2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

Guaranteed Output Bounds Using Performance Integral Quadratic Constraints

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***Abstract*— This paper adopts the dissipativity approach to robustness analysis using integral quadratic constraints (IQCs). The nominal part of the uncertain system is a discrete-time, linear time-varying system. Generalized performance criteria are defined using time-domain IQCs. The derived robust performance theorem allows for incorporating available knowledge about the disturbance sets by means of signal IQCs. A novel way to compute point-wise bounds on the performance outputs is proposed. The developed results are illustrated by examples.**

I. Introduction

In this paper, we deal with the robust performance problem for an uncertain system $(G, \mathbf{\Delta})$ formed by the interconnection of a nominal system G and a perturbation operator Δ that is assumed to lie within a pre-specified set $\mathbf{\Delta}$. System G is assumed to be a discrete-time, linear time-varying (LTV) system. The integral quadratic constraint (IQC) framework is used to derive the robust performance theorem. The classical results for IQC analysis are derived in [1], [2] and are proved using a homotopy argument. The recent works of [3], [4] prove alternative results using dissipativity arguments. In the dissipativity approach, the IQCs are defined in the time domain using a stable, dynamic, time-invariant filter and a constant symmetric matrix. The advantage of the dissipativity approach and the time-domain characterization of the IQCs is that the robustness results can be extended to cover more general classes of nominal systems G. For instance, the results in [5], which apply when G is a discrete-time, linear time-invariant (LTI) system, have been extended in [6] to the case of a discrete-time, LTV nominal system G, and in [7] to the case of a distributed system G formed by the interconnection of multiple discrete-time, LTV subsystems.

The classical results of [1], [2] allow for using signal IQCs to describe the sets of exogenous disturbances that affect the uncertain system. Examples of signal IQCs are given in [8], [9]. However, signal IQCs have not yet been incorporated into the IQC analysis results derived using the dissipativity approach. One of the contributions of the present work is bridging this gap. Namely, the first novelty of the derived robust performance theorem is that it is proved using a time-domain argument and it permits restricting the set of allowable input signals using signal IQCs. The second novelty is that the derived theorem allows for the performance criterion to be defined in terms of a time-domain, time-varying IQC: the performance IQC is defined in terms of a stable, dynamic, time-varying filter and a sequence of time-varying symmetric matrices. This general definition of the performance measure includes as special cases the standard robust ℓ_2-gain performance level and the robust $\mathcal{D}$-to-ℓ_2-gain performance level when the disturbance inputs are restricted to a subset $\mathcal{D}$ of ℓ_2. The third novelty of this work consists of exploiting the derived theorem to develop a new way for computing useful point-wise bounds on the performance outputs. Finally, the paper gives examples that showcase the usefulness of incorporating signal IQCs into the analysis and compare the proposed method for computing point-wise output bounds with the one in [6].

The paper is structured as follows. Section II gives the time-domain characterizations of IQCs, signal IQCs, and performance IQCs, and derives the robust performance theorem. Section III gives the novel method for computing point-wise bounds on the performance outputs. Section IV presents the illustrative examples. The paper concludes with Section V.

Notation

$\mathbb{N}_0$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote the sets of nonnegative integers, real vectors of dimension n, and real $n \times m$ matrices, respectively. We write $X \succeq 0$ ($X \succ 0$) to indicate that the symmetric matrix X is positive semi-definite (positive definite). $0_{n\times m}$ denotes an $n\times m$ zero matrix. The subscripts are dropped when the dimensions n and m do not pertain to the discussion. The diagonal augmentation of the matrices $A_1, \ldots, A_n$ is denoted by $\mathrm{diag}(A_1, \ldots, A_n)$.

$\mathbb{RL}_\infty^{m\times n}$ denotes the space of $m \times n$ real, rational, matrix-valued functions with no poles on the unit circle. $\mathbb{RH}_\infty^{m\times n}$ denotes the subspace of functions in $\mathbb{RL}_\infty^{m\times n}$ with no poles outside the unit disk. $\Pi^\sim$ denotes the para-Hermitian conjugate of $\Pi \in \mathbb{RL}_\infty^{m\times n}$ and is defined as $\Pi^\sim(z) = \Pi^T(z^{-1})$.

The Hilbert space ℓ_2^n is the space of all real, vector-valued sequences $w = (w(0), w(1), \ldots)$, where $w(k) \in \mathbb{R}^n$ for all $k \in \mathbb{N}_0$, that have a finite ℓ_2-norm defined as $\|w\|_{\ell_2}^2 := \sum_{k=0}^{\infty} w(k)^T w(k) < \infty$. The inner product associated with ℓ_2^n is defined by $\langle u, v\rangle = \sum_{k=0}^{\infty} u(k)^T v(k)$ for all $u, v \in \ell_2^n$. Given $T \in \mathbb{N}_0$ and a vector-valued sequence w, we define the finite-horizon truncation of w as follows: $w_{[0,T]}(k) = w(k)$ for $k \le T$ and $w_{[0,T]}(k) = 0$ for $k > T$. The extended space ℓ_{2e}^n is defined as the space of all vector-valued sequences w whose finite-horizon truncations $w_{[0,T]}$ are in ℓ_2^n for all $T \in \mathbb{N}_0$. We often use the simplified symbols ℓ_2 and ℓ_{2e}.

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This work was supported by the Office of Naval Research under Award No. N00014-18-1-2627, the National Science Foundation (NSF) under Grant No. CMMI-1351640, and the Center for Unmanned Aircraft Systems (C-UAS), an NSF Industry/University Cooperative Research Center (I/UCRC) under NSF Grant No. CNS-1650465 along with significant contributions from C-UAS industry members. D. Abou Jaoude acknowledges the support of the University Research Board at the American University of Beirut.

978-1-5386-8266-1/$31.00 ©2020 AACC

II. Robust Performance Theorem

Consider the uncertain system $(G, \mathbf{\Delta})$ in Figure 1, which consists of the interconnection of a nominal, stable, discrete-time LTV system G and a bounded causal perturbation operator $\Delta : \ell_{2e} \to \ell_{2e}$ that lies in some pre-specified set $\mathbf{\Delta}$. Let $k \in \mathbb{N}_0$ denote the discrete time-step. The nominal system equations are given by $x_G(0) = 0$ and

$$\begin{aligned} x_G(k+1) &= A_G(k)x_G(k) + B_{G1}(k)\vartheta(k) + B_{G2}(k)d(k), \\ \varphi(k) &= C_{G1}(k)x_G(k) + D_{G11}(k)\vartheta(k) + D_{G12}(k)d(k), \\ e(k) &= C_{G2}(k)x_G(k) + D_{G21}(k)\vartheta(k) + D_{G22}(k)d(k). \end{aligned} \quad (1)$$

System $(G, \mathbf{\Delta})$ is thus described by the equations in (1) and

$$\vartheta = \Delta(\varphi), \quad (2)$$

where $\Delta \in \mathbf{\Delta}$. We assume that $\sup_{\Delta \in \mathbf{\Delta}} \|\Delta\| < \infty$, where $\|\Delta\|$ denotes the ℓ_2-induced norm of Δ. Definition 1 formalizes the notions of well-posedness and robust stability for system $(G, \mathbf{\Delta})$.

Definition 1: The uncertain system $(G, \mathbf{\Delta})$ is well-posed if for all $\Delta \in \mathbf{\Delta}$ and $d \in \ell_{2e}$, there exist unique solutions x_G, φ, ϑ, and e in ℓ_{2e} that satisfy (1)-(2) and causally depend on d. The system is robustly stable if it is well-posed and for all $\Delta \in \mathbf{\Delta}$ and $d \in \ell_2$, the system equations (1)-(2) admit unique solutions in ℓ_2 and further define a bounded causal mapping from d to e.

We use IQCs to describe the uncertainty set $\mathbf{\Delta}$ and the disturbance set $\mathcal{D} \subseteq \ell_2$ in which the operators Δ and exogenous disturbances d are assumed to lie, respectively. Let $\Psi \in \mathbb{RH}_\infty^{n_r \times (n_\varphi + n_\vartheta)}$ and $S = S^T \in \mathbb{R}^{n_r \times n_r}$ for some integer $n_r > 0$. $\mathbf{\Delta}$ satisfies the IQC defined by (Ψ, S), or $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$ for short, if for $x_\Psi(0) = 0$ and all $\varphi \in \ell_2$, $\vartheta = \Delta(\varphi)$, and $\Delta \in \mathbf{\Delta}$, the following condition holds:

$$\begin{aligned} &\sum\nolimits_{k=0}^{\infty} r(k)^T S r(k) \geq 0, \text{ where} \\ x_\Psi(k+1) &= A_\Psi x_\Psi(k) + B_{\Psi 1}\varphi(k) + B_{\Psi 2}\vartheta(k), \\ r(k) &= C_\Psi x_\Psi(k) + D_{\Psi 1}\varphi(k) + D_{\Psi 2}\vartheta(k), \end{aligned} \quad (3)$$

for all $k \in \mathbb{N}_0$. Let $\Theta \in \mathbb{RH}_\infty^{n_m \times n_d}$ and $U = U^T \in \mathbb{R}^{n_m \times n_m}$ for some positive integer n_m. $\mathcal{D}$ satisfies the signal IQC defined by (Θ, U), or $\mathcal{D} \in \mathrm{sig\,IQC}(\Theta, U)$ for simplicity, if for $x_\Theta(0) = 0$ and all $d \in \mathcal{D}$, the following condition holds:

$$\begin{aligned} &\sum\nolimits_{k=0}^{\infty} m(k)^T U m(k) \geq 0, \text{where} \\ x_\Theta(k+1) &= A_\Theta x_\Theta(k) + B_\Theta d(k), \\ m(k) &= C_\Theta x_\Theta(k) + D_\Theta d(k), \end{aligned} \quad (4)$$

for all k in $\mathbb{N}_0$. We also use IQCs to define performance criteria.

Definition 2: Let Ξ be a stable, discrete-time, LTV system, and $\{W(k)\}_{k \in \mathbb{N}_0}$ be a sequence of uniformly bounded symmetric matrices. System $(G, \mathbf{\Delta})$ defined by (1)-(2) satisfies the performance IQC defined by $(\Xi, \{W(k)\}_{k \in \mathbb{N}_0})$ if the uncertain system is robustly stable and for $x_\Xi(0) = 0$ and all $d \in \mathcal{D}$ and $\Delta \in \mathbf{\Delta}$, the following condition holds:

$$\begin{aligned} &\sum\nolimits_{k=0}^{\infty} p(k)^T W(k) p(k) \geq 0, \text{ where} \\ x_\Xi(k+1) &= A_\Xi(k)x_\Xi(k) + B_{\Xi 1}(k)d(k) + B_{\Xi 2}(k)e(k), \\ p(k) &= C_\Xi(k)x_\Xi(k) + D_{\Xi 1}(k)d(k) + D_{\Xi 2}(k)e(k), \end{aligned} \quad (5)$$

for all $k \in \mathbb{N}_0$.

For instance, let $p = \begin{bmatrix} d \\ e \end{bmatrix}$, i.e., let Ξ be a static, time-invariant operator defined by the sequence of matrices

$$\Xi(k) = \mathrm{diag}\,(I, I) \text{ for all } k \in \mathbb{N}_0. \quad (6)$$

In this case, $D_{\Xi 1}(k) = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $D_{\Xi 2}(k) = \begin{bmatrix} 0 \\ I \end{bmatrix}$ for all $k \in \mathbb{N}_0$, and the rest of the state-space matrices in (5) have at least one zero dimension and so are nonexistent. Then, if

$$W(k) = \mathrm{diag}\left(\gamma^2 I, -I\right) \text{ for all } k \in \mathbb{N}_0, \quad (7)$$

it follows that $\sum_{k=0}^{\infty} p(k)^T W(k) p(k) = \gamma^2 \|d\|_{\ell_2}^2 - \|e\|_{\ell_2}^2$. Thus, if the uncertain system $(G, \mathbf{\Delta})$ satisfies the performance IQC defined by (6) and (7), it follows that $\|e\|_{\ell_2} \leq \gamma \|d\|_{\ell_2}$ for all $d \in \mathcal{D}$ and $\Delta \in \mathbf{\Delta}$. This performance IQC corresponds to the standard robust performance criterion, and if satisfied, the uncertain system $(G, \mathbf{\Delta})$ is said to have a robust $\mathcal{D}$-to-ℓ_2-gain performance level of γ. If $\mathcal{D} = \ell_2$ and no signal IQC is used to describe the set $\mathcal{D}$, we say that the uncertain system $(G, \mathbf{\Delta})$ has a robust ℓ_2-gain performance level of γ.

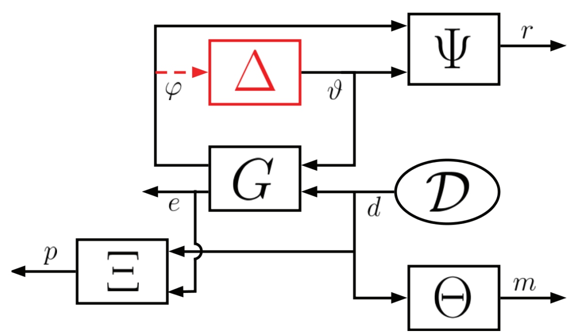

Fig. 1. Uncertain system $(G, \mathbf{\Delta})$ to be analyzed. The figure also shows the filters Ψ, Θ, and Ξ used for defining the IQC describing $\mathbf{\Delta}$, the signal IQC describing $\mathcal{D}$, and the performance IQC, respectively.

To derive the robustness analysis theorem, we form an augmented, stable, discrete-time, LTV system H that maps (ϑ, d) to (r, m, p) and is equivalent to the one shown in Figure 1 after removing the block and connection in red. For all $k \in \mathbb{N}_0$, system H is described by $x_H(0) = 0$ and

$$\begin{bmatrix} x_H(k+1) \\ r(k) \\ m(k) \\ p(k) \end{bmatrix} = \begin{bmatrix} A_H(k) & B_{H1}(k) & B_{H2}(k) \\ C_{H1}(k) & D_{H11}(k) & D_{H12}(k) \\ C_{H2}(k) & D_{H21}(k) & D_{H22}(k) \\ C_{H3}(k) & D_{H31}(k) & D_{H32}(k) \end{bmatrix} \begin{bmatrix} x_H(k) \\ \vartheta(k) \\ d(k) \end{bmatrix}, \quad (8)$$

where

$$\begin{aligned} &C_{H1}(k) = \begin{bmatrix} D_{\Psi 1}C_{G1}(k) & C_\Psi & 0 & 0 \end{bmatrix}, \quad D_{H22}(k) = D_\Theta, \\ &D_{H11}(k) = D_{\Psi 1}D_{G11}(k) + D_{\Psi 2}, \quad D_{H12}(k) = D_{\Psi 1}D_{G12}(k), \\ &C_{H2}(k) = \begin{bmatrix} 0 & 0 & C_\Theta & 0 \end{bmatrix}, \quad D_{H31}(k) = D_{\Xi 2}(k)D_{G21}(k), \\ &D_{H32}(k) = D_{\Xi 1}(k) + D_{\Xi 2}(k)D_{G22}(k), \quad D_{H21}(k) = 0, \\ &C_{H3}(k) = \begin{bmatrix} D_{\Xi 2}(k)C_{G2}(k) & 0 & 0 & C_\Xi(k) \end{bmatrix}, \end{aligned} \quad (9)$$

$$A_H(k) = \begin{bmatrix} A_G(k) & 0 & 0 & 0 \\ B_{\Psi 1} C_{G1}(k) & A_\Psi & 0 & 0 \\ 0 & 0 & A_\Theta & 0 \\ B_{\Xi 2}(k) C_{G2}(k) & 0 & 0 & A_\Xi(k) \end{bmatrix},$$

$$B_{H1}(k) = \begin{bmatrix} B_{G1}(k) \\ B_{\Psi 1} D_{G11}(k) + B_{\Psi 2} \\ 0 \\ B_{\Xi 2}(k) D_{G21}(k) \end{bmatrix}, \; x_H(k) = \begin{bmatrix} x_G(k) \\ x_\Psi(k) \\ x_\Theta(k) \\ x_\Xi(k) \end{bmatrix},$$

$$B_{H2}(k) = \begin{bmatrix} B_{G2}(k) \\ B_{\Psi 1} D_{G12}(k) \\ B_\Theta \\ B_{\Xi 1}(k) + B_{\Xi 2}(k) D_{G22}(k) \end{bmatrix}.$$

Theorem 1: Consider the uncertain system $(G, \mathbf{\Delta})$ defined by (1)-(2), and suppose that

1. the uncertain system $(G, \mathbf{\Delta})$ is robustly stable;
2. $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$;
3. $\mathcal{D} \in \mathrm{sig\,IQC}(\Theta, U)$;
4. there exists a sequence $\{P(k)\}_{k \in \mathbb{N}_0}$ of uniformly bounded symmetric matrices such that

$$\begin{aligned} &\begin{bmatrix} A_H^T(k) \\ B_{H1}^T(k) \\ B_{H2}^T(k) \end{bmatrix} P(k+1) \begin{bmatrix} A_H(k) & B_{H1}(k) & B_{H2}(k) \end{bmatrix} \\ &\qquad - \mathrm{diag}\left(P(k), 0, 0\right) \\ &\quad + \begin{bmatrix} C_{H1}^T(k) \\ D_{H11}^T(k) \\ D_{H12}^T(k) \end{bmatrix} S \begin{bmatrix} C_{H1}(k) & D_{H11}(k) & D_{H12}(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} C_{H2}^T(k) \\ D_{H21}^T(k) \\ D_{H22}^T(k) \end{bmatrix} U \begin{bmatrix} C_{H2}(k) & D_{H21}(k) & D_{H22}(k) \end{bmatrix} \\ &- \begin{bmatrix} C_{H3}^T(k) \\ D_{H31}^T(k) \\ D_{H32}^T(k) \end{bmatrix} W(k) \begin{bmatrix} C_{H3}(k) & D_{H31}(k) & D_{H32}(k) \end{bmatrix} \preceq 0 \end{aligned} \tag{10}$$

for all $k \in \mathbb{N}_0$, where the state-space matrix-valued functions $A_H(\cdot)$, $B_{H1}(\cdot)$, and so on are defined in (9).

Then, the uncertain system $(G, \mathbf{\Delta})$ satisfies the performance IQC defined by $(\Xi, \{W(k)\}_{k \in \mathbb{N}_0})$.

Proof: Since the uncertain system $(G, \mathbf{\Delta})$ is robustly stable, then for all $\Delta \in \mathbf{\Delta}$ and $d \in \ell_2$, there exist unique solutions x_G, φ, ϑ, and e to (1)-(2) that are in ℓ_2 and causally depend on d. Using d, φ, ϑ, and e in ℓ_2, define x_Ψ, x_Θ, x_Ξ, r, m, and p in ℓ_2 that satisfy (3), (4), (5), and thus (8). Pre- and post-multiplying (10) by $\begin{bmatrix} x_H^T(k) & \vartheta(k)^T & d(k)^T \end{bmatrix}$ and its transpose, we get

$$\begin{aligned} &x_H^T(k+1) P(k+1) x_H(k+1) - x_H^T(k) P(k) x_H(k) \\ &+ r(k)^T S r(k) + m(k)^T U m(k) - p(k)^T W(k) p(k) \le 0. \end{aligned}$$

This inequality is summed from $k = 0$ to $k = N$ to yield

$$\begin{aligned} &x_H^T(N+1) P(N+1) x_H(N+1) + \sum_{k=0}^{N} r(k)^T S r(k) \\ &+ \sum_{k=0}^{N} m(k)^T U m(k) - \sum_{k=0}^{N} p(k)^T W(k) p(k) \le 0, \end{aligned}$$

where the fact $x_H(0) = 0$ is used to simplify the resulting inequality. Taking the limit as $N \to \infty$ and using the facts that $\sum_{k=0}^{\infty} r(k)^T S r(k) \ge 0$ since $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$, $\sum_{k=0}^{\infty} m(k)^T U m(k) \ge 0$ since $\mathcal{D} \in \mathrm{sig\,IQC}(\Theta, U)$, and $\lim_{N \to \infty} x_H(N+1) = 0$ since x_G, x_Ψ, x_Θ, and x_Ξ are in ℓ_2, it follows that $\sum_{k=0}^{\infty} p(k)^T W(k) p(k) \ge 0$, i.e., system $(G, \mathbf{\Delta})$ satisfies the desired performance IQC. ■

The works of [5], [6] deal with the robustness analysis problem for uncertain systems where the nominal system is discrete-time LTI and discrete-time LTV, respectively, and the performance measure is the robust ℓ_2-gain performance level. When the performance IQC is defined by (6)-(7), Theorem 1 improves on the results therein in that it incorporates the available information about $\mathcal{D}$ into the analysis, namely, $\mathcal{D} \in \mathrm{sig\,IQC}(\Theta, U)$, thereby reducing conservatism. In this case, we speak of a robust $\mathcal{D}$-to-ℓ_2-gain performance level.

To derive a robust stability result to check for condition 1 in Theorem 1, the results in [6] need to be reworked along the lines of [3]. We will state the result but omit the proof.

Theorem 2: Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_\varphi + n_\vartheta) \times (n_\varphi + n_\vartheta)}$ be factorized as $\Pi(z) = \Psi^\sim(z) S \Psi(z)$, where $\Psi \in \mathbb{RH}_\infty^{n_r \times (n_\varphi + n_\vartheta)}$ and $S = S^T \in \mathbb{R}^{n_r \times n_r}$ for some integer $n_r > 0$. Suppose that Π is partitioned as in $\Pi = [\Pi_{ij}]_{i,j=1,2}$, where $\Pi_{11} \in \mathbb{RL}_\infty^{n_\varphi \times n_\varphi}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{n_\vartheta \times n_\vartheta}$ and satisfy $\Pi_{11}(e^{j\omega}) \succeq 0$ and $\Pi_{22}(e^{j\omega}) \preceq 0$ for all $\omega \in [-\pi, \pi]$. Then, the uncertain system $(G, \mathbf{\Delta})$ defined by (1)-(2) is robustly stable if

1. the uncertain system $(G, \mathbf{\Delta})$ is well-posed;
2. $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S)$;
3. there exist a sequence $\{P(k)\}_{k \in \mathbb{N}_0}$ of uniformly bounded symmetric matrices and $\epsilon > 0$ such that

$$\begin{aligned} &\begin{bmatrix} \bar{A}_H^T(k) \\ \bar{B}_H^T(k) \end{bmatrix} P(k+1) \begin{bmatrix} \bar{A}_H(k) & \bar{B}_H(k) \end{bmatrix} - \mathrm{diag}\left(P(k), 0\right) \\ &\qquad + \begin{bmatrix} \bar{C}_H^T(k) \\ \bar{D}_H^T(k) \end{bmatrix} S \begin{bmatrix} \bar{C}_H(k) & \bar{D}_H(k) \end{bmatrix} \preceq -\epsilon I \end{aligned} \tag{11}$$

for all $k \in \mathbb{N}_0$, where

$$\bar{A}_H(k) = \begin{bmatrix} A_G(k) & 0 \\ B_{\Psi 1} C_{G1}(k) & A_\Psi \end{bmatrix}, \; \bar{D}_H(k) = D_{\Psi 1} D_{G11}(k) + D_{\Psi 2},$$

$$\bar{B}_H(k) = \begin{bmatrix} B_{G1}(k) \\ B_{\Psi 1} D_{G11}(k) + B_{\Psi 2} \end{bmatrix}, \; \bar{C}_H(k) = \begin{bmatrix} D_{\Psi 1} C_{G1}(k) & C_\Psi \end{bmatrix}.$$

If system $(G, \mathbf{\Delta})$ is well-posed, the assumptions on Π in Theorem 2 hold, and the performance IQC is defined by (6)-(7), it is possible to employ the strict versions of the linear matrix inequalities (LMIs) in (10), i.e., replace 0 on the right-hand side of the LMIs by $-\epsilon I$ with $\epsilon > 0$, to conclude that the system is robustly stable. Then, the robust stability condition in Theorem 1 can be relaxed, as it would be guaranteed by the remaining and aforementioned conditions: the existence of solutions to the strict versions of the LMIs in (10) implies the existence of solutions to the LMIs in (11). However, the same cannot be concluded when general performance measures are considered, and so in general we need to separately verify that system $(G, \mathbf{\Delta})$ is robustly stable before applying Theorem 1.

The previous discussion applies for general time-varying nominal systems and performance IQCs. We conclude this section with a brief discussion on the special case of eventually periodic systems and performance IQCs.

Definition 3: A matrix-valued sequence $\{P(k)\}_{k\in\mathbb{N}_0}$ is (h,q)-eventually periodic, for some integers $h\geq 0$ and $q>0$, if $P(k+h+q\eta)=P(k+h)$ for all $k,\eta\in\mathbb{N}_0$. A discrete-time LTV system is (h,q)-eventually periodic if all its state-space matrix sequences are (h,q)-eventually periodic.

Proposition 1: Suppose that the matrix-valued sequence $\{W(k)\}_{k\in\mathbb{N}_0}$ and the systems G and Ξ are (h,q)-eventually periodic. Then, there exists a sequence $\{P(k)\}_{k\in\mathbb{N}_0}$ of uniformly bounded symmetric matrices such that the strict version of (10) holds for all $k\in\mathbb{N}_0$ if and only if there exists an (h,q)-eventually periodic sequence of symmetric matrices, $\{P_{h,q}(k)\}_{k\in\mathbb{N}_0}$, that satisfies the strict version of (10) for all $k\in\mathbb{N}_0$.

Proof: This result is proved using similar arguments to the ones in [6], [10], [11]. ■

When the nominal system G and the performance IQC defined by $(\Xi,\{W(k)\}_{k\in\mathbb{N}_0})$ are (h,q)-eventually periodic, i.e., Ξ and $\{W(k)\}_{k\in\mathbb{N}_0}$ are (h,q)-eventually periodic, the sequences $\{A_H(k)\}_{k\in\mathbb{N}_0}$, $\{B_{H1}(k)\}_{k\in\mathbb{N}_0}$, and so on defined from the state-space matrices in (9) will be (h,q)-eventually periodic as well. Proposition 1 states that, in this case, if a strict version of (10) is considered, then it suffices to only check a finite sequence of the (strict) LMIs defined in (10) for $k=0,1,\ldots,h+q-1$, along with the constraint $P(h+q)=P(h)$. If G and the performance IQC are $(0,q)$-eventually periodic, it can be shown using a similar averaging technique to the ones in [10], [12] that the result of Proposition 1 holds for (10). Time-invariant systems and sequences are $(0,1)$-eventually periodic, and so in the case of time-invariant nominal systems and performance IQCs, the problem reduces to finding a symmetric matrix P satisfying a single nonstrict LMI.

III. Computing Point-wise Output Bounds

In this section, we build on a mathematical trick from [13] to give a novel application of Theorem 1. Let v be a scalar-valued signal in ℓ_2^1. The discrete-time Fourier transform of v is defined as $\hat{v}(e^{j\omega})=\sum_{k=0}^{\infty}v(k)e^{-j\omega k}$ for all $\omega\in[-\pi,\pi]$. If $\hat{v}$ is given, then $v(k)$ is computed from $v(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{v}(e^{j\omega})e^{j\omega k}d\omega$ for all $k\in\mathbb{N}_0$. Moreover,

$$|v(k)|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\hat{v}(e^{j\omega})\right|\left|e^{j\omega k}\right|d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\hat{v}(e^{j\omega})\right|d\omega$$

for all $k\in\mathbb{N}_0$. Given $v\in\ell_2^1$, let the set $\mathcal{D}_v$ be defined as

$$\mathcal{D}_v=\left\{d_v\in\ell_2^1:\hat{d}_v(e^{j\omega})=\|d_v\|_{\ell_2}e^{j\phi(\omega)}\right\}\subseteq\ell_2^1,\quad(12)$$

where $\hat{v}(e^{j\omega})=|\hat{v}(e^{j\omega})|e^{j\phi(\omega)}$. Let u^* denote the complex conjugate of u. The spectrum of $d_v\in\mathcal{D}_v$ is constant since $|\hat{d}_v(e^{j\omega})|^2=\|d_v\|_{\ell_2}^2$ for all $\omega\in[-\pi,\pi]$, and we have

$$\langle v,d_v\rangle=\sum_{k=0}^{\infty}v(k)d_v(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{v}(e^{j\omega})\left(\hat{d}_v(e^{j\omega})\right)^*d\omega$$

$$=\frac{\|d_v\|_{\ell_2}}{2\pi}\int_{-\pi}^{\pi}|\hat{v}(e^{j\omega})|d\omega\quad\text{for all } d_v\in\mathcal{D}_v,$$

where Parseval's theorem is used on the first line. It then follows that $\|d_v\|_{\ell_2}|v(k)|\leq\langle v,d_v\rangle$ for all $k\in\mathbb{N}_0$ and $d_v\in\mathcal{D}_v$. For each $v\in\ell_2^1$, the set $\mathcal{D}_v$ is a subset of the set $\mathcal{W}$ of scalar-valued, white noise signals defined as follows:

$$\mathcal{W}=\left\{w\in\ell_2^1:|\hat{w}(e^{j\omega})|^2=\|w\|_{\ell_2}^2\text{ for all }\omega\in[-\pi,\pi]\right\}.\quad(13)$$

$\mathcal{W}$ satisfies $\mathcal{W}\in\operatorname{sig}\operatorname{IQC}(\Theta_{\mathcal{W}},U_{\mathcal{W}})$ [9], where the stable filter $\Theta_{\mathcal{W}}$ and the symmetric matrix $U_{\mathcal{W}}$ are defined by

$$\begin{aligned}
&A_{\Theta,\mathcal{W}}=-\operatorname{diag}(a_1,\ldots,a_N),\quad &&B_{\Theta,\mathcal{W}}=\begin{bmatrix}1&\cdots&1\end{bmatrix}^T,\\
&C_{\Theta,\mathcal{W}}=\begin{bmatrix}I\\0_{1\times N}\end{bmatrix},&&D_{\Theta,\mathcal{W}}=\begin{bmatrix}0_{N\times1}\\1\end{bmatrix},\qquad(14)\\
&U_{\mathcal{W}}=\begin{bmatrix}0_{N\times N}&y\\y^T&2y_0\end{bmatrix},&&y=\begin{bmatrix}y_1&\cdots&y_N\end{bmatrix}^T.
\end{aligned}$$

In (14), $y_0\geq0$ and $a_i\in(-1,1)$ for $i=1,\ldots,N$. Vector-valued sequences are also considered in [9], but in such cases the notion of whiteness is defined in an average sense.

Theorem 3 gives a novel method to compute bounds on $|e(k)|$ for all $k\in\mathbb{N}_0$, $d\in\mathcal{D}$, and $\Delta\in\mathbf{\Delta}$, where e is a scalar performance output of system $(G,\mathbf{\Delta})$ defined by (1)-(2).

Theorem 3: Consider the augmented uncertain system $(G_e,\mathbf{\Delta})$ shown in Figure 2, where $G_e=\begin{bmatrix}G&0\end{bmatrix}$ and system G is defined by (1). Assume that the exogenous disturbance d lies in a set $\mathcal{D}$, the exogenous disturbance d_e lies in the set $\mathcal{W}$ defined in (13), and the uncertainty operator Δ lies in the set $\mathbf{\Delta}$. If the augmented uncertain system $(G_e,\mathbf{\Delta})$ satisfies the performance IQC defined by $(\Xi_e,\{W_e(k)\}_{k\in\mathbb{N}_0})$, where Ξ_e is a static time-invariant operator defined by

$$\Xi_e(k)=\begin{bmatrix}D_{\Xi_1}(k)&D_{\Xi_2}(k)\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}I&0\\0&1\\0&0\end{bmatrix}&\begin{bmatrix}0\\0\\1\end{bmatrix}\end{bmatrix}\text{ for all }k\in\mathbb{N}_0,$$

$$\text{and }W_e(k)=\begin{bmatrix}\gamma^2I&0&0\\0&1&-1/2\\0&-1/2&0\end{bmatrix}\text{ for all }k\in\mathbb{N}_0,$$

then the uncertain system $(G,\mathbf{\Delta})$ defined by (1)-(2) is robustly stable, and its performance output e satisfies

$$|e(k)|\leq2\gamma\|d\|_{\ell_2}\text{ for all }k\in\mathbb{N}_0,\ d\in\mathcal{D},\text{ and }\Delta\in\mathbf{\Delta}.$$

Proof: Since $G_e=\begin{bmatrix}G&0\end{bmatrix}$, it follows that

$$\begin{bmatrix}\varphi\\e\end{bmatrix}=G_e\begin{bmatrix}\vartheta\\d\\d_e\end{bmatrix}=G\begin{bmatrix}\vartheta\\d\end{bmatrix}.$$

Thus, one sees that if $\vartheta=\Delta(\varphi)$, where $\Delta\in\mathbf{\Delta}$, the signals ϑ, φ, e in Figure 2 satisfy (1)-(2). Since the uncertain system $(G_e,\mathbf{\Delta})$ satisfies the performance IQC defined by $(\Xi_e,\{W_e(k)\}_{k\in\mathbb{N}_0})$, it follows by definition that it is robustly stable, and so system $(G,\mathbf{\Delta})$ is also robustly stable.

For the given Ξ_e, p_e in Figure 2 satisfies

$$p_e(k)=\Xi_e(k)\begin{bmatrix}d(k)\\d_e(k)\\e(k)\end{bmatrix}=\begin{bmatrix}d(k)\\d_e(k)\\e(k)\end{bmatrix}\text{ for all }k\in\mathbb{N}_0.$$

Note that, for the given Ξ_e, the state-space matrices in (5) all have at least one zero dimension except $D_{\Xi 1}(k)$ and $D_{\Xi 2}(k)$. Moreover,

$$\sum_{k=0}^{\infty} p_e^T(k) W_e(k) p_e(k) = \gamma^2 \|d\|_{\ell_2}^2 + \|d_e\|_{\ell_2}^2 - \langle d_e, e \rangle \geq 0$$

for all $d \in \mathcal{D}$, $d_e \in \mathcal{W}$, and $\Delta \in \mathbf{\Delta}$. Since $e \in \ell_2^1$ and $\mathcal{D}_e \subseteq \mathcal{W}$ for all $e \in \ell_2^1$, where $\mathcal{D}_e$ is defined similarly to $\mathcal{D}_v$ in (12), we can conclude from the previous inequality that

$$\|d_e\|_{\ell_2} |e(k)| \leq \langle d_e, e \rangle \leq \gamma^2 \|d\|_{\ell_2}^2 + \|d_e\|_{\ell_2}^2$$

for all $k \in \mathbb{N}_0$, $d \in \mathcal{D}$, $d_e \in \mathcal{D}_e$, and $\Delta \in \mathbf{\Delta}$. For all $0 \neq d_e \in \mathcal{D}_e$, it follows that

$$|e(k)| \leq \frac{\gamma^2 \|d\|_{\ell_2}^2}{\|d_e\|_{\ell_2}} + \|d_e\|_{\ell_2} \text{ for all } k \in \mathbb{N}_0.$$

Choosing $\|d_e\|_{\ell_2} = \gamma \|d\|_{\ell_2}$ to minimize the right-hand side of this inequality, we get $|e(k)| \leq 2\gamma \|d\|_{\ell_2}$ for all $k \in \mathbb{N}_0$, $d \in \mathcal{D}$, and $\Delta \in \mathbf{\Delta}$. ■

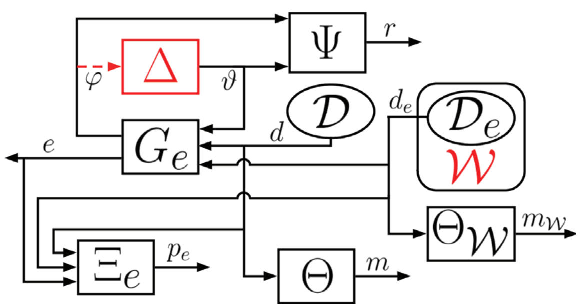

Fig. 2. Augmented system used for computing point-wise output bounds.

IV. Illustrative Examples

A. Using signal IQCs to reduce conservatism

This section illustrates the importance of using signal IQCs to describe the sets of exogenous disturbances affecting the uncertain system. We revisit the example from [5], where γ_1 is computed such that $\|e\|_{\ell_2} \leq \gamma_1 \|d\|_{\ell_2}$ for all $d \in \ell_2$. Here, we assume that the disturbance is a white noise signal, i.e., $d \in \mathcal{W}$, and show how Theorem 1 can be used to compute $\gamma_2 \leq \gamma_1$ such that $\|e\|_{\ell_2} \leq \gamma_2 \|d\|_{\ell_2}$ for all $d \in \mathcal{W}$.

For all $k \in \mathbb{N}_0$, the equations of the LTI system G are

$$\begin{aligned} x_G(k+1) &= -0.5 x_G(k) + 0.5\vartheta(k) + 0.4 d(k), \\ \varphi(k) &= 2.5 x_G(k) + 0\vartheta(k) + 0.6 d(k), \\ e(k) &= 2 x_G(k) + 0\vartheta(k) + 0.9 d(k). \end{aligned} \tag{15}$$

$\mathbf{\Delta}$ satisfies $\mathbf{\Delta} \in \mathrm{IQC}(\Psi_1, M)$ and $\mathbf{\Delta} \in \mathrm{IQC}(\Psi_2, M)$, where $M = \mathrm{diag}(1, -1)$, Ψ_1 is a dynamic system, and Ψ_2 is a constant matrix. In the convex optimization problems to be solved, we consider that $\mathbf{\Delta} \in \mathrm{IQC}(\Psi, S(\lambda))$, where Ψ is constructed from Ψ_1 and Ψ_2, $S(\lambda) = \mathrm{diag}(\lambda_1 M, \lambda_2 M)$, and the decision variables λ_1 and λ_2 satisfy $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. For all $k \in \mathbb{N}_0$, Ψ is described by

$$x_\Psi(k+1) = -0.3 x_\Psi(k) + 1.3\varphi(k) + 0\vartheta(k),$$

$$r(k) = \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \end{bmatrix} x_\Psi(k) + \begin{bmatrix} 0.2 \\ 0 \\ -0.5 \\ 0 \end{bmatrix} \varphi(k) + \begin{bmatrix} 0 \\ -0.1 \\ 0.3 \\ 1.7 \end{bmatrix} \vartheta(k).$$

From [5], the uncertain system $(G, \mathbf{\Delta})$ is robustly stable. We use Theorem 1 to compute γ_1. For this purpose, we use the standard performance IQC $(\Xi, \{W(k)\}_{k \in \mathbb{N}_0})$ defined in (6) and (7). For the computation of γ_1, we assume that the disturbance set $\mathcal{D} = \ell_2$. Appealing to Proposition 1 and its subsequent discussion, γ_1 is obtained by solving the following semidefinite program (SDP):

$$\gamma_1^2 = \min_{P, \lambda_1, \lambda_2} \gamma^2, \tag{16}$$

$$\begin{aligned} \text{subject to: } & P = P^T, \ \lambda_1 \geq 0, \ \lambda_2 \geq 0, \\ & \text{and (10) for } k = 0 \text{ and } P(1) = P(0) = P. \end{aligned}$$

The obtained value of γ_1 is 5.01. Since it is assumed that $d \in \mathcal{W} \subseteq \ell_2$, where $\mathcal{W}$ is defined in (13), we use the fact that $\mathcal{W} \in \mathrm{sig\,IQC}(\Theta_\mathcal{W}, U_\mathcal{W})$, where $\Theta_\mathcal{W}$ and $U_\mathcal{W}$ are defined in (14), to obtain a lower robust performance level γ_2. In (14), we choose $N = 4$, $a_1 = 0.7$, $a_2 = 0.1$, $a_3 = -0.5$, and $a_4 = -0.9$. The resulting robust $\mathcal{W}$-to-ℓ_2-gain performance level $\gamma_2 = 3.45$, which represents a 31% improvement over $\gamma_1 = 5.01$. Incorporating the signal IQC into the analysis increases the size of the SDP in (16); e.g., the decision variables $y_0, \ldots, y_N$ and the constraint $y_0 \geq 0$ are added and the size of P is increased. In the SDP defined in (16), the number of constraints is 6, the dimension of the SDP variable is 4, the number of SDP blocks is 1, and the dimension of the linear variable is 2. In the augmented SDP obtained after adding the signal IQC, these values are 29, 8, 1, and 3, respectively. The SDPs are solved using SDPT3 [14] combined with YALMIP [15]. The solution times are 0.96 sec and 1.02 sec, respectively, where the computations are carried out in MATLAB 9.5 on a Lenovo Thinkpad laptop with quad-core Intel Core i7-8650U, 1.90GHz processors, and 16GB of RAM running Windows 10.

B. Using signal IQCs to compute point-wise output bounds

In this section, we give multiple ways to compute bounds on $|e(k)|$. One bound is readily available from Section IV-A: $|e(k)|^2 \leq \|e\|_{\ell_2}^2 \leq \gamma_1^2 \|d\|_{\ell_2}^2$, i.e., $|e(k)| \leq \gamma_1 \|d\|_{\ell_2}$, for all $k \in \mathbb{N}_0$ and $d \in \ell_2$. Since system $(G, \mathbf{\Delta})$ is robustly stable, the internal and output signals causally depend on the inputs. Any finite-horizon truncation $d_{[0,T]}$ of $d \in \ell_2$ is also in ℓ_2, and so $|e(T)| \leq \gamma_1 \|d_{[0,T]}\|_{\ell_2}$ for all $T \in \mathbb{N}_0$. Note that, in this section, no signal IQC is used to describe the disturbance set $\mathcal{D}$. If a signal IQC is to be used, it must first be ensured that any finite-horizon truncation of the disturbance still lies in the assumed disturbance set. That is, it needs to be proved that $d \in \mathcal{D}$ implies that $d_{[0,T]} \in \mathcal{D}$ for all $T \in \mathbb{N}_0$.

A second method to compute a bound on $|e(T)|$ for some $T \in \mathbb{N}_0$ is given in [6]. It assumes that the nominal system G is a finite-horizon LTV system. For $k \leq T$, the equations for x_G and φ are defined as in (15). The performance output equation is given by $e(k) = 0$ for $k < T$ and

$$e(T) = 2 x_G(T) + 0\vartheta(T) + 0.9 d(T).$$

For $k > T$, all the state-space matrices of the nominal system become zeros. System G just defined is a finite-horizon LTV system with time horizon T, i.e., (h, q)-eventually periodic

LTV system with $h-1=T$ and $q=1$. The state-space matrices in the periodic part are zeros. The performance IQC used is the one defined in (6) and (7). Thus, applying the method of [6] amounts to solving the following SDP:

$$\gamma_3^2 = \min_{P(k),\lambda_1,\lambda_2} \gamma^2, \tag{17}$$

$$\text{subject to:} \quad P(k) = P(k)^T \text{ and (10) for } k = 0,\dots,h, \quad P(h+1) = P(h),\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,$$

where the strict version of LMI (10) is imposed. By solving problem (17) at a value of T of interest, we get $|e(T)| \leq \gamma_3(T)\|d_{[0,T]}\|_{\ell_2}$. From Table I, it is seen that γ_3 increases with increasing values of T and converges to 2.68. The size of the SDP in (17) increases with T. When $T=10$, the number of constraints is 39, the dimension of the SDP variable is 48, the number of SDP blocks is 12, the dimension of the linear variable is 2, and the solution time is 1.04 sec. For $T=25$, these values are 84, 108, 27, 2, and 1.07 sec, respectively. For $T=50$, the corresponding values are 159, 208, 52, 2, and 1.16 sec, respectively. That is, the number of constraints, the dimension of the SDP variable, and the number of SDP blocks scale with $h+q=T+2$. Exact complexity expressions can be obtained by formulating the dual problem to (17); see, for instance, [16], [17] for complexity analysis of SDPs that appear in model reduction problems.

Finally, the method of Section III is applied to compute γ_4 such that $|e(T)| \leq 2\gamma_4\|d_{[0,T]}\|_{\ell_2}$ for all $T \in \mathbb{N}_0$. Theorem 1 is applied to the augmented system $(G_e, \mathbf{\Delta})$ in Figure 2 with the performance IQC defined in Theorem 3. We use the signal IQC defined in (14) to constrain the input d_e to $\mathcal{W}$, and choose $N=4$, $a_1=0.3$, $a_2=0.6$, $a_3=-0.3$, and $a_4=-0.5$. We obtain $\gamma_4=1.5$, i.e., $|e(T)| \leq 3\|d_{[0,T]}\|_{\ell_2}$ for all T. In contrast with the method of [6], this method uses an LTI nominal system and eliminates the need of forming the finite-horizon system. Also, the SDP solved to compute γ_4 is smaller than the SDP in (17) for large values of T: the number of constraints is 29, the dimension of the SDP variable is 9, the number of SDP blocks is 1, the dimension of the linear variable is 3, and the solution time is 1.05 sec. However, $2\gamma_4 > \gamma_3(T)$ for all the considered values of T, albeit the increase in bound is less than 12% for $T \geq 10$.

To conclude, the method of Section III can be used as a simple and fast means for obtaining point-wise output bounds that are valid for all time-steps. Tighter bounds at specific time-steps T can be obtained by applying the method of [6], which involves dealing with time-varying nominal systems and is more computationally intensive for large values of T. All the solution times reported here are small and comparable, even though the SDP problem sizes are different. The reason is that we are dealing with illustrative, small-scale examples. Nonetheless, the considered examples demonstrate the following two points: 1) incorporating signal IQCs into the analysis reduces conservatism but results in increased computational costs; and 2) the method given in Section III may still be appealing even if the associated bound is more conservative than the one obtained from the method of [6].

TABLE I
γ_3 VERSUS T.

T	1	5	10	15	20	25	50
γ_3	1.745	2.452	2.643	2.678	2.682	2.683	2.683

V. CONCLUSION

This paper extends recent works that employ the dissipativity approach to IQC analysis in two respects: 1) it defines the performance criterion in terms of a general, time-domain, time-varying IQC, and 2) allows for using signal IQCs to characterize the sets of disturbance signals, which renders the robustness analysis results less conservative. The paper shows how to use signal IQCs and performance IQCs in a novel way to compute point-wise output bounds. The usefulness of the proposed methods is showcased using simple illustrative examples.

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