we focus on shortest-path multi-interdictor games, which we describe in detail next.

2.2. Shortest path multi-interdictor games

As the name suggests, *shortest path multi-interdictor (SPMI)* games involve agents or interdictors whose adversaries are interested in the shortest path between source-target node pairs on a network. Single-interdictor shortest path interdiction games have been well studied in the literature, such as Fulkerson and Harding (1977), Israeli and Wood (2002). Our work builds upon the existing results (such as the linear programming reformulation) and extend to the situation where there are multiple, noncooperative interdictors, who act in advance to increase the length of the shortest path of their respective adversaries by interdicting (in particular, lengthening) arcs on the network.

To describe these games formally, we build upon the setup of the general multi-interdictor game described in Section 2.1. Each agent $f \in \mathcal{F}$ has a target node $t^f \in V$ that it wishes to protect from an adversary at source node $s^f \in V$ by maximizing the length of the shortest path between the two nodes. The agents achieve this goal by committing some resources (e.g. monetary spending) to increase the individual arc lengths on the network: the decision variable x_{uv}^f represents the contribution of agent $f \in \mathcal{F}$ towards lengthening arc $(u, v) \in A$. The arc length $d_{uv}(x^f, x^{-f})$ of arc $(u, v) \in A$ in the aftermath network depends on the decisions of all the agents.

We consider two types of interdiction. The first type of interdiction is *continuous and additive*, i.e. each interdictor's decisions for arcs are continuous variables and the net effect on the arc is the

sum of the effects of all the interdiction taken together. In particular,

$$X^f := \{x^f \in \mathbb{R}^{|A|} : x_{uv}^f \geq 0 \quad \forall (u, v) \in A\}.$$

The arc lengths after an interdiction strategy $(x^1, \dots, x^F)$ has been executed are

$$d_{uv}(x^1, \dots, x^F) = d_{uv}^0 + \sum_{f \in \mathcal{F}} x_{uv}^f \quad \forall (u, v) \in A, \quad (3)$$

where d_{uv}^0 denotes the initial length of the arc (u, v) and x_{uv}^f captures how much agent f extends the length of the arc. We assume that $d_{uv}^0 \geq 0$ for all $(u, v) \in A$.

The second type of interdiction is *discrete and binary*, i.e. each interdicator decides whether or not to interdict an arc, and the arc length increases by a fixed amount as long as at least one interdicator acts on it. In this case,

$$X^f := \{x^f \in \mathbb{R}^{|A|} : x_{uv}^f \in \{0, 1\} \quad \forall (u, v) \in A\}$$

and the arc lengths in the aftermath network are

$$d_{uv}(x^1, \dots, x^F) = d_{uv}^0 + e_{uv} \max_{f \in \mathcal{F}} x_{uv}^f \quad \forall (u, v) \in A, \quad (4)$$

where $e_{uv} \in \mathbb{R}_{\geq 0}$ is the fixed extension of arc (u, v) . We can easily reformulate (4) to an equivalent set of linear constraints using standard techniques in integer programming involving the use of an auxiliary variable to model the max function.¹

Let $P^f = \{p_1^f, p_2^f, \dots, p_{k_f}^f\}$ be the set of $s^f - t^f$ paths available to agent $f \in \mathcal{F}$. The length of a path $p \in P^f$ is given by

$$d_p(x^1, \dots, x^F) = \sum_{(u,v) \in p} d_{uv}(x^1, \dots, x^F), \quad (5)$$

where $d_{uv}(x^1, \dots, x^F)$ is as defined in Eq. (3) for continuous interdiction, and as defined in (4) for the discrete case. The optimization problem for each interdicting agent $f \in \mathcal{F}$ is then:

$$\begin{aligned} & \text{maximize} \quad \theta^f(x^f, x^{-f}) \equiv \min_{p \in P_f} d_p(x^f, x^{-f}) \\ & \text{subject to} \quad \sum_{\substack{(u,v) \in A \\ x^f \in X^f}} c_{uv}^f x_{uv}^f \leq b^f, \end{aligned} \quad (6)$$

Under continuous interdiction and the assumptions that X^f is nonempty and convex, and contains non-negativity constraints on x^f , the feasible action set for agent f , given by $\{x^f \in X^f \mid \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f\}$ is then convex and compact. To rule out uninteresting cases, we also assume that the feasible set for each agent is also nonempty (meaning that each agent has the budget to interdict some arcs). Moreover, given an x^{-f} , the objective function in (6) is the minimum of a set of affine functions of x^f , and therefore continuous in x^f . Thus, by Weierstrass's extreme value theorem, each agent has an optimal strategy given the strategies of the other agents. Note, however, that the objective function in (6) is not differentiable with respect to x^f in general.

For SPMI games with discrete interdiction, the feasible strategy set for each agent is finite. Therefore an optimal solution to each agent's problem always exists with a given x^{-f} . In the following section, we analyze the existence and uniqueness of pure strategy Nash Equilibria for SPMI games, under both continuous and discrete settings.

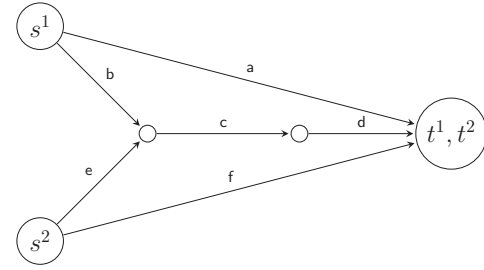


Fig. 1. Network topology for the SPMI game in Example 1.

Table 1
Network data for Example 1.

Arc tag	Initial length	Arc extension	Cost to agent 1	Cost to agent 2
a	7	0.5	3	20
b	0	2	6	20
c	0	1.5	5	20
d	0	6	15	15
e	0	1	20	20
f	1	6	15	15

3. Game structure and analysis

3.1. Existence of equilibria

We first consider the existence of a Nash equilibrium of an SPMI game when interdiction decisions are continuous. The key is to show that the objective function in (6) $\theta^f(x^f, x^{-f})$ is concave in x^f , despite the fact that it is not differentiable.

Proposition 1. Assume that each agent f has the budget to interdict at least one arc, i.e. $b^f \geq \min_{(u,v) \in A} c_{uv}^f$. The SPMI game under continuous interdiction – each agent $f \in \mathcal{F}$ solves the problem (6) with $d_p(x^f, x^{-f})$ defined as in (5) and (3) – has a pure strategy Nash equilibrium.

Proof. Based on the assumption, the feasible region in (6) is nonempty, convex and compact. With a fixed x^{-f} , the objective function of agent f is the minimum of a finite set of affine functions in x^f , and therefore, is concave with respect to x^f (Cf. Boyd & Vandenberghe, 2004). Consequently, the SPMI game belongs to the class of “concave games”, introduced in Rosen (1965), and it is shown in Rosen (1965) that a pure-strategy Nash equilibrium always exists for a concave game. $\square$

Under discrete interdiction, the existence of a PNE is not always guaranteed when different interdicator are competing against different adversaries. We illustrate the nonexistence of PNE in Example 1 below.

Example 1. Consider the network given in Fig. 1.

In this game, there are two agents – agent 1 and agent 2 – who are attempting to maximize the lengths of the $s^1 - t^1$ paths and $s^2 - t^2$ paths respectively. Note that $t^1 = t^2$. The data for the problem, including initial arc lengths, cost of interdiction and arc extensions are given below in Table 1.

Suppose that the budgets are $b^1 = 8$ and $b^2 = 15$. As a result, agent 1 can either interdict the arcs a , b and c one at a time, or the arcs a and c simultaneously. Similarly, agent 2 can either interdict arc d or arc f . Thus, agent 1 has four feasible pure strategies and agent 2 has two feasible pure strategies. The strategy tuples along with the corresponding payoffs for each agent are summarized in Table 2. It is easy to verify that for any joint strategy profile, there is a player who would prefer to deviate unilaterally. Therefore, this instance of the SPMI game does not possess a PNE. $\square$

¹ It is possible to also consider discrete and additive interdiction, as well as continuous interdiction where the effect on each arc is the maximum of each agent's efforts. Both these types can be modeled using integer variables. We note that all of the results in this paper regarding discrete and binary interdiction can be applied to these other interdiction types.

Table 2
Payoff combinations for Example 1.

Agent 1 / Agent 2	None	d	f
None	0,0	0,7	0,0
a	0,0	6,1	0,0
c	1.5,1	7,1	1.5,1.5
(a, c)	1.5, 1	7.5,1	1.5,1.5
b	2,0	7,1	2,0

In the previous example, the agents have a common target node, but different source nodes. However, in the class of games in which the interdiction have a common adversary, i.e., when each agent maximizes the shortest path between a common source-target pair, we can show that SPMI games under discrete interdiction possess a PNE.

Consider the SPMI game where each agent is trying to maximize the shortest path lengths between nodes s and t . Since the objective function of each agent is the same, we can write the following centralized optimization problem to maximize the shortest $s - t$ path distance subject to the individual agents' budget constraints. Let P^{st} be the set of $s - t$ paths in the network. The centralized optimization problem is:

$$\begin{aligned} & \max_x \min_{p \in P^{st}} d_p(x^1, x^2, \dots, x^F) \\ & \text{subject to} \quad \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f \quad \forall f \in \mathcal{F}, \\ & \quad x_{uv}^f \in \{0, 1\} \quad \forall (u, v) \in A, f \in \mathcal{F}. \end{aligned} \quad (7)$$

The feasible solution space of the above problem is finite under individual agents' budget constraints. Therefore, the centralized problem always has a maximum. Furthermore, an optimal solution to this problem is a PNE of the SPMI game as we show in the following result.²

Proposition 2. Suppose the source and target for each agent in an SPMI game under discrete interdiction, are the same. Let x^* denote an optimal solution of the centralized problem (7). Then x^* is a PNE to the SPMI game.

Proof. Assume the contrary, and suppose that there is an agent h for whom there exists a feasible deviation x^h that strictly increases the path distance from the source to the target. By assumption, x^h is feasible for the budgetary constraints for agent h . Therefore, $\bar{x} \equiv (x^h, x^{*-h})$ is feasible for (7) with a strictly larger objective value. Clearly this is a contradiction to the optimality of x^* for (7). $\square$

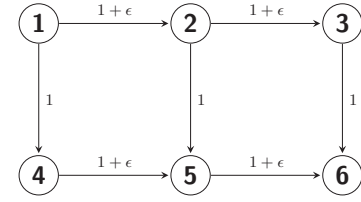
3.2. Non-uniqueness of equilibria

Establishing sufficient conditions for an SPMI game to have a unique equilibrium is quite difficult. However, it is easy to find simple instances of SPMI games for which multiple equilibria exist. We give two such examples below.

Example 2. Consider the following instance, based on the network in Fig. 2. There are 2 agents: agent 1 has an adversary with source node 1 and target node 5; agent 2 has an adversary with source node 1 and target node 6. The initial arc lengths are 0, interdiction is continuous, and the interdiction costs are the same for both agents and are given in the arc labels in Fig. 2. Both agents have a budget of 1.

Consider the case when $\epsilon = 2$. In this case, it is straightforward to see that the source-target path lengths for each agent must be equal at an equilibrium: if the path lengths are unequal, an agent could improve its objective function by equalizing the path lengths.

² Note that the converse statement is not true in general; namely, an equilibrium of the SPMI game may not be an optimal solution of the centralized problem (7)

**Fig. 2.** Network topology for the SPMI game in Example 2.

Therefore, in this example, any combination of decision variables that results in a shortest path length of $2/3$ for each agent is a Nash equilibrium, and there is a continuum of such decision variable combinations. Indeed, some of such equilibria are given in Table 3 in Section 5. $\square$

Example 3. Under discrete interdiction on the same underlying network, an interesting situation occurs when $\epsilon = 0$, both agents have a budget of 1, and the arc extensions are all set to 1. In this case, an equilibrium occurs when the arcs (1,4) and (1,2) are interdicted by one agent each. However, there exist equilibria that have inferior objective values for both agents. Indeed, the extreme case of neither agent interdicting any arc can easily be seen to be an equilibrium. This equilibrium in fact is a social utility minimizer over the set of feasible action combinations for the two agents. $\square$

4. Computing a Nash equilibrium

In this section we discuss algorithms to compute equilibria of SPMI games. While the general formulation with each agent solving (6) is sufficient for showing the existence of equilibria of SPMI games, computing an equilibrium using this formulation is difficult, mainly due to the bilevel nature of each agent's objective function (the 'min' in the objective functions). To overcome this difficulty, we first use a well-known reformulation of shortest path problems (through total unimodularity and linear programming duality) to formulate the SPMI game as a generalized Nash equilibrium problem (GNEP) in Section 4.1. Under continuous interdiction, we show in Section 4.2 that the resulting GNEP can be written as a linear complementarity problem (LCP), and the specific LCP has favourable properties, allowing the use of Lemke's pivoting algorithm with guaranteed convergence to a solution (as opposed to a secondary ray). Under discrete interdiction, we present in Section 4.3 the best-response based distributed algorithm and the conditions under which such an algorithm converges to an equilibrium. Finally, in Section 4.4, the single-level formulation (as opposed to the max-min formulation) of the centralized interdiction problem is provided, which is needed for computing average efficiency losses.

4.1. GNEP Formulation

In (6), each agent's objective function $\theta^f(x^f, x^{-f})$ involves its adversary's shortest path problem, which can be written as an integer program as follows:

$$\begin{aligned} & \theta^f(x^f, x^{-f}) \\ & = \left(\begin{array}{l} \min_{z^f} \quad \sum_{(u,v) \in A} z_{uv}^f d_{uv}(x^f, x^{-f}) \\ \text{s.t.} \quad \sum_{\{v \in V \mid (u,v) \in A\}} z_{uv}^f - \sum_{\{v \in V \mid (v,u) \in A\}} z_{vu}^f = \begin{cases} 1 & \text{if } u = s^f \\ 0 & \text{if } u \neq s^f, t^f \\ -1 & \text{if } u = t^f \end{cases} \\ z_{uv}^f \in \{0, 1\} \quad \forall (u, v) \in A \end{array} \right), \end{aligned} \quad (8)$$

where the binary variables z_{uv}^f represents whether an arc $(u, v) \in A$ is in the shortest $s^f - t^f$ path. Although the inner minimization problem is an integer program with binary variables, the constraint

Table 3
Multiple equilibria for the instance of the SPMI game in Example 2.

x_0	x_N^1	x_N^2	p_1	p_2
(0, 0, 0, 0, 0, 0, 0)	(0, 0, 0.5, 0.5, 0, 0, 0)	(0, 0, 0.1667, 0.1667, 0.6667, 0, 0)	0.6667	0.6667
(0.2, 0.2, 0, 0, 0, 0, 0)	(0, 0, 0.6, 0.4, 0, 0, 0)	(0, 0, 0.0667, 0.2667, 0.6667, 0, 0)	0.6667	0.6667
(0, 0, 0, 0, 0.2, 0.2)	(0, 0, 0.4, 0.6, 0, 0, 0)	(0, 0, 0.2667, 0.0667, 0.6667, 0, 0)	0.6667	0.6667
(0, 0, 0, 0, 0.3, 0.3)	(0, 0, 0.35, 0.65, 0, 0, 0)	(0, 0, 0.3167, 0.0167, 0.6667, 0, 0)	0.6667	0.6667
(0.3, 0.3, 0, 0, 0, 0, 0)	(0, 0, 0.65, 0.35, 0, 0, 0)	(0, 0, 0.0167, 0.3167, 0.6667, 0, 0)	0.6667	0.6667
(0.25, 0.25, 0, 0, 0, 0, 0)	(0, 0, 0.625, 0.375, 0, 0, 0)	(0, 0, 0.0417, 0.2917, 0.6667, 0, 0)	0.6667	0.6667
(0, 0, 0, 0, 0.25, 0.25)	(0, 0, 0.375, 0.625, 0, 0, 0)	(0, 0, 0.2917, 0.0417, 0.6667, 0, 0)	0.6667	0.6667
(0, 0, 0, 0, 0.15, 0.15)	(0, 0, 0.425, 0.575, 0, 0, 0)	(0, 0, 0.2417, 0.0917, 0.6667, 0, 0)	0.6667	0.6667
(0.15, 0.15, 0, 0, 0, 0, 0)	(0, 0, 0.575, 0.425, 0, 0, 0)	(0, 0, 0.0917, 0.2417, 0.6667, 0, 0)	0.6667	0.6667

matrix is totally unimodular (e.g., Schrijver (1998)), rendering the integer program equivalent to its linear programming relaxation. Therefore, once the interdiction variables ($x^1, \dots, x^F$) are fixed, we can use linear programming duality to transform the inner minimization problem to a maximization problem (Israeli & Wood, 2002) and reformulate agent f 's optimization problem (8) as:

$$\begin{aligned}
 & \underset{x^f, y^f}{\text{maximize}} && y_{tf}^f - y_{sf}^f \\
 & \text{subject to} && y_v^f - y_u^f \leq d_{uv}(x^f, x^{-f}) \quad \forall (u, v) \in A, \\
 & && \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f, \\
 & && x^f \in X^f, \\
 & && y_v^f \geq 0 \quad \forall v \in V.
 \end{aligned} \tag{9}$$

Following Israeli and Wood (2002), we have reversed the indicated signs for the dual variables y and thus may interpret the term $y_{tf}^f - y_{sf}^f$ as the post interdiction shortest path length from s^f to a node t^f . Since only the differences $y_v^f - y_u^f$ across arcs (u, v) are relevant to the formulation (9), we may always replace y_u^f by $y_u^f - y_{sf}^f$ for each $u \in V$ to obtain a feasible solution with equal objective value. This allows us to restrict the y^f variables to be non-negative. In addition, it also allows us to restrict the y^f variables to be integral if the underlying network data is integral, since at optimality all path lengths must also be integral. Moreover, as we show below, we can construct upper bounds on the y^f variables.

Remark 1. When interdiction is continuous, budgetary constraints imply that the maximum length of any path in the aftermath network is bounded above by

$$\bar{Y} = \sum_{(u,v) \in A} d_{uv}^0 + |A| \cdot F \cdot \max_{f \in \mathcal{F}, (u,v) \in A} \left\{ \frac{b^f}{c_{uv}^f} \right\}.$$

On the other hand, when interdiction is discrete, the length of any path in the aftermath network is bounded above by $\bar{Y} = \sum_{(u,v) \in A} (d_{uv}^0 + e_{uv})$. Therefore, we can then add the constraints $0 \leq y_u^f \leq \bar{Y}$ for all $u \in V$ to the problem (9) without changing its optimal value. Note that the feasible region in (9) is always non-empty as long as the set $\{x^f \in X^f \mid \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f\}$ is nonempty. Hence, the boundedness of the y variable ensures the existence of an optimal solution of Problem (9) by the Weierstrass extreme value theorem, regardless if the x variable is of continuous or discrete values.

The formulation (9) gives us some insight into the structure of strategic interactions among agents in an SPMI game. Note that the objective function for each agent $f \in \mathcal{F}$ only depends on variables indexed by f (in particular, y_{sf}^f and y_{tf}^f). However, the constraint set for each agent f is parameterized by other agents' variables x^{-f} , which leads to a generalized Nash equilibrium problem.

More generally speaking, let $\chi^f = (x^f, y^f)$. We can describe the feasible set of (9) as a mapping $\Xi^f(\chi^{-f})$ from $\mathbb{R}^{(n-n_f)}$ to $\mathbb{R}^{n_f}$, where $n := \sum_{f \in \mathcal{F}} n_f$. In contrast, in a regular Nash equilibrium

problem, each agent's feasible action space is a fixed set. Parameterized by the other agents' decisions χ^{-f} , each agent $f \in \mathcal{F}$ in a GNEP solves the following problem:

$$\begin{aligned}
 & \underset{\chi^f}{\text{maximize}} && \theta^f(\chi^f, \chi^{-f}) \\
 & \text{subject to} && \chi^f \in \Xi^f(\chi^{-f}).
 \end{aligned} \tag{10}$$

In this specific situation, we have that

$$\begin{aligned}
 \theta^f(\chi^f, \chi^{-f}) &= \theta^f(\chi^f) = y_{tf}^f - y_{sf}^f, \quad \text{and} \\
 \Xi^f(\chi^{-f}) &= \left\{ \chi^f = (x^f, y^f) \mid \begin{cases} y_v^f - y_u^f \leq d_{uv}(x^f, x^{-f}) \quad \forall (u, v) \in A, \\ \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f, \\ 0 \leq y_u^f \leq \bar{Y} \quad \forall u \in V, \\ x^f \in X^f \end{cases} \right\}.
 \end{aligned} \tag{11}$$

Note that $\chi = (\chi^1, \dots, \chi^F) \in \mathbb{R}^n$, where $n = F \times (|V| + |A|)$.

Let $\Omega(\chi)$ denote the Cartesian product of the feasibility sets of the agents corresponding to decisions $\chi = (\chi^1, \dots, \chi^F)$, i.e.,

$$\Omega(\chi) := \Xi^1(\chi^{-1}) \times \Xi^2(\chi^{-2}) \times \dots \times \Xi^F(\chi^{-F}). \tag{12}$$

For a simultaneous-move game with each agent solving problem (10), a generalized Nash equilibrium is defined below.

Definition 1. A vector $\chi_N = (\chi_N^1, \dots, \chi_N^F) \in \Omega(\chi_N)$ is a pure-strategy generalized Nash equilibrium (PGNE) if for each agent $f \in \mathcal{F}$,

$$\theta^f(\chi_N^f, \chi_N^{-f}) \geq \theta^f(\chi^f, \chi_N^{-f}), \quad \forall \chi^f \in \Xi^f(\chi_N^{-f}). \tag{13}$$

Based on the above definitions and discussions, it is easy to see that if (x, y) is an equilibrium to an SPMI game formulated as a GNEP using both primal and dual variables (11), then x must be an equilibrium to the SPMI game using only the primal variables (6). Such a relationship is formally stated below.

Proposition 3. Suppose that $\chi = (x, y) \in \mathbb{R}^{F \times (|A| + |V|)}$ is a PGNE of the GNEP where each agent solves (11). Then x is a PNE of the SPMI game where each agent solves (6). $\square$

The GNEP reformulation of an SPMI game can facilitate computation of a PNE of such games, as will be discussed in detail in the following two subsections.

4.2. Equilibrium-based approach through linear complementarity formulation

We first focus on SPMI games with continuous interdiction. For continuous games in general (i.e., games of continuous decision variables), a prevailing approach to compute a Nash equilibrium is through stacking the first-order optimality conditions (aka the KKT conditions) of each player's optimization problem to form a single complementarity problem (CP). However, as pointed in Facchinei and Kanzow (2010a), doing so for a GNEP in general leads to a CP that does not have nice properties to facilitate computation. On the other hand, it is known that a GNEP is equivalent

to a (finite-dimensional) quasi-variational inequality (QVI) (Harker, 1991). However, there are few efficient algorithms available to find a solution of a QVI. Despite the difficulties for computing a GNEP in general, we show that for SPMI games with continuous interdiction, by stacking the KKT conditions of (9), we will obtain an LCP with favorable properties. Under such properties, the well-known Lemke method is guaranteed to find a solution of the LCP after a finite number of iterations (given that the LCP is nondegenerate). Furthermore, it is shown in Adler and Verma that LCPs solvable by the Lemke method belongs to the complexity class *PPAD* (Polynomial-time Parity Argument Directed), which is between *P* and *NP*. Hence, our problem represents a special instance of GNEPs that avoids many of the theoretical and computational difficulties of general GNEPs, as documented in Dorsch, Jongen, and Shikhman (2013).

Before presenting the LCP formulation, we first introduce some basic notations and definitions. Formally, given a vector $q \in \mathbb{R}^d$ and a matrix $\mathbf{M} \in \mathbb{R}^{d \times d}$, a linear complementarity problem $\text{LCP}(q, \mathbf{M})$ consists of finding a decision vector $w \in \mathbb{R}^d$ such that $0 \leq w \perp q + \mathbf{M}w \geq 0$, where the $\perp$ sign means that $w^T(q + \mathbf{M}w) = 0$ and is referred to as the complementarity condition. Any w that satisfies $w \geq 0$ and $q + \mathbf{M}w \geq 0$ is called a feasible solution, and the $\text{LCP}(q, \mathbf{M})$ is itself said to be feasible if such a w exists. A feasible w that satisfies the complementarity condition is called a *solution* of the LCP, and the set of such solutions is denoted by $\text{SOL}(q, \mathbf{M})$. An LCP is said to be solvable if it has a solution. A thorough exposition of the theory and algorithms for LCPs can be found in Cottle, Pang, and Stone (2009).

In order to present the LCP reformulation of the SPMI game, we introduce the following notation. Let $|V| = n$ and $|A| = m$. Denote by $\mathbf{G}$ the arc-node incidence matrix of the graph G . Further let $\mathbf{I}$ denote an identity matrix, and $\mathbf{0}$ be a vector or a matrix of all zeros with appropriate dimensions. The objective function coefficients for the LP (9) are given by the vector $\phi^f \in \mathbb{R}^{m+n}$, defined as follows:³

$$\phi^f = \begin{bmatrix} \mathbf{0}_m \\ v^f \end{bmatrix}, \quad \text{where} \quad v_u^f = \begin{cases} 1 & \text{if } u = s^f \\ 0 & \text{if } u \neq s^f, t^f \\ -1 & \text{if } u = t^f \end{cases}. \quad (14)$$

The right hand sides of the constraints in (9) can be represented by the vector $r^f(x^{-f}) \in \mathbb{R}^{m+1}$:

$$r^f(x^{-f}) = \begin{bmatrix} -d^0 \\ -b^f \end{bmatrix} - \sum_{\substack{f' \in \mathcal{F} \\ f' \neq f}} \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m \times n} \\ \mathbf{0}_m^T & \mathbf{0}_n^T \end{bmatrix} \begin{bmatrix} x^{f'} \\ y^{f'} \end{bmatrix}. \quad (15)$$

The left hand sides for the constraints are represented by the matrix $\mathbf{A}^f \in \mathbb{R}^{(m+1) \times (m+n)}$:

$$\mathbf{A}^f = \begin{bmatrix} \mathbf{I}_m & \mathbf{G} \\ -c^f & \mathbf{0}_n^T \end{bmatrix}. \quad (16)$$

Hence, each agent f 's LP (9) can be restated as follows:⁴

$$\begin{aligned} & \underset{x^f, y^f}{\text{minimize}} && \phi^{fT} \begin{bmatrix} x^f \\ y^f \end{bmatrix} \\ & \text{subject to} && \mathbf{A}^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq r^f(x^{-f}), \\ & && \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0. \end{aligned} \quad (17)$$

Let the dual variables for the LP (17) be (λ^f, β^f) , where λ^f are the multipliers for the arc potential constraints, β^f the multiplier for the budgetary constraint. The KKT conditions for (17) are given by the following system:

$$0 \leq \begin{bmatrix} x^f \\ y^f \end{bmatrix} \perp \phi^f - \mathbf{A}^{fT} \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, \quad (18)$$

$$0 \leq \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \perp -r^f(x^{-f}) + \mathbf{A}^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0. \quad (19)$$

Note that by stacking all agents' KKT conditions together, it does not directly give an LCP, as the vector $r^f(x^{-f})$ in (19) contains other agents' variables x^{-f} . To derive the LCP formulation of the overall game, we first expand the $r^f(x^{-f})$ in (19) term as follows:

$$0 \leq \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \perp \begin{bmatrix} d^0 \\ b^f \end{bmatrix} + \mathbf{A}^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} + \sum_{\substack{f' \in \mathcal{F} \\ f' \neq f}} \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m \times n} \\ \mathbf{0}_m^T & \mathbf{0}_n^T \end{bmatrix} \begin{bmatrix} x^{f'} \\ y^{f'} \end{bmatrix} \geq 0. \quad (20)$$

By stacking (18) and (20) together for all agents, and after straightforward algebraic derivations, we can obtain the following LCP for the SPMI game:

$$0 \leq \underbrace{\begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^F \end{bmatrix}}_w \perp \underbrace{\begin{bmatrix} \tilde{q}^1 \\ \tilde{q}^2 \\ \vdots \\ \tilde{q}^F \end{bmatrix}}_q + \underbrace{\begin{bmatrix} \mathbf{M}^1 & \tilde{\mathbf{M}}^2 & \dots & \tilde{\mathbf{M}}^F \\ \tilde{\mathbf{M}}^1 & \mathbf{M}^2 & \dots & \tilde{\mathbf{M}}^F \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}^1 & \tilde{\mathbf{M}}^2 & \dots & \mathbf{M}^F \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^F \end{bmatrix} \geq 0, \quad (21)$$

where for $f = 1, \dots, F$, $w^f = (x^{fT}, y^{fT}, \lambda^{fT}, \beta^{fT})^T$, $q^f = (\phi^{fT}, d^{0T}, b^{fT})^T$, and the diagonal block matrices $\mathbf{M}^f$ and off-diagonal block matrices $\tilde{\mathbf{M}}^f$ in the big matrix $\mathbf{M}$ are as follows:

$$\begin{aligned} \mathbf{M}^f &= \begin{bmatrix} \mathbf{0}_{(m+n) \times (m+n)} & -\mathbf{A}^{fT} \\ \mathbf{A}^f & \mathbf{0}_{(m+1) \times (m+1)} \end{bmatrix}, \\ \tilde{\mathbf{M}}^f &= \begin{bmatrix} \mathbf{0}_{m \times m} & \mathbf{0}_{m \times n} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times 1} \\ \mathbf{0}_{n \times m} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times m} & \mathbf{0}_{n \times 1} \\ \mathbf{I}_m & \mathbf{0}_{m \times n} & \mathbf{0}_{m \times m} & \mathbf{0}_{m \times 1} \\ \mathbf{0}_{1 \times m} & \mathbf{0}_{1 \times n} & \mathbf{0}_{1 \times m} & 0 \end{bmatrix}. \end{aligned} \quad (22)$$

With a closer look, in the $\mathbf{M}^f$ matrix, the top two blocks correspond to Eq. (18). Since on the right side of the complementarity condition in (18), only the dual variables (λ^f, β^f) appear, not the primal variables (x^f, y^f) , hence the top left all-zero matrix (corresponding to the primal variables), and the top right block matrix $-\mathbf{A}^{fT}$ (corresponding to the dual variables). The situation is reversed in Eq. (20), which corresponds to the lower two blocks in $\mathbf{M}^f$. The sole non-zero block matrix in $\tilde{\mathbf{M}}^f$ can be seen in Eq. (20), the coefficient matrix before $(x^{f'}, y^{f'})$.

With the definition of the vector q and the matrix $\mathbf{M}$ in (21), we simply use $\text{LCP}(q, \mathbf{M})$ to refer to the overall complementarity problem. $\text{LCP}(q, \mathbf{M})$ is equivalent to the corresponding (continuous) SPMI game in the following sense: if the collection $(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^F)$, where $\tilde{x}^f = (\tilde{x}^f, \tilde{y}^f)$, is a Nash equilibrium to the SPMI game, then there exist corresponding multipliers $(\tilde{\lambda}^f, \tilde{\beta}^f)$ such that $\tilde{w} = (\tilde{w}^1, \tilde{w}^2, \dots, \tilde{w}^F)$ with $\tilde{w}^f = (\tilde{x}^{fT}, \tilde{y}^{fT}, \tilde{\lambda}^{fT}, \tilde{\beta}^{fT})^T$, solves the $\text{LCP}(q, \mathbf{M})$ in (21). This is so because each $(\tilde{x}^f, \tilde{y}^f)$ must be an optimal solution to agent f 's problem (9), parameterized by $\tilde{x}^{-f}$. Since the constraints in (9) are all linear, the linear constraint qualification automatically holds and hence, there exist multipliers $(\tilde{\lambda}^f, \tilde{\beta}^f)$, together with $(\tilde{x}^f, \tilde{y}^f)$, that satisfy the KKT conditions (18) and (19), which leads to the $\text{LCP}(q, \mathbf{M})$.

³ Note that we change the objective in (9) to minimization first, just to conform to the convention that KKT conditions are usually written with respect to minimization problems.

⁴ Note that we switched from maximization in (9) to minimization in (17) simply to conform to the convention that the KKT systems are usually written with respect to minimization problems.

Conversely, if $\tilde{w}$, as above defined, is a solution to the LCP(q , $\mathbf{M}$), then each tuple $(\tilde{x}^f, \tilde{y}^f, \tilde{\lambda}^f, \tilde{\beta}^f)$ must satisfy the KKT conditions (18) and (19), with the corresponding $\tilde{x}^{-f}$. Again since (9) is an LP, the KKT conditions are sufficient for optimality, and hence $(\tilde{x}^f, \tilde{y}^f)$ are optimal solutions of the LP (9), parameterized by $\tilde{x}^{-f}$, and the collection of the optimal solutions $(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^F)$, with $\tilde{x}^f = (\tilde{x}^f, \tilde{y}^f)$, form a Nash equilibrium of the SPMI game.

In the following, we discuss algorithms to solve the LCP in (21). Methods for solving LCPs fall broadly into two categories: (i) pivotal methods such as Lemke's algorithm, and (ii) iterative methods such as splitting schemes and interior point methods. The former class of methods are finite when applicable, while the latter class converge to solutions in the limit. In general, the applicability of these algorithms depends on the structural properties of the matrix $\mathbf{M}$.

The following two lemmas establish that LCP(q , $\mathbf{M}$) for the SPMI game possesses two desired properties that allow us to use Lemke's pivotal algorithm. Recall that a matrix $\mathbf{M} \in \mathbb{R}^{d \times d}$ is said to be *copositive* if $x^T \mathbf{M} x \geq 0$ for all $x \in \mathbb{R}_+^d$. Similarly, given a set $K \in \mathbb{R}^d$, the set K^* denotes the *dual cone* of K ; i.e. $K^* = \{y \in \mathbb{R}^d : y^T x \geq 0, \forall x \in K\}$.

Lemma 1. *The matrix $\mathbf{M}$ defined as in (21) is copositive.*

Lemma 2. *Let the vector q and the matrix $\mathbf{M}$ be as defined in (21). Then $q \in (\text{SOL}(\mathbf{0}, \mathbf{M}))^*$.*

The proofs of the two lemmas are presented in Appendix A. With Lemma 1 and 2, we can apply the following result from Cottle et al. (2009).

Theorem 1 (Cottle et al. (2009), Theorem 4.4.13). *If $\mathbf{M}$ is copositive and $q \in (\text{SOL}(\mathbf{0}, \mathbf{M}))^*$, then the Lemke's method will compute a solution (with finite termination), if the problem is nondegenerate.⁵ □*

As discussed earlier, the LCP approach is not applicable for discrete SPMI games due to the presence of integer variables. In the following subsection, we present a decentralized approach that works for both discrete and continuous SPMI games.

4.3. Decentralized approach through best-response dynamics

Best-response-based algorithms are based upon arguably the most natural idea of finding a Nash equilibrium, which is for the players to take turns to solve their own optimization problem, while keeping other players' actions fixed at a certain level. Such algorithms are also known as the Gauss-Seidel iterative approach or the diagonalization scheme. We refer to such an approach as decentralized, which is applicable to both continuous and discrete SPMI games. While not necessarily more computationally efficient, decentralized algorithms indeed have several advantages over equilibrium-based algorithms, such as the Lemke's method discussed in the previous subsection. First, an equilibrium-based approach relies on the first-order optimality conditions of each agent's optimization problem. Such conditions are not available in discrete games, where agents' problems contain discrete variables. A decentralized approach can nevertheless be applied to discrete games, as each agent's problem can be solved as an integer program, without relying on explicit optimality conditions. Second, a decentralized algorithm may provide insight on how a particular equilibrium is achieved among agents' strategic interactions. Such insight is particularly useful when multiple equilibria exist, as is

the case for many GNEPs. It is known (for example, Myerson, 1978) that a game may possess unintuitive Nash equilibria that would never be a realistic outcome. Third, a decentralized algorithm can naturally lead to multithreaded implementations that can take advantage of a high performance computing environment. In addition, different threads in a multithreaded implementation may be able to find different equilibria of a game, making such an algorithm particularly suitable for computationally quantifying the average efficiency loss of noncooperative strategies.

The biggest drawback of decentralized algorithms is the lack of convergence property in general; that is, they are usually heuristic algorithms. In the following, however, we prove that if a best-response-based algorithm, in the form of Gauss-Seidel, converges, then the limit is indeed a Nash equilibrium of the SPMI game (either discrete or continuous). In addition, if the SPMI game is of common source-target pairs, then the Gauss-Seidel algorithm (or a regularized version for continuous SPMI games) indeed converges to a Nash equilibrium.

4.3.1. Gauss-Seidel Algorithm (Algorithm 1)

We first present the basic form of a best response based algorithm. The idea is simple: starting with a particular feasible decision vector $\chi_0 = (\chi_0^1, \chi_0^2, \dots, \chi_0^F) \in \Omega(\chi_0)$, solve the optimization problem of a particular agent, say, agent 1, with all of the other agents' actions fixed. Assume that an optimal solution exists to this optimization problem, and denote it as χ_1^{1*} . The next agent, say, agent 2, solves its own optimization problem, with the other agents' actions fixed as well, but with χ_0^1 replaced by χ_1^{1*} . Such an approach is often referred to as a diagonalization scheme or a Gauss-Seidel iteration, and for the remainder of this paper we use the latter name to refer to this simple best-response approach.

Consider applying the Gauss-Seidel iteration to a GNEP, with each agent solving the optimization problem (10), denoted as $\mathcal{P}(\chi^{-f})$. The Gauss-Seidel iterative procedure is presented in Algorithm 1 below.

Algorithm 1 Gauss-Seidel Algorithm for a GNEP.

Initialize. Choose $\chi_0 = (\chi_0^1, \dots, \chi_0^F)$ with $\chi_0^f \in \Xi^f(\chi_0^{-f}) \forall f \in \mathcal{F}$.
Set $k \leftarrow 0$.

Step 1:

for $f = 1, 2, \dots, F$ **do**

Set $\chi_{k,f} \leftarrow (\chi_{k+1}^1, \dots, \chi_{k+1}^{f-1}, \chi_k^f, \dots, \chi_k^F)$;

Solve $\mathcal{P}(\chi_{k,f}^{-f})$ to obtain an optimal solution $\chi_{k,f}^*$;

if $\theta^f(\chi_{k,f}^*, \chi_{k,f}^{-f}) > \theta^f(\chi_k^f, \chi_k^{-f})$ **then** Set $\chi_{k+1}^f \leftarrow \chi_{k,f}^*$;

else Set $\chi_{k+1}^f \leftarrow \chi_k^f$;

end if

end for

Set $\chi_{k+1} \leftarrow (\chi_{k+1}^1, \dots, \chi_{k+1}^F)$.

Set $k \leftarrow k + 1$.

if χ_k satisfies termination criteria, **then STOP**; **else GOTO** Step 1.

Note that updates in agent f 's decisions occur at iteration k only if there is a strict increase in the agent's payoff at the iteration. The algorithm can be directly applied to compute an equilibrium of an SPMI game with discrete interdiction. For finite termination, we fix a tolerance parameter ϵ and use the following stopping criterion:

$$\|\chi_k - \chi_{k-1}\| \leq \epsilon. \quad (23)$$

Proposition 4. *Suppose that the Gauss-Seidel algorithm (Algorithm 1) is applied to the SPMI game with discrete interdiction, and the termination criterion (23) is used with $\epsilon < 1$. If the algorithm terminates at χ_k , then χ_k is an equilibrium to this problem.*

⁵ When the LCP is degenerate, cycling in Lemke's method can indeed occur, as shown in Section 4.9 of Cottle et al. (2009). However, as indicated in Eaves (1971), when ambiguity arises in choosing the index to exit the basis, just randomly choose an index to leave the basis. The finite convergence of the Lemke's method still holds.

Proof. Since the variables χ_k are integral for discrete interdiction problems, choosing $\epsilon < 1$ for the termination criterion will ensure that the algorithm terminates only when successive outer iterates are equal. Consequently, by the assumption, $\chi_{k-1} = \chi_k$ at termination. This also implies that $\chi_{k-1,f}^{-f} = \chi_k^{-f}$ for $f = 1, \dots, F$. By construction of χ_k , we must then have

$$\chi_k^f = \underset{\chi^f \in \Xi^f(\chi_k^{-f})}{\operatorname{argmin}} \theta^f(\chi^f, \chi_k^{-f}).$$

Clearly, χ_k must then be an equilibrium. $\square$

Even though there is no guarantee that the algorithm will in fact converge, we note that in the discrete case, it is possible to detect when the algorithm fails to converge. Recall that $\Xi^f(\chi^{-f}) \subseteq K^f$ for each agent $f \in \mathcal{F}$, where K^f is defined below.

$$K^f = \left\{ (\chi^f, y^f) \mid \begin{array}{l} \sum_{(u,v) \in V} c_{uv}^f x_{uv}^f \leq b^f, \\ 0 \leq y_u^f \leq \bar{y} \quad \forall u \in V \end{array} \right\}. \quad (24)$$

Clearly, the set $\prod_{f=1}^F K^f$ is finite. Any intermediate point χ_k generated by Algorithm 1 must certainly satisfy the budgetary constraints on x_k^f and the bound constraints on y_k^f for each agent f . Therefore $\chi_k \in \prod_{f=1}^F K^f$. In other words, the set of possible points χ_k generated by Algorithm 1 lies in a finite set. This means that if the algorithm fails to converge, it must generate a sequence that contains at least one cycle. The existence of such cycles in non-convergent iterate paths can then be used to detect situations in which the algorithm fails to converge.

For the subclass of such games with common source-target pairs, we can in fact prove that the best response dynamics in Algorithm 1 always terminates in a PNE in a finite number of steps.

Proposition 5. Consider an SPMI game with discrete interdiction with common source-target pairs, and assume that the initial arc lengths d and arc extensions e are integral. Suppose that Algorithm 1 is applied to such a problem, and the termination criteria (23) is used with $\epsilon < 1$. Then the algorithm will terminate finitely at an equilibrium.

Proof. Denote the common source node as s , and the common target node as t . The set of joint feasible strategies in χ under the given assumptions is a finite set. Moreover, all the agents attempt to minimize the common objective, namely the s - t path length. Note that at any iteration k at which an update occurs for any agent's decision, there must then be a strict increase in the s - t path length. Thus there cannot exist cycles in the sequence $\{\chi_k\}$. Furthermore, since the set of joint feasible strategies is finite, the sequence must terminate at some point χ^* . It is easy to show that χ^* must be an equilibrium (cf. Proposition 4). $\square$

4.3.2. Regularized Gauss–Seidel Algorithm (Algorithm 2)

One disadvantage of the “naïve” Gauss–Seidel algorithm described above is that for continuous GNEPs, it can fail to converge to an equilibrium. However, Facchinei, Piccialli, and Scandrone (2011) showed that under certain assumptions, we can overcome this issue by adding a regularization term to the individual agent's problem solved in a Gauss–Seidel iteration.

The regularized version of the optimization problem for agent $f \in \mathcal{F}$ is

$$\begin{aligned} & \underset{\chi^f}{\operatorname{maximize}} && \theta^f(\chi^f, \chi^{-f}) - \tau \|\chi^f - \bar{\chi}^f\|^2 \\ & \text{subject to} && \chi^f \in \Xi^f(\chi^{-f}), \end{aligned} \quad (25)$$

where τ is a positive constant. Here the regularization term is evaluated in relation to a candidate point $\bar{\chi}^f$. Note that the point $\bar{\chi}^f$ and the other agents' decision variables χ^{-f} are fixed

when the problem (25) is solved. We refer to problem (25) as $\mathcal{R}(\chi^{-f}, \bar{\chi}^f)$. The regularized Gauss–Seidel procedure, herein referred to as Algorithm 2, is very similar to Algorithm 1, except that

Algorithm 2 Gauss–Seidel Algorithm for a GNEP.

Initialize. Choose $\chi_0 = (\chi_0^1, \dots, \chi_0^F)$ with $\chi_0^f \in \Xi^f(\chi_0^{-f}) \forall f \in \mathcal{F}$.
Set $k \leftarrow 0$.
Step 1:
for $f = 1, 2, \dots, F$ **do**
 Set $\chi_{k,f} \leftarrow (\chi_{k+1}^1, \dots, \chi_{k+1}^{f-1}, \chi_k^f, \dots, \chi_k^F)$;
 Solve $\mathcal{R}(\chi_{k,f}^{-f}, \chi_k^f)$ to obtain an optimal solution $\chi_{k,f}^*$;
 Set $\chi_{k+1}^f \leftarrow \chi_{k,f}^*$;
end for
Set $\chi_{k+1} \leftarrow (\chi_{k+1}^1, \dots, \chi_{k+1}^F)$.
Set $k \leftarrow k + 1$.
if χ_k satisfies termination criteria, **then STOP**; **else GOTO** Step 1.

$\mathcal{R}(\chi_{k,f}^{-f}, \chi_k^f)$ is solved in each iteration k instead of $\mathcal{P}(\chi_{k,f}^{-f})$.⁶

This version of the algorithm, along with its convergence proof, was originally presented in Facchinei et al. (2011) to solve GNEPs with shared constraints. The difficulty here that prevents us from showing convergence lies in the fact that we are dealing with GNEPs with non-shared constraints. As a result, any intermediate points resulting from an agent's best responses need not to be feasible in the other agents' problems. Consequently, we use Algorithm 2 only as a heuristic algorithm to solve SPMI games under continuous interdiction. Nevertheless, we can show that if Algorithm 2 converges, then the resulting point is an equilibrium to the SPMI game.

Proposition 6. Let $\{\chi_k\}$ be the sequence generated by applying Algorithm 2 to the SPMI problem under continuous interdiction, wherein each agent solves the regularized version of (9). Suppose $\{\chi_k\}$ converges to $\bar{\chi}$. Then $\bar{\chi}$ is an equilibrium to the SPMI problem.

The proof of this proposition is almost identical to that of Theorem 4.3 in Facchinei et al. (2011). However, we do want to point out one key difference in the proof. In Proposition 6, we need to assume that the entire sequence $\{\chi_k\}$ converges to $\bar{\chi}$. This is a strong assumption in the sense that it also requires that all the intermediate points $\chi_{k,f}$ in Algorithm 2 to converge to $\bar{\chi}$, a key to proving that $\bar{\chi}$ is indeed an equilibrium. In contrast, for GNEPs with shared constraints, this assumption may be weakened because the intermediate points $\chi_{k,f}$ and therefore the cluster points of the sequence generated by the algorithm are guaranteed to be feasible. The complete proof of Proposition 6 is presented in Appendix B.

Similar to discrete SPMI games, the convergence of Algorithm 2 is guaranteed for continuous-interdiction SPMI games with common source-target pairs. The key fact that allows us to prove this stronger result is that by dropping the dependence of the variables y on the agents $f \in \mathcal{F}$, any unilateral deviation in the shared variables y results in a solution that remains feasible in the other agents' optimization problems. The convergence result is formally stated below.

Proposition 7. Consider applying Algorithm 2 to the SPMI problem under continuous interdiction with common source-target pairs, where each agent solves the regularized version of (9). Let $\{\chi_k\}$ be

⁶ Another difference between the algorithms, in addition to $\mathcal{P}(\chi_{k,f}^{-f})$ versus $\mathcal{R}(\chi_{k,f}^{-f})$, is that in Algorithm 2, we do not insist that the decisions only be updated if there is a strict decrease in the objective value. Such a requirement in Algorithm 1 is to ensure finite termination (if it does not cycle).

the sequence generated by the algorithm. If $\bar{\chi}$ is a cluster point of this sequence, then it also solves the SPMI problem. $\square$

We conclude this section with a brief discussion on the social welfare maximizing solution where a centralized decision-maker computes the optimal interdiction plan jointly for all agents.

4.4. Social welfare maximization

When analyzing the SPMI game from a centralized decision-making perspective, we assume that the global obstruction function is utilitarian, i.e., the sum of the shortest s^f – t^f path lengths over all the agents $f \in \mathcal{F}$. We also assume that the resources are pooled among all the agents, resulting in a common budgetary constraint.⁷ The case where some resources cannot be pooled can be easily handled by including each player's constraints on such resources explicitly in (26). Thus the centralized problem for SPMI games can be given as follows:

$$\begin{aligned} & \underset{x, y}{\text{maximize}} && \sum_{f \in \mathcal{F}} (y_{t^f}^f - y_{s^f}^f) \\ & \text{subject to} && y_v^f - y_u^f \leq d_{uv}(x^f, x^{-f}) \quad \forall (u, v) \in A, f \in \mathcal{F}, \\ & && \sum_{f \in \mathcal{F}} \sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq \sum_{f \in \mathcal{F}} b^f \\ & && 0 \leq y_u^f \leq \bar{Y} \quad \forall u \in V, f \in \mathcal{F}, \\ & && x^f \in X^f \quad \forall f \in \mathcal{F}. \end{aligned} \quad (26)$$

Since y^f is bounded for all $f \in \mathcal{F}$, the feasibility set for (26) is compact. Thus a globally optimal solution exists regardless of whether x^f is continuous or discrete for all $f \in \mathcal{F}$. In the continuous case, Weierstrass's extreme value theorem applies since all the functions are continuous and the x^f variables are bounded due to the non-negativity and budgetary constraints. In the discrete case, there are only a finite number of values that the x^f variables can take.

5. Numerical results

We use the algorithms presented in the previous section to study several instances of SPMI games. The decentralized algorithms were implemented in MATLAB R2019a with native solvers from the optimization toolbox. The LCP formulation for the SPMI game with continuous interdiction was solved using the MATLAB interface for the complementarity solver PATH (Ferris & Munson, 1999). Computational experiments were carried out on a desktop workstation with a quad-core Intel Core i7 processor and 16 GHz of memory running Windows 7.

In the implementation of the decentralized algorithm, for SPMI games with discrete interdiction, we used Algorithm 1. For SPMI games with continuous interdiction, we followed a strategy of trying the “naïve” Gauss–Seidel algorithm (aka Algorithm 1) first. If it failed to converge in 1000 outer iterations, we then applied Algorithm 2 with a positive value of τ for the regularization term.

5.1. Computing equilibria

First, we applied the decentralized algorithm to Example 2 in Section 3.2, which is an SPMI game with continuous interdiction. In particular, the network is given in Fig. 2 and there are 2 agents: agent 1 has an adversary with source node 1 and target node 5, and agent 2 has an adversary with source node 1 and target node 6. Both agents have an interdiction budget of 1. The initial arc

lengths are 0, and the interdiction costs are equal for both agents and are given as the arc labels in Fig. 2, with $\epsilon = 2$. We set the regularization parameter $\tau = 0.01$. We were able to obtain a solution within an accuracy of 10^{-6} in 3 outer iterations.

Furthermore, we obtained multiple Nash equilibria by varying the starting point of the algorithm. All the equilibria obtained resulted in the same shortest path lengths for each agent. Some of the equilibria obtained are given in Table 3. The column x_0 represents the starting interdiction vector for each agent, the columns x_N^1 and x_N^2 give the equilibrium interdiction vectors for agents 1 and 2, respectively. The seven components in the vectors of x_0 , x_N^1 and x_N^2 represent the interdiction actions at each of the seven arcs in Fig. 2, with the arcs being ordered as follows: first, the top horizontal arcs (1,2) and (2,3), then the vertical arcs (1,4), (2,5) and (3,6), and finally the bottom horizontal arcs (4,5) and (5,6). The remaining two columns in Table 3, p_1 and p_2 , give the shortest path lengths for agents 1 and 2 respectively, at an equilibrium x_N .

Example 4. To test the algorithm on larger-scale problems, we expanded the instance in Example 2 to larger network sizes and numbers of agents. For F agents, the graph contains $2(F + 1)$ nodes with the arcs as shown in Fig. 3. The source node for all agents is a_1 . The target node for a given agent f is b_{f+1} . The initial arc lengths are all assumed to be zero. The interdiction costs are the same for all the agents and are given as the arc labels in Fig. 3. All the agents have an interdiction budget of 1. The cost parameter ϵ is chosen as 2. For discrete interdiction on these graphs, the arc extensions are assumed to be length 1.

The running time and iterations required to compute equilibria for these instances are summarized in Table 4. The first five columns in the table give the number of outer iterations and run-time for Algorithm 2 over these instances with continuous interdiction. The results indicate that the running time for the Lemke's method increases monotonically with the problem size. However, the running time for the decentralized method depends not just on the problem size but also on the number of outer iterations. In general, there is no correlation between these two parameters, as the algorithm is observed to converge in relatively few iterations even for some large problem instances. Also the results in Table 4 indicate that the running time for the decentralized algorithm grows quite slowly with respect to problem sizes. This is in stark contrast to the rapid increase in running time observed for the LCP approach as the problem size increases. For solving the LCP, as a benchmark against the Lemke's method, we also reformulate the LCP into a mixed integer linear program (MILP) using the modeling technique in Pardalos and Rosen (1988), and solve it using the native solver in MATLAB (intlinprog). The solution time is also reported in Table 4. It is interesting to note that the solution time explodes for the MILP approach even for quite small problem sizes. Indeed for instances with 10 or more players, the MILP approach failed to yield an optimal solution within the pre-set time limit of 30 minutes.

It must be noted that the order in which the individual agent problems are solved in the Gauss–Seidel algorithm plays an important role. The algorithm failed to converge for certain orderings of the agents, but succeeded in finding equilibria quickly for the same instance with other orderings. For instance, for a network of size 25, solving the agent problems (with continuous interdiction) in their natural order $\{1, 2, \dots, 25\}$ resulted in the failure of the “naïve” version of the algorithm to converge even after 1000 outer iterations. However, with a randomized agent order, the algorithm converged in as few as 13 iterations. It is encouraging to note that for the same agent order that resulted in the failure of the naïve version, the regularized method converged to a GNE within 394 outer-iterations with a runtime of 28 wall-clock seconds.

⁷ We have deliberately chosen to pool the budgetary constraints in the social welfare maximization problem (26), in contrast to the individual agents' budgetary constraints in the decentralized model. Our justification is that if centralized decision-making is possible, the central planner is usually able to allocate resources efficiently with respect to the entire system (i.e., pooling the resources together), which would reach the best possible social outcome.

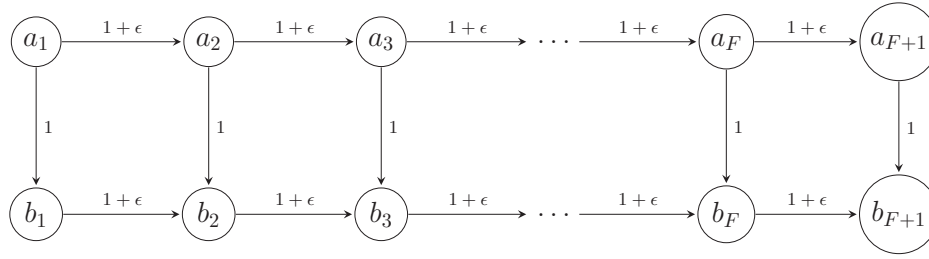


Fig. 3. Network structure for SPMI Example 4.

Table 4
Number of iterations and running times for SPMI Example 4.

	Continuous interdiction			Discrete interdiction		
	Decentralized		LCP Lemke	LCP MIP	Decentralized	
# Agents	# Iters	Runtime (s)	Runtime (s)	Runtime (s)	# Iters	Runtime (s)
3	3	0.0877	0.0079	1.3400	3	0.1184
5	3	0.0205	0.0290	5.7781	5	0.1776
7	3	0.2271	0.0650	22.9441	3	0.3628
10	5	0.0290	0.1833	–	3	0.1627
15	11	0.1103	0.7534	–	3	0.2419
20	5	0.0723	2.1106	–	3	0.3164
25	13	0.2609	4.8167	–	3	0.4005
30	15	0.4070	10.2256	–	3	0.5155
35	10	0.3605	17.7387	–	3	0.5948
40	41	1.7485	30.2382	–	3	0.7387
45	12	0.6601	48.6280	–	3	0.8794
50	12	0.7981	75.0420	–	3	1.0385

Table 5
Number of iterations and running time for SPMI Example 5.

Instance	# Nodes	# Arcs	# Agents	Continuous Interdiction		Discrete Interdiction	
				Decentralized	LCP	Decentralized	
# Iters	Runtime (s)	Runtime (s)	# Iters	Runtime (s)			
RussianRail 44	200	2	2	0.0466	0.1357	4	0.5967
			5	0.5365	0.5118	7	4.1210
			10	2.2557	16.1721	15	16.7551
SiouxFalls 24	76	2	2	0.0605	0.0221	3	0.1636
			5	0.2437	0.1578	5	0.7615
			10	1.7338	1.0771	13	2.8040
EasternMass74	258	2	3	0.0754	0.2170	8	4.4554
			5	0.1877	1.3622	8	6.5500
			10	0.4912	7.6393	35	54.5876
ChicagoSketch 33	2950	2	3	1.6353	27.1994	3	69.8861
			5	12.3496	151.4421	10	557.4518
			10	11.8890	836.2663	–	–

Example 5. We further present numerical experiments that illustrate the efficacy of the algorithms in Section 4 for solving practical problems. The problem sets are drawn from several different application settings. The RussianRail problem represents a Soviet era rail network presented in early network interdiction literature (Harris & Ross, 1955; Schrijver, 2002). The SiouxFalls network is based on a road traffic assignment problem presented in LeBlanc, Morlok, and Pierskalla (1975). The EasternMass and ChicagoSketch networks are sourced from equilibrium traffic assignment problems on aggregated highway networks. Data for the SiouxFalls network, as well as the EasternMass and ChicagoSketch networks were obtained from the Transportation Network Test Problem Set (Bar-Gera, 2002). The data generated for the experiments is available at https://github.com/harisreekumaran/dspi_games.

The results of our experiments are summarized in Table 5. For each instance, the size of the network, as well as the number of interdictors is given. For continuous interdiction, we were able to achieve convergence of Algorithm 2 with $\tau = 0$ for all instances.

On the other hand, for our largest instance (the ChicagoSketch network with 10 interdictors), the decentralized algorithm failed to reach convergence in our pre-set time limit of 30 minutes. In particular, for this instance, the bottleneck is the MIP solver that calls to solve each interdictor's problem.

5.2. Measuring inefficiency of equilibria

As mentioned earlier, one of the goals of this work is to quantify the inefficiency of an equilibrium of a decentralized multi-interdictor game – a decentralized solution to problem (1) – relative to a centrally planned optimal solution – an optimal solution to problem (2). A commonly used measure of such inefficiency is the *price of anarchy*. Formally speaking, let $\mathcal{N}_I$ be the set of all equilibria corresponding to a specific instance I . (In the context of multi-interdictor games, an instance consists of the network, obstruction functions, interdiction budgets, and costs, i.e., $I = (\mathcal{F}, G, \theta, X, b, c)$.) For the same instance I , let $(x^{1*}, \dots, x^{F*})$ denote a globally optimal solution to the centralized problem (26).

Then the price of anarchy of the instance I is defined as

$$p(I) := \max_{(x_N^1, \dots, x_N^F) \in \mathcal{N}_I} \frac{\theta^c(x^{1*}, \dots, x^{F*})}{\theta^c(x_N^1, \dots, x_N^F)}. \quad (27)$$

Let $\mathcal{I}$ be the set of all instances of a game. We assume implicitly that for all $I \in \mathcal{I}$, the set $\mathcal{N}_I$ is nonempty and a globally optimal solution to the centralized problem exists. By convention, p is set to 1 if the worst equilibrium as well as the globally optimal solution to the centralized problem both have zero objective value. If the worst equilibrium has a zero objective value while the optimal value of the centralized problem is positive, p is set to be infinity. In addition to the price of anarchy for an instance of a game, we also define the worst-case price of anarchy over all instances of the game (denoted as *w.p.o.a*) as follows:

$$w.p.o.a := \sup_{I \in \mathcal{I}} p(I). \quad (28)$$

We wish to study the efficiency loss of the class of SPMI games. We first show that in general, the worst-case price of anarchy cannot be bounded from above. We do so by demonstrating that given any candidate upper bound on the worst-case price of anarchy, we can construct an instance that invalidates the bound.

Consider the specific instance of the problem as depicted in Fig. 3. Recall that there are F agents and the source-target pair for agent f is (a_1, b_{f+1}) . Note that all paths for all agents contain either the arc (a_1, a_2) or the arc (a_1, b_1) . Then one feasible solution to the centralized problem is for each agent to interdict both these arcs by $1/(2 + \epsilon)$ for a total cost of 1. In this case, the length of both arcs become $F/(2 + \epsilon)$, giving a shortest path length of $F/(2 + \epsilon)$ for each agent. Note that this is not an equilibrium solution as agent 1 can deviate unilaterally to interdict arcs (a_1, b_1) and (a_2, b_2) by $1/2$ to obtain a shortest path length of $(F + \epsilon/2)/(2 + \epsilon)$.

A Nash equilibrium to this instance is given by the following solution. Agent f interdicts the vertical arcs $(a_1, b_1), \dots, (a_f, b_f)$ by $1/(f + 1)$ and the arc (a_{f+1}, b_{f+1}) by $f/(f + 1)$. Each agent then has a shortest path length of $F/(F + 1)$. Note that all the $s^f - t^f$ paths are of equal length for every agent. Therefore diverting any of the budget to any vertical arcs will result in unequal path lengths and a shorter shortest path for any agent. Obviously, diverting the budget to interdict any of the horizontal arcs is cost inefficient because of their higher interdiction cost at $1 + \epsilon$. Thus no agent has an incentive to deviate from this solution.

We now have a feasible solution to the centralized problem that has an objective value of $F/(2 + \epsilon)$ for each agent, and a Nash equilibrium that has an objective value of $F/(F + 1)$ for each agent. Therefore, by its definition in (28), the worst-case price of anarchy for the SPMI game depicted in Fig. 3 must be at least $(F + 1)/(2 + \epsilon)$.

The observation above implies that given any fixed candidate upper bound on the worst-case price of anarchy for the general class of SPMI games, under continuous interdiction, we can easily compute a tuple (F, ϵ) , which gives us an instance of the problem that breaks the bound.

While the worst-case price of anarchy provides a way to measure the inefficiency of equilibria, there are two major difficulties associated with this metric. First, it is well-known that the worst-case price of anarchy can be a very conservative measure of efficiency loss, since the worst case may only happen with pathological instances. Second, explicit theoretical bounds on the worst-case price of anarchy may be difficult to obtain for general classes of games. In fact most of the related research has focused on identifying classes of games for which such bounds may be derived. In this work, we show how our proposed decentralized algorithms can be used to numerically study the *average-case efficiency loss* (denoted by *a.e.l*). Let $\mathcal{I}' \subset \mathcal{I}$ denote a finite subset of instances, and let $|\mathcal{I}'|$

denote the cardinality of the the set $\mathcal{I}'$. Then

$$a.e.l(\mathcal{I}') := \frac{1}{|\mathcal{I}'|} \sum_{I \in \mathcal{I}'} p(I). \quad (29)$$

In other words, the average-case efficiency loss is the average value of $p(I)$ as defined in (27) over a set of sampled instances $\mathcal{I}' \subset \mathcal{I}$ of a game.

Using the decentralized algorithm and its potential to find multiple equilibria by starting at different points, we numerically study the efficiency loss of decentralized interdiction strategies in SPMI games. We focus first on Example 4, with the underlying network represented in Fig. 3. We compute lower bounds on the worst-case price of anarchy (denoted by *w.p.o.a*) and average efficiency losses for the same network topology with varying number of agents using the regularized Gauss-Seidel algorithm. The instances $\mathcal{I}'$ we consider are obtained by varying ϵ uniformly in the range of (1.5, 10). For the purpose of comparison, the numerical results are plotted in Fig. 4 below. Note that the average-case efficiency loss is much lower than the lower bounds for worst-case price of anarchy. For the particular graph structure under consideration, we observe that the average efficiency loss also grows at a much lower rate. However this observation cannot be generalized to other graph structures and such patterns may only be discernible by applying a decentralized computational framework as the one we presented.

Example 6. We further tested the decentralized algorithms for continuous interdiction on randomly generated graphs to study average efficiency losses of equilibria of SPMI games on networks with different topologies. For the graphs we randomly generated, the input parameters include the number of nodes and the density of a graph, which is the number of arcs divided by the maximum possible number of arcs. The number of agents was chosen randomly from the interval $(0, |V|/2)$, and one such number is chosen per vertex set size. Source-target pairs were chosen at random for each interdictor. Fixing the vertex set, we populated the arc set by successively generating source-target paths for the agents until the desired density was reached. We thus ensured connectivity between the source-target pairs for each agent. Costs, initial arc lengths and interdiction budgets were chosen from continuous uniform distributions. Arc interdiction costs were assigned uniformly in the range $[1, 5]$. The budget for each agent f was chosen uniformly from the interval $[b^f/10, b^f/2]$, where $b^f = \sum_{a \in A} c_a^f$. The initial length of each arc was chosen uniformly from $[1, 5]$.

For each combination of vertex set size, number of agents, and graph density, we generated 25 random instances by drawing values from the uniform distributions described above for the various network parameters. These instances constitute the set $\mathcal{I}'$ over which we compute the average efficiency loss, as well as lower bounds for worst-case price of anarchy. For each instance, we used 10 different random permutations of the agents to run the decentralized algorithms in an attempt to compute multiple equilibria. The lower bound on the worst-case price of anarchy for the game was computed as the highest price of anarchy over these 25 instances. The average efficiency loss over these instances was also computed. The results are summarized in Table 6. Our experiments indicate that the average efficiency loss and the worst-case price of anarchy tend to grow as the number of nodes and number of agents increases; on the other hand, these measures of efficiency loss sometimes do not appear to be monotonically increasing or decreasing with respect to the density of the underlying network.

One final note is that in calculating the results under the *a.e.l* and *w.p.o.a* columns in Table 6, we use the formulation (26), in which the budget constraint is pooled among all the agents. This is to reflect the situation where a central controller may have the freedom to allocate budget among agents. Such a shared budget

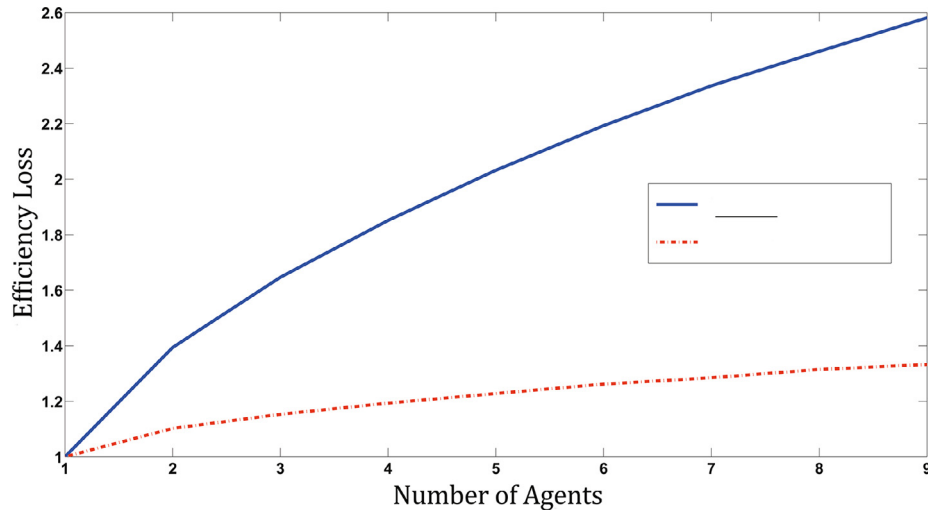


Fig. 4. Efficiency loss with respect to the number of agents.

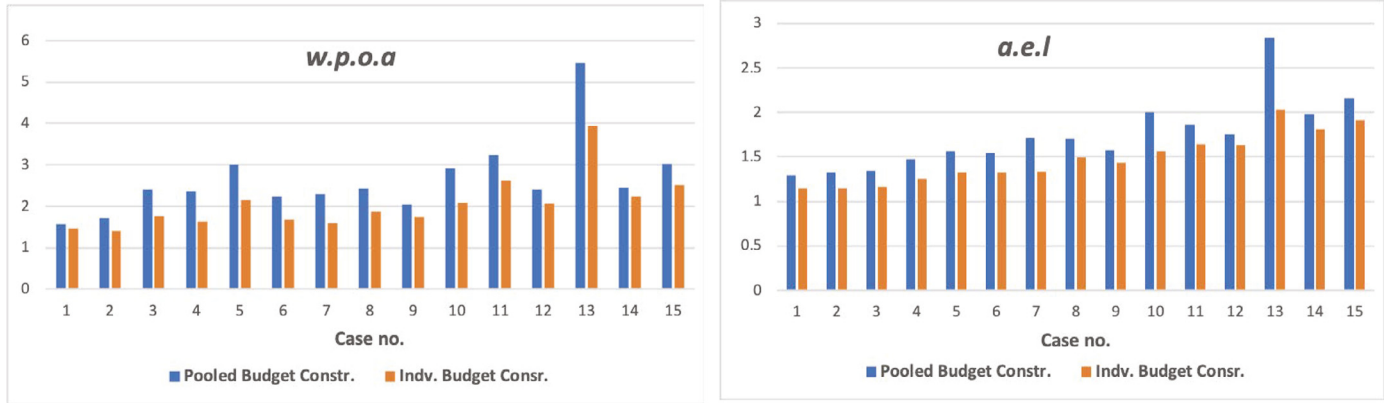


Fig. 5. Efficiency loss comparison: pooled versus individual budget constraint.

Table 6
SPMI continuous interdiction - random graphs.

Case no.	# Nodes	# Agents	Density	Avg. Run Time (second)	# Avg Iters.	<i>a.e.l</i>	<i>w.p.o.a</i>
1	5	3	0.25	0.0986	3	1.2918	1.5745
2	5	3	0.50	0.1171	4	1.3253	1.7269
3	5	3	0.75	0.1001	3	1.3440	2.4119
4	10	3	0.25	0.1146	3	1.4703	2.3675
5	10	3	0.50	0.2374	7	1.5593	3.0107
6	10	3	0.75	0.2750	8	1.5417	2.2395
7	15	4	0.25	0.2726	6	1.7091	2.2971
8	15	4	0.50	0.9492	21	1.7053	2.4351
9	15	4	0.75	0.8188	18	1.5746	2.0428
10	20	5	0.25	1.0981	19	1.9997	2.9176
11	20	5	0.50	1.4970	25	1.8588	3.2292
12	20	5	0.75	1.4883	25	1.7540	2.4137
13	25	7	0.25	2.6811	33	2.8349	5.4640
14	25	7	0.50	4.0756	47	1.9835	2.4575
15	25	7	0.75	4.3384	48	2.1579	3.0267

constraint, however, is different than the individual budget constraint in (9). One may question if the reported efficiency loss in Table 6 is due to decentralization or due to the individual versus pooled budget constraint. To provide some insight (from numerical perspective), we also solved the centralized problem (26) with the pooled budget constraint replaced by individual budget constraints: $\sum_{(u,v) \in A} c_{uv}^f x_{uv}^f \leq b^f$, for all $f \in \mathcal{F}$. The corresponding results of both *w.p.o.a* and *a.e.l* are presented in Fig. 5.

Since the feasible region in (26) with individual budget constraints is a subset of the region with a pooled budget constraint,

the pooled-constraint problem will always yield a no-lower optimal objective function value than the individual-constraint problem, and hence a higher value of *w.p.o.a*, as seen in Fig. 5. We also see that in certain cases, (such as in Case 13), the difference in the budget constraint does lead to notable differences in both the worst-case price of anarchy and the average efficiency loss. However, the overall trend is the same; that is, when the *w.p.o.a* is high for the pooled-budget case, it is also high for the individual-budget constraint case. Also the average efficiency losses are very similar for most of the cases.

6. Conclusions and Future Work

In this work, we introduced decentralized multi-interdictor games and provided formulations for one such class of games – shortest path multi-interdictor games. We analyzed the theoretical properties of SPMI games: in particular, we gave conditions for the existence of equilibria and examples where multiple equilibria exist. Specifically, we proved the existence of equilibria for general SPMI games under continuous interdiction. On the other hand, for the discrete counterpart, we provide an example where a pure-strategy equilibrium does not exist. However, for the subclass of problems with common source-target pairs, we are able to provide an existence guarantee.

We also showed that the SPMI game under continuous interdiction is equivalent to a linear complementarity problem, which can be solved by Lemke's algorithm. This constitutes a convergent centralized method to solve such problems. We also presented decentralized heuristic algorithms to solve SPMI games under both continuous and discrete interdiction. Finally, we used these algorithms to numerically evaluate the worst case and average efficiency loss of SPMI games.

There are other classes of network interdiction games that can be studied using the same framework we have developed, where the agents' obstruction functions are related to the maximum flow or minimum cost flow in the network. Establishing theoretical results and studying the applicability of the decentralized algorithms to other classes of decentralized network interdiction games are natural and interesting extensions of this work.

In our study of SPMI games, we also made the assumption that the games have complete information structure; that is, the normal form of the game – the set of agents, agents' feasible action spaces, and their objective functions – is assumed to be common knowledge to all agents. In addition, we made the implicit assumption that all input data are deterministic. However, data uncertainty and lack of observability of other agents' preferences or actions are prevalent in real-world situations. For such settings, we need to extend our work to accommodate games with exogenous uncertainties and incomplete information.

One might also be interested in designing interventions to reduce the loss of efficiency resulting from decentralized control. This leads to the topic of mechanism design. Such a line of work also defines a very important and interesting future research direction.

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Appendix A. Proofs pertaining to the LCP formulation

Proof of Lemma 1.. Let $w \in \mathbb{R}_+^{2m+n+1}$. Using the block structure of M given in (21), $w^T M w$ can be decomposed as follows:

$$w^T M w = \sum_{f=1}^F w^{fT} M^f w^f + \sum_{f=1}^F \sum_{\substack{f'=1 \\ f' \neq f}}^F w^{fT} \tilde{M}^{f'} w^{f'}. \quad (30)$$

We analyze the terms in the two summations separately. First consider $w^{fT} M^f w^f$ for any agent f . Let the dual variables (λ^f, β^f) be

collectively denoted by δ^f . We have

$$w^{fT} M^f w^f = \begin{bmatrix} \chi^{fT} & \delta^{fT} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{A}^{fT} \\ \mathbf{A}^f & \mathbf{0} \end{bmatrix} \begin{bmatrix} \chi^f \\ \delta^f \end{bmatrix} = -\chi^{fT} \mathbf{A}^{fT} \delta^f + \delta^{fT} \mathbf{A}^f \chi^f = 0. \quad (31)$$

Now consider any term of the form $w^{fT} \tilde{M}^{f'} w^{f'}$:

$$w^{fT} \tilde{M}^{f'} w^{f'} = \begin{bmatrix} \chi^{fT} & y^{fT} & \lambda^{fT} & \beta^{fT} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \chi^{f'} \\ y^{f'} \\ \lambda^{f'} \\ \beta^{f'} \end{bmatrix} = \begin{bmatrix} \chi^{fT} & y^{fT} & \lambda^{fT} & \beta^{fT} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \chi^{f'} \\ \mathbf{0} \end{bmatrix} = \lambda^{fT} \chi^{f'}. \quad (32)$$

Combining (31) and (32) we obtain

$$w^T M w = \sum_{f=1}^F \sum_{\substack{f'=1 \\ f' \neq f}}^F \lambda^{fT} \chi^{f'}. \quad (33)$$

Since λ^f 's and $\chi^{f'}$'s are the elements of w , $w \geq 0$ clearly implies that $w^T M w \geq 0$. $\square$

Proof of Lemma 2.. First note that $\text{SOL}(\mathbf{0}, \mathbf{M}) \neq \emptyset$ for any $\mathbf{M}$, since $\mathbf{0}$ is always a solution to $\text{LCP}(\mathbf{0}, \mathbf{M})$. Now consider a vector $w \in \text{SOL}(\mathbf{0}, \mathbf{M})$; i.e. $\mathbf{0} \leq w \perp \mathbf{0} + \mathbf{M}w \geq \mathbf{0}$. We prove that $q^T w \geq 0$. Observe that $q^T w$ can be decomposed as follows:

$$q^T w = \sum_{f=1}^F \bar{q}^{fT} w^f = \sum_{f=1}^F \left(\phi^{fT} \begin{bmatrix} \chi^f \\ y^f \end{bmatrix} + d^{0T} \lambda^f + b^f \beta^f \right) = \sum_{f=1}^F \left[(y_{sf}^f - y_{tf}^f) + d^{0T} \lambda^f + b^f \beta^f \right]. \quad (34)$$

The last two terms in the last equality above, $d^{0T} \lambda^f$ and $b^f \beta^f$, are non-negative for $f = 1, \dots, F$ because $w \in \text{SOL}(\mathbf{0}, \mathbf{M})$ implies that $\lambda^f, \beta^f \geq 0$, and by assumption $d^0, b^f \geq 0$ for each $f = 1, \dots, F$.

Now we focus on the first term in the last equality of (34): $\sum_{f=1}^F (y_{sf}^f - y_{tf}^f)$. Since w is given in $\text{SOL}(\mathbf{0}, \mathbf{M})$, then w must be feasible to $\text{LCP}(\mathbf{0}, \mathbf{M})$; i.e., $\mathbf{M}w \geq \mathbf{0}$, which implies that the following must be true (they are simply feasibility constraints after setting b^f and d^0 to 0):

$$\left. \begin{aligned} \sum_{a \in A} c_a^f x_a^f &\leq 0 \\ y_u^f - y_v^f + \sum_{f=1}^F x_{u,v}^f &\geq 0 \quad \forall (u, v) \in A \end{aligned} \right\} \text{ for } f = 1, \dots, F. \quad (35)$$

Recall that $c_a^f \geq 0$ for all $a \in A$ and $f = 1, \dots, F$ by assumption. Therefore, (35) implies that $x^f = 0$ for any agent f . It is easy to see that in this case, we must have

$$y_u^f - y_v^f \geq 0 \quad \forall (u, v) \in A, \text{ for } f = 1, \dots, F. \quad (36)$$

Now consider any s^f - t^f path p . By assumption, there must be at least one such path for each agent f . By summing up the inequalities (36) over the arcs in the path p , we obtain the desired result. In other words,

$$\sum_{(u,v) \in p} y_u^f - y_v^f = y_{s^f}^f - y_{t^f}^f \geq 0. \quad (37)$$

Summing up over the agents, we thus have shown that $q^T w \geq 0$ for any $w \in \text{SOL}(\mathbf{0}, \mathbf{M})$. $\square$

Appendix B. Proof of Proposition 6

We prove the proposition in two steps. We first show that $\bar{\chi}$ is feasible to each player's problem. Since by assumption $\chi_k \rightarrow \bar{\chi}$, we must have $\chi_k^f \rightarrow \bar{\chi}^f$ and

$$\lim_{k \rightarrow \infty} \|\chi_{k+1}^f - \chi_k^f\| = 0, \quad \forall f \in \mathcal{F}. \quad (38)$$

By construction of $\chi_{k,f}$, (38) implies that

$$\lim_{k \rightarrow \infty} \chi_{k,f} = \bar{\chi}, \quad \forall f \in \mathcal{F}. \quad (39)$$

Consider $\chi_{k,f+1} = (\chi_{k+1}^1, \dots, \chi_{k+1}^f, \chi_k^{f+1}, \chi_k^F)$. By Step 1 of Algorithm 2, we must have

$$\chi_{k+1}^f \in \Xi^f(\chi_{k,f+1}^{-f}). \quad (40)$$

Note that by (38) and (39), $\chi_{k+1}^f \rightarrow \bar{\chi}^f$ and $\chi_{k,f+1}^{-f} \rightarrow \bar{\chi}^{-f}$. The set $\Xi^f(\chi_{k,f+1}^{-f})$ is defined by linear inequalities parametrized by $\chi_{k,f+1}^{-f}$. Thus we may utilize continuity properties of this set valued mapping, and take limits on (40) to obtain

$$\bar{\chi}^f \in \Xi^f(\bar{\chi}^{-f}). \quad (41)$$

In other words, $\bar{\chi}$ is feasible for every agent's optimization problem (10).

We complete the proof by showing that for each agent $f \in \mathcal{F}$

$$\theta^f(\bar{\chi}^f, \bar{\chi}^{-f}) \geq \theta^f(\chi^f, \bar{\chi}^{-f}), \quad \forall \chi^f \in \Xi^f(\bar{\chi}^{-f}).$$

For the purpose of establishing a contradiction, suppose that there is an agent $\bar{f}$ and a vector $\bar{\xi}^{\bar{f}} \in \Xi^{\bar{f}}(\bar{\chi}^{-\bar{f}})$ such that

$$\theta^{\bar{f}}(\bar{\chi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}) < \theta^{\bar{f}}(\bar{\xi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}).$$

Let $d^{\bar{f}} = (\bar{\xi}^{\bar{f}} - \bar{\chi}^{\bar{f}})$. Then by the subdifferentiality inequality for concave functions we must have

$$\theta^{\bar{f}}(\bar{\chi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}; d^{\bar{f}}) > 0. \quad (42)$$

Our proof relies on constructing a contradiction to (42). To do so, we first construct a sequence $\xi_k^{\bar{f}}$ that is feasible to agent $\bar{f}$'s problem at the k -th iteration, such that $\xi_k^{\bar{f}} \rightarrow \bar{\xi}^{\bar{f}}$.

Using the linearity of the functions that define the set valued mapping $\Xi^{\bar{f}}(\cdot)$ we can show its inner semicontinuity relative to its domain (cf. Rockafellar & Wets, 1998 Chapter 5). Because $\bar{\chi}^{-\bar{f}} \in \text{dom}(\Xi^{\bar{f}}(\cdot))$, we then have

$$\liminf_{\xi^{-\bar{f}} \rightarrow \bar{\chi}^{-\bar{f}}} \Xi(\xi^{-\bar{f}}) \supseteq \Xi(\bar{\chi}^{-\bar{f}}), \quad (43)$$

where the limit in (43) is given by the following:

$$\liminf_{\xi^{-\bar{f}} \rightarrow \bar{\chi}^{-\bar{f}}} \Xi(\xi^{-\bar{f}}) = \left\{ u^{\bar{f}} \mid \forall \xi_k^{-\bar{f}} \rightarrow \bar{\chi}^{-\bar{f}}, \exists u_k^{\bar{f}} \rightarrow u^{\bar{f}} \text{ with } u_k^{\bar{f}} \in \Xi^{\bar{f}}(\xi_k^{-\bar{f}}) \right\}. \quad (44)$$

By assumption, $\bar{\xi}^{\bar{f}} \in \Xi^{\bar{f}}(\bar{\chi}^{-\bar{f}})$. By (39) we also have $\chi_{k,f}^{-\bar{f}} \rightarrow \bar{\chi}^{-\bar{f}}$. Eq. (44) then allows us to construct a sequence $\xi_k^{\bar{f}} \in \Xi^{\bar{f}}(\chi_{k,f}^{-\bar{f}})$ such that $\xi_k^{\bar{f}} \rightarrow \bar{\xi}^{\bar{f}}$ as $k \rightarrow \infty$.

Denote by Φ^f the regularized objective function for agent f 's subproblem. In other words,

$$\Phi^f(\chi^f, \chi^{-f}, z) = \theta^f(\chi^f, \chi^{-f}) - \tau \|\chi^f - z\|^2.$$

We then have

$$\Phi^f(\chi^f, \chi^{-f}, z; d^f) = \theta^f(\chi^f, \chi^{-f}; d^f) - 2\tau(\chi^f - z)^T d^f.$$

Note that $\chi_{k+1}^{\bar{f}}$ is obtained by solving the problem $\mathcal{R}(\chi_{k,f}^{-\bar{f}}, \chi_k^{\bar{f}})$. In other words, $\chi_{k+1}^{\bar{f}}$ maximizes $\Phi^{\bar{f}}(\cdot, \chi_{k,f}^{-\bar{f}}, \chi_k^{\bar{f}})$ over the set $\Xi^{\bar{f}}(\chi_{k,f}^{-\bar{f}})$.

Applying first order optimality conditions, setting $z = \chi_k^{\bar{f}}$ and $d^f = \xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}}$, we obtain the following:

$$\begin{aligned} \Phi^{\bar{f}}(\chi_{k+1}^{\bar{f}}, \chi_{k,f}^{-\bar{f}}, \chi_k^{\bar{f}}; (\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}})) &= \theta^{\bar{f}}(\chi_{k+1}^{\bar{f}}, \chi_{k,f}^{-\bar{f}}; (\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}})) \\ &\quad + 2\tau(\chi_{k+1}^{\bar{f}} - \chi_k^{\bar{f}})^T (\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}}) \\ &\leq 0. \end{aligned} \quad (45)$$

Passing to the limit $k \rightarrow \infty, k \in K$ and using (39) we obtain

$$0 \geq \theta^{\bar{f}}(\bar{\chi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}; (\bar{\xi}^{\bar{f}} - \bar{\chi}^{\bar{f}})),$$

which contradicts (42).

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