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Improved asymptotic expressions for the eigenvalues of Laplace's tidal equations

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ABSTRACT

Laplace's tidal equations govern the angular dependence of oscillations in stars when uniform rotation is treated within the so-called traditional approximation. Using a perturbation expansion approach, I derive improved expressions for the eigenvalue associated with these equations, valid in the asymptotic limit of large spin parameter q. These expressions have a relative accuracy of order q^{-3} for gravito-inertial modes, and q^{-1} for Rossby and Kelvin modes; the corresponding absolute accuracy is of order q^{-1} for all three mode types. I validate my analysis against numerical calculations, and demonstrate how it can be applied to derive formulae for the periods and eigenfunctions of Rossby modes.

Key words: hydrodynamics – waves – methods: analytical – methods: numerical – stars: oscillations – stars: rotation.

1 INTRODUCTION

Laplace's tidal equations (TEs) arise in the theory of stellar oscillations when uniform rotation is treated within the traditional approximation of rotation (TAR). Introduced by Eckart (1960), the TAR neglects the horizontal component of the rotation angular velocity vector $\mathbf{\Omega}$ when evaluating the Coriolis force. Together with the adiabatic and Cowling (1941) approximations, the TAR restores the separability of the oscillation equations in the three spherical coordinates (r, θ, ϕ). The resulting radial (r) equations appear the same as in the non-rotating case except that terms $\ell(\ell+1)$, where ℓ is the spherical harmonic degree, are replaced by a separation constant λ. This constant is found as the eigenvalue of the associated polar (θ) equations, a second-order system of differential equations and boundary conditions comprising the eponymous TEs first formulated by Laplace (1832).

Because it greatly simplifies inclusion of the Coriolis force, the TAR is commonly adopted in studies of waves, oscillations, and tides in rotating stars (e.g. Lee & Saio 1987; Bildsten, Ushomirsky & Cutler 1996; Papaloizou & Savonije 1997; Townsend 2005; Bouabid et al. 2013; Fuller & Lai 2014; Szewczuk & Daszyńska-Daszkiewicz 2017; Li et al. 2019). Typically, the TEs are solved numerically using standard techniques such as shooting, relaxation, or spectral expansion (see, respectively, Bildsten et al. 1996; Fuller & Lai 2014; Townsend 2003b). However, towards large spin parameter $q \equiv 2\Omega/\omega$, where $\Omega \equiv |\mathbf{\Omega}|$ and ω is the angular oscillation frequency in the co-rotating reference frame, the TEs approach an asymptotic limit where they become amenable to analytic solution. Building on earlier work in the geophysical literature (e.g. Matsuno 1966; Lindzen 1967), Townsend (2003a, hereafter T03) derives approximate expressions for the eigenvalue λ and associated eigenfunctions (known as Hough functions) of the TEs in this limit. These expressions are useful as initial guesses in the aforementioned numerical techniques; they simplify creating interpolating tables for fast TAR implementations; and they provide the basis for estimating oscillation frequencies in rotating stars.

In this paper, I reprise the T03 analysis with the twin goals of extending the asymptotic expressions for λ to higher order in q^{-1}, and of strengthening the mathematical rigor. Section 3 derives the new expressions, and Section 4 validates them by comparison against numerical calculations. Section 5 then summarizes and discusses the results of the paper.

2 LAPLACE'S TIDAL EQUATIONS

Within the TAR and accompanying approximations discussed in TO3, the components of the displacement perturbation $\boldsymbol{\xi}$ for a mode with integer azimuthal order m may be expressed in the co-rotating frame as

$$\xi_r = Y_r(r)\,\Theta(\theta)\,\mathrm{e}^{\mathrm{i}(m\phi-\omega t)}, \tag{1}$$

$$\xi_\theta = \frac{Y_\perp(r)}{\sin\theta}\,\widehat{\Theta}(\theta)\,\mathrm{e}^{\mathrm{i}(m\phi-\omega t)}, \tag{2}$$

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$$\xi_\phi = \frac{Y_\perp(r)}{\mathrm{i}\sin\theta}\,\widetilde{\Theta}(\theta)\,\mathrm{e}^{\mathrm{i}(m\phi-\omega t)}. \tag{3}$$

Here, Y_r and $Y_\perp$ are found by solving the radial parts of the oscillation equations (see 13–16 of T03). The Hough functions Θ, $\widehat{\Theta}$, $\widetilde{\Theta}$ are likewise obtained by solving the tidal equations

$$\left[\left(1-\mu^2\right)\frac{\mathrm{d}}{\mathrm{d}\mu} + mq\mu\right]\Theta = \left(q^2\mu^2-1\right)\widehat{\Theta}, \tag{4}$$

$$\left[\left(1-\mu^2\right)\frac{\mathrm{d}}{\mathrm{d}\mu} - mq\mu\right]\widehat{\Theta} = \left[\lambda\left(1-\mu^2\right)-m^2\right]\Theta, \tag{5}$$

$$\widetilde{\Theta} = m\Theta - q\mu\widehat{\Theta}, \tag{6}$$

where $\mu \equiv \cos\theta$. To avoid unphysical displacement perturbations, the Hough functions for non-axisymmetric modes ($m \neq 0$) must vanish at the poles ($\mu = \pm 1$), and similarly for the μ derivatives of the Hough functions for axisymmetric modes ($m = 0$). Note that TEs (4)–(6) appear slightly different from the presentation in TO3, due to my sign choice in the exponential terms of equations (1)–(3); this choice means that modes with $m > 0$ ($m < 0$) propagate in the prograde (resp. retrograde) direction in the co-rotating frame.

Eliminating $\widehat{\Theta}$ between equations (4) and (5) leads to a second-order form for the TEs,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\frac{1-\mu^2}{1-q^2\mu^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\mu}\right] + \left\{mq\frac{1+q^2\mu^2}{\left(1-q^2\mu^2\right)^2} + \lambda - \frac{m^2}{\left(1-\mu^2\right)\left(1-q^2\mu^2\right)}\right\}\Theta = 0; \tag{7}$$

this is equivalent to the presentation by Bildsten et al. (1996), who were the first explicitly to invoke the TEs in a stellar oscillation context. An alternative second-order form can be obtained by instead eliminating Θ, yielding

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\frac{1-\mu^2}{\lambda\left(1-\mu^2\right)-m^2}\frac{\mathrm{d}\widehat{\Theta}}{\mathrm{d}\mu}\right] - \left\{mq\frac{\lambda\left(1+\mu^2\right)-m^2}{\left[\lambda\left(1-\mu^2\right)-m^2\right]^2} + \frac{\lambda q^2\mu^2}{\lambda\left(1-\mu^2\right)-m^2} - \frac{1}{1-\mu^2}\right\}\widehat{\Theta} = 0. \tag{8}$$

This latter form provides the starting point for the asymptotic expressions I derive in the following section.

3 ASYMPTOTIC EXPRESSIONS

In the limit $|q| \to \infty$, solutions to the TEs can be classified according to the behaviour of the eigenvalue λ:

(i) For gravito-inertial (g-i) modes, $\lambda \propto q^2$
(ii) For Rossby (r) modes, which are retrograde ($mq < 0$), $\lambda \propto q^0$;
(iii) For Kelvin modes, which are prograde ($mq > 0$), $\lambda \propto q^0$.

In the following subsections, I derive asymptotic expressions for λ for these three mode types, in the form of power-series expansions in $w \equiv q^{-1}$. My approach is inspired by quantum mechanical perturbation theory; the analysis is complicated by the non-linearity of equation (8) in λ; but likewise simplified by the guaranteed non-degeneracy of the eigenvalues (see e.g. Homer 1992). For each mode type, the expansion extends to as high an order in w as appears possible while keeping the analysis relatively simple.

3.1 Gravito-inertial modes

For g-i modes, I re-parametrize the TEs to use an eigenvalue α and independent variable σ, where

$$\alpha^2 = \lambda\, w^2, \qquad \sigma^2 = \frac{\alpha}{w^2}\,\mu^2. \tag{9}$$

The second-order form (8) of the TEs becomes

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{\alpha-w^2\sigma^2}{\alpha^2-w^2\alpha\sigma^2-w^2m^2}\frac{\mathrm{d}\widehat{\Theta}}{\mathrm{d}\sigma}\right] - \left[mw\frac{\alpha^2+w^2\alpha\sigma^2-w^2m^2}{\left[\alpha^2-w^2\alpha\sigma^2-w^2m^2\right]^2} + \frac{\alpha\sigma^2}{\alpha^2-w^2\alpha\sigma^2-w^2m^2} - \frac{\alpha}{\alpha-w^2\sigma^2}\right]\widehat{\Theta} = 0. \tag{10}$$

I then expand α and $\widehat{\Theta}$ as power series in w:

$$\alpha = \sum_{i=0}^{\infty}\alpha_i\, w^i, \qquad \widehat{\Theta} = \sum_{j=0}^{\infty}\widehat{\Theta}_j\, w^j, \tag{11}$$

where the sequence of coefficients $\{\alpha_0, \alpha_1, \ldots\}$ and functions $\{\widehat{\Theta}_0, \widehat{\Theta}_1, \ldots\}$ will be determined. With these expansions, equation (10) has a leading order w^0 and may be expressed as

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathcal{L}_i^{\mathrm{g}}\,\widehat{\Theta}_j\, w^{i+j} = 0, \tag{12}$$

where $\{\mathcal{L}_0^{\rm g}, \mathcal{L}_1^{\rm g}, \ldots\}$ are a sequence of second-order linear differential operators that depend on the $\{\alpha_i\}$ but not on w. The first four terms in this sequence are

$$\mathcal{L}_0^{\rm g} = \frac{1}{\alpha_0}\left[\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2 + \alpha_0\right], \tag{13}$$

$$\mathcal{L}_1^{\rm g} = \frac{1}{\alpha_0^2}\left[-\alpha_1\left(\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2\right) - m\right], \tag{14}$$

$$\mathcal{L}_2^{\rm g} = \frac{1}{\alpha_0^3}\left[\left(m^2 + \alpha_1^2 - \alpha_0\alpha_2\right)\left(\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2\right) + 2m\alpha_1 + \alpha_0^2\sigma^2 - \alpha_0\sigma^4\right], \tag{15}$$

$$\mathcal{L}_3^{\rm g} = \frac{1}{\alpha_0^4}\left[\left(2\alpha_0\alpha_1\alpha_2 - 3m^2\alpha_1 - \alpha_1^3 - \alpha_0^2\alpha_3\right)\left(\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2\right) - m^3 - 3m\alpha_1^2 + 2m\alpha_0\alpha_2 - \left(3m\alpha_0 + \alpha_0^2\alpha_1\right)\sigma^2 + 2\alpha_0\alpha_1\sigma^4\right]. \tag{16}$$

Equation (12) holds for all possible choices of w, and so the coefficient of each power of w must vanish. This condition leads to a sequence of coupled differential equations, with the first four (labelled by their corresponding power of w) being

$$w^0: \qquad \mathcal{L}_0^{\rm g}\,\widehat{\Theta}_0 = 0, \tag{17}$$

$$w^1: \qquad \mathcal{L}_0^{\rm g}\,\widehat{\Theta}_1 + \mathcal{L}_1^{\rm g}\,\widehat{\Theta}_0 = 0, \tag{18}$$

$$w^2: \qquad \mathcal{L}_0^{\rm g}\,\widehat{\Theta}_2 + \mathcal{L}_1^{\rm g}\,\widehat{\Theta}_1 + \mathcal{L}_2^{\rm g}\,\widehat{\Theta}_0 = 0, \tag{19}$$

$$w^3: \qquad \mathcal{L}_0^{\rm g}\,\widehat{\Theta}_3 + \mathcal{L}_1^{\rm g}\,\widehat{\Theta}_2 + \mathcal{L}_2^{\rm g}\,\widehat{\Theta}_1 + \mathcal{L}_3^{\rm g}\,\widehat{\Theta}_0 = 0. \tag{20}$$

In the following sections, I solve these equations in order.

3.1.1 w^0 equation

I write equation (17) explicitly as

$$\frac{1}{\alpha_0}\left[\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2 + \alpha_0\right]\widehat{\Theta}_0 = 0. \tag{21}$$

The boundary conditions at the poles require that $\widehat{\Theta}_0 \rightarrow 0$ as $\sigma \rightarrow \pm\infty$. Solutions satisfying this constraint can be found only when

$$\alpha_0 = 2s + 1, \tag{22}$$

for integer meridional order[1] $s \geq 0$, and can be written

$$\widehat{\Theta}_0 = c_0\,\psi_s, \tag{23}$$

where c_0 is an arbitrary constant and ψ_s is a normalized Hermite function. Appendix A defines these functions and presents some identities that will prove useful in the subsequent analysis.

3.1.2 w^1 equation

I now use the normalized Hermite functions as a basis to expand $\widehat{\Theta}_1$ as

$$\widehat{\Theta}_1 = \sum_{k=0}^{\infty} c_{1,k}\,\psi_k, \tag{24}$$

where the sequence of coefficients $\{c_{1,0}, c_{1,1}, \ldots\}$ will be determined. Taking the inner product between equation (18) and ψ_t (for arbitrary t) then yields

$$\sum_{k=0}^{\infty} c_{1,k}\left\langle\psi_t, \mathcal{L}_0^{\rm g}\,\psi_k\right\rangle + c_0\left\langle\psi_t, \mathcal{L}_1^{\rm g}\,\psi_s\right\rangle = 0. \tag{25}$$

Using the relations presented in Appendix A, the inner products appearing here evaluate as

$$\left\langle\psi_t, \mathcal{L}_0^{\rm g}\,\psi_k\right\rangle = \frac{1}{2s+1}\left[(2s+1) - (2k+1)\right]\delta_{t,k}, \tag{26}$$

$$\left\langle\psi_t, \mathcal{L}_1^{\rm g}\,\psi_k\right\rangle = \frac{1}{(2s+1)^2}\left[\alpha_1(2k+1) - m\right]\delta_{t,k}. \tag{27}$$

[1] Section 4 of T03 discusses the mappings between s and other mode indices.

Setting $t = s$, equation (25) solves to give

$$\alpha_1 = \frac{m}{2s+1}. \tag{28}$$

Likewise, with $t \neq s$ it gives

$$c_{1,t} = 0. \tag{29}$$

The coefficient $c_{1,s}$ is unconstrained and can be set to an arbitrary value; this affects the overall normalization of $\widehat{\Theta}$, but is otherwise unimportant. Therefore, I choose $c_{1,s} = 0$ so that equation (29) remains true for all t.

3.1.3 w^2 equation

Proceeding as before, I expand $\widehat{\Theta}_2$ as

$$\widehat{\Theta}_2 = \sum_{k=0}^{\infty} c_{2,k}\, \psi_k, \tag{30}$$

Taking the inner product between equation (19) and ψ_t then yields

$$\sum_{k=0}^{\infty} c_{2,k} \left\langle \psi_t, \mathcal{L}_0^{\rm g}\, \psi_k \right\rangle + \sum_{k=0}^{\infty} c_{1,k} \left\langle \psi_t, \mathcal{L}_1^{\rm g}\, \psi_k \right\rangle + c_0 \left\langle \psi_t, \mathcal{L}_2^{\rm g}\, \psi_s \right\rangle = 0. \tag{31}$$

Using the relations presented in Appendix A, the inner product in the third term evaluates as

$$\begin{aligned}\left\langle \psi_t, \mathcal{L}_2^{\rm g}\, \psi_k \right\rangle = &-\frac{\sqrt{k(k-1)(k-2)(k-3)}}{4(2s+1)^2}\,\delta_{t,k-4} + (s-k+1)\frac{\sqrt{k(k-1)}}{(2s+1)^2}\,\delta_{t,k-2} + \frac{1}{4(2s+1)^5}\left\{-1-6k^2(2s+1)^3\right.\\ &+2s\left[-1+2s\left(3-4m^2+10s+8s^2\right)\right]+2k\left[(2s+1)^3(4s-1)-8m^2\left(1+2s+2s^2\right)\right]+4(2k+1)(2s+1)^3\,\alpha_2\Big\}\,\delta_{t,k}\\ &+(s-k-1)\frac{\sqrt{(k+1)(k+2)}}{(2s+1)^2}\,\delta_{t,k+2} - \frac{\sqrt{(k+1)(k+2)(k+3)(k+4)}}{4(2s+1)^2}\,\delta_{t,k+4}.\end{aligned} \tag{32}$$

Setting $t = s$, equation (31) solves to give

$$\alpha_2 = \frac{1+2s(s+1)\left[1+8m^2-4s(s+1)\right]}{4(2s+1)^3}. \tag{33}$$

Likewise, with $t \neq s$ it gives

$$c_{2,t} = \frac{c_0}{4(2s+1)}\left[\frac{\sqrt{s(s-1)(s-2)(s-3)}}{8}\,\delta_{t,s-4} - \sqrt{s(s-1)}\,\delta_{t,s-2} - \sqrt{(s+1)(s+2)}\,\delta_{t,s+2} - \frac{\sqrt{(s+1)(s+2)(s+3)(s+4)}}{8}\,\delta_{t,s+4}\right]. \tag{34}$$

Note that this expression is not required in the subsequent analysis; I include it here for the sake of completeness (but see the closing comments in Section 5). Similarly to before, the coefficient $c_{2,s}$ is unconstrained and can be set to zero, so that the expression remains true for all t.

3.1.4 w^3 equation

Again proceeding as before, I expand $\widehat{\Theta}_3$ as

$$\widehat{\Theta}_3 = \sum_{k=0}^{\infty} c_{3,k}\, \psi_k, \tag{35}$$

Taking the inner product between equation (20) and ψ_t then yields

$$\sum_{k=0}^{\infty} c_{3,k} \left\langle \psi_t, \mathcal{L}_0^{\rm g}\, \psi_k \right\rangle + \sum_{k=0}^{\infty} c_{2,k} \left\langle \psi_t, \mathcal{L}_1^{\rm g}\, \psi_k \right\rangle + \sum_{k=0}^{\infty} c_{1,k} \left\langle \psi_t, \mathcal{L}_2^{\rm g}\, \psi_k \right\rangle + c_0 \left\langle \psi_t, \mathcal{L}_3^{\rm g}\, \psi_s \right\rangle = 0. \tag{36}$$

Using the relations presented in Appendix A, the inner product in the fourth term evaluates as

$$\begin{aligned}\left\langle \psi_t, \mathcal{L}_3^{\rm g}\, \psi_k \right\rangle = &\, m\frac{\sqrt{k(k-1)(k-2)(k-3)}}{2(2s+1)^4}\,\delta_{t,k-4} - m(4s-2k+3)\frac{\sqrt{k(k-1)}}{(2s+1)^4}\,\delta_{t,k-2}\\ &+\frac{1}{2(2s+1)^7}\left\{-m+6mk^2(2s+1)^3+4mk\left[4m^2\left(1+s+s^2\right)-(2s+1)^2\left(1+4s+7s^2\right)\right]\right.\\ &-12ms-4ms^2\left[14-4m^2(s+2)+s\left(29+24s+4s^2\right)\right]+2(2k+1)(2s+1)^5\,\alpha_3\Big\}\,\delta_{t,k}\\ &-m(4s-2k-1)\frac{\sqrt{(k+1)(k+2)}}{(2s+1)^4}\,\delta_{t,k+2} + m\frac{\sqrt{(k+1)(k+2)(k+3)(k+4)}}{2(2s+1)^4}\,\delta_{t,k+4}.\end{aligned} \tag{37}$$

Setting $t = s$, equation (36) solves to give

$$\alpha_3 = m\frac{1 + 2s(s+1)\left[7 - 8m^2 + 20s(s+1)\right]}{2(2s+1)^5}. \tag{38}$$

Likewise, with $t \neq s$ it gives

$$c_{3,t} = -\frac{m\,c_0}{4(2s+1)^3}\left[\frac{\sqrt{s(s-1)(s-2)(s-3)}}{4}\,\delta_{t,s-4} - (2s+3)\sqrt{s(s-1)}\,\delta_{t,s-2}\right.$$
$$\left. + (2s-1)\sqrt{(s+1)(s+2)}\,\delta_{t,s+2} - \frac{\sqrt{(s+1)(s+2)(s+3)(s+4)}}{4}\,\delta_{t,s+4}\right]. \tag{39}$$

Once again, I include this expression for the sake of completeness. The coefficient $c_{3,s}$ is unconstrained and can be set to zero, so that the equation remains true for all t.

3.1.5 Eigenvalues for gravito-inertial modes

As the final step, I combine the expressions for $\alpha_0, \ldots, \alpha_3$ given in equations (22), (28), (33), and (38) with the relationship (9) between α and λ, and transform from w back to q to obtain the asymptotic eigenvalues for gravito-inertial modes as

$$\lambda = q^2\left[(2s+1) + \frac{m}{2s+1}q^{-1} + \frac{1 + 2s(s+1)\left[1 + 8m^2 - 4s(s+1)\right]}{4(2s+1)^3}q^{-2} + m\frac{1 + 2s(s+1)\left[7 - 8m^2 + 20s(s+1)\right]}{2(2s+1)^5}q^{-3} + \mathcal{O}(q^{-4})\right]^2. \tag{40}$$

This can be compared against an equivalent expression obtained by a Taylor-series expansion of the positive root in TO3's equation (36):

$$\lambda_{\rm TO3} = q^2\left[(2s+1) + \frac{m}{2s+1}q^{-1} + \frac{4m^2 s(s+1)}{(2s+1)^3}q^{-2} + \mathcal{O}(q^{-3})\right]^2. \tag{41}$$

(I have also corrected for the different m sign convention). The two expressions differ at the third term in brackets, indicating that the T03 expression for λ has a relative accuracy of order q^{-1}, and an absolute accuracy of order q.

3.2 Rossby modes

For r modes, I repeat the analysis of the preceding section but now re-parametrizing via

$$\alpha^2 = \lambda, \qquad \sigma^2 = \frac{\alpha}{w}\,\mu^2. \tag{42}$$

The second-order form (8) of the TEs then becomes

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\frac{\alpha - w\sigma^2}{w\alpha^2 - w^2\alpha\sigma^2 - wm^2}\frac{\mathrm{d}\widehat{\Theta}}{\mathrm{d}\sigma}\right] - \left[mw\frac{\alpha^2 + w\alpha\sigma^2 - m^2}{[w\alpha^2 - w^2\alpha\sigma^2 - wm^2]^2} + \frac{\alpha\sigma^2}{w\alpha^2 - w^2\alpha\sigma^2 - wm^2} - \frac{\alpha}{\alpha - w\sigma^2}\right]\widehat{\Theta} = 0. \tag{43}$$

With the power-series expansions (11) for α and $\widehat{\Theta}$, and under the ansatz that $\alpha_0^2 \neq m^2$, this equation has a leading order w^{-1} and may be expressed as

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathcal{L}_i^{\rm r}\,\widehat{\Theta}_j\,w^{i+j-1} = 0. \tag{44}$$

The first two terms in the sequence of differential operators $\{\mathcal{L}_0^{\rm r}, \mathcal{L}_1^{\rm r}, \ldots\}$ are

$$\mathcal{L}_0^{\rm r} = \frac{1}{\alpha_0^2 - m^2}\left[\alpha_0\left(\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \sigma^2\right) - m\right], \tag{45}$$

$$\mathcal{L}_1^{\rm r} = \frac{1}{(\alpha_0^2 - m^2)^2}\left[\left(m^2\sigma^2 - \alpha_0^2\alpha_1 - m^2\alpha_1\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \sigma^2\right) + 2m^2\sigma\frac{\mathrm{d}}{\mathrm{d}\sigma} + m^4 + \alpha_0^4 + 2m\alpha_0\alpha_1 - 2m^2\alpha_0^2 - 3m\alpha_0\sigma^2 + \left(m^2 - \alpha_0^2\right)\sigma^4\right]; \tag{46}$$

and the resulting sequence of coupled differential equations is now

$$w^{-1}: \qquad \mathcal{L}_0^{\rm r}\,\widehat{\Theta}_0 = 0, \tag{47}$$

$$w^{0}: \qquad \mathcal{L}_0^{\rm r}\,\widehat{\Theta}_1 + \mathcal{L}_1^{\rm r}\,\widehat{\Theta}_0 = 0. \tag{48}$$

In the following sections I solve these equations in order.

3.2.1 w^{-1} equation

I write equation (47) explicitly as

$$\frac{1}{\alpha_0^2 - m^2}\left[\alpha_0\left(\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \sigma^2\right) - m\right]\widehat{\Theta}_0 = 0. \tag{49}$$

Similarly to Section (3.1.1), solutions satisfying the boundary constraint can be found only when

$$\alpha_0 = -\frac{m}{2s+1} \tag{50}$$

for integer meridional order $s \geq 1$[2], and can be written

$$\widehat{\Theta}_0 = c_0\,\psi_s, \tag{51}$$

where c_0 is an arbitrary constant.

Equation (50) indicates that α and m have opposite signs in the limit $w \to 0$. My definition (42) of σ requires that α and w (or q) share the same sign, because σ would otherwise be imaginary. It therefore follows that m and q must have opposite signs for r modes: these modes are necessarily retrograde.

3.2.2 w^0 equation

Proceeding as before, I expand $\widehat{\Theta}_1$ as

$$\widehat{\Theta}_1 = \sum_{k=0}^{\infty} c_{1,k}\,\psi_k. \tag{52}$$

Taking the inner product between equation (48) and ψ_t then yields

$$\sum_{k=0}^{\infty} c_{1,k}\left\langle \psi_t, \mathcal{L}_0^{\mathrm{r}}\,\psi_k\right\rangle + c_0\left\langle \psi_t, \mathcal{L}_1^{\mathrm{r}}\,\psi_s\right\rangle = 0. \tag{53}$$

The inner products appearing here evaluate as

$$\left\langle \psi_t, \mathcal{L}_0^{\mathrm{r}}\,\psi_k\right\rangle = (s-k)\frac{(2s+1)}{2ms(s+1)}\,\delta_{t,k} \tag{54}$$

$$\begin{aligned}\left\langle \psi_t, \mathcal{L}_1^{\mathrm{r}}\,\psi_k\right\rangle = {} & (2s+1)^2\frac{\sqrt{k(k-1)(k-2)(k-3)}}{16m^2s(s+1)}\,\delta_{t,k-4} + (2s+1)^2(3s-k+2)\frac{\sqrt{k(k-1)}}{16m^2s^2(s+1)^2}\,\delta_{t,k-2} \\ & - \frac{1}{16m^2s^2(s+1)^2}\left\{k(2s+1)^2\left[2s(s-2)-1\right] + 2k^2(2s+1)^2\left(1+s+s^2\right) - s^2\left[16m^2(s+1)^2 - 3(2s+1)^2\right]\right. \\ & \left. - 4(2s+1)^2\left[k+s^2+2ks(s+1)\right]\alpha_1\right\}\,\delta_{t,k} + (2s+1)^2(3s-k)\frac{\sqrt{(k+1)(k+2)}}{16m^2s^2(s+1)^2}\,\delta_{t,k+2} \\ & + (2s+1)^2\frac{\sqrt{(k+1)(k+2)(k+3)(k+4)}}{16m^2s(s+1)}\,\delta_{t,k+4}.\end{aligned} \tag{55}$$

Setting $t = s$, equation (53) solves to give

$$\alpha_1 = -\frac{1 + 2s(s+1)\left[1 + 8m^2 - 4s(s+1)\right]}{4(2s+1)^3}. \tag{56}$$

Likewise, with $t \neq s$ it gives

$$c_{1,t} = -\frac{(2s+1)\,c_0}{8m}\left[\frac{\sqrt{s(s-1)(s-2)(s-3)}}{4}\,\delta_{t,s-4} + \frac{\sqrt{s(s-1)}}{s}\,\delta_{t,s-2} - \frac{\sqrt{(s+1)(s+2)}}{s+1}\,\delta_{t,s+2} - \frac{\sqrt{(s+1)(s+2)(s+3)(s+4)}}{4}\,\delta_{t,s+4}\right]. \tag{57}$$

The coefficient $c_{1,s}$ is unconstrained and can be set to zero, so that this expression remains true for all t.

3.2.3 Eigenvalues for Rossby modes

As the final step, I combine the expressions for α_0, α_1 given in equations (50) and (56) with the relationship (42) between α and λ, and transform from w back to q to obtain the asymptotic eigenvalues for Rossby modes as

$$\lambda = \left[-\frac{m}{2s+1} - \frac{1 + 2s(s+1)\left[1 + 8m^2 - 4s(s+1)\right]}{4(2s+1)^3}q^{-1} + \mathcal{O}(q^{-2})\right]^2. \tag{58}$$

[2]The $s = 0$ case must be ruled out because it violates the ansatz $\alpha_0^2 \neq m^2$.

This can be compared against an equivalent expression obtained by a Taylor-series expansion of the negative root in TO3's equation (36):

$$\lambda_{\rm TO3} = \left[-\frac{m}{2s+1} - \frac{4m^2 s(s+1)}{(2s+1)^3} q^{-1} + \mathcal{O}(q^{-2}) \right]^2 \tag{59}$$

(again, I have corrected for the different m sign convention). The two expressions differ at the second term in brackets, indicating that the T03 expression for λ has a relative and absolute accuracy of order q^0.

3.3 Kelvin modes

For Kelvin modes, I repeat the analysis of the preceding section but now adopting the ansatz[3] $\alpha_0 = m$. With the power-series expansions (11) for α and $\widehat{\Theta}$, equation (43) then has a leading order w^{-2} and may be expressed as

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathcal{L}_i^{\rm k} \, \widehat{\Theta}_j \, w^{i+j-2} = 0. \tag{60}$$

The first term in the sequence of differential operators $\{\mathcal{L}_0^{\rm k}, \mathcal{L}_1^{\rm k}, \ldots\}$ is

$$\mathcal{L}_0^{\rm k} = \frac{1}{(\sigma^2 - 2\alpha_1^2)} \left[(2\alpha_1 - \sigma^2) \left(\frac{{\rm d}^2}{{\rm d}\sigma^2} - \sigma^2 \right) + 2\sigma \frac{\rm d}{{\rm d}\sigma} - (\sigma^2 + 2\alpha_1) \right], \tag{61}$$

and the first differential equation is

$$w^{-2}: \qquad \mathcal{L}_0^{\rm k} \, \widehat{\Theta}_0 = 0. \tag{62}$$

A solution to this equation satisfying the boundary constraint can be found only when

$$\alpha_1 = \frac{1}{4}, \tag{63}$$

and can be written

$$\widehat{\Theta}_0 = c_0 \, \psi_1, \tag{64}$$

where c_0 is an arbitrary constant.

Combining the $\alpha_0 = m$ ansatz with equation (63), the asymptotic eigenvalues for Kelvin modes are

$$\lambda = \left[m + \frac{1}{4} q^{-1} + \mathcal{O}(q^{-2}) \right]^2. \tag{65}$$

A Taylor-series expansion of TO3's equation (55), with the usual correction for the different m sign convention, leads to the same result, and so the latter is confirmed to have a relative and absolute accuracy of order (at least) q^{-1}.

Equation (65) can also be derived by setting $s = -1$ in the r-mode expression (58), underscoring the assignment of a nominal meridional order $s = -1$ to Kelvin modes (see e.g. Gill 1982). Using the same reasoning as in Section 3.2.1, m and q must have the same signs for Kelvin modes: these modes are necessarily prograde.

4 VALIDATION

To validate the preceding analysis, Fig. 1 compares the eigenvalue expressions (40), (58), and (65) against numerical calculations, for azimuthal orders $-2 \leq m \leq 2$ and selected meridional orders s of each mode type. Each panel plots $\log_{10}|\epsilon|$ as a function of $\log_{10} q$, where ϵ is the difference between asymptotic and numerical eigenvalues. For evaluating the numerical eigenvalues, I leverage the `eval_lambda` tool bundled with release 6.0 and later[4] of the open-source GYRE stellar oscillation code (Townsend & Teitler 2013; Townsend, Goldstein & Zweibel 2018). In brief, this tool solves the TEs using the spectral matrix approach described by Townsend (2003b), implemented via a Sturm Sequence method (e.g. Barth, Martin & Wilkinson 1967). Initial eigenvalue brackets are established using the asymptotic expressions themselves. At a given q, the matrix dimension N is determined dynamically by repeated doubling until λ converges to a fixed value in 64-bit floating-point precision.

Each panel reveals a scaling $|\epsilon| \propto q^{-2}$ towards larger q. This is the expected behaviour of the asymptotic expressions, which all claim an absolute accuracy of order q^{-1}. For the gravito-inertial modes, the noise appearing for $\log_{10} q \gtrsim 3.5$ is due to the effects of rounding errors on the numerical eigenvalues, rather than any issue with the asymptotic ones.

To explore how the eigenvalue expressions perform towards larger m and s, Fig. 2 plots $\log_{10}|\delta|$ as a function of m, evaluated at fixed spin parameter $q = 10^3$ for selected meridional orders of each mode type. Here, $\delta \equiv \epsilon/\lambda$ is the relative difference between asymptotic and numerical eigenvalues. For the gravito-inertial modes, $|\delta|$ tends to decrease towards larger s, but increase towards larger $|m|$. For the Rossby modes, the opposite trend is seen with respect to s, while $|\delta|$ becomes independent of azimuthal order towards large $|m|$. Finally, for the Kelvin modes

[3] A solution to equation (43) can also be found with $\alpha_0 = -m$; however, this ultimately leads to a Θ that diverges as $|\sigma| \to \infty$, and is therefore unphysical.
[4] Available for download at https://github.com/rhdtownsend/gyre

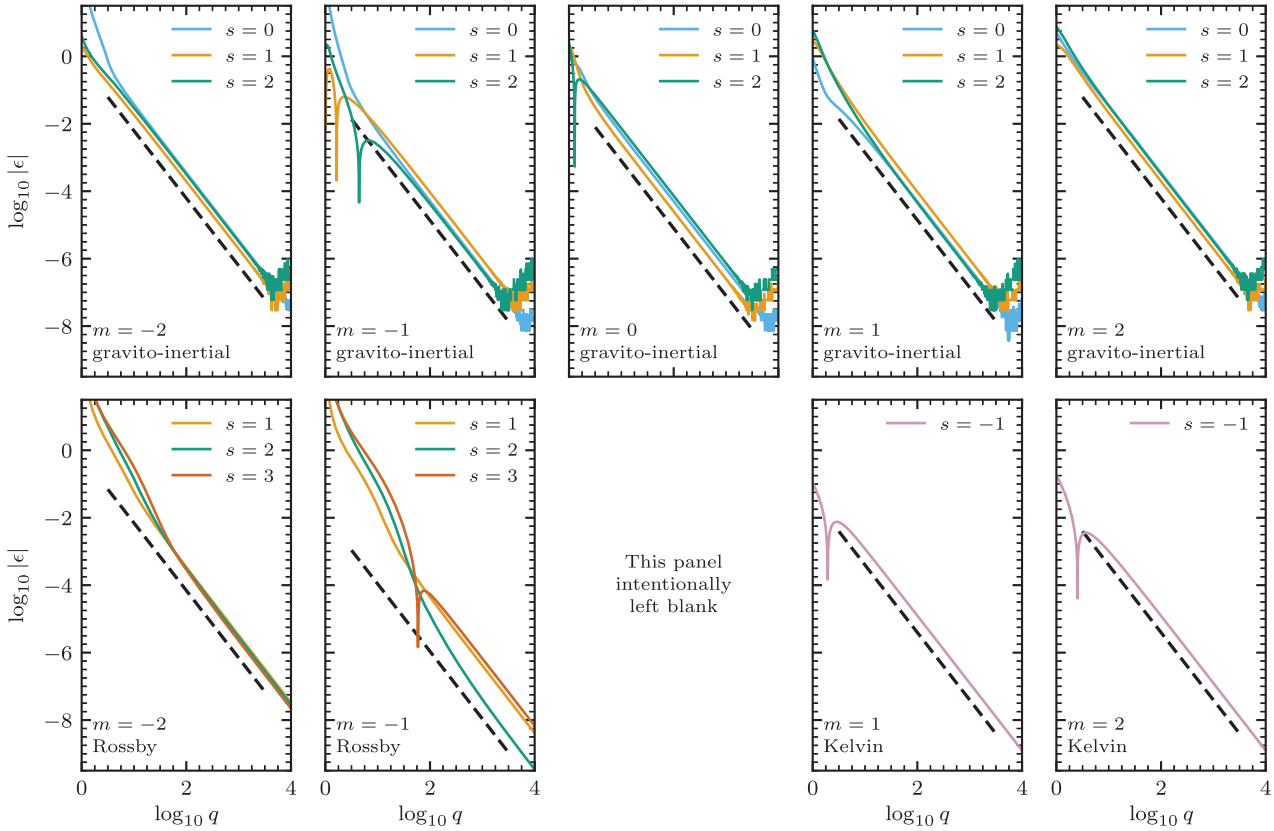

Figure 1. Log–log plots of the difference ϵ between asymptotic and numerical eigenvalues as a function of spin parameter q, for various azimuthal orders m and meridional orders s of each mode type. The black dashed lines indicate the scaling $|\epsilon| \propto q^{-2}$.

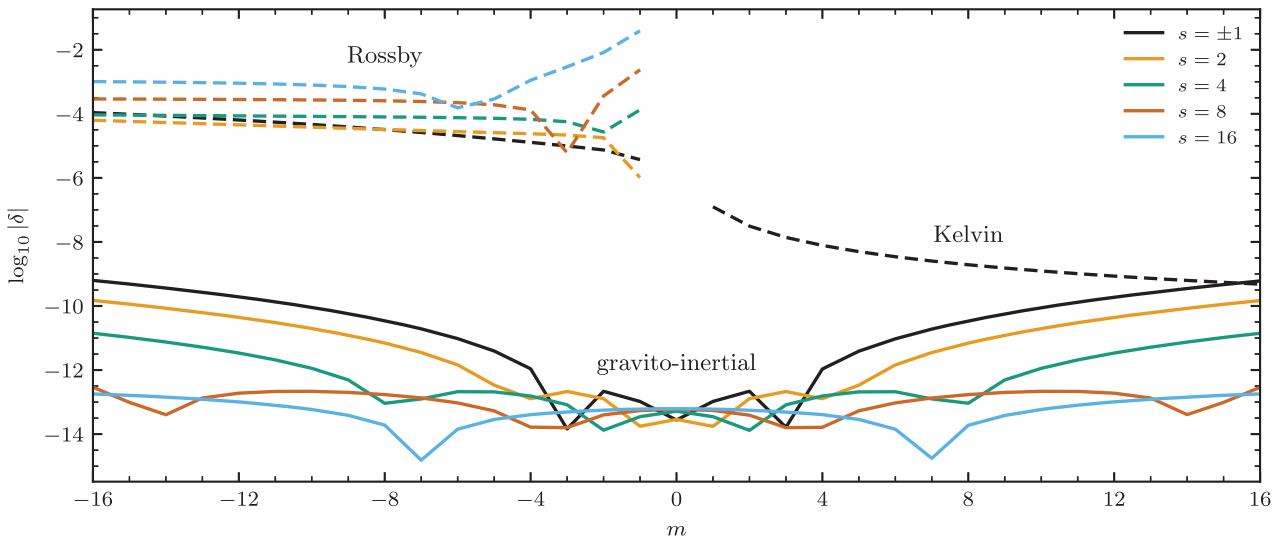

Figure 2. Log–log plots of the relative difference δ between asymptotic and numerical eigenvalues as a function of spin parameter m, evaluated at $q = 10^3$ and for various meridional orders s of each mode type. Gravito-inertial modes are plotted with solid lines, and Rossby and Kelvin modes with dashed lines.

$|\delta|$ decreases towards larger $|m|$. The detailed reasons for these different behaviours lie beyond the scope of this paper (since their elucidation would require extending the asymptotic expressions to higher order in q^{-1}).

5 SUMMARY AND DISCUSSION

The principal results of this paper are the improved asymptotic expressions for the eigenvalues of Laplace's tidal equations. For gravito-inertial modes (equation 40), the new expression has a relative (absolute) accuracy of order q^{-3} (q^{-1}), and extends two orders in q^{-1} further than the

corresponding TO3 result. For Rossby modes (equation 58), the new expression has an accuracy (both relative and absolute) of order q^{-1}, one order in q^{-1} further than T03. For Kelvin modes (equation 65), the new expression also has an accuracy of order q^{-1} – the same as T03, but the latter did not formally establish the order of correctness.

As one example application of these expressions, consider the approximate formula

$$P \equiv \frac{2\pi}{\omega} \approx \frac{\Pi_0}{\sqrt{\lambda}} \left(n + \frac{1}{2} \right) \tag{66}$$

governing the co-rotating frame periods of low-frequency modes trapped in the radiative zone between a convective core and a convective surface layer. In this expression, which is derived from radial asymptotic analysis within the TAR (see e.g. Bouabid et al. 2013), n is the mode radial order and Π_0 the asymptotic g-mode period spacing. Combining with equation (58), I solve to obtain an explicit expression for Rossby-mode periods,

$$P \approx -\frac{(2s+1)\Pi_0}{m} \left(n + \frac{1}{2} \right) - \pi \frac{1 + 2s(s+1)\left[1 + 8m^2 - 4s(s+1)\right]}{4(2s+1)^2 m\,\Omega} \tag{67}$$

Thus, for rotation sufficiently rapid that equation (58) provides a reasonable approximation, a sequence of Rossby modes with the same m and s and consecutive n should exhibit a uniform period spacing within the co-rotating frame, equal to $(2s + 1)/|m|$ times the asymptotic g-mode period spacing. This result may prove useful in analysing recent identifications of these modes in γ Doradus stars (e.g. Van Reeth, Tkachenko & Aerts 2016; Li et al. 2019).

On a closing note, although this paper focuses on the eigenvalues of Laplace's tidal equations, my analysis can also be used to construct asymptotic expressions for the corresponding eigenfunctions. For instance, combining equations (11), (51), (52), and (57) leads to an expression for Rossby-mode Hough functions,

$$\begin{aligned} \widehat{\Theta} = c_0 \Bigg\{ \psi_s - \frac{(2s+1)}{8m} \Bigg[& \frac{\sqrt{s(s-1)(s-2)(s-3)}}{4} \psi_{s-4} + \frac{\sqrt{s(s-1)}}{s} \psi_{s-2} - \frac{\sqrt{(s+1)(s+2)}}{s+1} \psi_{s+2} \\ & - \frac{\sqrt{(s+1)(s+2)(s+3)(s+4)}}{4} \psi_{s+4} \Bigg] q^{-1} + \mathcal{O}(q^{-2}) \Bigg\}, \end{aligned} \tag{68}$$

to accompany the eigenvalue expression (58). The equivalent expression in TO3 (equation 32, ibid) included only the first term in the braces.

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DATA AVAILABILITY STATEMENT

The numerical data used to validate the asymptotic expressions are available on request from the author.

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APPENDIX A: NORMALIZED HERMITE FUNCTIONS

The normalized Hermite functions ψ_j ($j = 0, 1, 2 \ldots$) are defined in terms of the Hermite polynomials H_j as

$$\psi_j(\sigma) = \frac{1}{\left(\sqrt{\pi} 2^j j!\right)^{1/2}} \exp\left(-\sigma^2/2\right)\ H_j(\sigma). \tag{A1}$$

(see e.g. Section 18.2 of Arfken, Weber & Harris 2013). They are orthonormal on the interval $[-\infty, \infty]$,

$$\left\langle \psi_j, \psi_k \right\rangle = \delta_{j,k}, \tag{A2}$$

where

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f(\sigma)\, g(\sigma)\, \mathrm{d}\sigma \tag{A3}$$

defines the inner product between the functions f and g, and $\delta_{j,k}$ is the Kronecker delta. As such, they form a complete orthogonal basis for square-integrable real functions.

The normalized Hermite functions obey the identities

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - \sigma^2\right) \psi_j = -(2j+1)\, \psi_j, \tag{A4}$$

$$\sigma \frac{\mathrm{d}\psi_j}{\mathrm{d}\sigma} = \frac{\sqrt{j(j-1)}}{2} \psi_{j-2} - \frac{1}{2}\, \psi_j - \frac{\sqrt{(j+1)(j+2)}}{2}\, \psi_{j+2}, \tag{A5}$$

$$\sigma^2 \psi_j = \frac{\sqrt{j(j-1)}}{2}\, \psi_{j-2} + \frac{2j+1}{2}\, \psi_j + \frac{\sqrt{(j+1)(j+2)}}{2}\, \psi_{j+2}; \tag{A6}$$

these are used extensively in evaluating the inner products appearing in Section (3).

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