LOCAL ASYMPTOTICS FOR ORTHONORMAL POLYNOMIALS ON THE UNIT CIRCLE VIA UNIVERSALITY

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ABSTRACT. Let μ be a positive measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Assume that in some subarc J, μ is absolutely continuous, while μ' is positive and continuous. Let $\{\phi_n\}$ be the orthonormal polynomials for μ. Using universality limits, we show that for appropriate $z \in J$,

$$\lim_{n\to\infty} \frac{\phi_n\left(z\left(1+\frac{u}{n}\right)\right)}{\phi_n(z)} = e^u,$$

uniformly for u in compact subsets of the plane.

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1. Results

Let μ be a finite positive Borel measure on $[-\pi, \pi)$ with infinitely many points in its support. Then we may define orthonormal polynomials

$$\phi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(z)\overline{\phi_m(z)}d\mu(\theta) = \delta_{mn}, \tag{1.1}$$

where $z = e^{i\theta}$. We shall usually assume that μ is *regular* in the sense of Stahl and Totik [11], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1. \tag{1.2}$$

This is true if for example $\mu' > 0$ a.e. in $[-\pi, \pi)$, but there are pure jump and pure singularly continuous measures that are regular. We denote the zeros of ϕ_n by $\{z_{jn}\}_{j=1}^{n}$. They lie inside the unit circle, and may not be distinct.

The nth reproducing kernel for μ is

$$K_n(z, u) = \sum_{j=0}^{n-1} \phi_j(z)\overline{\phi_j(u)}.$$

One of the key limits in random matrix theory, the so-called universality limit [1], [3], [4], [5], [9], [15], [16] can be cast in the following form for measures on the unit circle [4, Thm. 6.3, p. 559]:

Theorem A
Let μ be a finite positive Borel measure on $[-\pi, \pi)$ that is regular. Let $J \subset (-\pi, \pi)$ be compact, and such that μ is absolutely continuous in an open set containing

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J. Assume moreover, that μ' is positive and continuous at each point of J. Then uniformly for $\theta \in J$ and a, b in compact subsets of the complex plane, we have

$$\lim_{n\to\infty} \frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right), z\left(1+\frac{i2\pi \bar{b}}{n}\right)\right)}{K_n(z,z)} = e^{i\pi(a-b)} \frac{\sin \pi (a-b)}{\pi (a-b)}, \tag{1.3}$$

There are several refinements and generalizations of this result (Totik, Simon...)

In this paper, we shall use the universality limit to establish "local" asymptotics for the ratio $\phi_n\left(z\left(1+\frac{u}{n}\right)\right)/\phi_n(z)$. Analogous results for orthogonal polynomials associated with measures on compact subsets of the real line were established in [6], [7]. In [6], we showed that if μ is a regular measure on $[-1,1]$ for which $\mu'(x)(1-x)^{-\alpha}$ has a finite positive limit as $x \to 1-$, then the orthonormal polynomials $\{p_n\}$ for μ satisfy, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{p_n\left(1-\frac{z^2}{2n^2}\right)}{p_n(1)} = \frac{J_\alpha^*(z)}{J_\alpha^*(0)},$$

where $J_\alpha^*(z) = J_\alpha(z)/z^\alpha$ is the normalized Bessel function of order α. In [7], we showed that if μ is a regular measure with compact support in the real line, and in some closed subinterval J of the support, μ is absolutely continuous, while μ' is continuous, then for points y_{jn} in a compact subset of J^o with $p_n'(y_{jn}) = 0$, we have

$$\lim_{n\to\infty} \frac{p_n\left(y_{jn} + \frac{z}{n\omega(y_{jn})}\right)}{p_n(y_{jn})} = \cos \pi z$$

uniformly in y_{jn} and for z in compact subsets of the plane. Here ω is the density of the equilibrium measure of the support.

The case of the unit circle turns out to be more difficult, because there is no obvious analogue of the point 1 at the endpoint of $[-1,1]$, or the local maximum point y_{jn} of $|p_n|$ inside the support. The derivative ϕ_n' of the orthonormal polynomial ϕ_n has all its zeros inside the unit circle. Moreover, $\left|\phi_n\left(e^{i\theta}\right)\right|$ might have only a few local maxima for $\theta \in [-\pi, \pi]$. For that reason, we shall use *paraorthogonal polynomials*

$$\phi_{n+1}(z;\beta) = z\phi_n(z) - \beta\phi_n^*(z) \tag{1.4}$$

where $|\beta| = 1$ and

$$\phi_n^*(z) = z^n \overline{\phi_n\left(\frac{1}{\bar{z}}\right)}$$

is the reverted polynomial. The paraorthogonal polynomial $\phi_{n+1}(z;\beta)$ has $n+1$ distinct simple zeros on the unit circle. Moreover, they interlace for different β. This is an easy consequence of the fact that

$$B_n(z) = \frac{z\phi_n(z)}{\phi_n^*(z)} \tag{1.5}$$

is a finite Blaschke product (ref. Simon [ ]). It is a consequence of universality limits that the zeros of $\{\phi_n(\cdot;\beta)\}$ exhibit "clock behavior" and this has been studied in detail by Simon and his collaborators. We shall use heavily use those results. Our main theorem is:

Theorem 1.1

This is a conseuqnce of a more general result:

Theorem 1.2
Let μ be a positive measure on the unit circle. Assume that $\{\zeta_n\}$ is a sequence of numbers on the unit circle, and that uniformly for a, b in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(\zeta_n\left(1+\frac{i2\pi a}{n}\right), \zeta_n\left(1+\frac{i2\pi \bar{b}}{n}\right)\right)}{K_n\left(\zeta_n, \zeta_n\right)} = e^{i\pi(a-b)} \frac{\sin \pi\left(a-b\right)}{\pi\left(a-b\right)}. \tag{1.6}$$

The following are equivalent:
(a)

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty; \ \sup_{n\geq 1} \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{\left|\zeta_n - z_{jn}\right|^2} < \infty.$$

(b) From every infinite sequence of positive integers, we can choose an infinite subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\in\mathcal{S}} \frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n\left(\zeta_n\right)} = e^u + Ce^{v/2} \sin\left(\frac{v}{2i}\right), \tag{1.7}$$

where

$$C = 2i \lim_{n\in\mathcal{S}} \left(\frac{\zeta_n}{n} \frac{\phi_n'\left(\zeta_n\right)}{\phi_n\left(\zeta_n\right)} - 1\right). \tag{1.8}$$

and C is bounded independently of the subsequence $\mathcal{S}$.

We note that it is possible to formulate a version of this theorem where μ is replaced at the nth stage by a measure μ_n so that we are handling varying measures, as was done in [6], [7] for measures on the real line.

We note that our proofs very heavily use the fact that there is a Christoffel-Darboux formula for orthogonal polynomials on the unit circle. Since such a formula is lacking for more general contours, it will be a significant challenge to extend the results of this paper to such a setting.

2. Proof of Theorem 1.2

We shall use the Christoffel-Darboux formula [9, p. 954], [12, p. 293]

$$K_n\left(z,t\right) = \sum_{k=0}^{n-1} \phi_k\left(z\right)\overline{\phi_k\left(t\right)} = \frac{\overline{\phi_n^*\left(t\right)}\phi_n^*\left(z\right) - \overline{\phi_n\left(t\right)}\phi_n\left(z\right)}{1-\bar{t}z}. \tag{2.1}$$

Let

$$H_n\left(z,t\right) = \frac{\phi_n^*\left(z\right)}{\phi_n\left(z\right)} - \frac{\phi_n^*\left(t\right)}{\phi_n\left(t\right)}. \tag{2.2}$$

Lemma 2.1
(a)

$$H_n\left(z,t\right) = \frac{t^n K_n\left(z, \frac{1}{t}\right)\left(1-\frac{z}{t}\right)}{\phi_n\left(t\right)\phi_n\left(z\right)}. \tag{2.3}$$

(b)

$$H_n(z,t) = H_n(z,u) + H_n(u,t). \tag{2.4}$$

Proof

(a) Now

$$\begin{aligned} & t^n K_n\left(z, \frac{1}{\bar{t}}\right)\left(1 - \frac{z}{t}\right) \\ = \; & t^n \left[\overline{\phi_n^*\left(\frac{1}{\bar{t}}\right)}\phi_n^*(z) - \overline{\phi_n\left(\frac{1}{\bar{t}}\right)}\phi_n(z)\right] \\ = \; & \phi_n(t)\phi_n^*(z) - \phi_n^*(t)\phi_n(z), \end{aligned}$$

so ( ) follows from the definition of $H_n(z,t)$.

(b) This is immediate from the definition of H_n. ■

Lemma 2.2

Let $\{\zeta_n\}$ be a sequence on the unit circle. The following are equivalent:

(a)

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty; \sup_{n\geq 1} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{|\zeta_n - z_{jn}|^2} < \infty. \tag{2.5}$$

(b) The functions $\left\{\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)}\right\}$ are uniformly bounded for u in compact subsets of $\mathbb{C}$.

Proof

(a)⇒(b)

Now

$$\begin{aligned} \log\left|\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)}\right| &= \sum_{j=1}^{n} \log\left|1 + \frac{u\zeta_n}{n(\zeta_n - z_{jn})}\right| \\ &= \frac{1}{2}\sum_{j=1}^{n} \log\left(1 + 2\operatorname{Re}\left(\frac{u\zeta_n}{n(\zeta_n - z_{jn})}\right) + \left|\frac{u\zeta_n}{n(\zeta_n - z_{jn})}\right|^2\right) \quad (2.6) \\ &\leq \operatorname{Re}\left(\frac{u\zeta_n}{n}\sum_{j=1}^{n}\frac{1}{\zeta_n - z_{jn}}\right) + \frac{|u|^2}{2n^2}\sum_{j=1}^{n}\frac{1}{|\zeta_n - z_{jn}|^2} \end{aligned}$$

Then given $R > 0$, we obtain from (2.5),

$$\sup_{n\geq 1}\sup_{|u|\leq R}\left|\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)}\right| < \infty.$$

(b)⇒(a)

Let

$$A = \sup_{n\geq 1}\sup_{|u|\leq 1}\log\left|\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)}\right|.$$

We use the fact that for each j,

$$\operatorname{Re}\left(\frac{\zeta_n}{\zeta_n - z_{jn}}\right) = \frac{1 - \operatorname{Re}\left(\zeta_n\overline{z_{jn}}\right)}{|\zeta_n - z_{jn}|^2} \geq 0,$$

so that setting $u=1$ above, we have

$$2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n-z_{jn}\right)}\right)+\left|\frac{1}{n\left(\zeta_n-z_{jn}\right)}\right|^2\geq 0.$$

Then also, for each j, we have from the identity (2.5) above

$$e^{2A}-1\geq 2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n-z_{jn}\right)}\right)+\left|\frac{1}{n\left(\zeta_n-z_{jn}\right)}\right|^2\geq 0.$$

Choose $C_1>0$ such that

$$\log\left(1+t\right)\geq C_1 t \text{ for } t\in[0,e^{2A}-1].$$

Then from (2.6),

$$\begin{aligned} A &\geq C_1\sum_{j=1}^{n}\left(2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n-z_{jn}\right)}\right)+\left|\frac{1}{n\left(\zeta_n-z_{jn}\right)}\right|^2\right)\\ &= 2C_1\operatorname{Re}\left(\zeta_n\sum_{j=1}^{n}\frac{1}{\zeta_n-z_{jn}}\right)+\frac{C_1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|\zeta_n-z_{jn}\right|^2}. \end{aligned}$$

As both terms are nonnegative, we obtain

$$\sup_{n\geq 1}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|\zeta_n-z_{jn}\right|^2}<\infty\ .$$

Next, we apply Cauchy's inequalities for derivatives to $f_n\left(u\right)=\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n\left(\zeta_n\right)}$. We obtain

$$\left|\frac{\zeta_n}{n}\sum_{j=1}^{n}\frac{1}{\zeta_n-z_{jn}}\right|=\left|f_n'\left(0\right)\right|\leq\sup_{|u|\leq 1}\left|f_n\left(u\right)\right|\leq e^A.$$

■

Proof of Theorem 1.2
(a)⇒(b)

By Lemma 2.2, the functions $\{f_n\left(u\right)\}=\left\{\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n\left(\zeta_n\right)}\right\}$ form a normal family. Assume that $\mathcal{S}$ is an infinite subsequence of integers such that

$$\lim_{n\in\mathcal{S}}f_n\left(u\right)=G\left(u\right),$$

uniformly for u in compact subsets of the plane, where G is an entire function. Let

$$\Delta_n=\frac{n\phi_n\left(\zeta_n\right)^2}{z^nK_n\left(\zeta_n,\zeta_n\right)}.$$

Then uniformly for u, v in compact sets, and u, v with $G(u), G(v)$ non-zero, Lemma 2.1 gives

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right), \zeta_n\left(1+\frac{v}{n}\right)\right) \\
= \; & \frac{\left(\zeta_n\left(1+\frac{v}{n}\right)\right)^n \frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)} n\left(1-\frac{\zeta_n\left(1+\frac{u}{n}\right)}{\zeta_n\left(1+\frac{v}{n}\right)}\right)}{\zeta_n\left[\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)}\frac{\phi_n\left(\zeta_n\left(1+\frac{v}{n}\right)\right)}{\phi_n(\zeta_n)}\right]} \\
= \; & \frac{e^v}{G(u)\,G(v)} \frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)} \left(\frac{v-u}{\zeta_n}\right)(1+o(1)).
\end{aligned}$$

Write $u = 2\pi i a$, $-\bar{v} = 2\pi i \bar{b}$ so that $v = -2\pi i b$. Here by the uniform convergence in ( ),

$$\begin{aligned}
\frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)} &= \frac{K_n\left(\zeta_n\left(1+\frac{i2\pi a}{n}\right), \zeta_n\left(1+\frac{i2\pi\bar{b}}{n}+O\left(\frac{1}{n^2}\right)\right)\right)}{K_n(\zeta_n,\zeta_n)} \\
&= e^{i\pi(a-b)}\mathbb{S}(a-b)+o(1) \\
&= e^{(u-v)/2}\mathbb{S}\left(\frac{u-v}{2\pi i}\right)+o(1),
\end{aligned}$$

so

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right), \zeta_n\left(1+\frac{v}{n}\right)\right) \\
= \; & \frac{e^v}{G(u)\,G(v)} e^{(u-v)/2}\mathbb{S}\left(\frac{u-v}{2\pi i}\right)(v-u)+o(1) \\
= \; & 2i\frac{e^{(u+v)/2}}{\zeta_n G(u)\,G(v)} \sin\left(\frac{v-u}{2i}\right)+o(1).
\end{aligned}$$

Now we use this in (2.4). We have

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right), \zeta_n\left(1+\frac{v}{n}\right)\right) \\
= \; & \Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right), \zeta_n\left(1+\frac{w}{n}\right)\right) + \Delta_n H_n\left(\zeta_n\left(1+\frac{w}{n}\right), \zeta_n\left(1+\frac{v}{n}\right)\right)
\end{aligned}$$

and hence for u, v, w with $G(u)\,G(v)\,G(w) \neq 0$,

$$\begin{aligned}
&\frac{e^{(u+v)/2}}{G(u)\,G(v)}\sin\left(\frac{v-u}{2i}\right) \\
= \; & \frac{e^{(u+w)/2}}{G(u)\,G(w)}\sin\left(\frac{w-u}{2i}\right) + \frac{e^{(w+v)/2}}{G(w)\,G(v)}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&G(w)\,e^{(u+v)/2}\sin\left(\frac{v-u}{2i}\right) \\
= \; & G(v)\,e^{(u+w)/2}\sin\left(\frac{w-u}{2i}\right) + G(u)\,e^{(w+v)/2}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

By analytic continuation, this holds for all u, v, w. Next, we note the elementary identity

$$\begin{aligned} & e^{w} e^{(u+v)/2} \sin\left(\frac{v-u}{2i}\right) \\ = \; & e^{v} e^{(u+w)/2} \sin\left(\frac{w-u}{2i}\right) + e^{u} e^{(w+v)/2} \sin\left(\frac{v-w}{2i}\right). \end{aligned}$$

Then subtracting the two, we have

$$\begin{aligned} & \left[G(w) - e^{w}\right] e^{(u+v)/2} \sin\left(\frac{v-u}{2i}\right) \\ = \; & \left[G(v) - e^{v}\right] e^{(u+w)/2} \sin\left(\frac{w-u}{2i}\right) + \left[G(u) - e^{u}\right] e^{(w+v)/2} \sin\left(\frac{v-w}{2i}\right). \end{aligned}$$

Now we set $w = 0$ and use $G(0) = 1$:

$$0 = -\left[G(v) - e^{v}\right] e^{u/2} \sin\left(\frac{u}{2i}\right) + \left[G(u) - e^{u}\right] e^{v/2} \sin\left(\frac{v}{2i}\right)$$

so that

$$\frac{G(v) - e^{v}}{e^{v/2} \sin\left(\frac{v}{2i}\right)} = \frac{G(u) - e^{u}}{e^{u/2} \sin\left(\frac{u}{2i}\right)}.$$

Then both sides are constant, so calling the right-hand side C,

$$G(v) = e^{v} + C e^{v/2} \sin\left(\frac{v}{2i}\right).$$

To determine C, we note that

$$G'(0) = 1 + \frac{C}{2i}.$$

In addition, we know that

$$G'(0) = \lim_{n \in \mathcal{S}} f_n'(0) = \lim_{n \in \mathcal{S}} \frac{\zeta_n}{n} \frac{\phi_n'(\zeta_n)}{\phi_n(\zeta_n)}$$

Thus

$$C = 2i \lim_{n \in \mathcal{S}} \left(\frac{\zeta_n}{n} \frac{\phi_n'(\zeta_n)}{\phi_n(\zeta_n)} - 1 \right).$$

(b)⇒(a)
Since C is bounded independently of the subsequence $\mathcal{S}$, the uniform convergence we are assuming gives that $\{f_n\}$ is uniformly bounded in compact subsets of the plane. Lemma 2.2 gives ( ). ■

3. Proof of Theorem 1.1

As we have noted, it is not trivial to verify the conditions ( ) in the case of the unit circle. We begin with some identities. Recall the notation ( ) and ( ), so

$$B_n(z) = \frac{z \phi_n(z)}{\phi_n^*(z)}.$$

Lemma 3.1

Let $|z| = 1$.
(a)

$$\operatorname{Re}\left[z\frac{\phi_n'(z)}{\phi_n(z)}\right] = \frac{n}{2} + \frac{1}{2}\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}. \tag{3.1}$$

(b)

$$\operatorname{Re}\left(\frac{z}{n}\frac{\phi_n'(z)}{\phi_n(z)} - 1\right) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\operatorname{Re}\left((z-z_{jn})\overline{z_{jn}}\right)}{|z-z_{jn}|^2}\right). \tag{3.2}$$

(c)

$$\left|\phi_{n+1}(z;\beta)\right|^2 = 2\left|\phi_n(z)\right|^2\left\{1-\operatorname{Re}\left\{\beta B_n(z)\right\}\right\}. \tag{3.3}$$

(d)

$$1-\beta B_n(z) = \frac{-\beta\phi_{n+1}(z;\beta)}{\phi_n^*(z)}. \tag{3.4}$$

(e)

$$\operatorname{Im}\left\{\frac{z\phi_{n+1}'(z;\beta)}{\phi_{n+1}(z;\beta)}\right\} = \operatorname{Im}\left\{\frac{z\phi_n'(z)}{\phi_n(z)}\right\} - \frac{1}{2}\left|B_1'(z)\right|\frac{\operatorname{Im}\left\{\beta B_n(z)\right\}}{1-\operatorname{Re}\left\{\beta B_n(z)\right\}}. \tag{3.5}$$

(f) If $\beta = \overline{B_n(w)}$,

$$\phi_{n+1}(z;\beta) = -\frac{\bar{\beta}}{\overline{\phi_n^*(w)}}\left(1-z\bar{w}\right)K_n(z,w). \tag{3.6}$$

(g)

$$\frac{\left|\operatorname{Im}\beta B_n(z)\right|}{\left|1-\operatorname{Re}\beta B_n(z)\right|} = \left(4\frac{\left|\phi_n(z)\right|^2}{\left|\phi_{n+1}(z;\beta)\right|^2} - 1\right)^{1/2}. \tag{3.7}$$

Proof
(a)

$$\begin{aligned}
&\frac{\phi_n^{*\prime}(z)}{\phi_n^*(z)} - \frac{\phi_n'(z)}{\phi_n(z)} \\
= \quad & \sum_{j=1}^{n}\frac{-\overline{z_{jn}}}{1-\overline{z_{jn}}z} - \sum_{j=1}^{n}\frac{1}{z-z_{jn}} \\
= \quad & -\frac{1}{z}\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}.
\end{aligned}$$

Thus

$$z\frac{\phi_n^{*\prime}(z)}{\phi_n^*(z)} - z\frac{\phi_n'(z)}{\phi_n(z)} = -\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}. \tag{3.8}$$

Next, we differentiate $\phi_n^*\left(e^{i\theta}\right) = e^{in\theta}\overline{\phi_n\left(e^{i\theta}\right)}$ to obtain

$$\phi_n^{*\prime}\left(e^{i\theta}\right)ie^{i\theta} = ine^{in\theta}\overline{\phi_n\left(e^{i\theta}\right)} + e^{in\theta}\overline{\phi_n'\left(e^{i\theta}\right)e^{i\theta}i}$$

so

$$z\phi_n^{*\prime}(z) = nz^n\overline{\phi_n(z)} - z^n\overline{\phi_n'(z)\,z}.$$

Dividing by $z^n\overline{\phi_n(z)} = \phi_n^*(z)$ gives

$$\frac{z\phi_n^{*\prime}(z)}{\phi_n^*(z)} = n - \overline{\left[\frac{\phi_n'(z)\,z}{\phi_n(z)}\right]},$$

so

$$z\frac{\phi_n'(z)}{\phi_n(z)} - \sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} = n - \overline{\left[\frac{\phi_n'(z)\,z}{\phi_n(z)}\right]}$$

$$\Rightarrow \quad 2\operatorname{Re}\left[z\frac{\phi_n'(z)}{\phi_n(z)}\right] = n + \sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}.$$

Together with ( ), this gives the result.
(b)

$$\begin{aligned}
\operatorname{Re}\left(\frac{z}{n}\frac{\phi_n'(z)}{\phi_n(z)} - 1\right) &= \frac{1}{2n}\sum_{j=1}^{n}\left(\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2} - 1\right) \\
&= \frac{1}{2n}\sum_{j=1}^{n}\left(\frac{1-|z_{jn}|^2 - \left(1+|z_{jn}|^2 - 2\operatorname{Re}(z\overline{z_{jn}})\right)}{|z-z_{jn}|^2}\right) \\
&= \frac{1}{2n}\sum_{j=1}^{n}\left(\frac{2\operatorname{Re}(z\overline{z_{jn}}) - 2|z_{jn}|^2}{|z-z_{jn}|^2}\right) \\
&= \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\operatorname{Re}((z-z_{jn})\,\overline{z_{jn}})}{|z-z_{jn}|^2}\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\left|\phi_{n+1}(z;\beta)\right|^2 &= \left(z\phi_n(z) - \bar\beta z^n\overline{\phi_n(z)}\right)\overline{\left(z\phi_n(z) - \bar\beta z^n\overline{\phi_n(z)}\right)} \\
&= 2|\phi_n(z)|^2 - 2\operatorname{Re}\left(z\phi_n(z)\,\overline{\bar\beta z^n\overline{\phi_n(z)}}\right) \\
&= 2|\phi_n(z)|^2 - 2\operatorname{Re}\left(z\phi_n(z)\,\beta z^{-n}\phi_n(z)^2\right) \\
&= 2|\phi_n(z)|^2\left\{1 - \operatorname{Re}\left\{\beta z\frac{\phi_n(z)}{z^n\overline{\phi_n(z)}}\right\}\right\} \\
&= 2|\phi_n(z)|^2\left\{1 - \operatorname{Re}\{\beta B_n(z)\}\right\}.
\end{aligned}$$

(d)

$$\begin{aligned}
1 - \beta B_n(z) &= \frac{\phi_n^*(z) - \beta z\phi_n(z)}{\phi_n^*(z)} \\
&= \frac{-\beta}{\phi_n^*(z)}\left(z\phi_n(z) - \bar\beta\phi_n^*(z)\right) \\
&= \frac{-\beta\phi_{n+1}(z;\beta)}{\phi_n^*(z)}.
\end{aligned}$$

(d) Next,

$$\ln\left|\phi_{n+1}(z;\beta)\right|^2 = \ln 2 + \ln|\phi_n(z)|^2 + \ln\{1 - \operatorname{Re}\{\beta B_n(z)\}\}$$

so

$$iz\frac{\partial}{\partial\theta}\ln\left|\phi_{n+1}\left(z;\beta\right)\right|^{2}=iz\frac{\partial}{\partial\theta}\ln\left|\phi_{n}\left(z\right)\right|^{2}+\frac{-\operatorname{Re}\left\{iz\beta B_{n}'\left(z\right)\right\}}{1-\operatorname{Re}\left\{\beta B_{n}\left(z\right)\right\}}iz$$

so

$$\frac{\partial}{\partial\theta}\ln\left|\phi_{n+1}\left(z;\beta\right)\right|^{2}=\frac{\partial}{\partial\theta}\ln\left|\phi_{n}\left(z\right)\right|^{2}+\frac{\operatorname{Im}\left\{z\beta B_{n}'\left(z\right)\right\}}{1-\operatorname{Re}\left\{\beta B_{n}\left(z\right)\right\}}.$$

Now a calculation shows that (Reference)

$$\frac{zB_{n}'\left(z\right)}{B_{n}\left(z\right)}=\left|B_{n}'\left(z\right)\right|$$

so

$$\frac{\partial}{\partial\theta}\ln\left|\phi_{n+1}\left(z;\beta\right)\right|^{2}=\frac{\partial}{\partial\theta}\ln\left|\phi_{n}\left(z\right)\right|^{2}+\left|B_{1}'\left(z\right)\right|\frac{\operatorname{Im}\left\{\beta B_{n}\left(z\right)\right\}}{1-\operatorname{Re}\left\{\beta B_{n}\left(z\right)\right\}}. \tag{3.9}$$

Next, if P is a polynomial,

$$\begin{aligned}\frac{\partial}{\partial\theta}\left|P\left(e^{i\theta}\right)\right|^{2}&=\frac{\partial}{\partial\theta}\left(P\left(e^{i\theta}\right)\overline{P\left(e^{i\theta}\right)}\right)\\&=P'\left(e^{i\iota}\right)ie^{i\theta}\overline{P\left(e^{i\theta}\right)}+P\left(e^{i\theta}\right)\overline{P'\left(e^{i\theta}\right)\left(ie^{i\theta}\right)}\\&=i\left|P\left(z\right)\right|^{2}\left\{\frac{zP'\left(z\right)}{P\left(z\right)}-\overline{\left[\frac{zP'\left(z\right)}{P\left(z\right)}\right]}\right\}\\&=-2\left|P\left(z\right)\right|^{2}\operatorname{Im}\left\{\frac{zP'\left(z\right)}{P\left(z\right)}\right\},\end{aligned}$$

so

$$\frac{\partial}{\partial\theta}\ln\left|P\left(e^{i\theta}\right)\right|^{2}=-2\operatorname{Im}\left\{\frac{zP'\left(z\right)}{P\left(z\right)}\right\}.$$

Substituing this in ( ) gives ( ).
(e)

$$\begin{aligned}\phi_{n+1}\left(z;\beta\right)&=z\phi_{n}\left(z\right)-\bar{\beta}\phi_{n}^{*}\left(z\right)\\&=\bar{\beta}\left[z\phi_{n}\left(z\right)\beta-\phi_{n}^{*}\left(z\right)\right]\\&=\bar{\beta}\left[z\phi_{n}\left(z\right)\overline{\left[\frac{\omega\phi_{n}\left(w\right)}{\phi_{n}^{*}\left(w\right)}\right]}-\phi_{n}^{*}\left(z\right)\right]\\&=\frac{\bar{\beta}}{\overline{\phi_{n}^{*}\left(w\right)}}\left[z\bar{w}\phi_{n}\left(z\right)\overline{\phi_{n}\left(w\right)}-\phi_{n}^{*}\left(z\right)\phi_{n}^{*}\left(w\right)\right]\\&=-\frac{\bar{\beta}}{\overline{\phi_{n}^{*}\left(w\right)}}\left(1-z\bar{w}\right)K_{n}\left(z,w\right)\end{aligned}$$

See [10, p. 116, (2.14.20)].
(f) Now

$$1-\operatorname{Re}\beta B_{n}\left(z\right)=\frac{\left|\phi_{n+1}\left(z;\beta\right)\right|^{2}}{2\left|\phi_{n}\left(z\right)\right|^{2}}.$$

Also

$$\left|1-\beta B_{n}\left(z\right)\right|^{2}=\frac{\left|\phi_{n+1}\left(z;\beta\right)\right|^{2}}{\left|\phi_{n}\left(z\right)\right|^{2}}.$$

Then

$$\begin{aligned}(\operatorname{Im}\beta B_n(z))^2 &= |1-\beta B_n(z)|^2 - (1-\operatorname{Re}\beta B_n(z))^2 \\ &= \frac{|\phi_{n+1}(z;\beta)|^2}{|\phi_n(z)|^2} - \frac{|\phi_{n+1}(z;\beta)|^4}{4|\phi_n(z)|^4}\end{aligned}$$

and then

$$\frac{(\operatorname{Im}\beta B_n(z))^2}{(1-\operatorname{Re}\beta B_n(z))^2} = 4\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z;\beta)|^2} - 1.$$

■

Next we turn to quantitative estimates. In the sequel, we assume that μ is regular on the unit circle, and that μ is absolutely continuous in some closed subarc J, while μ' is positive and continuous there. J_1 will denote a subarc of the interior of J.

Lemma 3.2
(a)

$$n\inf\{1-|z_{jn}| : z_{jn}\in J_1\}\to\infty \text{ as } n\to\infty.$$

(b)

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}\frac{|\phi_n(z)|^2}{n}\mu'(z)=1.$$

(c)

$$\operatorname{Re}\left(\frac{z\phi_n'(z)}{n\phi_n(z)}-1\right)=\frac{1}{2}\left(\frac{1}{|\phi_n(z)|^2\mu'(z)(1+o(1))}-1\right).$$

(d)

$$\sup_{t\in I}|\phi_{n+1}(t;\beta)|\,|\phi_n(w)|\,\mu'(w)=2+o(1).$$

In particular, if z gives a local max of $\phi_{n+1}(t;\beta)$,

$$|\phi_{n+1}(z;\beta)|\,|\phi_n(w)|\,\mu'(w)=2+o(1).$$

(e) If z gives a local max of $\phi_{n+1}(t;\beta)$ and we use universality, then

$$\left|\operatorname{Im}\left\{\frac{z\phi_n'(z)}{\phi_n(z)}\right\}\right|=\frac{1+o(1)}{2}\frac{\left(|\phi_n(w)|\,|\phi_n(z)|^2\mu'(w)^2-1\right)^{1/2}}{|\phi_n(z)|^2\mu'(z)}.$$

Proof
(a) Suppose for infinitely many j, we have

$$1-|z_{jn}|\le C/n.$$

Write

$$z_{jn}=z\left(1+\frac{i2\pi a_n}{n}\right).$$

We can assume that in a subsequence $a_n\to a$. Let

$$t_{jn}=1/\overline{z_{jn}}.$$

Then

$$\begin{aligned} t_{jn} &= z\left(1+\frac{\overline{i2\pi a_n}}{n}\right)^{-1} \\ &= z\left(1+\frac{i2\pi\overline{a_n}}{n}+O\left(\frac{1}{n^2}\right)\right). \end{aligned}$$

Now

$$K_n\left(z_{jn},t_{jn}\right)=\frac{\overline{\phi_n^*\left(t_{jn}\right)}\phi_n^*\left(z_{jn}\right)-\overline{\phi_n\left(t_{jn}\right)}\phi_n\left(z_{jn}\right)}{1-\bar{t}z}=0$$

but from universality,

$$\begin{aligned} \lim_{n\to\infty}\frac{K_n\left(z_{jn},t_{jn}\right)}{K_n\left(z,z\right)} &= \\ \lim_{n\to\infty}\frac{K_n\left(z\left(1+\frac{i2\pi a_n}{n}\right),z\left(1+\frac{i2\pi\overline{a_n}}{n}\right)\right)}{K_n\left(z,z\right)} &= e^{i\pi(a-a)}\mathbb{S}\left(0\right)=1, \end{aligned}$$

a contradiction.

(b) Now uniformly for a in compact sets,

$$\lim_{n\to\infty}\frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right),z\right)}{K_n\left(z,z\right)}=e^{i\pi a}\mathbb{S}\left(a\right),$$

so

$$\lim_{n\to\infty}\frac{\overline{\phi_n^*\left(z\right)}\phi_n^*\left(z\left(1+\frac{i2\pi a}{n}\right)\right)-\overline{\phi_n\left(z\right)}\phi_n\left(z\left(1+\frac{i2\pi a}{n}\right)\right)}{\left[1-\bar{z}\left(z\left(1+\frac{i2\pi a}{n}\right)\right)\right]K_n\left(z,z\right)}=e^{i\pi a}\mathbb{S}\left(a\right)$$

so as $K_n\left(z,z\right)/n\to\mu'\left(z\right)$,

$$\begin{aligned} &\lim_{n\to\infty}\left[\overline{\phi_n^*\left(z\right)}\phi_n^*\left(z\left(1+\frac{i2\pi a}{n}\right)\right)-\overline{\phi_n\left(z\right)}\phi_n\left(z\left(1+\frac{i2\pi a}{n}\right)\right)\right]\mu'\left(z\right) \\ &= -2\pi iae^{i\pi a}\mathbb{S}\left(a\right)=-2ie^{i\pi a}\sin\pi a. \end{aligned}$$

Also, differentiating, as we can,

$$\begin{aligned} &\lim_{n\to\infty}\left[\overline{\phi_n^*\left(z\right)}\phi_n^{*\prime}\left(z\left(1+\frac{i2\pi a}{n}\right)\right)-\overline{\phi_n\left(z\right)}\phi_n'\left(z\left(1+\frac{i2\pi a}{n}\right)\right)\right]\frac{2\pi iz}{n}\mu'\left(z\right) \\ &= -\frac{d}{da}\left[2ie^{i\pi a}\sin\pi a\right]=2\pi e^{i\pi a}\sin\pi a-2i\pi e^{\iota\pi a}\cos\pi a, \end{aligned}$$

or

$$\begin{aligned} &\lim_{n\to\infty}\left[\overline{\phi_n^*\left(z\right)}\phi_n^{*\prime}\left(z\left(1+\frac{i2\pi a}{n}\right)\right)-\overline{\phi_n\left(z\right)}\phi_n'\left(z\left(1+\frac{i2\pi a}{n}\right)\right)\right]\frac{z}{n}\mu'\left(z\right) \\ &= -ie^{i\pi a}\sin\pi a-e^{\iota\pi a}\cos\pi a, \end{aligned}$$

so that taking $a=0$,

$$\lim_{n\to\infty}\left[\overline{\phi_n^*\left(z\right)}\phi_n^{*\prime}\left(z\right)-\overline{\phi_n\left(z\right)}\phi_n'\left(z\right)\right]\frac{z}{n}\mu'\left(z\right)=-1$$

so

$$\lim_{n\to\infty}\left[\frac{z\phi_n^{*\prime}\left(z\right)}{\phi_n^*\left(z\right)}-\frac{z\phi_n'\left(z\right)}{\phi_n\left(z\right)}\right]\frac{\left|\phi_n\left(z\right)\right|^2}{n}\mu'\left(z\right)=-1.$$

Then

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1-\left|z_{jn}\right|^2}{\left|z-z_{jn}\right|^2}\frac{\left|\phi_n\left(z\right)\right|^2}{n}\mu'\left(z\right)=1.$$

(c) Then

$$\begin{aligned}\operatorname{Re}\left(\frac{z\phi_n'(z)}{n\phi_n(z)}-1\right) &= \frac{1}{2n}\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}-\frac{1}{2}\\ &= \frac{1}{2}\left(\frac{1}{|\phi_n(z)|^2\mu'(z)(1+o(1))}-1\right).\end{aligned}$$

(d) For $t\in I$,

$$\begin{aligned}\left|\phi_{n+1}(t;\beta)\right| &= \left|-\overline{\left[\frac{\beta}{\phi_n^*(w)}\right]}(1-\bar{w}t)K_n(t,w)\right|\\ &\le \frac{C}{|\phi_n(w)|}.\end{aligned}$$

Moreover, if we take $w=e^{i\alpha}$ and $t=e^{i(\alpha+2\pi a_n/n)}$, we have

$$\begin{aligned}\left|\phi_{n+1}(t;\beta)\right| &= \frac{1}{|\phi_n^*(w)|}\left|\left(1-e^{i2\pi a_n/n}\right)K_n\left(e^{i(\alpha+2\pi a_n/n)},e^{i\alpha}\right)\right|\\ &= \frac{1}{|\phi_n^*(w)|}\left|\frac{2\pi a_n}{n}K_n\left(e^{i(\alpha+2\pi a_n/n)},e^{i\alpha}\right)\right|\\ &= \frac{1}{|\phi_n^*(w)|}\left|\frac{2\pi a_n}{n}K_n\left(e^{i\alpha},e^{i\alpha}\right)\frac{K_n\left(e^{i(\alpha+2\pi a_n/n)},e^{i\alpha}\right)}{K_n\left(e^{i\alpha},e^{i\alpha}\right)}\right|\\ &= \frac{2\left|\sin\pi a_n\right|+o(1)}{|\phi_n(w)|\,\mu'(w)(1+o(1))}\end{aligned}$$

so

$$\sup_{t\in I}\left|\phi_{n+1}(t;\beta)\right||\phi_n(w)|\,\mu'(w)=2+o(1).$$

As z maximizes $\phi_{n+1}(t;\beta)$, so this also gives

$$\left|\phi_{n+1}(z;\beta)\right||\phi_n(w)|\,\mu'(w)=2+o(1).$$

(e) We know that

$$0=\operatorname{Im}\left\{\frac{z\phi_{n+1}'(z;\beta)}{\phi_{n+1}(z;\beta)}\right\}=\operatorname{Im}\left\{\frac{z\phi_n'(z)}{\phi_n(z)}\right\}-\frac{1}{2}|B_1'(z)|\frac{\operatorname{Im}\{\beta B_n(z)\}}{1-\operatorname{Re}\{\beta B_n(z)\}}.$$

and

$$\frac{|\operatorname{Im}\beta B_n(z)|}{|1-\operatorname{Re}\beta B_n(z)|}=\left(4\frac{|\phi_n(z)|^2}{\left|\phi_{n+1}(z;\beta)\right|^2}-1\right)^{1/2}.$$

Thus

$$\begin{aligned}\left|\operatorname{Im}\left\{\frac{z\phi_n'(z)}{\phi_n(z)}\right\}\right| &= \frac{1}{2}|B_1'(z)|\left(4\frac{|\phi_n(z)|^2}{\left|\phi_{n+1}(z;\beta)\right|^2}-1\right)^{1/2}\\ &= \frac{1}{2}\left(\sum_{j=1}^{n}\frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}\right)\left(4\frac{|\phi_n(z)|^2}{\left|\phi_{n+1}(z;\beta)\right|^2}-1\right)^{1/2}\\ &= \frac{1}{2}\frac{1}{|\phi_n(z)|^2\mu'(z)(1+o(1))}\left(4\frac{|\phi_n(z)|^2}{\left|\phi_{n+1}(z;\beta)\right|^2}-1\right)^{1/2}.\end{aligned}$$

Using the above,

$$\begin{aligned}\left|\operatorname{Im}\left\{\frac{z\phi_n'(z)}{\phi_n(z)}\right\}\right| &= \frac{1}{2}\frac{1}{|\phi_n(z)|^2\mu'(z)(1+o(1))}\left(|\phi_n(w)||\phi_n(z)|^2\mu'(w)^2-1\right)^{1/2}\\ &= \frac{1+o(1)}{2}\frac{1}{|\phi_n(z)|^2\mu'(z)(1+o(1))}\left(|\phi_n(w)||\phi_n(z)|^2\mu'(w)^2-1\right)^{1/2}\end{aligned}$$

So we want both

$$|\phi_n(z)|^2|\mu'(z)| = 1+o(1) \text{ and } |\phi_n(w)|^2\mu'(w) = 1+o(1).$$

■

4. The Converse

Suppose now that

$$\lim_{n\to\infty}\frac{\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\phi_n(\zeta_n)} = e^u$$

uniformly for u in compact sets. Then

$$\begin{aligned}&\frac{\phi_n^*\left(\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)}{\phi_n^*(\zeta_n)}\\ =\;& \frac{\left(\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)^n\overline{\phi_n\left(\frac{1}{\overline{\zeta_n\left(1+\frac{\bar{v}}{n}\right)}}\right)}}{\zeta_n\overline{\phi_n\left(\frac{1}{\overline{\zeta_n}}\right)}}\\ =\;& e^{\bar{v}+o(1)}\overline{\left[\frac{\phi_n\left(\zeta_n\left(1-\frac{v}{n}+O\left(\frac{1}{n^2}\right)\right)\right)}{\phi_n(\zeta_n)}\right]}\\ =\;& e^{\bar{v}+o(1)-\bar{v}} = 1+o(1),\end{aligned}$$

uniformly for z in compact sets. Then

$$\begin{aligned}&\frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right),\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)}{K_n(\zeta_n,\zeta_n)}\\ =\;& \frac{\overline{\phi_n^*\left(\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)}\phi_n^*\left(\zeta_n\left(1+\frac{u}{n}\right)\right)-\overline{\phi_n\left(\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)}\phi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\left(1-\overline{\left[z\left(1+\frac{\bar{v}}{n}\right)\right]}\left[z\left(1+\frac{u}{n}\right)\right]\right)K_n(\zeta_n,\zeta_n)}\\ =\;& \frac{|\phi_n^*(\zeta_n)|^2(1+o(1))-|\phi_n(\zeta_n)|^2(1+o(1))e^{v+u}}{\left[-\left(\frac{u+v}{n}\right)+O\left(\frac{1}{n^2}\right)\right]K_n(\zeta_n,\zeta_n)}\\ =\;& \frac{|\phi_n(\zeta_n)|^2}{\frac{1}{n}K_n(\zeta_n,\zeta_n)}\frac{1-e^{v+u}+o(1)}{\left[-(u+v)+O\left(\frac{1}{n}\right)\right]}.\end{aligned}$$

If we have the standard Christoffel function asymptotics

$$\frac{1}{n}K_n(\zeta_n,\zeta_n) = \mu'(\zeta_n)^{-1}+o(1),$$

then we have $\frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right),\zeta_n\left(1+\frac{\bar{v}}{n}\right)\right)}{K_n(\zeta_n,\zeta_n)}$This is suprising! It then gives

$$|\phi_n(\zeta_n)|^2\mu'(\zeta_n) = 1+o(1).$$

Now if we set

$$u = 2\pi i a \text{ and } \bar{v} = 2\pi i \bar{b} \Rightarrow v = -2\pi i b,$$

then we obtain

$$\begin{aligned} \frac{1-e^{v+u}}{-(u+v)} &= \frac{1-e^{2\pi i(a-b)}}{-2\pi i(a-b)} \\ &= \frac{e^{\pi i(a-b)}\sin(\pi(a-b))}{\pi(a-b)} \\ &= e^{\pi i(a-b)}\mathbb{S}(a-b) \end{aligned}$$

so

$$\begin{aligned} & \frac{K_n\left(\zeta_n\left(1+\frac{2\pi i a}{n}\right), \zeta_n\left(1+\frac{2\pi i \bar{b}}{n}\right)\right)}{K_n(\zeta_n, \zeta_n)} \\ = \; & |\phi_n(\zeta_n)|^2 \mu'(\zeta_n) e^{\pi i(a-b)}\mathbb{S}(a-b) + o(1). \end{aligned}$$

Now if we do have the standard universality, this does indeed give

$$|\phi_n(\zeta_n)|^2 \mu'(\zeta_n) = 1 + o(1).$$

References

[1] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[2] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[3] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Maths. Research Notices, 30(2002), 1575-1600.

[4] Eli Levin and D.S. Lubinsky, *Universality Limits in the Bulk for Varying Measures,* Advances in Mathematics, 219(2008), 743-779.

[5] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915-939.

[6] D.S. Lubinsky, *Pointwise Asymptotics for Orthonormal Polynomials at the Endpoints of the Interval via Universality,* International Maths Research Notices, https://doi.org/10.1093/imrn/rny042.

[7] D.S. Lubinsky, *Local Asymptotics for Orthonormal Polynomials in the Interior of the Support via Universality*, to appear in Proc. Amer. Math. Soc.

[8] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer, New York, 1997.

[9] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.

[10] B. Simon, *Szegö's Theorem and its Descendants*, Princeton University Press, Princeton, 2011.

[11] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[12] G. Szegő, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, 1975.

[13] V. Totik, *Polynomial Inverse Images and Polynomial Inequalities,* Acta Math., 187(2001), 139-160.

[14] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d'Analyse Math., 81(2000), 283-303.

[15] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361-391.

[16] V. Totik, *Universality Under Szegő's Condition*, Canadian Math. Bulletin. 59 (2016), 211–224.

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