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On differential operators over a map, thick morphisms of supermanifolds, and symplectic micromorphisms ☆

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Dedicated to the memory of Alexandre Mikhailovich Vinogradov

ABSTRACT

We recall the notion of a differential operator over a map (in linear and non-linear settings) and consider its versions such as formal $\hbar$-differential operators over a map. We study constructions and examples of such operators, which include pullbacks by thick morphisms and operators arising as quantization of symplectic micromorphisms.

Contents

1. Introduction

The notion of a “differential operator over a map” (or, in algebraic version, over an algebra homomorphism) is not new. It can be traced to Gabriel [1, Exposé VIIA], and can be seen as a natural extension of the algebraic definition of a differential operator on a scheme or a commutative algebra by Grothendieck [3, §16.8]. (For the latter notion, see also Vinogradov [8] and [9], and also Koszul [6].)

However, in spite of its being very “natural”, this notion is missing from standard texts. Recently constructions appeared such as *thick morphisms* between manifolds or supermanifolds (due to the second author,

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see [15]) that provide examples of differential operators over maps, or versions or modifications thereof. The purpose of this paper is to review this central notion and its variants, which include non-linear operators, formal, pseudo- and $\hbar$-(formal, pseudo-) versions. We give constructions and examples of such operators for $\mathbb{R}^n$ and for (super)manifolds. In particular, we consider operators arising as quantization of symplectic micromorphisms introduced in a recent work by Cattaneo, Dherin and Weinstein [2] and compare them with pullbacks by quantum thick morphisms [13–15].

We dedicate this work to A. M. Vinogradov (1938–2019), a remarkable man and mathematician, friendship with whom we shall always treasure in our memories.

2. Differential operators over maps and related concepts

Let $\varphi\colon\; M_1 \to M_2$ be a smooth map of differentiable manifolds, which in local coordinates is expressed as $y^i = \varphi^i(x)$. Then a *differential operator over a map* φ of *order* $\leq k$ is a linear operator $L\colon\; C^\infty(M_2) \to C^\infty(M_1)$ that in local coordinates can be written as

$$L(g) = \sum_{|\alpha|\leq k} L_\alpha(x)\, \partial_y^\alpha(g)(\varphi(x))\,. \tag{1}$$

(Here α is a multi-index, $\partial_y^\alpha = \partial_{y^1}^{\alpha_1} \dots \partial_{y^m}^{\alpha_m}$, $|\alpha| = \alpha_1 + \ldots + \alpha_m$.) In other words, we take a function of variables y^i, differentiate it with respect to y^i and substitute in the result the variables y^i as functions of x^a (as given by the map φ), and then take a linear combination of these derivatives-followed-by-substitution with the coefficients depending on x.

(Everywhere in this section we speak about C^∞ functions and maps, but it is equally possible to consider real-analytic or complex-analytic functions or formal power series.)

Example 1. A *vector field over a map* (or *along a map*) gives an example of a differential operator over a map of order ≤ 1. Such a vector field over a map $\varphi\colon\; M_1 \to M_2$ is defined as a section of $\varphi^*(TM_2) \to M_1$, i.e. a map $Y\colon\; M_1 \to TM_2$ such that $Y(x) \in T_{\varphi(x)}M_2$. As an operator on functions, in coordinates,

$$Y = Y^i(x)\, \frac{\partial}{\partial y^i}_{|y=\varphi(x)}\,. \tag{2}$$

Alternatively, a vector field Y over a map φ can be understood as an infinitesimal variation of φ, i.e. a "map" (depending of a formal parameter ε, $\varepsilon^2 = 0$) $\varphi_\varepsilon\colon\; M_1 \to M_2$, $\varphi_\varepsilon(x) := \varphi(x) + \varepsilon Y(x)$. A particular example is the velocity of a parameterized curve, which is a vector field $d\gamma/dt$ over $\gamma\colon\; (a, b) \to M$. Another particular example arises when there is a family of maps $\varphi_t\colon\; M_1 \to M_2$ and the derivative $Y_t := \partial\varphi_t/\partial t$ is a vector field over φ_t for each t. (This vector field over a map Y_t appears in differential geometry for example in Cartan homotopy formula for differential forms.) About vector fields over maps see also [11].

An algebraic version of the same concept can be formulated as follows. Let $\alpha\colon\; A \to B$ be an algebra homomorphism of commutative algebras. Then *differential operators over an algebra homomorphism* α (shortly: *d.o.'s* over α) of order $\leq k$ (or kth order) are defined inductively by the following conditions. A *differential operator over* α *of order zero* is a linear map $L\colon\; A \to B$ satisfying

$$L(aa') = \alpha(a)\, L(a') \tag{3}$$

for all $a, a' \in A$. If A, B are algebras with a unit and α preserves units, one can see that such an L acts as

$$L(a) = L(a1) = \alpha(a)L(1) = \alpha(a)b\,, \tag{4}$$

where $b = L(1) \in B$, i.e. L is the combination of the action of the homomorphism α and a multiplication operator. Now for $k > 0$, a linear map $L\colon\ A \to B$ is a *differential operator over* α *of order* k if for all $a, a' \in A$,

$$L(aa') = \alpha(a)\, L(a') + L_1(a') \tag{5}$$

where $L_1\colon\ A \to B$ is a differential operator over α of order $k-1$ (depending on $a \in A$).

Example 2. One can see that a first order differential operator L over an algebra homomorphism $\alpha\colon\ A \to B$ satisfying $L(1) = 0$ is nothing but a *derivation over* α, i.e. satisfies the Leibniz rule

$$L(a_1 a_2) = L(a_1)\alpha(a_2) + \alpha(a_1)\, L(a_2)\,, \tag{6}$$

and conversely. Such operators L define infinitesimal variations of algebra homomorphisms, $\alpha_\varepsilon = \alpha + \varepsilon\, L\colon\ A \to B$. (This is an algebraic version of a vector field over a smooth map.)

A version of the same definition for superalgebras includes signs: *L is a differential operator over a superalgebra homomorphism* $\alpha\colon\ A \to B$ between commutative superalgebras if for all $a, a' \in A$

$$L(aa') = (-1)^{\tilde{a}_1 \tilde{L}} \alpha(a)\, L(a') + L_1(a')\,, \tag{7}$$

where L_1 is order $k-1$.

Like in the usual case, one can show that for algebras of smooth functions the algebraic definition and the coordinate definition give the same notion. If for (super)manifolds M_1 and M_2 we denote by $\mathrm{DO}^k(M_1 \overset{\varphi}{\to} M_2)$ the set of all kth order differential operators over a smooth map $\varphi\colon\ M_1 \to M_2$ and denote

$$\mathrm{DO}^k(M_1, M_2) = \bigcup_{\varphi\colon\ M_1 \to M_2} \mathrm{DO}^k(M_1 \overset{\varphi}{\to} M_2) \tag{8}$$

and use the similar notation for algebras, then

$$\mathrm{DO}^k(M_1 \overset{\varphi}{\to} M_2) = \mathrm{DO}^k(C^\infty(M_2) \overset{\varphi^*}{\to} C^\infty(M_1))\,, \tag{9}$$

$$\mathrm{DO}^k(M_1, M_2) = \mathrm{DO}^k(C^\infty(M_2), C^\infty(M_1))\,. \tag{10}$$

We shall refer to the map $\varphi\colon\ M_1 \to M_2$ for an operator $L \in \mathrm{DO}(M_1, M_2)$ as the *core* or *support* of L.

One can check that differential operators over maps with matching source and target can be composed, so if $L \in \mathrm{DO}^k(M_1 \overset{\varphi_{21}}{\to} M_2)$ and $K \in \mathrm{DO}^\ell\,(M_2 \overset{\varphi_{32}}{\to} M_3)$, then

$$L \circ K \in \mathrm{DO}^{k+\ell}\,(M_1 \overset{\varphi_{32}\circ\varphi_{21}}{\to} M_3)\,. \tag{11}$$

Therefore we obtain a category whose arrows are differential operators over maps. Denote it DO. It contains as a subcategory the (opposite to the) usual category of (super)manifolds and smooth maps, if one identifies a map $\varphi\colon\ M_1 \to M_2$ with the zero-order differential operator over itself $L = \varphi^*$.

The category DO is not additive, as one cannot always add elements of $\mathrm{DO}(M_1, M_2)$, unless they are over the same map φ. So the category DO is not a so straightforward generalization of the usual algebra of differential operators $\mathrm{DO}(M)$ for a fixed manifold M (which are operators over the identity map). Still, it makes sense to ask about “generators” of the category DO, i.e. such arrows that generate all other arrows by compositions and sums (when they are defined). (Similar to the description of the algebra of polynomial differential operators on $\mathbb{R}^n$ as the Weyl algebra.) It seems that as such generators one can take: (1) all pull-backs φ^* by maps $\varphi\colon\ M \to N$ for all M, N; (2) all vector fields over maps Y, for all $\varphi\colon\ M \to N$; and (3) all operators of multiplication by functions $f \in C^\infty(M)$, for all M. There are the "Heisenberg-type" relation

$$Y \circ g = (-1)^{\tilde{Y}\tilde{g}}\varphi^*(g) \circ Y + Y(g) \tag{12}$$

and the relation $\varphi^* \circ g = \varphi^*(g) \circ \varphi^*$.

It is not of great difficulty to generalize the above definitions to the case of operators between modules over commutative superalgebras or, in the differential-geometric setting, to operators acting on sections of (super) vector bundles. In the expression in local coordinates (1), this would amount to consider matrix coefficients. We shall not go in this direction further. Instead we shall discuss two particular variations of our theme: non-linear operators and $\hbar$-formal operators.

We say that a mapping $L\colon\ C^\infty(M_2) \to C^\infty(M_1)$ is a *non-linear differential operator over a smooth map* $\varphi\colon\ M_1 \to M_2$ of order $\leq k$ if L sends a function $g \in C^\infty(M_2)$ to a function $L(g) = f \in C^\infty(M_1)$ that in local coordinates is expressed as a polynomial in partial derivatives $\partial_y^\alpha g$ with $|\alpha| \leq k$ evaluated at $y = \varphi(x)$ with the coefficients depending on x:

$$L(g)(x) = P\left(x, \partial g, \partial^2 g, \dots, \partial^k g\right)_{|y=\varphi(x)}\,, \tag{13}$$

where by $\partial^r g$ we denote the whole collection of partial derivatives $\partial_y^\alpha g(y)$ for all α with $|\alpha| = r$. The right-hand side of (13) is a polynomial in $\partial g(\varphi(x)), \partial^2 g(\varphi(x)), \dots, \partial^k g(\varphi(x))$. (Linear operators considered above are of course a particular case, when the polynomial P in (13) is linear in the derivatives.)

One can say this equivalently by introducing a bundle $J^k(M_1 \overset{\varphi}{\to} M_2)$ over the manifold M_1, as the pull-back bundle

$$J^k(M_1 \overset{\varphi}{\to} M_2) := \varphi^*\left(J^k(M_2)\right)$$

of the jet bundle $J^k(M_2) = J^k(M, \mathbb{R})$ over M_2. Then a non-linear differential operator of order $\leq k$ over φ is a fiberwise-polynomial function on the total space $J^k(M_1 \overset{\varphi}{\to} M_2)$. Again, as in the linear case, this can be further generalized to sections of (possibly non-linear) fiber bundles instead of scalar functions. Everywhere where we say manifold, we actually can say supermanifold (or graded manifold, see [16]).

Coming back to the linear case, an important variation of the definition above is as follows.

Let $\hbar$ be a formal parameter to which we shall refer to as "Planck's constant". Consider smooth functions on (super)manifolds that depend on $\hbar$ as formal power series (with non-negative powers only). For them we use the notation $C_\hbar^\infty(M)$. (Other classes can be also useful, such as e.g. formal oscillatory exponentials with coefficients from $C_\hbar^\infty(M)$.) Now $\mathbb{C}[[\hbar]]$ is the ground ring instead of $\mathbb{C}$. (From this point, it is convenient to work with complex-valued functions.)

For a given map $\varphi\colon\ M_1 \to M_2$, we say that a linear (i.e. $\mathbb{C}[[\hbar]]$-linear) operator

$$L\colon\ \ C_\hbar^\infty(M_2 \to C_\hbar^\infty(M_1)$$

is an *$\hbar$-differential operator over φ of order* $\leq k$ (abbreviation: $\hbar$-d.o.) if for every $g \in C_\hbar^\infty(M_2)$,

$$L \circ g - (-1)^{\tilde{g}\tilde{L}}\varphi^*(g) \circ L = -i\hbar L_1\,, \tag{14}$$

where $L_1\colon\ \ C_\hbar^\infty(M_2 \to C_\hbar^\infty(M_1)$ is an $\hbar$-differential operator over φ of order $\leq k-1$ and all $\hbar$-differential operator over φ of order ≤ 0 are zero.

Example 3. As before, we can see that an $\hbar$-d.o. of order 0 has the form $L = f_0 \cdot \varphi^*$, where $f_0 = L(1)$, so at this stage no difference arises. However, for an $\hbar$-d.o. of order ≤ 1, we can deduce from the definition that every such operator has the form

$$L = -i\hbar D + f_0 \cdot \varphi^*\,,$$

where $D\colon\ C^\infty_\hbar(M_2) \to C^\infty_\hbar(M_1)$ is a derivation over φ^*, i.e. a vector field over φ, and $f_0 \in C^\infty_\hbar(M_1)$, $f_0 = L(1)$.

In general, we can deduce that in local coordinates an $\hbar$-differential operator over a map φ of order $\leq k$ has the form

$$L(g) = \sum_{|\alpha|\leq k} L_\alpha(x)\,(-i\hbar\partial_y)^{\alpha}(g)(\varphi(x))\,, \tag{15}$$

where the coefficients $L_\alpha(x)$ are power series in $\hbar$. In other words, we have a special case of (1) with an extra condition that every partial derivative $\partial/\partial y^a$ carries a factor of $-i\hbar$. Denote by $\hat{p}^\alpha_y$ the operator $\varphi^* \circ (-i\hbar\partial_y)^{\alpha}$. (Warning: $\hat{p}^\alpha_y$ is *not* a product!) Then

$$L = \sum_{|\alpha|\leq k} L_\alpha(x)\,\hat{p}^\alpha_y \tag{16}$$

is a general form of an $\hbar$-d.o. over φ. This expression of course depends on a choice of local coordinates, and it is not difficult to deduce a transformation law for the symbols $\hat{p}^\alpha_y$, as well as commutation relations with functions on M_2.

There is an important observation similar to the one made in [7]: it is possible to introduce a grading into the space of $\hbar$-d.o.'s over a map (besides filtration given by order). We define the *degree* of an $\hbar$-differential operator over φ by the following rules:

$$\deg \hat{p}^\alpha_y := |\alpha|\,,\quad \deg\hbar = 1\,,\quad \deg f(x) = 0 \tag{17}$$

(compare with the definition of total degree of an $\hbar$-d.o. on a manifold M in [7, §3.2]).

Proposition 1. *The degree defined by* (17) *does not depend on a choice of local coordinates on* M_1 *and* M_2*.*

Proof. We have $\hat{p}^\alpha_y = \varphi^* \circ (\hat{p}_1^{\alpha_1} \dots \hat{p}_m^{\alpha_m})$, where $\hat{p}_i = -i\hbar\partial/\partial y^i$. Under a change of coordinates, each operator $\hat{p}_i$ becomes a linear combination (with coefficients independent of $\hbar$) of similar operators relative "new" coordinate system. When we move the coefficients to the left, we use the Leibniz rule and at each step we lose one operator $\hat{p}_{i'}$ but gain one factor of $-i\hbar$. So the total degree does not change. □

Strictly speaking, degree is well-defined only on operators "of finite type", i.e. those whose coefficients are polynomials in $\hbar$. By taking infinite sums of such operators $L_{[k]}$, $\deg L_{[k]} = k$, of all degrees $k = 0, 1, 2, \dots$,

$$L = L_{[0]} + L_{[1]} + L_{[2]} + \dots \tag{18}$$

we arrive at the notion of *formal $\hbar$-differential operators over a map φ*. (In the next section we shall push this further to obtain "pseudodifferential operators over a smooth map".) We shall denote the space of all formal $\hbar$-d.o.'s over $\varphi\colon\ M_1 \to M_2$ by $\mathrm{DO}_\hbar(M_1 \overset{\varphi}{\to} M_2)$ and the space of all $\hbar$-d.o.'s over all maps from M_1 to M_2 by $\mathrm{DO}_\hbar(M_1, M_2)$,

$$\mathrm{DO}_\hbar(M_1, M_2) = \bigcup_{\varphi\colon\, M_1 \to M_2} \mathrm{DO}_\hbar(M_1 \overset{\varphi}{\to} M_2)\,.$$

By $\mathrm{DO}_\hbar{}^{[k]}(M_1 \overset{\varphi}{\to} M_2)$ and $\mathrm{DO}_\hbar{}^{[k]}(M_1, M_2)$ we denote the corresponding spaces of operators of degree k, so

$$\mathrm{DO}_\hbar\,(M_1, M_2) = \prod_{k=0}^{+\infty} \mathrm{DO}_\hbar{}^{[k]}(M_1, M_2)$$

The same argument as in the proof of Proposition 1 shows that modulo $\hbar$, the operators $\hat{p}_y^\alpha$ behave under a change of coordinates as products of commuting variables. Indeed, we have a product of operators $\hat{p}_i$ followed by the substitution $y = \varphi(x)$. If we change coordinates on M_2, we obtain that for a single such operator,

$$\hat{p}_i = \frac{\partial y^{i'}}{\partial y^i}(y)\,\hat{p}^{i'}\,.$$

Hence, for the transformation formula for a product $\hat{p}_{i_1}\dots p_{i_k}$, we need to move the coefficients of the Jacobi matrix to the left of all "new" $\hat{p}_{i'}$. Each time, by using the commutation relation, we gain an extra term proportionate to $\hbar$. Hence modulo $\hbar$ (and taking into account the substitution $y = \varphi(x)$) we arrive at the transformation law of the product of commuting variables p_i, where for each variable we have

$$p_i = \frac{\partial y^{i'}}{\partial y^i}\big(\varphi(x)\big)\,p_{i'}\,.$$

This is exactly the transformation law for fiber coordinates in the bundle $\varphi^*(T^*M_2) \to M_1$. We have arrived at the following statement.

Theorem 1. *To every formal $\hbar$-differential operator L over a map $\varphi\colon\, M_1 \to M_2$ we can canonically assign a function $H = \sigma(L)$ on the bundle $\varphi^*(T^*M_2)$ by setting $\hbar = 0$ in the coefficients in (16) and replacing the operators $\hat{p}_y^\alpha$ by the monomials p^α, where p_i are fiberwise coordinates. The function $\sigma(L)$ is a formal power series in p_i and a polynomial on p_i if L is an $\hbar$-d.o. over φ.* □

For the simplicity of notation we use $C^\infty(T^*M)$ and similar also for functions that are formal power series along the fibers.

The function $\sigma(L)$, a power series or a polynomial in the fiber variables on $\varphi^*T^*M_2$, is called the *principal symbol* of L. It is a new notion. (Compare with the definitions of the principal symbol of a formal $\hbar$-differential operator on a manifold in [7] and in [15].)

Suppose we have maps $\varphi_{21}\colon\, M_1 \to M_2$ and $\varphi_{32}\colon\, M_2 \to M_3$ and operators

$$L \in \mathrm{DO}_\hbar(M_1 \overset{\varphi_{21}}{\to} M_2)\,, \quad K \in \mathrm{DO}_\hbar(M_2 \overset{\varphi_{32}}{\to} M_3)\,.$$

We have the composition

$$L \circ K \in \mathrm{DO}_\hbar(M_1 \overset{\varphi_{32}\circ\varphi_{21}}{\to} M_3)\,.$$

What can be said about the principal symbols? We know what to expect in the classical situation of a single manifold. To be able to say that "the principal symbols multiply", we need actually to introduce the corresponding multiplication. The problem is that they are functions on different bundles. However, they possess nice functorial properties. Namely, if $H \in C^\infty(\varphi_{21}^*T^*M_2)$ and $F \in C^\infty(\varphi_{32}^*T^*M_3)$, there are the *pull-back* $\varphi_{21}^*(F) \in C^\infty(\varphi_{31}^* T^*M_3)$ and the *push-forward* $\varphi_{32*}(H) \in C^\infty(\varphi_{31}^* T^*M_3)$. In a self-explanatory notation for the position and momentum variables, $H = H(x_1, p_2)$, $F = F(x_2, p_3)$ and

$$\varphi_{32*}(H) = H\Big(x_1, \frac{\partial x_3}{\partial x_2} p_3\Big) \quad \text{and} \quad \varphi_{21}^*(F) = F\big(x_2(x_1), p_3\big)\,.$$

We define the *product* of functions $H \in C^\infty(\varphi_{21}^* T^*M_2)$ and $F \in C^\infty(\varphi_{32}^* T^*M_3)$ to be a function $HF = FH$ (in the supercase $HF = FH(-1)^{\tilde{F}\tilde{H}}$) on the bundle $\varphi_{31}^* T^*M_3$, where $\varphi_{31} := \varphi_{32} \circ \varphi_{21}$, given by

$$H \cdot F := \varphi_{32*}(H)\, \varphi_{21}^*(F)\,, \tag{19}$$

where at the right-hand side is the usual product of functions on $\varphi_{31}^* T^*M_3$.

Theorem 2. *For formal $\hbar$-differential operators over maps,*

$$\sigma(L \circ K) = \sigma(L) \cdot \sigma(K)\,. \tag{20}$$

Proof. Directly by the definitions of the principal symbol and the product (19). □

Suppose we have a commutative diagram of smooth maps:

$$\begin{array}{ccc} M_1 & \xrightarrow{\varphi_{21}} & M_2 \\ {\scriptstyle \psi_{31}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{42}} \\ M_3 & \xrightarrow[\varphi_{43}]{} & M_4 \end{array}\,, \tag{21}$$

so $\psi_{42} \circ \varphi_{21} = \varphi_{43} \circ \psi_{31}$, and suppose we have formal $\hbar$-d.o.'s L_{12} over φ_{21}, L_{34} over φ_{43}, K_{13} over ψ_{31}, and K_{24} over ψ_{42}. Since the diagram (21) is commutative, the compositions $L_{12} \circ K_{24}$ and $K_{13} \circ L_{34}$ are defined over the same map, so can be compared. Consider the difference

$$\Delta = L_{12} \circ K_{24} - (-1)^{\tilde{L}\tilde{K}} K_{13} \circ L_{34} \tag{22}$$

(we assume that the parities agree so that $\tilde{L}_{12} = \tilde{L}_{34} =: \tilde{L}$ and $\tilde{K}_{24} = \tilde{K}_{13} =: \tilde{K}$). It is not particularly interesting if we do not assume any relation between the operators. Suppose further that

$$\psi_{42*}\left(\sigma(L_{12})\right) = \psi_{31}^*\left(\sigma(L_{34})\right) \quad \text{and} \quad \varphi_{21}^*\left(\sigma(K_{24})\right) = \varphi_{43*}\left(\sigma(K_{13})\right)\,. \tag{23}$$

It follows that $\sigma(\Delta) = 0$, by the commutativity of the product of symbols. Hence Δ is divisible by $\hbar$. We can define an analog of the Poisson bracket, by

$$\{H_{12}, H_{34}\,;\, F_{24}, F_{13}\} := \sigma\left(\frac{i}{\hbar}\Delta\right)\,, \tag{24}$$

where we denoted $H_{12} = \sigma(L_{12})$, $H_{34} = \sigma(L_{34})$, and $F_{24} = \sigma(K_{24})$, $F_{13} = \sigma(K_{13})$. We hope to investigate this operation elsewhere.

3. Constructions and examples

In this section we consider constructions leading to (formal, $\hbar$-) differential operators over maps. We note that in the same way as familiar differential operators on $\mathbb{R}^n$ or on a manifold, differential operators over maps can be defined by integral formulas using various forms of "full symbol calculus". Consider first the simplest case of maps between Cartesian spaces. Let $\varphi\colon\ \mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ be a smooth map. Then by the definition of a formal $\hbar$-differential operator over φ, every such operator $L\colon\ C^{\infty}_{\hbar}(M_2)\to C^{\infty}_{\hbar}(M_1)$ can be expressed as

$$L(g)(x_1)=\int\limits_{\mathbb{R}^{2n_2}} dx_2\,\bar{d}p_2\; e^{\frac{i}{\hbar}(\varphi(x_1)-x_2)p_2}\,H_{\hbar}(x_1,p_2)\,g(x_2)\,. \tag{25}$$

Indeed, such an integral amounts to the $\hbar$-Fourier transform of the function $g(x_2)$, followed by the multiplication by $H_{\hbar}(x_1,p_2)$ and then followed by the inverse $\hbar$-Fourier transform with the substitution $y_2=\varphi(x_1)$, which is exactly the application of a (formal, $\hbar$-) differential operator over a map φ such as given by (15). Here we use standard notations such as $\bar{d}p$ for denoting coordinate volume element normalized so that it contains all numerical factors depending on dimension arising in inversion formulas for $\hbar$-Fourier transform. (In the supercase, we use similar notation e.g. $\bar{D}p$ etc.) Here $H_{\hbar}(x_1,p_2)$ is a formal series (in p_i and $\hbar$) of the form

$$H_{\hbar}(x_1,p_2)=\sum_{k=0}^{+\infty}\Big(H_0^{i_1\dots i_k}(x_1)p_{i_1}\dots p_{i_k}+(-i\hbar)H_1^{i_1\dots i_{k-1}}(x_1)p_{i_1}\dots p_{i_{k-1}}+\dots+H_k^0(x_1)\Big)\,. \tag{26}$$

The coefficients $H^{i_1\dots i_k}(x_1)$ etc. do not depend on $\hbar$. We refer to the function $H_{\hbar}(x_1,p_2)$ as the *full symbol* of L. If we need a notation, we shall write $\sigma_{\text{full}}(L)$. One can find the full symbol of L by the formula

$$\sigma_{\text{full}}(L)=e^{-\frac{i}{\hbar}\varphi(x_1)p_2}L(e^{\frac{i}{\hbar}x_2p_2})\,. \tag{27}$$

It is clear that instead of formal power series one can consider functions from different classes as long as the integral makes sense and this will give various types of "$\hbar$-pseudodifferential operators over a map φ". They all will be of course particular examples of Fourier integral operators [4].

The full symbol given by (26) and (27) and the principal symbol defined in the previous section are related by

$$\sigma(L)=\sigma_{\text{full}}(L)_{|\hbar=0}\,, \tag{28}$$

or, in terms of the expansion (26),

$$\sigma(L)=\sum_{k=0}^{+\infty}H_0^{i_1\dots i_k}(x_1)p_{i_1}\dots p_{i_k}\,. \tag{29}$$

Everything above can be done in the supercase, replacing $\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ by $\mathbb{R}^{n_1|m_1}\to\mathbb{R}^{n_2|m_2}$. It makes no principle difference, so we do not dwell on that.

Generalization from Cartesian spaces to (super)manifolds of formulas (25)–(27) can be done in two different ways. In the first approach, one can simply consider integral formulas such as (25) in coordinate domains and require that they specify an operator independent on a choice of coordinates. If we write the same formula as (25) for manifolds M_1 and M_2 and a map $\varphi\colon\ M_1\to M_2$ as

$$L(g)(x_1)=\int\limits_{T^*M_2} dx_2\,\bar{d}p_2\; e^{\frac{i}{\hbar}(\varphi(x_1)-x_2)p_2}\,H_{\hbar}(x_1,p_2)\,g(x_2)\,, \tag{30}$$

then the function $H_\hbar(x_1, p_2)$ will not be an invariantly-defined function on $\varphi^*T^*M_2$,[1] but instead will have a non-trivial transformation law, similar with the transformation law for full symbols of (pseudo)differential operators on manifolds. Another approach can be based on choosing an extra structure on manifolds in question such as a connection and a volume element and/or a metric. Then the "non-invariance" of integral formulas will be packed instead of a dependence on a choice of coordinates into a dependence of a choice such an extra structure (e.g. connection). Let us give the corresponding formulas. (Note that there may be slightly non-equivalent ways for writing them, and we use full symbol calculus for pseudodifferential operators on Riemannian manifolds built in [10] as prototype.)

Let $H \in C^\infty_\hbar(\varphi^*T^*M_2)$ for a map $\varphi\colon\ M_1 \to M_2$. Define the operator $\hat{H}\colon\ C^\infty_\hbar(M_2) \to C^\infty_\hbar(M_1)$ by the formula

$$(\hat{H}g)(x_1) = \int\limits_{T^*_{\varphi(x_1)}M_2 \times T_{\varphi(x_1)}M_2} dv_2\, \bar{d}p_2\; e^{-\frac{i}{\hbar}v_2 p_2}\, H(x_1, p_2)\, g(\exp_{\varphi(x_1)} v_2) \tag{31}$$

We do not place $\hbar$ explicitly in the notation for H, in part for distinction with $H_\hbar$ in formula (30) (but H still is a power series in $\hbar$). Here exp is the exponential mapping defined by a connection on M_2. By a change of variables $x_2 = \exp_{\varphi(x_1)} v_2$, it is possible to rewrite (31) also as

$$(\hat{H}g)(x_1) = \int\limits_{M_2 \times T^*_{\varphi(x_1)}M_2} dx_2\, \bar{d}p_2\; \mu(x_1, x_2)\, e^{\frac{i}{\hbar}\exp^{-1}_{\varphi(x_1)}(x_2)\, p_2}\, H(x_1, p_2)\, g(x_2) \tag{32}$$

or

$$(\hat{H}g)(x_1) = \int\limits_{T^*M_2} dx_2\, \bar{d}p_2\; \mu(x_1, x_2)\, e^{-\frac{i}{\hbar}\exp^{-1}_{x_2}(\varphi(x_1))\, p_2}\, H(x_1, \tau(\varphi(x_1), x_2)p_2)\, g(x_2) \tag{33}$$

Here $\mu(x_1, x_2)$ is some Jacobian function arising from a change of variables and $\tau(x_2, x_2')$ is the parallel translation along the geodesic joining x_2 and x_2'.

Define a class of functions on φ^*T^*M that are power series in momentum variables and $\hbar$ together. In particular, this class includes polynomials. We will continue to use the notation $C^\infty_\hbar(\varphi^*T^*M)$ meaning this class.

Strictly speaking, the exponential mapping ceases being invertible for large tangent vectors, so one may wish to insert some bump function into the integrals to take care of that (e.g. like it is done in [10]). However, for the functions H that we consider it is not necessary, since their $\hbar$-Fourier transform is supported at the graph $x_2 = \varphi(x_1)$.

Proposition 2. *For every $H \in C^\infty_\hbar(\varphi^*T^*M_2)$, the operator given by (31), (32) (33) is a formal $\hbar$-differential operator over φ.*

Proof. Use normal coordinates centered at $x_2 = \varphi(x_1)$ and express the integral in these coordinates. □

Example 4. Consider an integral operator defined by the formula

$$L(g)(x_1) = \int\limits_{T^*M_2} dx_2\, \bar{d}p_2\; e^{\frac{i}{\hbar}(S(x_1, p_2)) - x_2 p_2} g(x_2)\,, \tag{34}$$

[1] But the function $H_0(x_1, p_2)$ obtained by setting $\hbar = 0$ will be a well-defined function on $\varphi^*T^*M_2$, hence the "non-invariance" of the full symbol can be seen as "quantum corrections".

where $S(x_1, p_2)$ is a power series in p_2. We call S a (quantum) *generating function*. Compare with (30): in (34) there is no "amplitude" $H(x_1, p_2)$ in front of the oscillating exponential, but instead of $\varphi(x_1)p_2$ in the exponential there is $S(x_1, p_2)$. If we express $S(x_1, p_2)$ as

$$S(x_1, p_2) = S^0_\hbar(x_1) + \varphi^i_\hbar(x_1)p_{2i} + S^+_\hbar(x_1, p_2)\,, \tag{35}$$

where $S^+_\hbar(x_1, p_2)$ contains terms of order ≥ 2 in p_2, it becomes possible to rewrite (34) in the same form as (30) as its special case.

Theorem 3 *([14]). The operator defined by* (34) *is a formal $\hbar$-differential operator over a map $\varphi\colon\ M_1 \to M_2$, of the form*

$$L = e^{\frac{i}{\hbar}S^0(x_1)}\left(e^{\frac{i}{\hbar}S^+(x_1, \frac{\hbar}{i}\frac{\partial}{\partial x_2})}\right)_{x_2=\varphi_\hbar(x_1)}. \tag{36}$$

Here $\varphi_\hbar$ is an *$\hbar$-perturbation of a smooth map* $\varphi\colon\ M_1 \to M_2$. It is given in local coordinates by formulas $y^i = \varphi^i_\hbar(x)$, where $\varphi^i_\hbar(x) = \varphi^i_0(x) + \hbar\varphi^i_1(x) + \ldots$ is a formal power series such that $\varphi^i_0(x) = \varphi^i(x)$ specify the initial map φ, and these local descriptions transform appropriately on the intersections of coordinate charts. With an abuse of language we speak simply of a map $\varphi_\hbar\colon\ M_1 \to M_2$ "depending on $\hbar$".

Operators (34) were introduced in [13] as "pullbacks by quantum thick morphisms". A distinctive feature of such operators is that in the classical limit obtained by the stationary phase method, see [15], they give pullbacks by classical *thick morphisms* Φ^*, introduced in [12], which are formal non-linear differential operators over smooth maps,

$$\Phi^*(g) = \Phi^*_{[0]}(g) + \Phi^*_{[1]}(g) + \Phi^*_{[2]}(g) + \ldots$$

where each summand $\Phi^*_{[k]}(g)$ is a non-linear differential operator over a map applied to g of order k in g, expansion over k, and among all such nonlinear operators pullbacks by thick morphisms are distinguished because they are **non-linear algebra homomorphisms**. By definition, a *non-linear algebra homomorphism* is a (formal) map of algebras or superalgebras such that its derivative for every element is an ordinary algebra homomorphism [15].

Theorem 4. *If a formal non-linear operator $L\colon\ C^\infty(M_2) \to C^\infty(M_1)$ is a non-linear algebra homomorphism, then $L = \Phi^*$, the pullback by some (unique) thick morphism $\Phi\colon\ M_1 \Rrightarrow M_2$.*

This statement was conjectured by the second author in [15] and it has been recently proved by H. Khudaverdian [5].

It is possible to specify quantum thick morphisms by an "invariant" generating function (depending on a connection). The corresponding formula will be

$$L(g)(x_1) = \int\limits_{T^*_{\varphi(x_1)}M_2\times T_{\varphi(x_1)}M_2} dv_2\,\bar{d}p_2\; e^{\frac{i}{\hbar}(S^0(x_1)+\bar{S}^{(+)}(x_1,p_2)-v_2p_2)}\, g(\exp_{\varphi(x_1)} v_2) \tag{37}$$

Here we had to explicitly identify the support map $\varphi\colon\ M_1 \to M_2$. The function $\bar{S}^{(+)}(x_1, p_2)$ is a global function on $\varphi^*T^*M_2$.

The last example that we shall consider is "quantization of symplectic micromorphisms" as introduced in [2]. A "symplectic micromorphism" in the terminology of Cattaneo, Dherin and Weinstein is a morphism between "symplectic microfolds"; a symplectic microfold is a germ of a symplectic manifold at a Lagrangian submanifold, by Weinstein's symplectic tubular neighborhood theorem it can be identified with the germ of a cotangent bundle. A *symplectic micromorphism* between such germs is defined as (the germ of) a canonical relation which is "close" to the relation corresponding to a map of bases. "Close" basically means that it can be specified by a generating function of the type $S(x_1, p_2)$, i.e. exactly the same type as in the definition of thick morphisms (classical or quantum).

We observe that symplectic micromorphisms and thick morphisms are very similar, the difference being like between a germ and a jet, and also in the presence of the term $S^0(x_1)$ in generating functions for thick morphisms.

A "quantization of a symplectic micromorphism" according to [2] is a linear integral operator of the form very close to (34) and (37):

$$L(g)(x_1) = \int\limits_{T^*_{\varphi(x_1)}M_2 \times T_{\varphi(x_1)}M_2} dv_2\, \bar{d}p_2\; e^{\frac{i}{\hbar}(\bar{S}^{(+)}(x_1,p_2) - v_2 p_2)}\, H(x_1, p_2)\, g(\exp_{\varphi(x_1)} v_2)\,. \tag{38}$$

(In [2] operators acting on half-densities are considered, but this makes no essential difference. Also, exp is not necessarily defined by a connection.) It is assumed that $\bar{S}^{(+)}(x_1, p_2)$ as a function of p_2 has zero of order two at p_2.

We see that the main difference of (38) with the pull-back by a quantum thick morphism given by (34) and (37) is the presence of the function $H(x_1, p_2)$, which is a genuine function on $\varphi^* T^* M_2$. Also, no term $S^0(x_1)$ in the exponential. The particular case of $\bar{S}^{(+)}(x_1, p_2) = 0$ is called in [2] a "quantization of cotangent lift". It is in fact the same as an $\hbar$-differential operator over φ written in an integral form as (30) or (31). Moreover, a closer look shows that the class of operators obtained by formula (38) is not different from the class of operators over a map. If $\hbar$ is treated as a formal parameter, one can see that only the Taylor expansions of $\bar{S}^{(+)}(x_1, p_2)$ and $H(x_1, p_2)$ play a role and we have the following statement. (See also remark below.)

Theorem 5. *If $\hbar$ is regarded as a formal parameter, then the class of operators obtained as quantization of symplectic micromorphisms coincides with the class of all formal $\hbar$-differential operators over smooth maps.*

If we take the viewpoint that thick morphisms are generalizations of ordinary maps, then one may consider "differential operators over thick morphisms". The practical difference is what emerges as their symbols: if we have for example

$$L(g)(x_1) = \left(e^{\frac{i}{\hbar} S^{+}(x_1, \frac{\hbar}{i}\frac{\partial}{\partial x_2})}\, H\Big(x_1, \frac{\hbar}{i}\frac{\partial}{\partial x_2}\Big)\, g(x_2) \right)_{x_2 = \varphi_\hbar(x_1)}\,,$$

then it is either $H(x_1, p_2)$, which may be polynomial in p_2, or $H(x_1, p_2)\, e^{\frac{i}{\hbar} S^{+}(x_1, p_2)}$.

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