SQUARE FUNCTIONS, NONTANGENTIAL LIMITS, AND HARMONIC MEASURE IN CODIMENSION LARGER THAN 1

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Abstract

We characterize the rectifiability (both uniform and not) of an Ahlfors regular set E of arbitrary codimension by the behavior of a regularized distance function in the complement of that set. In particular, we establish a certain version of the Riesz transform characterization of rectifiability for lower-dimensional sets. We also uncover a special situation in which the regularized distance is itself a solution to a degenerate elliptic operator in the complement of E. This allows us to precisely compute the harmonic measure of those sets associated to this degenerate operator and prove that, in sharp contrast with the usual setting of codimension 1, a converse to Dahlberg's theorem must be false on lower-dimensional boundaries without additional assumptions.

1. Introduction

The beginning of the twenty-first century has brought a series of long-sought-after results enlightening connections between the scale-invariant geometric, analytic, and PDE properties of sets. Among the most celebrated ones were the Riesz transform characterizations of uniform rectifiability (see, e.g., [9], [20], [22]) and full description of the sets for which the harmonic measure is absolutely continuous with respect to the Lebesgue measure, in terms of uniform rectifiability along with a certain topological condition (see [1]). Both of these results rest on, and have been surrounded by, a plethora of important related advancements. We do not intend to review the related literature in the present work, but we point out that virtually the entire theory has been restricted to the $(n-1)$-dimensional boundaries of domains in $\mathbb{R}^n$. The question of a possible extension of these results to lower-dimensional sets has become one of the central open problems in the subject ever since.

DUKE MATHEMATICAL JOURNAL
Advance publication—final volume, issue, and page numbers to be assigned.

DOI 10.1215/00127094-2020-0048
Received 28 August 2018. Revision received 27 May 2020.
2010 *Mathematics Subject Classification*. Primary 28A75; Secondary 42B20, 42B37, 35R35, 35J70.

In [5], the first and third author (together with Joseph Feneuil) introduced a regularized distance function $D_{\mu,\alpha}$ (see (1.3) and (1.5) below) as a tool in a long program to characterize uniformly rectifiable sets of codimension greater than 1 by the behavior of certain (degenerate) elliptic operators in the complement of that set. In this article, we provide necessary and sufficient conditions for both the rectifiability and uniform rectifiability (see Definition 1.1) of a d-Ahlfors regular set E in terms of the oscillation of $|\nabla D_{\mu,\alpha}|$ in the complement of E. These results are new even in the classical context of codimension 1. However, most notably, for lower-dimensional sets, and for special values of involved parameters, they provide an unexpected version of the Riesz transform characterization. We will discuss the details after appropriate definitions.

We also discover a surprising situation in which the distance function itself is a solution to the degenerate elliptic operator introduced by [5]. This allows us to compute Green's function *explicitly* and compare the associated harmonic measure (see (1.11) below) to the Hausdorff measure *no matter how irregular E is*—a situation unheard of in codimension 1. In particular, as we mentioned above, recently, as a culmination of a long line of research starting with the work of F. and M. Riesz [21], the results of Azzam, Hofmann, Martell, Mourgoglou, and Tolsa [1] showed that the harmonic measure supported on a codimension 1 set is nice if and only if the set itself is nice. More precisely, assuming a "quantitative openness" condition on $\Omega \subset \mathbb{R}^n$ and condition (1.1) on $\partial\Omega$ (for $d = n-1$), the harmonic measure of Ω supported on $\partial\Omega$ is regular if and only if $\partial\Omega$ is uniformly rectifiable and a weak connectivity condition holds inside of Ω. Our result shows that the analogous characterization fails brutally in the codimension greater than 1 situation described above (see below for further discussion).

In order to more precisely state our results (and the analogous work in codimension 1), let us introduce some notation and notions. We are given an Ahlfors regular set $E \subset \mathbb{R}^n$, with $n \geq 2$, of any dimension $d < n$, and an Ahlfors regular measure μ on E. Recall that a measure μ is d-Ahlfors regular if

$$C^{-1} r^d \leq \mu\big(B(Q,r)\big) \leq C r^d \quad \forall Q \in E, \forall 0 < r < \operatorname{diam}(E). \tag{1.1}$$

A set E is d-Ahlfors regular if $\mathcal{H}^d|_E$ is a d-Ahlfors regular measure or, equivalently, if it supports some d-Ahlfors regular measure.

The most salient class of regularity for us is uniform rectifiability. Recall that a set $E \subset \mathbb{R}^n$ is d-rectifiable (with $d \in \mathbb{N}$ necessarily) if

$$\begin{gathered}\text{there exist countably many Lipschitz functions}\\ f_i : \mathbb{R}^d \to \mathbb{R}^n \qquad \text{such that} \qquad \mathcal{H}^d\Big(E \setminus \bigcup_i f_i(\mathbb{R}^d)\Big) = 0.\end{gathered} \tag{1.2}$$

Uniform rectifiability is a quantitative version of the following (cf. [8]).

Definition 1.1
Let d be an integer, and let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set. We say that E is d*-uniformly rectifiable* (with constants $\theta > 0$, $L > 0$) if it has big pieces of Lipschitz images. That is, for any $x \in E$, $\mathrm{diam}(E) > r > 0$ there exists an $f : \mathbb{R}^d \to \mathbb{R}^n$ which is L-Lipschitz and such that

$$\mathcal{H}^d\big(E \cap B(x,r) \cap f\big(\mathbb{R}^d \cap B(0,r)\big)\big) \geq \theta r^d.$$

The definition above is just one of several (equivalent) definitions of uniform rectifiability which we chose because we like its geometric flavor. In our work here, we will introduce as needed (and use) other characterizations of uniform rectifiability (e.g., using β or α numbers). One of our goals is to provide another characterization of uniform rectifiability using the regularized distance to E. Note that the sets considered in this paper satisfying (1.1) or Definition 1.1 will always be unbounded. This choice simplifies many of our theorems and proofs and comports with prior work in [6] and [5].

1.1. Regularized distances and (uniform) rectifiability
As above, let E be a d-Ahlfors regular set, and let μ be a d-Ahlfors regular measure whose support is E. In some cases, μ will be the restriction to E of the d-dimensional Hausdorff measure $\mathcal{H}^d$, but not always.

For each $\alpha > 0$ (often fixed in the argument), we define first a function $R \equiv R_{\mu,\alpha}$ on $\Omega = \mathbb{R}^n \setminus E$ by

$$R(x) = R_{\mu,\alpha}(x) = \int_{y\in E} |x-y|^{-d-\alpha}\, d\mu(y), \tag{1.3}$$

where the convergence comes from the Ahlfors regularity of μ. In fact, a simple estimate with dyadic annuli shows that

$$C^{-1}\delta(x)^{-\alpha} \leq R(x) \leq C\delta(x)^{-\alpha} \quad \text{for } x \in \mathbb{R}^n \backslash E, \tag{1.4}$$

where we set $\delta(x) = \mathrm{dist}(x,E)$ and C depends on μ and α. After this, we define $D = D_{\mu,\alpha}$ (suppressing the dependence on μ, α when it is clear from context or unimportant) by

$$D(x) \equiv D_{\mu,\alpha}(x) = R_{\mu,\alpha}^{-1/\alpha}(x) \quad \text{for } x \in \mathbb{R}^n \backslash E. \tag{1.5}$$

These distances were first introduced by [5] to study degenerate elliptic PDE, but in the first part of our paper we are more concerned with the analytic properties of ∇D, which we think of as analogous to the Riesz transform (though it is regularized by the presence of α in the kernel). Indeed,

$$\begin{aligned}\nabla D_{\mu,\alpha}(x) &= -\frac{1}{\alpha}\Big(\int_{y\in E}|x-y|^{-d-\alpha}\,d\mu(y)\Big)^{-\frac{1}{\alpha}-1}\int_{y\in E}\nabla_x\big(|x-y|^{-d-\alpha}\big)\,d\mu(y)\\ &= \frac{d+\alpha}{\alpha}\Big(\int_{y\in E}|x-y|^{-d-\alpha}\,d\mu(y)\Big)^{-\frac{1}{\alpha}-1}\\ &\quad\times\int_{y\in E}\frac{x-y}{|x-y|^{d+\alpha+2}}\,d\mu(y), \end{aligned}\tag{1.6}$$

for every $x\in\Omega$. Setting formally $\alpha=-1$ above and properly reinterpreting the integrals would transform the latter term into the classical Riesz transform. However, our α is always a positive number, so that the resultant expression, while analogous, is actually a quite surprising extension of the concept of the Riesz transform. One of the main discoveries of this article is that $\nabla D_{\mu,\alpha}$ carries rich geometric information, similar to the original Riesz transform, for sets of *arbitrary* dimension (not necessarily $n-1$).

To measure the oscillation of $\nabla D_{\mu,\alpha}$ in a scale-invariant way we define

$$F(x)\equiv F_{\mu,\alpha}(x)=\delta(x)\big|\nabla\big(|\nabla D|^2\big)(x)\big|=\delta(x)\Big(\sum_{k=1}^{n}\Big|\frac{\partial}{\partial x_k}\big(|\nabla D|^2\big)(x)\Big|^2\Big)^{1/2}\tag{1.7}$$

for $x\in\Omega\equiv\mathbb{R}^n\backslash E$. This is a dimensionless quantity, or rather, in crude terms, it is easy to see that D is Lipschitz and $|\nabla(|\nabla D|^2)(x)|$ is bounded by $\delta(x)^{-1}$. We say that D satisfies the *usual square function estimates* (USFE for short) when

$$F(x)^2\delta(x)^{-n+d}\,dx\quad\text{is a Carleson measure on }\Omega.\tag{1.8}$$

Let us recall the definition of a Carleson measure (which is intimately linked to uniform rectifiability and will be used several times), first on $E\times\mathbb{R}_+$ (the standard case) and then on Ω (as needed above).

Definition 1.2
Let $E\subset\mathbb{R}^n$ be d-Ahlfors regular. We say that $\nu(x,r)$ is a Carleson measure on $E\times\mathbb{R}_+$ if there exists a $C>0$ such that, for every $X\in E$ and $R>0$, we have

$$\nu\big(B(X,R)\cap E\times[0,R]\big)\le CR^d.\tag{1.9}$$

Similarly, $\mathscr{G}$ is a Carleson subset of $E\times(0,+\infty)$ if the measure $\mathbf{1}_{\mathscr{G}}(x,r)\times\frac{d\mathcal{H}^d(x)\,dr}{r}$ is a Carleson measure on $E\times\mathbb{R}_+$.

Definition 1.3
Let $E\subset\mathbb{R}^n$ be d-Ahlfors regular. We say that $\lambda(x)$ is a Carleson measure on $\Omega\equiv\mathbb{R}^n\backslash E$ if there exists a $C\ge 0$ such that

$$\lambda\big(\Omega \cap B(X,R)\big) \leq CR^d$$

for all $X \in E$ and $R > 0$.

Similarly, $Z \subset \Omega$ is a Carleson set if $\mathbf{1}_Z(x)\delta(x)^{-n+d}\,dx$ is a Carleson measure on Ω.

Notice that, for the moment, we use the definition of F that is the simplest for us to use; taking $F(x) = |\nabla(|\nabla D|)(x)|$ instead will give the same results (see, e.g., Corollary 2.2).

The results of Sections 2 and 4 characterize uniform rectifiability through the USFE. In particular, we show the following.

THEOREM 1.4
Let $n \geq 2$ and $0 < d < n$ (a priori d is not necessarily an integer). A d-Ahlfors regular set in $\mathbb{R}^n$ equipped with a d-Ahlfors regular measure μ is uniformly rectifiable if and only if $F_{\mu,\alpha}$ satisfies the USFE for some $\alpha > 0$.

In fact, a slightly weaker condition than the USFE, which we call “weak USFE,” implies that d is an integer and E is uniformly rectifiable. We note that this characterization is new even in codimension 1 and in some instances may be easier to check than previous conditions involving square functions (due to David and Semmes [8]) and singular integrals (see. e.g., [9], [20], [22]). Indeed, while it is often hard to check the L^2 boundedness of singular operators, given a set E, $F_{\mu,\alpha}$ can be computed fairly explicitly. We also remark the parallel with the “classical” USFE, involving a Carleson measure condition similar to the above for the second derivative of the Newtonian kernel, that is, the gradient of the kernel of the classical Riesz transform (see [9]). Just as ∇D in (1.6) resembles the Riesz transform only formally, our expressions here with $\alpha > 0$ are, of course, different, both intuitively (we really think of them as derivatives of a regularized distance) and factually (these are not classical singular integrals). Most importantly, the results here apply to the lower-dimensional setting while the (obvious extension of) the classical USFE is known to fail for sets of dimension lower than $n-1$ (see [9, p. 267]), and no lower-dimensional analogue of this characterization has been known thus far.

We would like to highlight a crucial component of the proof that the USFE implies uniformly rectifiability: Corollary 3.2. There we show that if $|\nabla D_{\mu,\alpha}|$ is constant, then E must be a d-affine space and μ a constant multiple of $\mathcal{H}^d|_E$. This corollary follows from Theorem 3.1, which is possibly of wider interest, and states that if the distance to a set E is a C^1 function, then the set E must be convex. As we were writing, we learned that this line of inquiry is related to results in convex analysis (see Section 3 for details).

We are also interested in the existence of nontangential limits for $|\nabla D|$. That is, the limit of $|\nabla D(x)|$ as x approaches a point $Q \in E$ without getting "too close" to E (see (5.3) for the definition of a *nontangential region* and Definition 5.2 for the definition of *nontangential convergence*). Nontangential limits are an important concept in harmonic analysis (e.g., the classical Fatou's theorem). In Section 5, we prove the following.

THEOREM 1.5
Let $n \geq 2$, and let $0 < d < n$ (with d not necessarily an integer). A d-Ahlfors regular set in $\mathbb{R}^n$ equipped with a d-Ahlfors regular measure μ is rectifiable if and only if the nontangential limit of $|\nabla D_{\mu,\alpha}|$ exists at μ-almost every point in E (for cones of every aperture).

Continuing to think of $\nabla D_{\mu,\alpha}$ as a slightly smoother version of Riesz transform, these results are in the vein of Tolsa's in [22], who shows that the existence of principle values of the Riesz transform is equivalent to rectifiability. We remark that the L^2 boundedness of the operator

$$\frac{d+\alpha}{\alpha}\Big(\int_{y\in E}|x-y|^{-d-\alpha}\,d\mu(y)\Big)^{-\frac{1}{\alpha}-1}\int_{y\in E}\frac{x-y}{|x-y|^{d+\alpha+2}}f(y)\,d\mu(y),$$

naturally associated to $\nabla D_{\mu,\alpha}$, is valid on all Ahlfors regular sets, using a simple domination by the Hardy–Littlewood maximal function. Thus, the celebrated result of [20] that says that, for an Ahlfors regular set E of codimension 1, E is uniformly rectifiable if and only if the Riesz kernel defines a bounded operator on $L^2(E, \mathcal{H}^{n-1})$ trivially fails in our case. Yet, much as in [22], the existence of the limits characterizes rectifiability, albeit these are different, nontangential limits.

Finally, let us remark that in both Theorems 1.4 and 1.5 we assume (implicitly) that d is an integer in one direction ("(uniform) rectifiability implies control on the oscillation of $|\nabla D|$"), whereas the fact that d is an integer is a corollary of the geometric conclusion in the other direction ("control on the oscillation of $|\nabla D|$ implies (uniform) rectifiability").

1.2. Harmonic measure in codimension 1 and greater
Associated to these distances is the degenerate elliptic PDE:

$$L_{\mu,\alpha}u \equiv -\operatorname{div}\Big(\frac{1}{D_{\mu,\alpha}^{n-d-1}}\nabla u\Big) = 0. \tag{1.10}$$

In [6], elliptic estimates and some potential theory were established for solutions of $L_{\mu,\alpha}$ in the complement of E. Most saliently for our purposes, it was shown that a maximal principle holds and that the Dirichlet problem could be solved for continuous data. Thus, for $X \in \Omega \equiv \mathbb{R}^n \setminus E$, we can define a harmonic measure $\omega^X \equiv \omega^X_{\mu,\alpha}$ as the measure given by the Riesz representation theorem with the property that if u_f is the unique solution to

$$\begin{aligned} L_{\mu,\alpha} u_f &= 0, \quad \text{in } \Omega, \\ u_f &= f, \quad \text{in } E, \end{aligned}$$

then

$$u_f(X) = \int_E f(Q)\, d\omega^X_{\mu,\alpha}(Q). \tag{1.11}$$

The distances $D_{\mu,\alpha}$ were introduced by [5] as a smooth replacement for $\delta(x)$; this smoothness was essential to the proof of the codimension greater than 1 analogue of Dahlberg's theorem in [5]. Recall that, for the Laplacian in codimension 1, Dahlberg proved that for Lipschitz domains the harmonic measure is quantitatively absolutely continuous (precisely, an A_∞-weight) with respect to surface measure (see [3]). In [5], the authors proved that in codimension greater than 1, $\omega_{\mu,\alpha} \in A_\infty(d\mathcal{H}^d|_E)$ when E is a graph with small Lipschitz constant. In a recent preprint, [7] (see also [12]), the first and third authors extended this result to all uniformly rectifiable sets E.

As mentioned above, the analogue of this program in codimension 1 has been a question of central interest for years, in particular because the behavior of harmonic measure supported on a set E has important consequences for the solutions of the Dirichlet problem in the complement of that set (see, e.g., [11]; for recent results in higher codimension, see [19]). Recently, Azzam, Hofmann, Martell, Mourgoglou, and Tolsa [1] have completed this program in codimension 1 for the Laplacian. To be precise, they start with an open set $\Omega \subset \mathbb{R}^n$ which satisfies the interior corkscrew condition (roughly this is used to rule out cusps in $\partial\Omega$ pointing into $\overline{\Omega}^c$). They further assume that $\partial\Omega$ is $(n-1)$-Ahlfors regular. In one direction, they show that an additional connectivity near the boundary assumption (known as the *weak local John condition*) on Ω combined with the $(n-1)$-uniform rectifiability of $\partial\Omega$ implies that the harmonic measure of Ω is quantitatively absolutely continuous with respect to $\mathcal{H}^{n-1}|_{\partial\Omega}$ (they use a condition known as *weak-*A_∞, which takes into account the pole of the harmonic measure and is natural due to the potentially nasty topology of Ω). The aforementioned work of [7] should be seen as a generalization of this result to higher codimension.

More impressively, uniform rectifiability and the weak local John condition are necessary and sufficient. Indeed, under the same assumptions of interior corkscrews and $(n-1)$-Ahlfors regularity of the boundary as above, [1] shows that if the harmonic measure in Ω is in weak A_∞, then it must be that $\partial\Omega$ is uniformly rectifiable and Ω

satisfies the weak local John condition. (That weak A_∞ implies uniform rectifiability was actually already known; cf. [13]. The contribution of [1] is to show that weak A_∞ implies the weak local John condition on Ω.) For a more precise description of these results and discussion on the interplay between these assumptions, we suggest the Introduction of [1].

Our initial goal was to connect the USFE and the existence of nontangential limits with the behavior of the harmonic measure $\omega_{\mu,\alpha}$ with the hopes of proving a higher codimension version of [13] and therefore characterizing uniform rectifiability by the behavior of $\omega_{\mu,\alpha}$ (note that the topological conditions above, namely, the interior corkscrew condition and the weak local John condition, are satisfied by $\Omega \equiv \mathbb{R}^n \setminus E$ whenever E is d-Ahlfors regular with $d < n-1$).

However, as mentioned above, such a result turns out to be completely false in some cases. There is a specific value of α, described in Section 6, where $D_{\mu,\alpha}$ itself is a solution of $L_{\mu,\alpha}u = 0$. In this scenario, we can explicitly compute $\omega_{\mu,\alpha}$ by showing that $D_{\mu,\alpha}$ is, in fact, Green's function with pole at infinity (see Definition 6.2 and Corollary 6.8).

More precisely, we show that $\omega_{\mu,\alpha}$ is proportional to $\sigma = \mathcal{H}^d|_E$ for *any Ahlfors regular set* E, including purely unrectifiable ones (see Theorem 6.7). Thus, in the case of "magic α," the converse to [7] (hence the higher codimension generalization of [13]) fails in the most spectacular way possible. We also show that, for any rectifiable set E, the harmonic measure for magic α is a constant multiple of σ (see Corollary 6.10). This is surprising for two reasons. First, for the Laplacian in codimension 1, under mild topological assumptions, the only set for which $\omega = \sigma$ is the half-space (see [17]; here ω is the usual harmonic measure for the Laplacian). Second, there are very few situations in codimension 1 in which the Poisson kernel $\frac{d\omega}{d\sigma}$ or Green's function can be precisely computed. Essentially only in the presence of lots of symmetry (e.g., the ball) or where there is an explicit conformal transformation from the ball (e.g., polygonal domains in $\mathbb{R}^2$) are the Poisson and Green's kernels known. Here, for magic α, we are able to compute Green's function with pole at infinity *for any Ahlfors regular set* and the Poisson kernel $\frac{d\omega_{\mu,\alpha}}{d\sigma}$ *for any rectifiable set* E.

We now expect that the situation for magic α is really exceptional, and hope to make precise how this is so in future investigations.

2. The square function estimate for uniformly rectifiable sets

In this section, we prove the direct results concerning the USFE.

THEOREM 2.1
Let $n \geq 2$, let $E \subset \mathbb{R}^n$ be a uniformly rectifiable set of dimension $d < n$ (so $d \in \mathbb{N}$), and let μ be a d-Ahlfors regular measure whose support is E. Then for each $\beta > 0$,

$D_{\mu,\beta}$ satisfies the USFE. That is, if $D_{\mu,\beta}$ is as in (1.3) and (1.5), and F is defined as in (1.7), then (1.8) holds.

See Corollary 2.2 concerning another, roughly equivalent, function $\widetilde{F}$. Notice that we do not assume that $d < n-1$ here, but we do not talk about the possible relations with the operator L either.

The main outline of the proof is as follows. It is clear that $F_{\nu,\beta} = 0$ when ν is a multiple of Hausdorff measure restricted to a plane (we call these measures *flat*; see (2.1) below). The key estimate (2.4) makes this quantitative: the size of $F_{\mu,\beta}$ can be estimated by the distance between μ and a well-chosen flat measure (this distance is measured by the α numbers; see (2.3)). Tolsa's characterization of uniform rectifiability using α numbers finishes the proof.

Proof

Fix $\beta > 0$ and μ as in the theorem statement. From now on, we will abuse notation and refer to $D_{\mu,\beta}$ as D (similarly for R and F). Occasionally, we will have to work with $D_{\nu,\beta}$ for some other measure ν. Here again we will suppress the dependence on β and just refer to D_ν.

Before we can introduce the key estimate alluded to above, we must introduce the Wasserstein distances and α-numbers. Let us denote by $\mathcal{F} = \mathcal{F}_d$ the set of flat measures;

$$\text{a flat measure is a positive multiple of the Lebesgue measure on an affine } d\text{-plane.} \tag{2.1}$$

We are interested in Wasserstein distances, which we define as follows. Given two positive measures μ and ν and a ball $B(x,r)$, we define $\mathcal{D}_{x,r}(\mu,\nu)$ by

$$\mathcal{D}_{x,r}(\mu,\nu) = r^{-d-1} \sup_{f\in\Lambda(x,r)} \left| \int_{B(x,r)} f(d\mu - d\nu) \right|, \tag{2.2}$$

where we denote by $\Lambda(x,r)$ the set of functions f that are 1-Lipschitz on $\mathbb{R}^n$ and vanish on $\mathbb{R}^n \setminus B(x,r)$. Notice that the normalization is such that $\mathcal{D}_{x,r}(\mu,\nu) \le C$ when μ and ν are Ahlfors regular, with a constant C that does not depend on x or r. Let us note that we will not require μ, ν to be probability measures, so it is misleading to say that $\mathcal{D}_{x,r}$ is a "distance."

We are especially interested in the numbers

$$\alpha(x,r) = \inf_{\nu\in\mathcal{F}} \mathcal{D}_{x,r}(\mu,\nu), \tag{2.3}$$

where $x \in \mathbb{R}^n$ and $r > 0$ are such that $B(x,r)$ meets E. These "α-numbers" measure the local Wasserstein distances from μ to flat measures. In the context of quantitative rectifiability, these numbers were introduced and widely used by Tolsa (see, e.g., [23]) to create a theory for measures that is analogous to Jones's β-numbers. We will use the fact that (see [23]), for any uniformly rectifiable set, E has Carleson measure estimates on the $\alpha(x,r)^2$ (see (2.5)).

We can now introduce the key estimate. Fix $x \in \Omega$, set $r_0 = \delta(x)$, and for $k \geq 0$, let $r_k \equiv 2^k r_0$. We will prove that, for $1 \leq i \leq n$,

$$\begin{aligned} \left|\partial_i\left(\left|\nabla D(x)\right|^2\right)\right| &= \left|\partial_i\left(\left|\nabla D(x)\right|^2\right) - \partial_i\left(\left|\nabla D_\nu(x)\right|^2\right)\right| \\ &\leq C\delta(x)^{-1}\sum_{l\geq 0} 2^{-(\beta+1)l}\alpha(y, 2^{l+6}r_0), \end{aligned} \tag{2.4}$$

for $y \in E \cap B(x, 16\delta(x))$, and where ν is a correctly chosen flat measure. The first equality comes from the easy fact (proved below) that, for any flat measure ν, $|\nabla D_\nu(x)|^2$ is constant.

Before we prove (2.4), let us see how the estimate implies the final result. Theorem 1.2 in [23] says (among other things) that when E is a d-dimensional uniformly rectifiable set and μ is an Ahlfors regular measure whose support is E,

$$\alpha(x,r)^2 \frac{d\mu(x)\, dr}{r} \quad \text{is a Carleson measure on } E \times \mathbb{R}_+, \tag{2.5}$$

which means that, for $X \in E$ and $R > 0$,

$$\int_{x\in E\cap B(X,R)} \int_{r=0}^{R} \alpha(x,r)^2 \frac{d\mu(x)\, dr}{r} \leq CR^d. \tag{2.6}$$

To be precise, in [23], the estimate (2.6) is not written in terms of the numbers $\alpha(x,r)^2$, but numbers $\alpha(Q)$ indexed by dyadic pseudocubes $Q \subset E$; however, the $\alpha(Q)$ and the $\alpha(x,r)$ mutually dominate each other for comparable values of $l(Q)$ and r, and it is a standard argument based on Fubini's theorem to go from the condition of [23] to (2.6). We skip the computation because it is both easy and done in [5, Lemma 5.9].

Notice that the function $\alpha(x,r)$ depends both on E and μ, and (2.6) contains information both on the geometry of E (the fact that it is close to a d-plane in most balls) and on the distribution of μ inside E. In fact, Tolsa's result is already significant when $E = \mathbb{R}^d$ and $d\mu = f\, d\lambda$ for some function f such that $C^{-1} \leq f \leq C$.

Now we claim that the Carleson measure estimate (1.8) follows from (2.6) and the key inequality (2.4); let us sketch this. First, we use the estimate (2.4) to estimate

$$\begin{aligned} &\int_{B(Q,R)} F^2(X)\delta(X)^{-n+d}\, dX \\ &\quad \leq C\int_{B(Q,R)} \fint_{B(X,16\delta(X))\cap E} a\big(Y, 2^6\delta(X)\big)^2\, d\mu(y)\delta^{-n+d}\, dX, \end{aligned}$$

where the a function represents the sum on the right-hand side of (2.4) (this notation is from Lemma 5.89 of [5]). Using a Whitney decomposition of $B(Q, R)$ and changing the order of integration, we can dominate the above integral by a sum over dyadic subcubes of $E \cap B(Q, 16R)$. Arguing as above or as in Lemma 5.9 in [5], we estimate

$$\int_{B(Q,R)} F^2(X)\delta(X)^{-n+d}\, dX \leq \int_0^{16R} \int_{B(Q,s)} a(x, 2^6 s)^2 \frac{d\mu(x)\, ds}{s}.$$

Finally, this last integral can be dominated by the one in (2.6) following the computation in Lemma 5.89 of [5].

Proving (2.4): To prove (2.4), recall that $F = F_{\mu,\beta}$ from (1.7) and $R = R_{\mu,\beta}$ from (1.3); we will need to compute their derivatives. Set $h(z) = |z|^{-d-\beta}$; this is a smooth function on $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$, and we denote by $\nabla^j h$ its iterated gradient (i.e., the collection of all its derivatives of order j). Notice that R is smooth on Ω, and

$$\nabla^j R(x) = \int_E \nabla^j h(x-y)\, d\mu(y). \tag{2.7}$$

Next, D is smooth on Ω, and

$$\nabla D(x) = -\frac{1}{\beta} R(x)^{-\frac{1}{\beta}-1} \nabla R(x) \tag{2.8}$$

and

$$\left|\nabla D(x)\right|^2 = \frac{1}{\beta^2} R(x)^{-\frac{2}{\beta}-2} \left|\nabla R(x)\right|^2 = \frac{1}{\beta^2} R(x)^{-\frac{2}{\beta}-2} \sum_j \left|\partial_j R(x)\right|^2 \tag{2.9}$$

and then

$$\begin{aligned}\partial_i\left(\left|\nabla D(x)\right|^2\right) = &-\frac{1}{\beta^2}\frac{2+2\beta}{\beta} R(x)^{-\frac{2}{\beta}-3} \partial_i R(x) \sum_j \left|\partial_j R(x)\right|^2 \\ &+ \frac{2}{\beta^2} R(x)^{-\frac{2}{\beta}-2} \sum_j \partial_j R(x) \partial_i \partial_j R(x).\end{aligned} \tag{2.10}$$

The precise structure of (2.10) is not so important; it suffices that we can compute (and bound) the errors we obtain from modifying the measure μ more or less explicitly.

The computations above are simpler when $F = F_\nu$ for a flat ν; let $\nu = \lambda \mathcal{H}^d_{|P}$ for some $\lambda > 0$. In this case,

$$R_\nu(x) = c_1 \lambda \delta_P^{-\beta}(x), \tag{2.11}$$

where $c_1 > 0$ is a constant that depends on d and α, and $\delta_P(x) \equiv \mathrm{dist}(x, P)$. In this case, (1.5) yields

$$D_\nu(x) = c_2 \lambda^{-1/\beta} \delta_P(x), \tag{2.12}$$

with $c_2 = c_1^{-1/\beta}$. It follows that $|\nabla D_\nu|^2 = c_2^2 \lambda^{-2/\beta}$, and $\partial_i(|\nabla D_\nu(x)|^2) = 0$ for $1 \le i \le n$.

Our next goal is to estimate the differences $|\nabla^j R(x) - \nabla^j R_\nu(x)|$, where $j \ge 0$ and ν is a well-chosen flat measure. In turn, these will allow us to estimate the difference $\partial_i(|\nabla D(x)|^2) - \partial_i(|\nabla D_\nu(x)|^2)$. Given the complexity of (2.10), we expect lots of terms, but they will all involve differences of the form $|\nabla^j R(x) - \nabla^j R_\nu(x)|$. We will start with the simplest case, $j = 0$.

Recall that $x \in \Omega$, $r_0 = \delta(x)$, and, for $k \ge 0$, $r_k = 2^k r_0$. Let φ be a (fixed) smooth bump function such that $0 \le \varphi \le 1$ on $\mathbb{R}^n$, φ is radial, $\varphi = 1$ on $B(0, 8r_0)$, $\varphi = 0$ on $\mathbb{R}^n \setminus B(0, 16r_0)$; then let $\varphi_0 = \varphi$, and, for $k \ge 1$, let $\varphi_k(x) = \varphi(2^{-k}x) - \varphi(2^{-k+1}x)$ (so that φ_k is supported on $A_k \equiv \overline{B}(0, 2^{k+4}r_0) \setminus B(0, 2^{k+2}r_0)$, where $A_0 = \overline{B}(0, 16r_0)$). Note that $\sum_{k \ge 0} \varphi_k = 1$.

Next, we will choose a flat measure ν_k which is nearly optimal for the definition of $\alpha(x, 32r_k)$. That is, we will let $\nu_k = \lambda_k \mathcal{H}^d_{|P_k}$ for some $\lambda_k > 0$ and some affine d-plane P_k, so that, in particular,

$$\mathcal{D}_{x,32r_k}(\mu, \nu_k) \le C\alpha(x, 32r_k), \tag{2.13}$$

where C depends on n, d, and the Ahlfors regularity constant for μ. As we shall see, at points and scales where μ is not well approximated by flat measures, rather than choosing the measure ν_k which minimizes the right-hand side of (2.13), we will prefer to make sure that we keep some control on λ_k and the support of ν_k.

To pick the ν_k, let $c > 0$ be small, to be chosen later. If $\alpha(x, 32r_k) \le c$, then we just pick ν_k so that $\mathcal{D}_{x,32r_k}(\mu, \nu_k) = \alpha(x, 32r_k)$. To see that such a minimizer exists (and is nice), recall that $\mu(B(x, 2r_k)) \ge C^{-1} r_k^d$, by Ahlfors regularity; if c is small enough, depending only on n, d, and the Ahlfors regularity constant for μ, then any flat measure η such that $\mathcal{D}_{x,32r_k}(\mu, \eta) \le 2\alpha(x, 32r_k)$ must be such that $\eta(B(x, 2r_k)) \ge (2C)^{-1} r_k^d$ (test (2.2) on a bump function centered at x). Hence (writing $\eta = \lambda \mathcal{H}^d_{|P}$ as above),

$$P \cap B(x, 2r_k) \ne \emptyset \qquad \text{and} \qquad C^{-1} \le \lambda \le C. \tag{2.14}$$

It is now easy to find a minimizing $\nu_k = \lambda_k \mathcal{H}^d_{|P_k}$, where (2.14) holds for P_k, λ_k. In the remaining case where $\alpha(x, 32r_k) \ge c$, we do not complicate our lives, and pick $\nu_k = \mathcal{H}^d_{|P_k}$, where P_k is any d-plane through $B(x, 2r_k)$. Then (2.13) and (2.14) hold trivially.

Set $\nu = \nu_0$ and $P = P_0$. By the translation invariance of our problem, we may assume that the origin lies in $P \cap B(x, 2r_0)$ (this will simplify our notation, because this way we do not need to translate our bump functions φ_k).

Recall that $R_\nu(x) = c_1\lambda_0\delta_P^{-\beta}(x)$ by (2.11). Recall, furthermore, the notation in (2.7). We want to estimate the difference

$$\big|R(x) - R_\nu(x)\big| = \Big|\sum_{k\geq 0}\int_{A_k}\varphi_k(y)h(x-y)(d\mu - d\nu)(y)\Big|; \tag{2.15}$$

by (1.3), $\sum_k \varphi_k = 1$ and $\operatorname{supp}\varphi_k \subset A_k$.

Notice that $\varphi_k(y)h(x-y)$ is Lipschitz in y, with a constant at most $Cr_k^{-d-\beta-1}$, and it vanishes outside of $B(0,2^{k+4}r_0) \subset B(x,2^{k+5}r_0)$; thus, by (2.2) (applied with $C^{-1}r_k^{d+\beta+1}\varphi_k(\cdot)h(x-\cdot) \in \Lambda(x,2^{k+5}r_0)$),

$$\Big|\int_{A_k}\varphi_k(y)h(x-y)(d\mu - d\nu)(y)\Big| \leq Cr_k^{-\beta}\mathcal{D}_{x,2^{k+5}r_0}(\mu,\nu). \tag{2.16}$$

For $k=0$, $\mathcal{D}_{x,2^{k+5}r_0}(\mu,\nu) = \mathcal{D}_{x,32r_0}(\mu,\nu) = \mathcal{D}_{x,32r_0}(\mu,\nu_0)$. For $k \geq 1$, we use intermediate measures. We start with

$$\begin{aligned}\mathcal{D}_{x,2^{k+5}r_0}(\mu,\nu) &\leq \mathcal{D}_{x,2^{k+5}r_0}(\mu,\nu_k) + \mathcal{D}_{x,2^{k+5}r_0}(\nu_k,\nu_0)\\ &\leq \mathcal{D}_{x,2^{k+5}r_0}(\mu,\nu_k) + \sum_{l=1}^{k}\mathcal{D}_{x,2^{k+5}r_0}(\nu_l,\nu_{l-1}),\end{aligned} \tag{2.17}$$

where the triangle inequality comes directly from the definition (2.2). We claim that

$$\mathcal{D}_{x,2^{k+5}r_0}(\nu_l,\nu_{l-1}) \leq C\alpha(x,2^{l+4}r_0), \tag{2.18}$$

because both measures ν_l and ν_{l-1} approximate μ well in $B(x,2^{l+4}r_0)$. The general idea is that since the two measures are flat measures associated to planes that pass near x (i.e., (2.14) holds), a good control on $B(x,2^{l+5}r_0)$ implies a good control on $B(x,2^{k+5}r_0)$. The proof is almost the same as for equation (5.83) in [5], so we leave it.

We return to (2.15): use (2.13), (2.16), (2.17), and (2.18) to obtain

$$\big|R(x) - R_\nu(x)\big| \leq C\sum_{k\geq 0}r_k^{-\beta}\sum_{0\leq l\leq k}\alpha(x,2^{l+5}r_0) \leq C\sum_{l\geq 0}r_l^{-\beta}\alpha(x,2^{l+5}r_0). \tag{2.19}$$

Notice that $\alpha(x,2^{l+5}r_0) \leq 2^{d+1}\alpha(y,2^{l+6}r_0)$ for every $y \in B(x,16r_0)$, just by (2.2), (2.3), and because $B(x,2^{l+5}r_0) \subset B(y,2^{l+6}r_0)$ and hence $\Lambda(x,2^{l+5}r_0) \subset \Lambda(y,2^{l+6}r_0)$ (recall, from (2.2), that $\Lambda(x,r)$ is the set of functions f that are 1-Lipschitz on $\mathbb{R}^n$ and vanish on $\mathbb{R}^n \setminus B(x,r)$). Thus,

$$\begin{aligned}\big|R(x) - R_\nu(x)\big| &\leq C\sum_{l\geq 0}r_l^{-\beta}\alpha(y,2^{l+6}r_0) = Cr_0^{-\beta}\sum_{l\geq 0}2^{-\beta l}\alpha(y,2^{l+6}r_0)\\ &= C\delta(x)^{-\beta}\sum_{l\geq 0}2^{-\beta l}\alpha(y,2^{l+6}r_0)\end{aligned} \tag{2.20}$$

for $y \in B(x,16r_0)$.

This was our estimate for R, but we have a similar estimate for the iterated derivatives of R. That is, we start from (2.7) instead of (1.3), and observe that we can compute as above, with an extra $|x-y|^{-j}$, which transforms into an extra $r_k^{-j} \leq C\delta(x)^{-j}$ in the estimates below. This yields

$$\begin{aligned} \left|\nabla^j R(x) - \nabla^j R_\nu(x)\right| &= \left|\int_E \nabla^j h(x-y)(d\mu - d\nu)(y)\right| \\ &\leq C\delta(x)^{-\beta-j} \sum_{l\geq 0} 2^{-(\beta+j)l} \alpha(y, 2^{l+6} r_0) \end{aligned} \tag{2.21}$$

for $y \in B(x, 16\delta(x))$. Observe also that a direct estimate with (2.7) yields

$$\left|\nabla^j R(x)\right| \leq C\delta(x)^{-\beta-j}. \tag{2.22}$$

Let us check that if we pick P_0, our initial plane, correctly, then we have a similar estimate for $R_{\nu,\beta}$, that is,

$$\left|\nabla^j R_\nu(x)\right| \leq C\delta(x)^{-\beta-j}. \tag{2.23}$$

Recall from the discussion below (2.13), we choose ν_0 such that $\mathcal{D}_{x,32r_0}(\mu,\nu_k) = \alpha(x, 32r_0)$ when $\alpha(x, 32r_k r) \leq c$. In this regime, we claim that, perhaps by choosing c a little bit smaller, the following inequality holds:

$$\operatorname{dist}(y, E) \leq 10^{-1} r_0 \quad \text{for } y \in P_0 \cap B(x, 16r_0). \tag{2.24}$$

Otherwise, pick $y \in P_0 \cap B(x, 16r_0)$, at distance at least $10^{-1} r_0$ from E, and choose a Lipschitz bump function f, supported in $B(y, 2 \cdot 10^{-2} r_0)$ so that $f = 10^{-2} r_0$ on $B(y, 10^{-2} r_0)$ and f is 1-Lipschitz. Then (2.2) yields $|\int f(d\mu - d\nu_0)| \leq c r_0^{d+1}$, while $\int f\, d\mu = 0$ (because E does not meet $B(y, 2\cdot 10^{-2} r_0)$) and $\int f\, d\nu_0 \geq 10^{-2} r_0 \nu(B(y, \tau r_0)) \geq C^{-1} r_0^{d+1}$ by (2.14) and because $y \in P_0$. If we take c small enough, then we get a contradiction that proves (2.24). We deduce from this that

$$|y - x| \geq \frac{r_0}{2} \quad \text{for } y \in P_0, \tag{2.25}$$

because either $y \in B(x, 16r_0)$ and we use the fact that $10^{-1} r_0 \overset{(2.24)}{\geq} \operatorname{dist}(y, E) \geq \operatorname{dist}(x, E) - |y - x_0| = r_0 - |y - x_0|$, or else $|y - x| \geq 16r_0$ anyway.

When $\alpha(x, 32r_0) \geq c$, we decided to pick any d-plane through $B(x, 2r_0)$, and we simply make sure that (2.25) holds when we do this. Once we have (2.25), (2.23) easily follows from (2.14) and the usual computations.

We may now return to our original formula (2.10). It says that $\partial_i(|\nabla D(x)|^2)$ is a sum of $2n$ terms, and we claim that because of (2.22), each of these terms is bounded from above by $C\delta(x)^{-1}$.

Indeed, if we did not have any derivatives, then we would simply obtain $CR(x)^{-\frac{2}{\beta}} = CD^2 \le C\delta(x)^2$ by (1.5) and (1.4). But we have three additional derivatives, which give an extra $\delta(x)^{-3}$. Altogether, the brutal estimate is $|\partial_i(|\nabla D(x)|^2)| \le C\delta(x)^{-1}$. By (2.23), we would have the same estimate when we replace R with $R_{\nu,\beta}$ in some places. Now we need to estimate $\partial_i(|\nabla D(x)|^2) - \partial_i(|\nabla D_{\nu,\beta}(x)|^2)$, which is a sum of terms like the above, except that now one of the terms of each product is replaced with the corresponding difference involving $|\nabla^j R(x) - \nabla^j R_{\nu,\beta}(x)|$. We use (2.21) for this difference (which allows us to multiply the estimate by a sum of α-numbers), keep the same estimates for the rest of each product, sum everything up, and obtain that

$$\begin{aligned}\big|\partial_i\big(\big|\nabla D(x)\big|^2\big)\big| &= \big|\partial_i\big(\big|\nabla D(x)\big|^2\big) - \partial_i\big(\big|\nabla D_{\nu,\beta}(x)\big|^2\big)\big| \\ &\le C\delta(x)^{-1}\sum_{l\ge 0} 2^{-(\beta+1)l}\alpha(y, 2^{l+6}r_0),\end{aligned}$$

for $y \in B(x, 16\delta(x))$. This is (2.4), the desired result. □

The attentive reader may ask why we raise $|\nabla D|$ to the second power in the definition of F (see (1.7)). Indeed, this is done mostly for aesthetic reasons (mainly so that (2.10) does not look so nasty). In the following corollary, we show that our result still holds if F is replaced by $\widetilde{F}$ (which is the same except that we do not square $|\nabla D|$).

COROLLARY 2.2
Theorem 2.1 is still valid when we replace $F(x)$ with

$$\widetilde{F}(x) = \delta(x)\big|\nabla\big(|\nabla D|\big)(x)\big|. \tag{2.26}$$

Proof
Noting that $|\nabla D| = (|\nabla D|^2)^{1/2}$, we see that $\widetilde{F}(x) = \frac{1}{2}F(x)|\nabla D|^{-1}$, at least for x such that $\nabla D(x) \ne 0$. Let $C_1 \ge 0$ be large, to be chosen later (depending on n, d, and the Ahlfors regularity constant for μ), and set $Z = \{x \in \Omega; |\nabla D| \le C_1^{-1}\}$. It is enough to control $\widetilde{F}(x)\mathbf{1}_Z(x)$, because we can use Theorem 2.1 for the rest of $\widetilde{F}$.

Even on Z, $\widetilde{F}$ is not as large as one may fear; for $1 \le i \le n$,

$$\begin{aligned}\big|\partial_i\big(|\nabla D|\big)(x)\big| &= \big|\partial_i\big(\sqrt{|\nabla D|^2}\big)(x)\big| = \frac{1}{2}\left|\frac{\partial_i(|\nabla D|^2)(x)}{|\nabla D(x)|}\right| \\ &= \left|\frac{\partial_i\nabla D(x)\cdot\nabla D(x)}{|\nabla D(x)|}\right| \le \big|\nabla^2 D(x)\big| \le C\delta(x)^{-1}\end{aligned} \tag{2.27}$$

by brutal computations, and at the end, (2.8) and (2.22). In particular, this implies that $\widetilde{F}$ is bounded uniformly on Z:

$$\big|\widetilde{F}(x)\big| = \delta(x)\big|\nabla\big|\nabla D(x)\big|\big| \le C, \quad \forall x \in Z. \tag{2.28}$$

In addition, we claim that Z itself is not large. Indeed, let $x \in Z$ be given; recall the notation used in the proof of Theorem 2.1, specifically that ν is a well-chosen flat measure so that $D_{\nu,\beta}(z) = c_2\lambda_0^{-1/\beta}\delta_P(z)$. Hence, by (2.14), $|\nabla D_{\nu,\beta}(x)| \geq C^{-1}$ and, by (2.8) (and (2.25)),

$$\left|\nabla R_{\nu,\beta}(x)\right| \geq C^{-1} R_{\nu,\beta}(x)^{\frac{1}{\beta}+1} \geq C^{-1}\delta(x)^{-1-\beta}. \tag{2.29}$$

On the other hand, $|\nabla D| \leq C_1^{-1}$ by definition of Z; hence, by (2.8) again,

$$\left|\nabla R(x)\right| \leq CC_1^{-1} R(x)^{\frac{1}{\beta}+1} \leq CC_1^{-1}\delta(x)^{-1-\beta}. \tag{2.30}$$

If we choose C_1 large enough, then we deduce from equations (2.29) and (2.30) that

$$\left|\nabla R_{\nu,\beta}(x) - \nabla R(x)\right| \geq c\delta(x)^{-1-\beta} \tag{2.31}$$

for some $c > 0$. Then by (2.21) (with $j = 1$),

$$\sum_{l\geq 0} 2^{-(\beta+j)l}\alpha(y, 2^{l+6}r_0) \geq C^{-1} \quad \text{for } y \in B\big(x, 16\delta(x)\big). \tag{2.32}$$

But we have seen earlier that the work of Tolsa [23] gives a Carleson estimate on the square of the sum (over dyadic cubes) of the left-hand side of (2.32) (see the discussion right before the beginning of the proof of (2.4)). This Carleson estimate implies by Chebyshev (and the same computations using Fubini that lead from (2.6) to (1.8); see the two paragraphs after (2.6)) that Z is a Carleson set. That is, there is a constant $C \geq 0$ such that, for $X \in E$ and $R > 0$,

$$\int_{B(X,R)\cap Z} \frac{d\mu(x)}{\delta(x)^{n-d}} \leq CR^d. \tag{2.33}$$

This immediately leads to a Carleson bound on $\widetilde{F}|_Z$,

$$\int_{B(X,R)\cap Z} \left|\widetilde{F}(x)\right|^2 \frac{d\mu(x)}{\delta(x)^{n-d}} \overset{(2.28)}{=} \int_{B(X,R)\cap Z} C^2 \frac{d\mu(x)}{\delta(x)^{n-d}} \leq CR^d. \tag{2.34}$$

This completes the proof of Corollary 2.2. $\square$

3. E is flat when $|\nabla D|$ is constant on Ω

To prove the converse to Theorem 2.1 (and later in Section 5 to study nontangential limits), we first prove the "limiting result": if F vanishes, that is, if $|\nabla D_{\mu,\beta}|$ is constant, then μ must be supported on a plane.

More precisely, we show in this section that in this case $D_{\mu,\beta}$ is a multiple of δ, E is a d-plane, and μ is a multiple of $\mathcal{H}^d|_E$.

We learned while writing the paper that a subset C of a Banach space X with the property that, for every $x \in X$, there is a unique closest point $c \in C$ to x is called a *Chebyshev set*. Chebyshev sets are well studied (see the survey [2]), and it is an old theorem, attributed to Bunt, that every Chebyshev set in Euclidean space is convex (see Theorem 13 in [2]). Invoking this result would allow us end the proof of Theorem 3.1 after (3.5). However, we include the whole argument for the sake of completeness. As an aside, it is apparently an interesting open question as to whether every Chebyshev set in a Hilbert space is convex.

THEOREM 3.1
Let E be a closed set in $\mathbb{R}^n$, and let D be a continuous nonnegative function on $\mathbb{R}^n$, which vanishes on E, is of class C^1 on $\Omega = \mathbb{R}^n \setminus E$, and such that $|\nabla D|$ is positive and constant on every connected component of Ω. Then E is convex. If, in addition, $|\nabla D| = 1$ on Ω, then $D(x) = \operatorname{dist}(x, E)$ for $x \in \mathbb{R}^n$.

Theorem 3.1 is stated as is so that we may apply it easily in the proof of Corollary 3.2. However, the discerning reader will notice that the theorem is really the combination of two separate facts: first, that a C^1 function vanishing on E with constant derivative on a connected component of $\mathbb{R}^n \setminus E$ is a constant multiple of $\operatorname{dist}(x, E)$ on that component, and second, the fact—mentioned in the Introduction—that if $\operatorname{dist}(x, E)$ is C^1 in $\mathbb{R}^n \setminus E$, then E is convex.

Note that the function D in Theorem 3.1 is not necessarily of the form $D_{\mu,\alpha}$ defined in (1.3). However, we will eventually apply the theorem to exactly those functions.

Proof
We start with the assumption that $|\nabla D| = 1$ on some connected component Ω_0 of Ω. Observe first that $D(x) > 0$ on Ω_0, because of our assumption that $|\nabla D| \neq 0$ (and that $D \geq 0$). We may of course assume that $\Omega_0 \neq \emptyset$. In our main case, when E is Ahlfors regular of dimension $d < n - 1$, $\Omega \equiv \mathbb{R}^n \setminus E$ is connected and dense in $\mathbb{R}^n$, so $\Omega_0 = \Omega$.

Set $v(x) = \nabla D(x)$ on Ω_0; this is a C^0 vector field that does not vanish, and we can use it to define a flow. That is, given $x \in \Omega_0$, we can define $\varphi(x, \cdot)$ to be the solution of the equation $\frac{\partial \varphi(x,t)}{dt} = -v(\varphi(x, t))$ such that $\varphi(x, 0) = x$, which is defined on a maximal (open) interval $I(x)$. By the chain rule (and $|\nabla D| \equiv 1$), $\partial_t D(\varphi(x, t)) = -1$ for $t \in I(x)$. Integrating this, we note that $D(\varphi(x, t)) = D(x) - t$ for $t \in I(x)$.

This solution can be extended as long as $\varphi(x, t)$ stays in Ω (equivalently, Ω_0), which means at least as long as $D(\varphi(x, t)) > 0$. So I contains $[0, D(x))$, and $\lim_{t \uparrow D(x)} D(\varphi(x, t)) = 0 \Rightarrow \varphi(x, D(x)) \in E$. To be precise, while the flow cannot be extended naturally to time $t = D(x)$, the limit $p(x) \equiv \lim_{t\uparrow D(x)} \varphi(x,t)$ exists and $p(x) \in E$.

Since δ and $\varphi(x,\cdot)$ are 1-Lipschitz, $t \mapsto \delta(\varphi(x,t))$ is 1-Lipschitz, and

$$\delta(x) = \delta\big(\varphi(x,0)\big) \leq D(x) + \lim_{t\uparrow D(x)} \delta\big(\varphi(x,t)\big) = D(x). \tag{3.1}$$

On the other hand, if $p_{\delta(x)}$ is a point of E that minimizes the distance to x, then the bound on the gradient of D implies that

$$D(x) = \big|D(p_{\delta(x)}) - D(x)\big| \leq |p_{\delta(x)} - x| = \delta(x),$$

where the first equality follows from the continuity of D at $p_{\delta(x)}$. That is, not only did we prove that

$$\delta(x) = D(x) \quad \text{for } x \in \Omega_0, \tag{3.2}$$

but we also learned that the flow follows straight lines. More precisely, setting

$$\Gamma_x = \big\{\varphi(x,t); 0 \leq t < D(x)\big\}, \tag{3.3}$$

we know that the length of Γ_x is $D(x) = \delta(x)$, and since $|p(x) - x| \geq \delta(x)$ by definition of $\delta(x)$, the fact that Γ_x goes from x to $p(x)$ and has a length $D(x)$ implies that it is the line segment $[x, p(x))$.

Let us pause to point out that we have already proved the second conclusion of the theorem: that if $|\nabla D(x)| \equiv 1$, then $D(x) = \operatorname{dist}(x, E)$. To prove the first part of the theorem, it will suffice to show that

$$p \text{ is 1-Lipschitz on } \overline{\Omega_0}. \tag{3.4}$$

Indeed, if we know (3.4), let us assume for the sake of contradiction that E is not convex. In particular, let points $a, b \in E$ be given, with $b \neq a$ such that there exists an $x \in (a,b)$ (the open line segment between a and b) that does not lie in E. Let Ω_0 denote the connected component of Ω that contains x, and define p on $\overline{\Omega_0}$, as above (if $|\nabla D| = c \neq 1$ on Ω_0, then we can always consider D/c without losing generality). Denote by I_0 the connected component of $(a,b) \cap \Omega$ that contains x; this is an interval $(a',b') \subset (a,b)$, which is contained in Ω_0 (because it is connected and contained in Ω), and $a', b' \in E$. By (3.4), the length of the arc $p(I_0)$ is at most $|b' - a'|$, and since p is the identity on E (this follows from $|p(x) - x| = \delta(x)$ and p continuous on $\overline{\Omega_0}$), we obtain that $p(I_0) = (a', b')$. In particular, $x \in p(I_0) \subset E$, which is a contradiction.

For the remainder of the proof we study p, aiming toward (3.4). We first check that $p(x)$ is the unique closest point in E to x.

Observe that $\nabla D(x) = -\frac{p(x)-x}{|p(x)-x|}$ for $x \in \Omega_0$. This is, for instance, because Γ_x has a tangent at x that points in the direction of $-v(x) = -\nabla D(x)$. But Γ_x also points in the direction of $p(x) - x$, since $\Gamma_x = [x, p(x))$. Let us deduce from this that

$$|y - x| > \delta(x) \quad \text{for } y \in E \setminus \{p(x)\}, \tag{3.5}$$

that is, that $p(x)$ is the only point of E that realizes the distance to x. Indeed, let $y \in E$ be such that $|y - x| = \delta(x)$, and observe that along $[y, x]$ the function $D(\xi)$ goes from 0 to $\delta(x)$; since $D(x)$ is 1-Lipschitz and the length of the segment is $\delta(x)$, integrating $\partial_t D((1-t)y + tx)$ from $t = 0$ to $t = 1$ implies that $\langle \nabla D(\xi), \frac{y-x}{|y-x|} \rangle = -1$, for all $\xi \in [x, y)$. Therefore, $\frac{y-x}{|y-x|}$ also points in the direction of $-v(x)$. We conclude that $y - x$ and $p(x) - x$ are two vectors which point in the same direction and have the same length, hence $y = p(x)$. The claim, (3.5), follows.

Let us extend p to $\overline{\Omega}_0$ by setting $p(x) = x$ when $x \in E$. We claim that p is continuous on $\overline{\Omega}_0$. Indeed, if $\{x_k\}$ in $\overline{\Omega}_0$ converges to x, then the sequence $\{p(x_k)\}$ is bounded (because $|p(x_k) - x_k| = \delta(x_k)$), and it is easy to see that any point of accumulation y of this sequence is such that $|y - x| = \lim_{k\to+\infty} |p(x_k) - x_k| = \delta(x)$, hence is equal to $p(x)$. That is, $\{p(x_k)\}$ converges to $p(x)$, as needed for the continuity of p.

To prove the higher regularity of p, we start by showing that if $L_+(p(x), x)$ is the closed half-line that starts from $p(x)$ and contains x, then

$$\begin{aligned} L_+\big(p(x), x\big) \subset \Omega_0 \cup \{p(x)\} \qquad &\text{and} \\ p(y) = p(x) \quad \text{for } y \in L_+\big(p(x), x\big). & \end{aligned} \tag{3.6}$$

To prove the first part of (3.6), first note that the half-open segment $[p(x), x)$ cannot contain a point in E (other than $p(x)$), otherwise that point would be closer to x than $p(x)$ is. Later in this argument we will show that the rest of $L_+(p(x), x)$ also cannot contain a point in E, which will complete the proof that $L_+(p(x), x) \subset \Omega_0 \cup \{p(x)\}$.

Note that if $y \in [p(x), x]$, then $p(y) = p(x)$ is immediate by the uniqueness of C^0 vector flows; that is, $y = \varphi(x, t)$ for some $t \in (0, D(x)]$, and therefore $\varphi(y, s) = \varphi(x, t + s)$ for all $s \in [0, D(x) - t] = [0, D(y)]$. But, as we have seen above, the point where the flow starting at y hits E is, by definition, $p(y)$. This implies that $p(y) = \varphi(y, D(y)) = \varphi(x, D(y) + t) = \varphi(x, D(x)) = p(x)$.

To prove (3.6) for $y \in L_+(p(x), x)\setminus[p(x), x]$ we must reverse the flow. For $x \in \Omega_0$, we define $\varphi_+(x, \cdot)$ to be reverse flow of φ; that is $\frac{\partial \varphi_+(x,t)}{dt} = v(\varphi_+(x, t))$, with the initial value $\varphi_+(x, 0) = x$. This function is defined on an interval $I_+ \subset [0, +\infty)$, and since we can check as we did for φ above that $D(\varphi_+(x, t)) = D(x) + t \geq D(x) > 0$ for $t \in I(x)$, and that we can extend the solution as long as $\varphi_+(x, t) \in \Omega_0$, it follows that $I(x) = [0, +\infty)$.

Let $x \in \Omega_0$ and $t_0 > 0$ be given, and set $y = \varphi_+(x, t_0)$ (note that $D(y) = D(x) + t_0 > 0$ so $y \notin E$). Notice that $\varphi_+(x, t_0 - t) = \varphi(y, t)$ for $0 \le t \le t_0$, because φ_+, φ come from reverse flows. This implies that $x \in \Gamma_y$ (recall the notation from (3.3)). But we know that Γ_y is a straight line from y to $p(y)$. From this, $p(y) = p(x)$ immediately follows; indeed, if $p(y) \in (p(x), x]$, then $\delta(x) \le |p(y) - x| < |p(x) - x|$, which is a contradiction. Similarly, if $p(x) \in (p(y), y]$, then the second part of (3.6) follows. Note that we have also shown that the whole ray $L_+(p(x), x) \setminus [p(x), x]$ is contained in the image of the flow of $t \mapsto \varphi_+(x, t)$. Since $D(\varphi_+(x, t)) \ge D(x) > 0$, this image is contained in Ω, which finishes the proof that $L_+(p(x), x) \subset \Omega_0 \cup \{p(x)\}$.

For $x \in \Omega_0$, denote by $P(x)$ the hyperplane through $p(x)$ which is orthogonal to $x - p(x)$. Then let $H(x)$ denote the half-space on the other side of $P(x)$. That is, set

$$H(x) = \big\{z \in \mathbb{R}^n; \big\langle z, x - p(x)\big\rangle \le \big\langle p(x), x - p(x)\big\rangle\big\}. \tag{3.7}$$

We claim that $E \subset H(x)$. To check this, we may assume that $p(x) = 0$ and $x = \lambda e_n$, where e_n is the last element of the canonical basis and $\lambda > 0$. By the discussion above, $p(te_n) = 0$ for every $t > 0$, and this means that $t = \operatorname{dist}(te_n, 0) \le \operatorname{dist}(te_n, z)$ for every $z \in E$. Write $z = ae_n + v$, with $v \perp e_n$; then $\operatorname{dist}(te_n, z)^2 = |(t-a)e_n - v|^2 = (t-a)^2 + |v|^2$ and we get that $t^2 \le (t-a)^2 + |v|^2$. We let t tend to $+\infty$ and get that $a \le 0$, hence $\langle z, x - p(x)\rangle = \langle z, x\rangle = a\lambda \le 0 = \langle p(x), x - p(x)\rangle$, which means that $z \in H(x)$, as needed.

We now turn to (3.4). Let $x, y \in \overline{\Omega_0}$ be given; we want to prove that

$$\big|p(x) - p(y)\big| \le |x - y|. \tag{3.8}$$

Without loss of generality, we may assume that $p(x) = 0$ and $x = \lambda e_n$ for some $\lambda \ge 0$. We have two inequalities that we can use, one from the fact that $p(y) \in H(x)$ (because $p(y) \in E$), which says that

$$\big\langle p(y), x\big\rangle \le 0, \tag{3.9}$$

and similarly another from the fact that $0 = p(x) \in H(y)$, that is, $0 = \langle p(x), y - p(y)\rangle \le \langle p(y), y - p(y)\rangle$, or equivalently

$$\big\langle p(y), y\big\rangle \ge \big|p(y)\big|^2. \tag{3.10}$$

Write $y = \mu e_n + y_0$ for some $y_0 \in e_n^\perp$, and first assume that $\mu \le 0$. Replacing x with 0 diminishes $|x - y|$ but does not change $|p(x) - p(y)|$; thus it is enough to prove (3.8) for $x = 0$. That is, we just need to show that $|p(y)| \le |y|$, which follows from (3.10) and Cauchy–Schwarz.

So we may assume that $\mu > 0$. Replacing x with μe_n diminishes $|x - y|$ but does not change $|p(x) - p(y)|$, so as before we may assume that $\lambda = \mu$. That is,

$y = \lambda e_n + y_0$. Now write $p(y) = ae_n + by_0 + z$, with $a, b \in \mathbb{R}$ and $z \in e_n^\perp \cap y_0^\perp$. Then $a \le 0$ by (3.9) and because $\lambda > 0$, (3.10) yields

$$\lambda a + b|y_0|^2 = \langle y, p(y)\rangle \ge \big|p(y)\big|^2 = a^2 + b^2|y_0|^2 + |z|^2. \tag{3.11}$$

Now

$$\big|p(x) - p(y)\big|^2 = \big|p(y)\big|^2 \le \lambda a + b|y_0|^2 \tag{3.12}$$

and for (3.8) we just need to know that $\lambda a + b|y_0|^2 \le |x - y|^2 = |y_0|^2$. Since $a \le 0$, we just need to check that $b \le 1$ or $y_0 = 0$. We return to (3.11), which says that

$$b(b-1)|y_0|^2 \le \lambda a - a^2 - |z|^2. \tag{3.13}$$

The right-hand side is nonpositive, so $y_0 = 0$ or else $b(b-1) \le 0$; this last case is impossible if $b > 1$, so finally (3.8) holds and p is 1-Lipschitz. □

Notice that there is a (less interesting) converse. If E is convex and $D(x) = \operatorname{dist}(x, E)$, then D is 1-Lipschitz, the point $p(x) \in E$ such that $|x - p(x)| = D(x)$ is unique, and it is not so hard to check that $\nabla D(x) = -\frac{p(x)-x}{|p(x)-x|}$ and so $|\nabla D(x)| \equiv 1$.

We now apply Theorem 3.1 to the situation where $D = D_{\mu,\alpha}$ is defined by (1.3).

COROLLARY 3.2

Let $0 < d < n$, and let μ be a d-dimensional Ahlfors regular measure supported on the closed set $E \subset \mathbb{R}^n$. Suppose that for some $\alpha > 0$, the function $D_{\mu,\alpha}$ defined by (1.3) and (1.5) is such that on $\Omega = \mathbb{R}^n \setminus E$, $|\nabla D_{\mu,\alpha}|$ is locally constant and positive. Then d is an integer, E is a d-plane, and the density of μ with respect to $\mathcal{H}^d_{|E}$ is constant. If $d < n-1$, then there is a constant $c > 0$ such that $D_{\mu,\alpha}(x) = c\operatorname{dist}(x, E)$ for $x \in \Omega$.

Proof

Let μ, E, and α satisfy the assumptions, and let $D = D_{\mu,\alpha}$. We observed earlier that D is smooth on Ω, and by (1.4) it is equivalent to δ on Ω, hence has a continuous extension to $\mathbb{R}^n$ such that $D(x) = 0$ on E. Then on each connected component of Ω there is a constant $c > 0$ such that $c^{-1}D$ satisfies the assumptions of Theorem 3.1; hence, E is convex and D is a constant multiple of δ on each of the connected components of Ω (at the end of this proof we show that the constant multiple must be the same on each component).

Next we check the geometric fact that if $d < n$ and E is a convex Ahlfors regular set of dimension d, then d is an integer and E is a subset of an affine d-space.

Denote by m the smallest integer greater than or equal to d. That is, $m = d$ if d is an integer, and $m = [d] + 1$ otherwise. First we check that $m = d$ (and d is an integer). Suppose that $0 \in E$. It is easy to find $m+1$ independent points in E, that is, points $x_0, \dots, x_m \in E$ that are not contained in any $(m-1)$-plane (a d-Ahlfors regular set cannot be a subset of an $(m-1)$-dimensional plane since $m-1<d$).

Since E is convex, it contains the convex hull of the $m+1$ points above, and in particular it contains an m-disk Δ. This forces $d \geq m$, and hence $d=m$. In addition, let P denote the affine d-plane that contains Δ; notice that $E \subset P$, because otherwise E contains a $(d+1)$-disk (by convexity again), and cannot be Ahlfors regular of dimension d.

We want to show now that E is all of P. We will show a slightly more general statement, that if $d<n$ and E is a convex, d-Ahlfors regular subset of a d-plane $P \subset \mathbb{R}^n$, and if $\delta(x) = \operatorname{dist}(x, E)$ is of class C^2 on all of $\mathbb{R}^n \setminus P$, then $E=P$. Since in the present situation $D = c\delta$ and $D \in C^\infty(\mathbb{R}^n \setminus E)$, we will conclude that $E=P$.

To see this, assume that $E \neq P$ and, without loss of generality, that zero is a boundary point of E, considered as a subset of P. That is, $0 \in E$ (because E is closed) and every ball around zero contains a point in $P \setminus E$. Let C be the set of points e such that $\lambda e \in E$ for some $\lambda > 0$. Since E is convex and contains zero, C is also the set of points e such that $\lambda e \in E$ for $\lambda > 0$ small, and then C is a convex cone. Next let e_1 lie in the interior of C; such a point exists because E contains an m-disk Δ (as above), and $-e_1 \notin \overline{C}$ because otherwise zero would be an interior point of E (note that if $-e_1 \in \partial C$, then we can perturb e_1 slightly to $e_2 \in C$ such that $-e_2 \in C$, which still gives a contradiction).

Let e_2 be a unit direction which is normal to P; we claim that δ is not C^2 in the direction e_1 at the point e_2. For small $\epsilon > 0$, we have $\epsilon e_1 \in E$; this is the definition of C. Thus $\delta(\epsilon e_1 + e_2) = 1$ for all small enough positive $\epsilon > 0$, and consequently, $\partial_{e_1} \delta(\epsilon e_1 + e_2) = \partial^2_{e_1 e_1} \delta(\epsilon e_1 + e_2) = 0$. On the other hand, the fact that $-e_1 \notin \overline{C}$ implies that there exists some $\theta > 0$ such that $e_1 \cdot x > -(1-\theta)\|x\|$ for all $x \in E$ (θ depends on the distance between $-e_1$ and $\overline{C}$). Let x_ϵ be the closest point in E to the point $e_2 - \epsilon e_1$; then

$$\begin{aligned}\delta^2(e_2 - \epsilon e_1) \equiv \|x_\epsilon - e_2 + \epsilon e_1\|^2 &= 1 + \epsilon^2 + \|x_\epsilon\|^2 + 2\epsilon\langle e_1, x_\epsilon\rangle \\ &\geq 1 + \big(\|x_\epsilon\| - \epsilon\big)^2 + 2\epsilon\theta\|x_\epsilon\|.\end{aligned}$$

After analyzing two cases, depending on the relative size of $\|x_\epsilon\|$ and $\varepsilon/2$, we find that

$$\delta(e_2 - \epsilon e_1) \geq 1 + c\epsilon^2.$$

Let $M_\epsilon = \sup_{t \in [0,\epsilon]} |\partial^2_{e_1 e_1} \delta(e_2 - t e_1)|$. If we assume that $\partial_{e_1}\delta$ is continuous at e_2 (i.e., that $\partial_{e_1}\delta(e_2) = 0$), then by the Taylor remainder theorem:

$$1 + c\epsilon^2 \leq \delta(e_2 - \epsilon e_1) \leq 1 + M_\epsilon \epsilon^2.$$

This implies that $\lim_{\epsilon\downarrow 0} M_\epsilon > c$ which in turn implies that δ is not C^2 at the point e_2. This contradicts the initial assumption that $E \neq P$.

We are left to prove the final claim: that μ must be a constant times $\mathcal{H}^d|_P$. Assume without losing generality that $0 \in P$ is a point of density for μ, with density $c_0 > 0$ (clearly everything is invariant under translation, but c_0 may depend on the point $0 \in P$). For $r_k \downarrow 0$, define the measure μ_k supported on P by $\mu_k(S) \equiv \frac{\mu(r_k S)}{r_k^d}$. Note that μ_k is still a d-Ahlfors regular measure supported on P. It is then easy to see that $\mu_k \rightharpoonup c_0\mathcal{H}^d|_P$ weakly as measures. By changing coordinates, $y = r_k z$, it is also clear that

$$
\begin{aligned}
D_{\mu,\alpha}(r_k x) &= \Big(\int_P \frac{d\mu(y)}{|r_k x - y|^{d+\alpha}}\Big)^{-1/\alpha} = \Big(\frac{1}{r_k^\alpha}\int_P \frac{d\mu(r_k z)}{r_k^d|x-z|^{d+\alpha}}\Big)^{-1/\alpha} \\
&= r_k D_{\mu_k,\alpha}(x), \forall x \in \mathbb{R}^n\backslash E.
\end{aligned}
$$

Since $D_{\mu,\alpha}(r_k x) = c\delta(r_k x) = c r_k \delta(x)$ for some $c > 0$ (which may depend on the component of Ω containing x), it follows that $D_{\mu_k,\alpha}(x) = c\delta(x)$. Letting $k \to \infty$ and using that $\mu_k \rightharpoonup c_0\mathcal{H}^d|_P$, we obtain that $D_{c_0\mathcal{H}^d|_P,\alpha}(x) = c\delta(x)$. However, by (2.12) for each $c > 0$ there is only one $\overline{c} > 0$ for which $D_{\overline{c}\mathcal{H}^d|_P,\alpha}(x) = c\delta(x)$. This implies that $c_0 = \overline{c}$, that is, that the density of μ with respect to $\mathcal{H}^d|_P$ is the same at all points of density in P. In addition, μ is independent of the connected component of Ω that contains x and thus the constant c is the same for all connected components of Ω. Therefore, $\mu = \overline{c}\mathcal{H}^d|_P$ and we are done. $\square$

Readers familiar with the concept of tangent measure will note that we essentially analyzed the tangent measures of μ at x_0 to obtain that μ has constant density with respect to $\mathcal{H}^d|_P$. This analysis was particularly easy in the case above, that is, when μ is an Ahlfors regular measure whose support is a d-plane. Later, in Section 5, we will need to understand the behavior of $D_{\mu,\alpha}(x)$ as $x \to E$ for more complicated sets E. In that section we will treat the concepts of tangent measure and blowup with more care and comprehensiveness.

4. A weak USFE implies the uniform rectifiability of E

In this section, we use the "endpoint result" of Section 3 to prove a (a priori slightly stronger) converse to Theorem 2.1 and Corollary 2.2. Let us note that, throughout this section, d is not assumed to be an integer (but will be forced to be so a posteriori).

THEOREM 4.1

Let $n \geq 1$ be an integer, and let $0 < d < n$ be given. Let μ be a d-dimensional Ahlfors regular measure supported on the closed set $E \subset \mathbb{R}^n$. Let $\alpha > 0$ be given, and define

$R = R_{\mu,\alpha}$, $D = D_{\mu,\alpha}$, $F = F_{\mu,\alpha}$, and $\widetilde{F} = \widetilde{F}_{\mu,\alpha}$ by (1.3), (1.5), (1.7), and (2.26), respectively. For $\varepsilon > 0$, set

$$Z(\varepsilon) = \{x \in \Omega; F(x) > \varepsilon\} \qquad \text{and} \qquad \widetilde{Z}(\varepsilon) = \{x \in \Omega; \widetilde{F}(x) > \varepsilon\}. \tag{4.1}$$

If for every $\varepsilon > 0$ $Z(\varepsilon)$ or $\widetilde{Z}(\varepsilon)$ is a Carleson set (see Definition 1.3), then d is an integer and E is uniformly rectifiable.

Notice that the USFE (applied to either F or $\widetilde{F}$) implies the Carleson condition on $Z(\varepsilon)$ (resp., $\widetilde{Z}(\varepsilon)$) in the statement, by Chebyshev; thus we will refer to the condition that $Z(\epsilon)$ (or $\widetilde{Z}(\epsilon)$) is a Carleson set as the *weak USFE*.

As is always the case with these types of results, what we will prove is that there is a constant $\varepsilon_0 > 0$ that depends on n, d, α, and the Ahlfors regularity constant for μ such that if $Z(\varepsilon_0)$ or $\widetilde{Z}(\varepsilon_0)$ is a Carleson set, then d is an integer and E is uniformly rectifiable. But this is not such a useful difference anyway, since ε_0 comes from a compactness argument and cannot be computed. However, it does mean that one should not be concerned about the constants associated to the Carleson set $Z(\varepsilon)$ potentially blowing up as $\varepsilon \downarrow 0$.

Before beginning the proof of Theorem 4.1, let us first check that it is enough to prove the theorem for Z. Indeed, recall from (1.7) and (2.26) that $F(x) = \delta(x)|\nabla(|\nabla D|^2)(x)|$ and $\widetilde{F}(x) = \delta(x)|\nabla(|\nabla D|)(x)|$. Thus, as observed at the beginning of the proof of Corollary 2.2, $F \le 2|\nabla D|\widetilde{F}$. By (2.8), (1.4), and (2.22) (note that this last estimate, while presented in the context of Theorem 2.1 uses only the d-Ahlfors regularity of μ),

$$|\nabla D|(x) \le C R(x)^{-\frac{1}{\alpha}-1}\big|\nabla R(x)\big| \le C\delta(x)^{1+\alpha}\big|\nabla R(x)\big| \le C. \tag{4.2}$$

So $F(x) \le C\widetilde{F}(x)$. If $F(x) > \varepsilon$, then $\widetilde{F}(x) > \varepsilon/C$. That is, $Z(\varepsilon) \subset \widetilde{Z}(\varepsilon/C)$. If $\widetilde{Z}(\varepsilon/C)$ is a Carleson set, then $Z(\varepsilon)$ is a Carleson set, and if we know the result for Z, then we can deduce the uniform rectifiability from this and get the result for $\widetilde{Z}$.

To prove the result for Z, we will show that the weak USFE (i.e., the condition that $Z(\varepsilon)$ is a Carleson set) implies that d is an integer and E satisfies the condition known as the bilateral weak geometric lemma (BWGL). The BWGL property, along with Ahlfors regularity, characterizes uniform rectifiability and so this will complete the proof. Let us quickly recall what the BWGL is (for a more comprehensive introduction to this and other characterizations of uniform rectifiability, see, e.g., [9]).

Recall the local normalized Hausdorff distances $d_{x,r}$ defined for $x \in \mathbb{R}^n$ and $r > 0$ by

$$\begin{aligned} d_{x,r}(E,F) = \frac{1}{r}\Big(&\sup\{\operatorname{dist}(y,F); y \in E \cap \overline{B}(x,r)\} \\ &+ \sup\{\operatorname{dist}(y,E); y \in F \cap \overline{B}(x,r)\}\Big), \end{aligned} \tag{4.3}$$

where E, F are closed sets that meet $\overline{B}(x,r)$ (we will not need the other cases). Using this distance for an integer $d > 0$, we can define a bilateral d-dimensional version of Jones's β numbers in [15], which we denote $\beta_b(x,r)$, as

$$\beta_b(x,r) \equiv \inf_P d_{x,r}(E,P), \tag{4.4}$$

where the infimum is taken over all affine d-planes P that meet $\overline{B}(x,r)$. These numbers measure, in a two-sided way, how close the set E is to being flat at the point x and scale $r > 0$. We can now state the BWGL.

Definition 4.2
Let $E \subset \mathbb{R}^n$ be a closed set, let d be a positive integer, and let $\beta_b(x,r)$ be defined with respect to E as in (4.4). Then E satisfies the condition known as the *bilateral weak geometric lemma* (BWGL) if the set $\mathcal{R}(\tau)$ defined by

$$\mathcal{R}(\tau) = \big\{(x,r) \in E \times (0,+\infty); \beta_b(x,r) \geq \tau\big\} \tag{4.5}$$

is a Carleson subset of $E \times (0,+\infty)$ for all $\tau > 0$. Recall that any $\mathcal{G} \subset E \times (0,+\infty)$ is a Carleson subset of E if there exists a $C > 0$ such that, for all $X \in \mathbb{R}^n$ and $R > 0$,

$$\int_{x \in E \cap B(X,R)} \int_{r \in (0,R]} \mathbf{1}_{\mathcal{G}}(x,r) \frac{d\mathcal{H}^d(x)\, dr}{r} \leq CR^d. \tag{4.6}$$

It is proved in [8] that if E is Ahlfors regular (of some integer dimension d) and satisfies the BWGL, then it is uniformly rectifiable (see also [9, Theorem I.2.4] for the statement). In fact, it is enough to show that $\mathcal{R}(\tau)$ is a Carleson set for a single $\tau > 0$, sufficiently small depending on the dimensions and the Ahlfors regularity constant for E (see [9, Remark II.2.5]).

To show that the BWGL holds, we will first replace the $Z(\epsilon)$'s with other similar sets $\mathcal{B}(\eta)$, which also satisfy a Carleson condition when the $Z(\epsilon)$'s do, and which are more amenable to a later compactness argument that will invoke Corollary 3.2.

LEMMA 4.3
Let $n \geq 2$, $d < n$, and let $E \subset \mathbb{R}^n$ support a d-Ahlfors regular measure μ. For $M \geq 1$ (a large constant, to be chosen later) and $x \in \Omega \equiv \mathbb{R}^n \setminus E$, define a big (Whitney) neighborhood of x as

$$W(x) = W_M(x) = \big\{y \in \Omega \cap B\big(x, M\delta(x)\big); \operatorname{dist}(y,E) \geq M^{-1}\delta(x)\big\}. \tag{4.7}$$

Define the bad set $\mathcal{B}(\eta) = \mathcal{B}_M(\eta)$ by

$$\mathcal{B}_M(\eta) = \big\{x \in \Omega; F(y) \geq \eta \text{ for some } y \in W_M(x)\big\}. \tag{4.8}$$

With this notation, if $Z(\varepsilon)$ is a Carleson set, then for each large enough M, $\mathscr{B}_M(3\varepsilon)$ is a Carleson set as well.

Thus, with our assumption that the weak USFE holds, each $\mathscr{B}_M(\eta)$ is a Carleson set.

Proof
This will be a relatively simple covering argument. Let $\tau \in (0,1)$ be small, to be chosen later (depending on ε). We define a very dense collection H_τ of points in Ω, which is a maximal subset of Ω with the property that $|x-y| \geq \tau \max\{\delta(x), \delta(y)\}$ when $x, y \in H_\tau$ are different.

The net H_τ is useful because F varies so slowly. Indeed, recalling the estimates below (2.25), $\delta(x)^{-1}F(x) = |\nabla(|\nabla D(x)|^2)| \leq C\delta(x)^{-1}$. The same argument, still based on the formula (2.10) and the estimate (2.22), yields

$$\big|\nabla\big(\delta(x)^{-1}F\big)\big|(x) \leq C\delta(x)^{-2}. \tag{4.9}$$

Let us use this to check that if τ is small enough (depending on ε), then

$$\big|F(x) - F(x')\big| \leq \varepsilon \quad \text{for } x, x' \in \Omega \text{ such that } |x'-x| \leq 4\tau\delta(x). \tag{4.10}$$

First observe that $\delta(x') \geq \delta(x) - |x'-x| \geq (1-4\tau)\delta(x) \geq \delta(x)/2$ and in fact $\delta(x)/2 \leq \delta(z) \leq 2\delta(x)$ for $z \in [x,x']$. Then, setting $G(x) = \delta(x)^{-1}F(x)$ just for the sake of the computation,

$$\begin{aligned}\big|F(x) - F(x')\big| &= \big|G(x)\delta(x) - G(x')\delta(x')\big| \\ &\leq \delta(x)\big|G(x) - G(x')\big| + G(x')\big|\delta(x) - \delta(x')\big| \\ &\leq C\delta(x)\big[|x'-x|\delta(x)^{-2}\big] + G(x')|x'-x| \leq C|x'-x|\delta(x)^{-1} \\ &\leq C\tau \leq \varepsilon,\end{aligned} \tag{4.11}$$

where we used (4.9) and the fact that $G(x') = \delta(x')^{-1}F(x') \leq C\delta(x')^{-1}$ by the estimate above (4.9). So (4.10) holds.

Now let $x \in \mathscr{B}(3\varepsilon)$ be given. This means that we can find $y \in W_M(x)$ such that $F(y) \geq 3\varepsilon$.

By maximality of H_τ, we can find $z \in H_\tau$ such that $|z-y| \leq \tau \max\{\delta(z), \delta(y)\}$ (otherwise, add y to H_τ). If τ is small enough, then the triangle inequality yields $\delta(y) \leq 2\delta(z)$ and so $|z-y| \leq 2\tau\delta(z)$.

If τ is small enough, then (4.11) implies that $F(z) \geq 2\varepsilon$. In fact, this stays true for all $w \in B(z, \tau\delta(z))$. Notice also that since $y \in W_M(x)$, $(2M)^{-1}\delta(z) \leq \delta(x) \leq 2M\delta(z)$, and also $|x-z| < 2M\delta(x) \leq 4M^2\delta(z)$. In short, $x \in V(z)$, where

$$V(z) = \big\{x \in \Omega \cap B\big(z, 4M^2\delta(z)\big); \delta(x) \geq (2M)^{-1}\delta(z)\big\}.$$

We are ready for the Carleson estimate. Recall (2.33), and set $d\sigma(x) = \delta(x)^{-n+d}\,dx$; we need to show that

$$A(X,R) := \int_{x\in\Omega\cap B(X,R)\cap\mathcal{B}(3\varepsilon)} d\sigma(x) \le CR^d \tag{4.12}$$

for $X \in E$ and $R > 0$. Let X and R be given. Observe that if $x \in \Omega \cap B(X,R) \cap \mathcal{B}(3\varepsilon)$, then any point $z \in H_\tau$ constructed as above lies in $H_\tau \cap B(X, 3MR)$, because $|z-x| \le 2M\delta(x)$ and $\delta(x) \le |x - X| \le R$. Furthermore, the argument above tells us that every $x \in \Omega \cap B(X,R) \cap \mathcal{B}(3\varepsilon)$ is in $V(z)$ for some $z \in H(\tau, X, M, R, \varepsilon) \equiv \{z \in H_\tau \cap B(X, 3MR); F(z) \ge 2\varepsilon\}$. Thus

$$\begin{aligned} A(X,R) &\le \sum_{z\in H(\tau,X,M,R,\varepsilon)} \int_{x\in V(z)} d\sigma(x) \le C \sum_{z\in H(\tau,X,M,R,\varepsilon)} \delta(z)^{-n+d} \big|V(z)\big| \\ &\le C \sum_{z\in H(\tau,X,M,R,\varepsilon)} \delta(z)^d \le C \sum_{z\in H(\tau,X,M,R,\varepsilon)} \sigma\big(B\big(z,\tau\delta(z)/10\big)\big) \\ &\le C\sigma\big(Z(\varepsilon)\cap B(X,4MR)\big) \le CR^d, \end{aligned} \tag{4.13}$$

by definition of σ (for the second inequality) and the fact that $V(z) \subset B(z, 4M^2\delta(z))$ (for the third one), and then because $\sigma(B(z,\tau\delta(z)/10)) \ge C^{-1}\delta(z)^d$ and the balls $B(z,\tau\delta(z)/10)$ are disjoint by definition of H_τ, and then finally (for the last line) since each $B(z,\tau\delta(z)/10)$ is contained in $Z(\epsilon) \cap B(X, 4MR)$, and by our Carleson estimate assumption on Z. The lemma follows. $\square$

As mentioned above, the set $\mathcal{B}_M(3\varepsilon)$ is defined in the right way to make it amenable to a compactness argument. In the following lemma we will show that if x is not in $\mathcal{B}_M(3\varepsilon)$, then the set E is relatively flat in a neighborhood of x of radius comparable to $\delta(x)$.

LEMMA 4.4
For each choice of $0 < d < n$, $\alpha > 0$, an Ahlfors regularity constant C_0, and constants $\eta > 0$ (small) and $N \ge 1$ (large), we can find $M \ge 1$ and $\varepsilon > 0$ such that if μ is Ahlfors regular (of dimension d, constant C_0, and support $E \subset \mathbb{R}^n$), and if $x \in \Omega \setminus \mathcal{B}_M(3\varepsilon)$, then d is an integer and there is a d-plane P such that $d_{x,N\delta(x)}(E,P) \le \eta$.

More explicitly, if d is not an integer, then we can find M and ε (depending on d too) such that $\Omega \setminus \mathcal{B}_M(3\varepsilon)$ is empty.

Proof
We will prove this by compactness. That is, let $0 < d < n$, C_0, $\alpha > 0$, N, and $\eta > 0$ be

given, and suppose that, for each $k \geq 0$, there is a set E_k, a d-Ahlfors regular measure μ_k with constant C_0 and whose support is E_k, which provide a counterexample with $M_k = 2^k$ and $\varepsilon_k = 2^{-k}$. That is, let F_k be defined as in (1.7) but adapted to E_k, μ_k, and α. We assume that there are points $x_k \in \Omega_k \equiv \mathbb{R}^n \setminus E_k$ that do not lie in the corresponding bad set $\mathcal{B}_{M_k}^{E_k}(3\varepsilon_k)$, that is,

$$\begin{aligned}&F_k(y) < 2^{-k}\\ &\quad \text{for all } y \in \Omega_k \cap B\big(x_k, 2^k \delta_{E_k}(x_k)\big) \text{ with } \delta_{E_k}(y) \geq 2^{-k}\delta_{E_k}(x_k),\end{aligned} \tag{4.14}$$

and yet for which the conclusion does not hold. That is, either d is not an integer, or d is an integer but there is no d-plane P_k such that $d_{x_k, N\delta(x_k)}(E_k, P_k) \leq \eta$. We want to reach a contradiction.

By translation and dilation invariance, we may assume that $x_k = 0$ and $\delta_{E_k}(x_k) = \operatorname{dist}(0, E_k) = 1$. We use the uniform Ahlfors regularity to replace $\{(E_k, \mu_k)\}$ with a subsequence for which μ_k converges (in the weak sense) to an Ahlfors regular measure μ_∞, and E_k converges (in the Hausdorff distance sense) to a closed set E_∞ (locally in $\mathbb{R}^n$). It is also easy to check that E_∞ is the support of μ_∞ and that μ_∞ is d-Ahlfors regular with a constant that depends only on C_0 and n. For more details (albeit in a slightly less general context), see the discussion before Lemma 5.1 below.

Additionally, $R_{\mu_k,\alpha} \to R_{\mu_\infty,\alpha}$ (uniformly on compact sets of $\Omega_\infty \equiv \mathbb{R}^n \setminus E_\infty$) and similarly for $D_{\mu_k,\alpha}$ and its derivatives. This follows from the weak convergence of the μ_k (actually a little work is necessary as $\partial^j h(x-y)$ is not compactly supported, but one can argue exactly as in Lemma 5.1 below). Because of this, and with hopefully obvious notation,

$$F_\infty(y) = \lim_{k \to +\infty} F_k(y) \tag{4.15}$$

for every $y \in \Omega_\infty$.

Let $W_k \equiv W_{2^k}(0)$ be as in (4.7) but associated to the set E_k. Clearly, any $y \in \Omega_\infty$ lies in W_k for k large, and so, by assumption, $F_k(y) \leq 2^{-k}$ for k large. Taking limits, (4.15) implies that $F_\infty(y) = 0$ for $y \in \Omega_\infty$ and, by (1.7), $|\nabla D_\infty|$ is locally constant. If by bad luck $|\nabla D_\infty| = 0$ on some connected component $\Omega_0 \subset \Omega_\infty$, then we also get that $D_\infty = 0$ on Ω_0 (because D_∞ vanishes on E); this is impossible, by the definition of D_∞ (cf. (1.4) and (1.5)). So $|\nabla D_\infty| \neq 0$ on Ω_0, and now Corollary 3.2 says that d is an integer and E_∞ is a d-plane.

Now recall that E_k converges to the d-plane E_∞; we thus get that, for k large, $d_{0,N}(E_k, E_\infty) \leq \eta$, which is the desired contradiction. Lemma 4.4 follows. $\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1
In view of Lemma 4.3, to prove Theorem 4.1, we just need to choose $\varepsilon = \varepsilon_\tau > 0$ such that

$$\begin{aligned}&\text{if } \mathcal{B}(3\varepsilon) \text{ is a Carleson set in } \Omega,\\ &\text{then } \mathcal{R}(\tau) \text{ is a Carleson set in } E \times (0, +\infty).\end{aligned} \tag{4.16}$$

Let us do this. For $(x, r) \in \mathcal{R}(\tau)$, we first use the Ahlfors regularity of E to choose $y \in \Omega \cap B(x, r/2)$ such that $\delta(y) \geq 2\kappa r$, where the constant $\kappa > 0$ depends on the dimensions and the Ahlfors regularity constant. The existence of y is standard; if we could not find it, then we would be able to find $C_n \kappa^{-n}$ balls B_j of radius $\kappa r/2$, centered on $E \cap B(x, r/2)$, and that are disjoint. This would yield

$$C_n^{-1} \kappa^{-n} (\kappa r/2)^d \leq C \sum_j \mathcal{H}^d (E \cap B_j) \leq C \mathcal{H}^d \big(E \cap B(x, r)\big) \leq C r^d,$$

a contradiction for κ small because $d < n$. Denote by $H(x, r)$ the ball $B(y, \kappa r)$. Then

$$|z - x| \leq r \qquad \text{and} \qquad \delta(z) \geq \kappa r \quad \text{for } z \in H(x, r). \tag{4.17}$$

Take $N = 10\kappa^{-1}$; this way, $B(z, N\delta(z))$ contains $B(x, r)$ for $z \in H(x, r)$. Let $\eta > 0$, to be chosen later in terms of τ, N, and choose $M = M_{N,\eta} > 0$ and $\varepsilon = \varepsilon_{N,\eta} > 0$ as in Lemma 4.4. That lemma says that if $z \in H(x, r) \setminus \mathcal{B}_M(3\varepsilon)$, then we can find a d-plane P such that $d_{z,N\delta(z)}(E, P) \leq \eta$. This also implies that $d_{x,r}(E, P) \leq r^{-1} N\delta(z) d_{z,N\delta(z)}(E, P) \leq N d_{z,N\delta(z)}(E, P) \leq N\eta$ (because $B(z, N\delta(z))$ contains $B(x, r)$ and by the definition (4.3)). We choose η so small that $N\eta < \tau$, and we get that $\beta_b(x, r) < \tau$. This contradicts the fact that $(x, r) \in \mathcal{R}(\tau)$; therefore, every $z \in H(x, r)$ lies in $\mathcal{B}_M(3\varepsilon)$.

Return to the proof of (4.16), and assume that $\mathcal{B}_M(3\varepsilon)$ is a Carleson set in Ω. Let $X \in E$ and $R > 0$ be given, and denote by $A(X, R)$ the left-hand side of (4.6), with $\mathcal{G} = \mathcal{R}(\tau)$. Since $|H(x, r)| \geq C^{-1} r^n$, we see that

$$\begin{aligned}A(X, R) \leq C &\int_{x \in E \cap B(X,R)} \int_{r \in (0,R]} \mathbf{1}_{\mathcal{R}(\tau)}(x, r) r^{-n}\\ &\times \Big(\int \mathbf{1}_{H(x,r)}(z)\, dz\Big) \frac{d\mathcal{H}^d(x)\, dr}{r}.\end{aligned} \tag{4.18}$$

Of course, we apply Fubini and integrate in x and r first.

Notice that $z \in B(X, 2R) \cap \mathcal{B}_M(3\varepsilon)$ and $|x - z| \leq r \leq \kappa^{-1}\delta(z)$, so we get that

$$A(X, R) \leq \int_{B(X,2R) \cap \mathcal{B}_M(3\varepsilon)} h(z)\, dz, \tag{4.19}$$

with

$$h(z) = \int_{x \in E \cap B(z, \kappa^{-1}\delta(z))} \int_{\delta(z) \le r \le R} \frac{d\mathcal{H}^d(x)\, dr}{r^{n+1}}$$

(because $r \ge |x - z| \ge \delta(z)$). The integral in r is at most $C\delta(z)^{-n}$. Then we integrate in x and get that $h(z) \le C\delta(z)^{d-n}$. Finally,

$$A(X, R) \le C \int_{B(X,2R) \cap \mathcal{B}_M(3\varepsilon)} \delta(z)^{d-n}\, dz \le CR^d, \tag{4.20}$$

by the assumption that $\mathcal{B}_M(3\varepsilon)$ is a Carleson set. □

5. Blowups and nontangential limits of $|\nabla D_\beta|$

Throughout this section, let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set with $d < n - 1$ and $n \ge 2$; this assumption is not strictly necessary for all our proofs, but without it we must be a bit more careful as to questions of topology and in any case it is the only scenario in which we are interested (we try to state when the result holds with $d < n$). Let μ be a d-Ahlfors regular measure supported on E.

We are interested in the behavior of $\nabla D_{\mu,\beta}$ near E. One convenient tool for studying this is the blowup procedure.

For $Q \in E$, $S \subset E$, $y \in \Omega$, and $r_i \downarrow 0$, we can define

$$\begin{aligned} E_{i,Q} &\equiv \frac{E - Q}{r_i}, \\ \mu_{i,Q}(S) &\equiv \frac{\mu(r_i S + Q)}{r_i^d}, \\ D_{i,\beta,Q}(y) &\equiv \frac{D_{\mu,\beta}(r_i y + Q)}{r_i}. \end{aligned} \tag{5.1}$$

When the point Q is unimportant or clear from context, we may abuse notation and refer simply to E_i, μ_i, and $D_{i,\beta}$. Note that the μ_i's are still Ahlfors regular (with the same constants as μ) and E_i is the support of μ_i. To explain the definition of D_i, let $y \in \mathbb{R}^n \backslash E_i$, which implies that $y = \frac{z-Q}{r_i}$ for some $z \in \mathbb{R}^n \backslash E$. Then we can calculate

$$\begin{aligned} D_{\mu_i,\beta}(y)^{-\beta} &\equiv \int_{E_i} \frac{d\mu_i(x)}{|x - y|^{d+\beta}} \overset{w = r_i x + Q \in E}{=} \int_E \frac{d\mu(w)}{r_i^d \left|\frac{w-Q}{r_i} - \frac{z-Q}{r_i}\right|^{d+\beta}} \\ &= \Big(\frac{D_{\mu,\beta}(z)}{r_i}\Big)^{-\beta}. \end{aligned} \tag{5.2}$$

As the μ_i's are uniformly Ahlfors regular, we know that, perhaps passing to a subsequence, we have $\mu_i \rightharpoonup \mu_\infty$. Since the μ_i's are uniformly Ahlfors regular, μ_∞ is also Ahlfors regular and its support E_∞ is the limit (in the Hausdorff distance sense) of the E_i. We want to show that R_i and D_i converge to R_∞ and D_∞.

LEMMA 5.1
Let E, μ be as above, and let $r_k \downarrow 0$ and $Q \in E$. With the notation and assumptions above, R_k, D_k, and their derivatives converge to R_∞, D_∞, and their derivatives, uniformly on every compact subset of $\Omega_\infty = \mathbb{R}^n \setminus E_\infty$.

Proof
Consider $\nabla^j R_k(x) = \int_{E_k} \nabla^j h(x-y)\, d\mu_k(y)$, as in (2.7), and fix a compact set $K \subset \Omega_\infty$. Also, let $\varepsilon > 0$ be given. Choose $R > 0$ large enough (depending on K, ε). Then there exists a smooth cutoff function $\varphi \equiv \varphi_{\varepsilon,R}$, supported in the large ball $B(0,R)$, and chosen close enough to $\chi_{B(0,R)}$ so that $\int |\nabla^j h(x-y)||1-\varphi(y)|\, d\mu_k(y) \le \varepsilon$ for all k, and for all $x \in K$.

Let $C \subset \mathbb{R}^n \backslash K$ be a closed set such that $E_k \subset C$ for all large enough K (such a C exists by the compactness of K and the Hausdorff convergence of $E_k \to E_\infty$). Then the functions $\{\nabla^j h(x-y)\varphi(y)\}_{x\in K}$, are uniformly bounded, as functions of y, in $\mathrm{Lip}(\overline{B(0,r)} \cap C)$ (this bound will depend on C, R, ε, but that is irrelevant). By the Arzelà–Ascoli theorem, this sequence (indexed by x) is precompact. Thus, we can find a finite collection of continuous functions $\{g_i\}$ supported in $B(0,2R)$ such that, for each $x \in K$, there is a g_i with $|g_i(y) - \nabla^j h(x-y)\varphi(y)| \le \varepsilon R^{-d}$ for all $y \in E_k \cap B(0,2R)$ (for any k large enough). Then by Ahlfors regularity, $\int |g_i(y) - \nabla^j h(x-y)\varphi(y)|\, d\mu_k + \int |g_i(y) - \nabla^j h(x-y)\varphi(y)|\, d\mu_\infty \le C\varepsilon$ for all k. Since each $\int g_i\, d\mu_k$ converges to $\int g_i\, d\mu_\infty$, we see that, for k large,

$$\Big|\int \nabla^j h(x-y)[d\mu_k - d\mu_\infty]\Big| \le C\varepsilon;$$

the conclusion (for $\nabla^j R_k(x)$) follows. The same estimates for $\nabla^j D$ follow as well, because on the compact set K we have uniform lower and upper bounds on the R_k (again for k large enough). This proves the lemma. □

Lemma 5.1 gives convergence on compact sets separated from E_∞. But we want to understand the convergence up the boundary. In order to do this, it will be convenient to introduce "nontangential access" regions, for reasons that we will make clear shortly. For $Q \in E$, $R > 0$, and $\eta \in (0,1)$, define

$$\Gamma_{R,\eta}(Q) = \big\{x \in \Omega \cap B(Q,R); \mathrm{dist}(x,E) \ge \eta|x-Q|\big\}. \tag{5.3}$$

Associated to these nontangential regions is the concept of a nontangential limit.

Definition 5.2
We say that f has a *nontangential limit* L at $Q \in E$ if there is some $\eta \in (0,1)$ such that

$$\lim_{R\downarrow 0} \sup_{x\in\Gamma_{R,\eta}(Q)} \big|f(x) - L\big| = 0.$$

We will denote this limit L by $\text{n.t.}\lim_{x\to Q} f(x)$, or even $\text{n.t.}\lim^{\eta}_{x\to Q} f(x)$ to be explicit.

Let E, Q, and $\{r_i\}$ be as in (5.1), and assume that the E_i's converge to E_∞. Let $\Gamma^\infty_{R,\eta}(0)$ be defined as in (5.3) but with respect to E_∞. Then, after a new sequence extraction, the sets $\frac{\Gamma_{Rr_i,\eta}(Q)-Q}{r_i}$ converge to a limit Γ, with $\Gamma^\infty_{R,\eta/2}(0) \supset \Gamma \supset \Gamma^\infty_{R,2\eta}(0)$. For the moment, we only know that D_i, R_i, and their derivatives, converge to D_∞, R_∞, and their derivatives, uniformly on compact subsets of $\mathbb{R}^n \setminus E_\infty$. If we want ∇D_i to converge to ∇D_∞ uniformly on compact subsets of $\mathbb{R}^n$, then it should at least converge uniformly on each $\Gamma^\infty_{R,\eta}(0)$, which roughly corresponds, after a change of variables, to ∇D having a nontangential limit at Q (in fact, for every small η).

In the following two theorems, we give a characterization of the existence of nontangential limits of $|\nabla D|$ at μ-a.e. point $Q \in E$. It turns out that the existence of this limit is intimately linked to the tangent measures of μ at Q (and thus the rectifiability of μ). We will assume some basic familiarity with tangent measures here; for more background we suggest Chapter 17 of [18].

THEOREM 5.3

Let E be d-Ahlfors regular and d-rectifiable (so necessarily $d \in \mathbb{N}$), let μ be a d-Ahlfors regular measure supported on E, and let $\beta > 0$. Then for μ-almost every $Q \in E$, the limit $\text{n.t.}\lim^{\eta}_{x\to Q} |\nabla D_{\mu,\beta}(x)|$ *exists for every $\eta > 0$.*

Proof

Notice first that it will be enough to show that, for *each* $\eta > 0$, the nontangential limit $\text{n.t.}\lim^{\eta}_{x\to Q} |\nabla D_{\mu,\beta}(x)|$ exists for μ-almost every $Q \in E$, because then the exceptional set of $Q \in E$ for which the limit fails to exist for *all* η is contained in the countable union of the exceptional sets for $\eta_i \equiv 2^{-i}$.

Let $x_i \in \Omega \equiv \mathbb{R}^n \backslash E$ be a sequence of points approaching $Q \in E$ non-tangentially (i.e. $x_i \in \Gamma_{R,\eta}(Q)$ for some $\eta \in (0,1)$, $R > 0$ and $x_i \to Q$). Let $r_i = |x_i - Q|$ and define E_i, μ_i, D_i as in (5.1).

By Lemma 5.1 (perhaps passing to a subsequence), $E_i \to E_\infty$ and $\mu_i \rightharpoonup \mu_\infty$ which is a d-Ahlfors regular measure supported on E_∞. Furthermore, $D_i \to D_\infty \equiv D_{\beta,\mu_\infty}$. This convergence happens uniformly on compacta inside of Ω_∞ in the C^∞ topology. Note that $X_i \equiv \frac{x_i - Q}{r_i} \in \Omega_i \cap \overline{B(0,1)}$. We also note (by the assumption that x_i is a nontangential sequence) that $\operatorname{dist}(X_i, E_i) \geq \eta$.

Passing to a subsequence, we may assume that $X_i \to X_\infty$, and then $X_\infty \in \Omega_\infty$ because $\operatorname{dist}(X_\infty, E_\infty) \geq \eta$ (recall that $r_i = |x_i - Q|$). Then by (5.2),

$$\big|\nabla D_\infty(X_\infty)\big| = \lim_i \big|\nabla D_i(X_i)\big| = \lim_i \big|\nabla D(x_i)\big|. \tag{5.4}$$

The reader may be worried because we only proved the existence of $\lim_i |\nabla D(x_i)|$ for a subsequence, but what will save us is that, for almost every choice of $Q \in E$, the left-hand side $L = |\nabla D_\infty(X_\infty)|$ does not depend on $\{x_i\}$ or the choice of subsequences. Then it will follow that all the accumulation points of $|\nabla D(x_i)|$, where $x_i \in \Gamma_{R,\eta}(Q)$ and x_i tends to Q, are equal to the number L (take a sequence $\{x_i\}$, so that $|\nabla D(x_i)|$ tends to a given accumulation point, and then proceed as above). The existence of the nontangential limit $\text{n.t.}\lim^{\eta}_{x\to Q} |\nabla D(x)| = L$ will follow.

So we look for $Q \in E$ such that $|\nabla D_\infty(X_\infty)|$ above does not depend on $\{x_i\}$, the choice of subsequences, or X_∞ for that matter.

Since E is rectifiable, E has an approximate tangent d-plane P' at almost every point $Q \in E$ (see Theorem 15.19 in [18]). Since E is Ahlfors regular, and by Exercise 41.21 in [4], for instance, P' is a true tangent plane, and any limit E_∞ that we get from extraction is the vector plane P parallel to P'. In addition, Theorem 16.5 in [18] says that (for almost every $Q \in E$) all the blowup limits of $\sigma = \mathcal{H}^d_{|E}$ are flat measures, and in fact of the form $\sigma_\infty = \mathcal{H}^d_{|P}$, because the density of σ is 1 almost everywhere. In addition, $\mu = f\sigma$ for some function f such that $C^{-1} \le f \le C$, and if Q is a Lebesgue density point for f, all the blowup limits of μ at Q are of the form $\mu_\infty = f(Q)\lambda_P$.

Thus, for almost every point $Q \in E$, we have no choice: in (5.4), $|\nabla D_\infty(X_\infty)|$ must be the constant value of $|\nabla D|$ associated to the plane P and the measure $\mu_\infty = f(Q)\lambda_P$. The existence of $\text{n.t.}\lim^{\eta}_{x\to Q} |\nabla D(x)|$, and Theorem 5.3, follows. $\square$

What follows is the converse to Theorem 5.3. However, we note that in order to prove the rectifiability of E, we need the nontangential limit to exist inside cones of *all* apertures, as opposed to checking the existence inside cones of any given aperture.

THEOREM 5.4

Let E be a set supporting the d-Ahlfors regular measure μ with $d < n$ (not necessarily an integer), and let $\beta > 0$. Assume that, for μ-almost every $Q \in E$, the nontangential limit $\text{n.t.}\lim^{\eta}_{x\downarrow Q} |\nabla D_{\mu,\beta}(x)|$ *exists for every aperture* $\eta \in (0,1)$. *Then d is an integer and E is d-rectifiable.*

Proof

We will show that at μ-almost every $Q \in E$, every tangent measure to μ is flat (i.e., is a multiple of the restriction of $\mathcal{H}^d$ to a d-plane). This implies that μ is d-rectifiable and, thus (since μ is Ahlfors regular), that E is d-rectifiable.

Let $Q \in E$ be a point such that the nontangential limit of $|\nabla D_{\mu,\beta}|$ exists for every aperture, and let $\{r_i\}$ be any sequence of positive numbers that tends to zero.

Then define E_i, μ_i and $D_i \equiv D_{\mu_i,\beta}$ as in (5.1). Lemma 5.1 shows that, passing to a subsequence if needed, we may assume that E_i tends to a limit E_∞, μ_i has a weak limit μ_∞, and D_i converges, uniformly on compact subsets of $\mathbb{R}^n \setminus E_\infty$, to $D_\infty \equiv D_{\mu_\infty,\beta}$.

We now want to show that $|\nabla D_\infty|$ is constant on $\Omega_\infty \equiv \mathbb{R}^n \setminus E_\infty$ and is equal to $L = \text{n.t.}\lim_{x\to Q} |\nabla D_{\mu,\beta}(x)|$. Let $Y, Z \in \Omega_\infty$, and set $\eta_Y = \operatorname{dist}(Y, E_\infty)/(2|Y|) \in (0,1)$, and similarly let $\eta_Z = \operatorname{dist}(Z, E_\infty)/(2|Z|) \in (0,1)$. We can assume that $\eta_Z \leq \eta_Y$ so that $\Gamma_{1,\eta_Y}(Q) \subseteq \Gamma_{1,\eta_Z}(Q)$. By the convergence of E_i to E_∞, we have $Z \in \Omega_i$ for i large enough and $\operatorname{dist}(Z, E_i) \geq \operatorname{dist}(Z, E_\infty)/2$. Therefore, $r_i Z + Q \in \Omega$ and $\operatorname{dist}(r_i Z + Q, E) = r_i \operatorname{dist}(Z, E_i) \geq r_i \operatorname{dist}(Z, E_\infty)/2 = r_i \eta_Z |Z|$. Thus, $r_i Z + Q \in \Gamma_{1,\eta_Z}(Q)$ for i large enough (where the cone is with respect to Ω). Similarly, $r_i Y + Q \in \Gamma_{1,\eta_Y}(Q)$ for i large enough (again where the cone is with respect to Ω). Observe that $L = \text{n.t.}\lim^{\eta_Z}_{x\to Q} |\nabla D_{\mu,\beta}(x)|$ because the nontangential convergence holds in every cone. We can then write

$$\begin{aligned}\left|\nabla D_{\mu_\infty,\beta}(Z)\right| &= \lim_{i\to+\infty} \left|\nabla D_{\mu,\beta}(r_i Z + Q)\right| = L = \lim_i \left|\nabla D_{\mu,\beta}(r_i Y + Q)\right| \\ &= \left|\nabla D_{\mu_\infty,\beta}(Y)\right|,\end{aligned}$$

where the first and last equalities follow from (5.1), $\mu_i \rightharpoonup \mu_\infty$, and Lemma 5.1.

We conclude that $|\nabla D_{\mu_\infty,\beta}|$ is constant on Ω_∞. If that constant is zero, then by the fact that $D_{\mu_\infty,\beta}$ vanishes on E_∞, we obtain $D_{\mu_\infty,\beta} \equiv 0$. This contradicts (1.4) and (1.5). Thus, $|\nabla D_{\mu_\infty,\beta}|$ is a nonzero constant on Ω_∞ and by Corollary 3.2, E_∞ is a d-dimensional affine space and μ_∞ is a constant multiple of $\mathcal{H}^d$ restricted to E_∞. Thus, μ_∞ is flat.

In the language of tangent measures, all the tangents to μ at Q are flat measures. Furthermore, by Ahlfors regularity, the upper density of μ is bounded away from infinity and the lower density of μ is bounded away from zero. Thus, we can invoke Theorem 17.6 in [18] and conclude that the support of μ is a d-rectifiable set. Since E is the support of μ, we are done. □

Finally, we can compute the nontangential limit of $|\nabla D_{\mu,\beta}|$ at a point at which the d-density of μ exists (call it $\Theta^d(\mu, Q)$) and E has a unique tangent plane (call it $T_Q E$). Blowing up at such a point gives $\mu_\infty = \Theta^d(\mu, Q)\mathcal{H}^d|_{T_Q E}$. Recalling that $D_{\mathcal{H}^d|_P,\beta} = c_{\beta,n,d}\,\delta_P$ for any plane P, we have

$$D_\infty = D_{\Theta^d(\mu,Q)\mathcal{H}^d|_{T_Q E},\beta} = c_{\beta,n,d}\,\Theta^d(\mu, Q)^{-1/\beta}\,\delta_{T_Q E}, \tag{5.5}$$

which implies that

$$\text{n.t.}\lim_{x\to Q} |\nabla D_{\mu,\beta}| = c_{\beta,n,d}\,\Theta^d(\mu, Q)^{-1/\beta}. \tag{5.6}$$

6. D_α for "magic α"

Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set, and let μ be a d-Ahlfors regular measure supported on E. If the numbers n, $d < n$ (not necessarily integer), and $\alpha > 0$ are such that

$$n = d + 2 + \alpha, \tag{6.1}$$

then it turns out that the function $D_{\mu,\alpha}$ defined in (1.5) is a solution of

$$L_{\mu,\alpha} u \equiv -\operatorname{div}\Big(\frac{1}{D_{\mu,\alpha}^{n-d-1}} \nabla u\Big) = 0$$

in $\Omega = \mathbb{R}^n \setminus E$ (throughout this section, α will satisfy (6.1) whereas $\beta > 0$ will be arbitrary; in particular, we will try to make it clear when we are assuming that $d < n-2$).

We can check this (in the classical sense) in Ω simply by differentiating the smooth function $D_{\mu,\alpha}$ (recall (2.8)),

$$\begin{aligned} L_{\mu,\alpha} D_{\mu,\alpha} &= -\operatorname{div}(D_{\mu,\alpha}^{-n+d+1} \nabla D_{\mu,\alpha}) = \frac{1}{\alpha} \operatorname{div}(D_{\mu,\alpha}^{-n+d+1} R_{\mu,\alpha}^{-\frac{1}{\alpha}-1} \nabla R_{\mu,\alpha}) \\ &= \frac{1}{\alpha} \operatorname{div}(D_{\mu,\alpha}^{-n+d+1} D_{\mu,\alpha}^{1+\alpha} \nabla R_{\mu,\alpha}) = \frac{1}{\alpha} \Delta R_{\mu,\alpha} \end{aligned} \tag{6.2}$$

by (1.5) and (6.1). Then by (1.3) (and (6.1)),

$$R_{\mu,\alpha}(x) \equiv \int_{y \in E} |x-y|^{-d-\alpha} \, d\mu(y) = \int_{y \in E} |x-y|^{2-n} \, d\mu(y); \tag{6.3}$$

we recognize the Green kernel (notice that $n > 2$ by (6.1)); hence, $L_{\mu,\alpha} D_{\mu,\alpha} = 0$ on Ω.

We want to say that $D_{\mu,\alpha}$ is "Green's function with pole at infinity" associated to the operator $L \equiv L_{\mu,\alpha}$ (indeed, it is a solution which behaves like distance to the boundary). To do so properly, however, we need to define Green's function with pole at infinity (and the corresponding harmonic measure). We will then show that in the complement of any d-Ahlfors regular set E these objects exist and are unique up to multiplication by a positive scalar. Throughout, we will use some of the elliptic regularity and potential theory studied in [6], in particular, we will assume that the reader is comfortable with the existence and properties of a Green's function and associated harmonic measure with finite pole.

Before we begin, we must recall the weighted Sobolev spaces introduced in [6]. Throughout this section, E will be a closed d-Ahlfors regular set and $\delta(x)$ will denote the distance from x to the closest point in E.

Definition 6.1 (see [6])
Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for some $d < n-1$ (not necessarily an integer). Set $w(x) \equiv \delta(x)^{-(n-d-1)}$, and define the *weighted Sobolev space*

$$W \equiv W_w^{1,2} \equiv \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \backslash E) : \nabla u \in L^2(\mathbb{R}^n \backslash E, w\, dx)\}.$$

We can then localize these Sobolev spaces: for any open $\mathcal{O} \subset \mathbb{R}^n$, we define

$$W_r(\mathcal{O}) = \{u \in L^1_{\mathrm{loc}}(\mathcal{O}), \varphi f \in W \text{ for all } \varphi \in C_0^\infty(\mathcal{O})\}.$$

It will be useful later to know that $w(x)$ is locally integrable. Indeed, it follows from Ahlfors regularity that

$$\left|\{x \in B(Q,r) \mid w(x) > \lambda\}\right| = \left|\{x \in B(Q,R) \mid \delta(x) < \lambda^{-\frac{1}{n-d-1}}\}\right| \leq C\lambda^{-\frac{n-d}{n-d-1}} R^d,$$

for all $R > 0$ and $Q \in E$, which in turn implies that

$$\begin{aligned}\int_{B(Q,R)} w(x)\, dx &\leq \int_0^{R^{-(n-d-1)}} \left|B(Q,R)\right| d\lambda + CR^d \int_{R^{-(n-d-1)}}^\infty \lambda^{-\frac{n-d}{n-d-1}}\, d\lambda \\ &= CR^n R^{-(n-d-1)} + CR^d \left[-(n-d-1)\lambda^{-\frac{1}{n-d-1}}\right]_{R^{-(n-d-1)}}^\infty \\ &\leq CR^{d+1}.\end{aligned}$$

(We thank a referee for pointing out a minor error in the previous version of this computation and for providing us with a fix.)

These Sobolev spaces are the setting in which the elliptic estimates and potential theory established in [6] hold. We can now define the Green's function and harmonic measure with pole at infinity.

Definition 6.2
Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for some $d < n-1$ (not necessarily an integer), let $\beta \in (0,1)$, and let μ be a d-Ahlfors regular measure supported on E. Let $\Omega = \mathbb{R}^n \backslash E$, $D \equiv D_{\mu,\beta}$ be as in (1.5), and let $L \equiv L_{\mu,\beta}$ be the associated degenerate elliptic operator. We say that u_∞, ω_∞ are the Green's function and harmonic measure with pole at infinity, respectively (associated to β, μ), if $u_\infty \in W_r(B(Q,R)) \cap C(\mathbb{R}^n)$ for every $Q \in E$ and $R > 0$ and the following holds:

$$\begin{aligned} u_\infty &> 0, \quad \text{in } \Omega, \\ u_\infty &= 0, \quad \text{on } E, \\ Lu_\infty &= 0, \quad \text{in } \Omega, \\ \int_\Omega D^{-(n-d-1)} \nabla u_\infty \cdot \nabla \varphi\, dX &= \int_E \varphi\, d\omega_\infty, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{6.4}$$

Before we can show the existence and uniqueness of these objects, we must recall the comparison principle for solutions, stated and proved in our setting in [6] (see also, e.g., [14] for the codimension 1 statement). Recall from [6] that there exists an $M > 1$ such that for $Q \in E$ and $r > 0$ there exists a point $A_r(Q)$ with

$$\left|A_r(Q) - Q\right| \le r \le M\delta\bigl(A_r(Q)\bigr). \tag{6.5}$$

We call $A_r(Q)$ a *corkscrew point* for Q at scale $r > 0$.

THEOREM 6.3 ([6, Theorem 11.146])
Let $Q \in E$, let $r > 0$, and let $X_0 = A_r(Q) \in \Omega$ be the corkscrew point for Q at scale r. Let $u, v \in W_r(B(Q, 2r))$ be nonnegative, not identically zero, solutions of $L_{\mu,\beta} u = L_{\mu,\beta} v = 0$ in $B(Q, 2r)$, $\beta > 0$, such that $Tu = Tv = 0$ on $E \cap B(Q, 2r)$ (where T is the trace operator defined in Theorem 3.4 in [6]). Then there exists a constant $C > 1$ depending on n, d, and Ahlfors regularity constants, such that

$$C^{-1}\frac{u(X_0)}{v(X_0)} \le \frac{u(X)}{v(X)} \le C\frac{u(X_0)}{v(X_0)}, \forall X \in \Omega \cap B(Q, r). \tag{6.6}$$

The comparison theorem leads naturally to Hölder regularity of quotients at the boundary. Our proof below is inspired by [10, Theorem 4.5], who show this regularity for solutions of a parabolic problem. The “usual” elliptic proof (cf. [14]) relies on interior approximating domains, which are difficult in the codimension greater than 1 setting because of the presence of boundaries with mixed dimension (see the discussion at the beginning of Section 11 of [6]).

COROLLARY 6.4
Let u, v, Q, r be as in Theorem 6.3. There exist $c > 0$, $\gamma \in (0, 1)$ (depending only on the Ahlfors regularity of E, n, d, and β) such that

$$\left|\frac{u(X)v(Y)}{u(Y)v(X)} - 1\right| \le c\left(\frac{\rho}{r}\right)^{\gamma}, \tag{6.7}$$

for all $X, Y \in B(Q, \rho) \cap \Omega$, as long as $\rho < r/4$.

Proof
We claim that there exists some $\theta \in (0, 1)$ (independent of Q and r) such that

$$\operatorname{osc}_{B(Q,r/2)} \frac{u}{v} \le \theta \operatorname{osc}_{B(Q,r)} \frac{u}{v}, \tag{6.8}$$

for all $r < r/2$. That the claim implies (6.7) follows from iterating (6.8) and appealing to Theorem 6.3.

Let $\inf_{B(Q,r)} \frac{u}{v} = c_1$ and $\sup_{B(Q,r)} \frac{u}{v} = c_2$, and replace u by $U = \frac{u - c_1 v}{c_2 - c_1}$ (if $c_2 = c_1$, then $u = c_1 v$ and the result is trivial). It follows that $L_{\mu,\beta} U = 0$ and $U \geq 0$ in $B(Q,r)$ and $U = 0$ on $E \cap B(Q,r)$. So we can apply Theorem 6.3 with $2r$ replaced by r. Also note that

$$0 \leq \frac{U}{v}(Z) \leq 1 = \operatorname{osc}_{B(Q,r)} \frac{U}{v}, \quad \forall Z \in B(Q,r) \cap \Omega.$$

Note that $\operatorname{osc}(U/v) = (c_2 - c_1)^{-1} \operatorname{osc}(u/v)$. So if estimate (6.8) holds for U, v, then it also holds for u, v.

Let $A_{r/2}(Q)$ be the corkscrew point for Q at scale $r/2$. If $\frac{U}{v}(A_{r/2}(Q)) < C^{-2}$ (where $C > 1$ is the constant from Theorem 6.3), then by Theorem 6.3 we would have

$$0 \leq \frac{U}{v}(Z) \leq C \frac{U}{v}\big(A_{r/2}(Q)\big) < \frac{1}{C}, \quad \forall Z \in B(Q,r/2) \cap \Omega,$$

which would imply that $\operatorname{osc}_{B(Q,r/2)} \frac{U}{v} < \frac{1}{C}$, and hence, the desired result (with $\theta = \frac{1}{C}$).

If, on the other hand, $\frac{U}{v}(A_{r/2}(Q)) > C^{-2}$, then we apply Theorem 6.3 to obtain

$$\inf_{B(Q,r/2)} \frac{U}{v} \geq C^{-1} \frac{U}{v}\big(A_{r/2}(Q)\big) > C^{-3}.$$

This implies that $\operatorname{osc}_{B(Q,r/2)} \frac{U}{v} \leq 1 - C^{-3}$ and so (6.8) holds with $\theta = 1 - C^{-3}$. □

Finally, using an argument inspired by [16, Lemma 3.7] and Corollary 3.2, we can show the existence and uniqueness of the Green's function and harmonic measure with pole at infinity.

LEMMA 6.5
For any E, β, μ as in Definition 6.2, there exist an associated harmonic measure and Green's function with pole at infinity. Furthermore, they are both unique up to multiplication by a positive scalar.

Proof
First we show the existence of Green's function with pole at infinity. Fix $Q \in E$, and let $X_i = A_{2^i}(Q) \in \Omega$ denote a corkscrew point for Q at scale 2^i (i.e., $M\delta(X_i) > |X_i - Q| \geq 2^i$; see (6.5)). Define (for $i > 1$) $g_i(X) \equiv \frac{g(X,X_i)}{g(X_1,X_i)}$, where $g(X,Y)$ is the Green's function for $L_{\beta,\mu}$ with pole at Y (cf. Section 10 in [6]). Similarly define $\omega_i(S) \equiv \frac{\omega^{X_i}(S)}{g(X_1,X_i)}$. These are somewhat arbitrary normalizations (recall that the Green's function that we want to construct will only be unique modulo a multiplicative function).

We claim that, for any $K \subset\subset \mathbb{R}^n$, there exists a $C > 0$ (depending on K) such that $g_i(X) < C$ for all $X \in K$ and $i > i_0(K) \geq 0$ large enough (so that X_i lies away from K). Indeed, this follows from the fact that $g_i(X_1) \equiv 1$, Harnack's inequality (see Lemma 8.42 in [6]), and the existence of Harnack chains in Ω (see Lemma 2.1 in [6]). From this it follows that the g_i's are uniformly Hölder continuous on compacta (see Lemmas 8.42 and 8.98 in [6]). Thus (after the extraction of a diagonal subsequence) we have that $g_i \to g_\infty$, where the convergence is uniform on compacta in the continuous topology. Note that in $K \subset\subset \Omega$ the equation is uniformly elliptic, so the uniform convergence on compacta also implies (again perhaps passing to a subsequence) that $\nabla g_i \to \nabla g_\infty$ pointwise almost everywhere. Finally, note that the uniform convergence implies that $g_\infty \geq 0$ in Ω and $g_\infty = 0$ on E. Furthermore, by Harnack's inequality and $g_\infty(X_1) = 1$, it must be that $g_\infty > 0$ in Ω.

For any $Q \in E$ and $R > 0$, if $i > i_0(Q, r) \geq 0$ is large enough, then we know that $X_i \notin B(Q, 4R)$. Thus, we can estimate

$$\int_{B(Q,R)} \left|\nabla_x g_i(x)\right|^2 w(x)\, dx \leq CR^{-2} \int_{B(Q,2R)} g_i(x)^2 w(x)\, dx \leq C_R,$$

where the first inequality follows from Lemma 8.47 in [6] (a Caccioppoli-type estimate) and the second inequality follows from the fact that $|g_i| < C_R$ on $B(Q, 2R)$ by the argument in the above paragraph (and the fact that $w(x)$ is locally integrable). Thus, the g_i's are in $W_r(B(Q, R))$ with uniformly controlled norms for all i large enough and, applying Fatou's lemma, we conclude that g_∞ is in $W_r(B(Q, R))$ for all $Q \in E$ and $R > 0$.

As the g_i's are in $W_r(B(Q, R))$ with uniformly controlled norms, it follows from the weak formulation of $L_{\mu,\beta} g_i = 0$ (and integration by parts) that $L_{\mu,\beta} g_\infty = 0$ in Ω.

We will now show that g_∞ is the unique positive solution to $L_{\mu,\beta}$ which vanishes on E and is in $W_r(B(Q, R))$ for all $Q \in E$ and $R > 0$ (up to scalar multiplication). Indeed, assume that there existed some other f which was positive in Ω, zero on E, in $W_r(B(Q, R))$ for all $R > 0$, $Q \in E$, and satisfied $L_{\mu,\beta} f = 0$. We can multiply f by a positive scalar such that $f(X_1) = 1$. Then by Corollary 6.4 applied at larger and larger scales, it is clear that $f(X) = g_\infty(X)$ for all $X \in \Omega$: starting from

$$\left|\frac{g_\infty(X)}{f(X)} - 1\right| \leq C\left(\frac{|X - X_1|}{R}\right)^\gamma,$$

take $R \to \infty$.

It is time to establish the existence of ω_∞; let $Q \in E$ and $R > 0$. If i is big enough, then by Lemma 11.78 in [6] we have $\omega^{X_i}(B(Q, R)) \leq CR^{d-1} g(X_i, A_R(Q))$, where $A_R(Q)$ is the corkscrew point for Q at scale R. Thus, $\omega_i(B(Q, R)) \leq CR^{d-1} g_i(A_R(Q)) \leq C_{R,Q} < \infty$ by Harnack's argument above. This implies that the sequence of measures $\{\omega_i\}$ is uniformly bounded on any compact set and so, perhaps passing to a subsequence, there is an ω_∞ such that $\omega_i \rightharpoonup \omega_\infty$.

We note that, by the definition of g_i and ω_i, we have

$$\int_\Omega D_{\mu,\beta}^{-(n-d-1)} \nabla g_i \nabla \varphi \, dx = \int_E \varphi \, d\omega_i,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n \setminus X_i)$ (cf. Section 9 in [6]). Fix φ, and let $i \to \infty$ on both sides; using the fact that $\omega_i \rightharpoonup \omega_\infty$ and $g_i \to g_\infty$ in $W_r(K)$ for any compact K, we obtain that

$$\int_\Omega D_{\mu,\beta}^{-(n-d-1)} \nabla g_\infty \nabla \varphi \, dx = \int_E \varphi \, d\omega_\infty,$$

as desired.

The uniqueness of ω_∞ then follows from its integral relationship with g_∞ (cf. (6.4)) and the uniqueness of g_∞. □

We shall now show that, for the magic value of $\alpha = n - d - 2$, D_α is the Green's function with pole at infinity.

COROLLARY 6.6
Let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set for $d < n-2$ (not necessarily an integer), and let μ be a d-Ahlfors regular measure supported on E. If $\alpha = n - d - 2$, then $D_{\mu,\alpha}$ is the Green's function with pole at infinity for E (cf. Definition 6.2).

We remark that the Green's function with pole at infinity is unique modulo a multiplicative constant, hence, strictly speaking, the corollary above assures that any such Green's function is either $D_{\mu,\alpha}$ or its multiple.

Proof
We remark that the uniqueness in Lemma 6.5 does not require that $D_{\mu,\alpha}$ verify the last condition in (6.4); the proof shows that a nonnegative solution to the degenerate operator which vanishes at the boundary and satisfies the correct growth condition is unique up to a scalar multiple.

We have seen earlier that $D_{\mu,\alpha}$ is a positive solution to the degenerate elliptic operator which vanishes on the boundary; because of Lemma 6.5, it suffices to show that $D_{\mu,\alpha} \in W_r(B(Q,R))$ for all $Q \in E$ and $R > 0$. However, we know from (4.2) that $|\nabla D_{\mu,\alpha}| \leq C$. Since $w(X)$ is locally integrable in $\mathbb{R}^n$, the desired result follows. □

The fact that we are able to explicitly write down the Green's function with pole at infinity for magic α allows us to easily compute and bound the associated harmonic measure. The next theorem shows that, for any Ahlfors regular set E and magic $\alpha = n - d - 2$, the harmonic measure $\omega_{\mu,\alpha}$ is comparable to surface measure. As a corollary, we have the analogous result for harmonic measure with finite pole. Thus, as mentioned in the Introduction, there is absolutely no converse to the theorem that $\omega^X \in A_\infty(\sigma)$ when E is uniformly rectifiable. In fact, our result holds even when $d < n - 2$ is not an integer.

THEOREM 6.7
Let $n \geq 3$, $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set (for $d < n - 2$, not necessarily an integer), and let μ be a d-Ahlfors regular measure supported on E. If $\alpha = n - d - 2$, then the harmonic measure with pole at infinity $\omega_{\mu,\alpha}$ is comparable to $\sigma \equiv \mathcal{H}^d|_E$; that is, there is a constant $C > 0$ (depending only on n, d and the Ahlfors regularity constants for μ and σ) such that if we normalize $\omega_{\mu,\alpha}$ as we did in the construction, then

$$C^{-1} \leq \frac{d\omega_{\mu,\alpha}}{d\sigma}(Q) \leq C, \quad \forall Q \in E. \tag{6.9}$$

For the rest of the section, we will use the notation $a \simeq b$ if there is a constant C, depending only on n, d and the Ahlfors regularity of μ, such that $C^{-1} \leq \frac{a}{b} \leq C$.

Proof
For the sake of brevity, we will write $\omega = \omega_{\mu,\alpha}$, $D = D_{\mu,\alpha}$. Recall Lemma 11.78 from [6]: there exists a $C > 0$ (depending on n, d, and the Ahlfors regularity constants of E, μ) such that if $Q \in E$, $r > 0$ and $X \in \Omega \setminus B(Q, 2r)$, then

$$C^{-1} r^{d-1} g(X, X_0) \leq \omega^X\big(B(Q,r)\big) \leq C r^{d-1} g(X, X_0), \tag{6.10}$$

where $X_0 = A_r(Q)$ is a corkscrew point for Q at scale r, $g(-, X_0)$ is the Green's function with pole at X_0 associated to $L_{\mu,\alpha}$, and ω^X is the harmonic measure with pole at X associated to $L_{\mu,\alpha}$. Divide (6.10) by $g(X_i, X_1)$ for $i > 1$, where $X_i = A_{4^i r}(Q)$, and take $X = X_i$. This yields

$$C^{-1} r^{d-1} \frac{g(X_i, X_0)}{g(X_i, X_1)} \leq \frac{\omega^{X_i}(B(Q,r))}{g(X_i, X_1)} \leq C r^{d-1} \frac{g(X_i, X_0)}{g(X_i, X_1)}.$$

Then we let i tend to $+\infty$; we claim that

$$C^{-1} r^{d-1} D(X_0) \leq \omega\big(B(Q,r)\big) \leq C r^{d-1} D(X_0). \tag{6.11}$$

Indeed, arguing as in Lemma 6.5, Harnack's inequality implies that $G_i(-) \equiv \frac{g(X_i,-)}{g(X_i,X_1)}$ are uniformly, in i, bounded on compacta, are all positive and harmonic in $\mathbb{R}^n \setminus (E \cup \{X_i\})$, and zero on E. Passing to a (subsequential) limit, we get that $G_i(-) \to G_\infty$, a function which satisfies the definition of a Green's function at infinity. Using the uniqueness of said function, we can conclude that $G_\infty = CD$ (in fact, we compute that $C = D(X_1)$). Similarly, the measures $\omega_i \equiv \frac{\omega^{X_i}}{g(X_i, X_1)}$ form a precompact sequence in the weak topology and, with the G_i's, satisfy the last line of (6.4). This equation is preserved under the uniform convergence of the G_i's and the weak limit of the ω_i's and, as such, $\omega_i \rightharpoonup \omega_\infty$, the harmonic measure with pole at infinity. With this convergence in mind, the inequality (6.11) follows from the prior offset inequality letting $i \to \infty$ (one also has to use the doubling of ω to see that $\omega(\overline{B(Q,r)}) \leq C\omega(B(Q,r))$).

From (6.11) the conclusion of the lemma is easy: notice that $D(X_0) \simeq \delta(X_0) \simeq r$. It follows that $\omega(B(Q,r)) \simeq r^d$ for any $Q \in E$ and any $r > 0$. □

There is an analogue to Theorem 6.7 for a harmonic measure with finite pole associated to $L_{\mu,\alpha}$ (where α is magic). The proof essentially follows from the boundedness of quotients and the comparability of the Green's function with the harmonic measure (Theorem 11.146 and Lemma 11.78 in [6], respectively).

COROLLARY 6.8

Let $n \geq 3$, $E \subset \mathbb{R}^n$ be an Ahlfors regular set of dimension d (not necessarily integer), and let μ be an Ahlfors regular measure with support E. Assume that $\alpha = n - d - 2 > 0$, and define $D = D_{\mu,\alpha}$ and $L = L_{\mu,\alpha}$ as above. Finally, let $X \in \Omega = \mathbb{R}^n \setminus E$ be given, and denote by ω^X the associated harmonic measure with pole at X. Set $R = \operatorname{dist}(X, E)$. Then there is a constant C, depending only on n, d, and the Ahlfors regularity constant for μ, such that

$$C^{-1}\mu(A) \leq R^d \omega^X(A) \leq C\mu(A)$$
$$\text{for every measurable set } A \subset E \cap B(X, 100R). \tag{6.12}$$

We remark that (6.12) is a correct, homogeneous, finite pole version of the statement that the harmonic measure is proportional to the Hausdorff measure. A reader might be more accustomed to seeing it as a strengthened version of the A^∞ condition: for all $Q \in E$, $X \in \Omega$, $R = \operatorname{dist}(X, E)$,

$$C^{-1}\frac{\mu(A)}{\mu(B(X,100R))} \leq \frac{\omega^X(A)}{\omega^X(B(X,100R))} \leq C\frac{\mu(A)}{\mu(B(X,100R))}, \tag{6.13}$$

for every measurable set $A \subset E \cap B(X, 100R)$.

Proof

Of course, there is nothing special about the radius $100R$, but the result for larger R could be obtained by a change of pole.

Rather than proving (6.12), we will find it more convenient to prove that, for all $Q \in E$, $r > 0$ such that $B(Q,r) \subset B(X,100R)$ and $r < R/4$,

$$C^{-1} r^d \leq R^d \omega^X\big(B(Q,r)\big) \leq C r^d. \tag{6.14}$$

By Lemma 11.78 in [6], we know that $\omega^X(B(Q,r)) \simeq r^{d-1} g(X, A_r(Q))$, where g is the Green's function associated to the operator $L_{\alpha,\mu}$. Note that $g(X,-)$ and $D(-)$ are both positive solutions to L in $B(Q,R/2)$ which vanish on $B(Q,R/2) \cap E$; thus, we can apply Theorem 6.3 and get that

$$\begin{aligned}\frac{\omega^X(B(Q,r))}{r^d} &\simeq \frac{g(X,A_r(Q))}{r} \simeq \frac{g(X,A_r(Q))}{D(A_r(Q))} \\ &\simeq \frac{g(X,A_{R/2}(Q))}{D(A_{R/2}(Q))} \simeq \frac{g(X,A_{R/2}(Q))}{R},\end{aligned} \tag{6.15}$$

so (6.14) will follow once we check that $g(X, A_{R/2}(Q)) \simeq R^{1-d}$.

Choose $Y \in \Omega$ so that $R/20 < |Y - X| < R/10$, and consider $g(X,Y)$. It is clear that $\operatorname{dist}(Y,E) \simeq R \simeq \operatorname{dist}(A_{R/2}(Q),E) \simeq |A_{R/2}(Q) - Y|$. Thus, by the existence of Harnack chains (see Lemma 2.1 in [6]), because we can find a chain from Y to $A_{R/2}(Q)$ that does not get close to X, and by Harnack's inequality, we have $g(X, A_{R/2}(Q)) \simeq g(X,Y)$.

Finally, by equations (10.89) and (10.96) in [6], we have $g(X,Y) \simeq R^{1-d}$. Plugging this into (6.15) gives (6.14); then (6.12) follows from the Lesbesgue differentiation theorem (or, equivalently, by a straightforward covering argument). □

When E is rectifiable the nontangential limit of $|\nabla D_{\mu,\alpha}|$ exists (see Theorem 5.3), and, much as in the codimension 1 setting, gives us the Poisson kernel (for magic α).

LEMMA 6.9

Let $n \geq 3$, and let $E \subset \mathbb{R}^n$ be a d-Ahlfors regular set with $d < n-2$ an integer. Assume that E is d-rectifiable, let μ be a d-Ahlfors regular measure whose support is E, and let $\alpha = n - d - 2$. Then for σ-a.e. $Q \in E$, the density of $\omega_{\mu,\alpha}$ is given by $\Theta^d(\mu, Q)$ modulo a multiplicative constant. To be precise, if we fix the constants so that the Green's function with pole at infinity is the function $D_{\mu,\alpha}$ that was constructed above, then there exist $c_{n,d} > 0$ and $\widetilde{c}_{n,d} > 0$ such that

$$\frac{d\omega_{\mu,\alpha}}{d\sigma}(Q) = \widetilde{c}_{n,d}\, \text{n.t.}\lim_{x \to Q} \big|\nabla D_{\mu,\alpha}(x)\big|^{-(n-d-2)} \overset{(5.6)}{=} c_{n,d}\, \Theta^d(\mu, Q).$$

Proof

For simplicity, write $D \equiv D_{\mu,\alpha}$, and let ω be the associated harmonic measure with pole at infinity. For any $\varphi \in C_c^\infty$, we know that

$$\int_\Omega D^{-(n-d-1)} \nabla D \cdot \nabla \varphi \, dx = \int_E \varphi \, d\omega. \tag{6.16}$$

Let $Q \in E$ be a point where the nontangential limit of $|\nabla D|$ exists and where there is a unique tangent to E and tangent measure for μ (call it μ_∞). Such a $Q \in E$ can be found σ-a.e. (by the theory of rectifiable sets and Theorem 5.3 above). Let φ be a smooth approximation of $\chi_{B(0,1)}$ and, for $r_i \downarrow 0$, define $\varphi_i(x) \equiv \frac{\varphi(\frac{x-Q}{r_i})}{r_i^d}$. Adapting notation as in (5.1), we get

$$\begin{aligned}\int_\Omega D^{-(n-d-1)} \nabla D \cdot \nabla \varphi_i \, dx &= \frac{1}{r_i^{d+1}} \int_\Omega D^{-(n-d-1)}(x) \nabla D(x) \cdot (\nabla \varphi)\Big(\frac{x-Q}{r_i}\Big) \, dx \\ &= \int_{\Omega_i} D_i^{-(n-d-1)}(y) \nabla D_i(y) \cdot \nabla \varphi(y) \, dy,\end{aligned}$$

by a change of variables $y = \frac{x-Q}{r_i}$.

We now take the limit in i. Recall that E has a unique tangent d-plane T at Q, and that there is a nontangential limit $L = \text{n.t.} \lim_{x \to Q} |\nabla D|$. In addition, by the discussion in Section 5 (see, in particular, (5.5) and (5.6)), D_i tends to $D_\infty(x) = L\delta_T(x)$, and this convergence happens uniformly up to T. The convergence of ∇D_i to ∇D_∞ is only uniform on compact sets of $\mathbb{R}^n \setminus T$, but close to T the integrals are controlled uniformly because the gradients are bounded, so we get that

$$\begin{aligned}&\lim_{i \to \infty} \int_\Omega D^{-(n-d-1)} \nabla D \cdot \nabla \varphi_i \, dx \\ &\quad = L^{-(n-d-2)} \int_{\mathbb{R}^n \setminus T} (\delta_T)^{-(n-d-1)} \nabla \delta_T \cdot \nabla \varphi \, dx.\end{aligned} \tag{6.17}$$

Split the integral on the right-hand side of (6.17) into two pieces: one on a neighborhood T_ε of radius $\varepsilon > 0$ around T, and the other outside of T_ε. The integral on T_ε goes to zero as $\varepsilon > 0$ goes to zero, by the Lipschitz character of δ_T and φ and the local integrability of $\delta^{-(n-d-1)}$. For the integral on $\mathbb{R}^n \setminus T_\varepsilon$, we can integrate by parts. Notice that δ_T is a distance to the d-tangent plane, hence, it is a radial function in a space with $n-d$ dimensions, and hence, δ_T^{-n+d+2} is harmonic. Then

$$\begin{aligned}\int_{\mathbb{R}^n \setminus T_\varepsilon} (\delta_T)^{-(n-d-1)} \nabla \delta_T \cdot \nabla \varphi \, dx &= \int_{\{x | \operatorname{dist}(x,T) = \varepsilon\}} \varepsilon^{-(n-d-1)} \varphi \, d\mathcal{H}^{n-1} \\ &= c_1 \int_T \int_{\epsilon \mathbb{S}^{n-d-1}} \varepsilon^{-(n-d-1)} \varphi \, d\mathcal{H}^{n-d-1} \, d\mathcal{H}^d \\ &= c_2 \int_T \varphi \, d\mathcal{H}^d + O(\varepsilon),\end{aligned} \tag{6.18}$$

where c_1 and c_2 are dimensional constants that we shall never need to compute. We let ε tend to zero, return to (6.17), and obtain

$$\lim_{i\to\infty}\int_\Omega D^{-(n-d-1)}\nabla D\cdot\nabla\varphi_i\,dx = c_2 L^{-(n-d-2)}\int_T \varphi\,d\mathcal{H}^d. \tag{6.19}$$

Now we let φ tend to $\chi_{B(0,1)}$ (as BV functions); the right-hand side tends to $c_2 L^{-(n-d-2)} V_d$, where V_d is the volume of the d-dimensional unit ball. For the left-hand side, notice that, by (6.16),

$$\int_\Omega D^{-(n-d-1)}\nabla D\cdot\nabla\varphi_i\,dx = \int_E \varphi_i\,d\omega = \int_E \varphi_i\frac{d\omega}{d\sigma}\,d\sigma.$$

When i tends to $+\infty$ and Q is a point of density for $\frac{d\omega}{d\sigma}$ (which is true σ-a.e.), the quantity $\int_E \varphi_i(Z)|\frac{d\omega}{d\sigma}(Z)-\frac{d\omega}{d\sigma}(Q)|\,d\sigma(Z)$ tends to zero; we are left with $\frac{d\omega}{d\sigma}(Q)\int_E\varphi_i\,d\sigma$. If Q is also a point of density 1 for σ (which is again true σ-a.e.), then $\int_E\varphi_i\,d\sigma$ tends to $\int_T\varphi\,d\mathcal{H}^d$. Now we let φ tend to $\chi_{B(0,1)}$ and get that the left-hand side of (6.19) tends to $c_3\frac{d\omega}{d\sigma}(Q)$, where c_3 may depend on how we normalize $\mathcal{H}^d$ with respect to the Lebesgue measure. Thus, (6.19) implies that $\frac{d\omega}{d\sigma}(Q) = c_4 L^{-(n-d-2)}$. This is the desired result. □

In the specific case where $\mu=\sigma\equiv\mathcal{H}^d|_E$ and E is d-rectifiable, (5.6) tells us that $\text{n.t.}\lim_{x\to Q}|\nabla D_{\sigma,\alpha}| = c_{n,d}\Theta^d(\sigma,Q)^{-1/\alpha} = c_{n,d}$ by the fact that $\Theta^d(\sigma,Q)=1$ for σ-a.e. Q in any d-rectifiable set. Thus, we can conclude that $\omega_{\sigma,\alpha}$ is proportional to σ.

COROLLARY 6.10
Let E, d, n, α be as in Lemma 6.9. Then there exists a constant $c>0$ such that $\omega_{\sigma,\alpha}=c\sigma$, where, as above, $\sigma=\mathcal{H}^d|_E$.

Acknowledgments. The authors want to thank Camille Labourie for pointing out Borwein's work [2]. They would also like to thank the referees, whose careful readings and comments greatly improved this paper.

David's work was partially supported by Simons Collaborations in Mathematics and Physical Sciences (MPS) grant 601941, Agence Nationale de la Recherche (ANR) GEOMETRYA project grant ANR-12-BS01-0014, and European Community Marie Curie MANET project grant 607643 and GHAIA project H2020 grant 777822. Engelstein's work was partially supported by a National Science Foundation (NSF) postdoctoral fellowship, NSF grant DMS-1703306, and by David Jerison's grant DMS-1500771. Mayboroda's work was partially supported by NSF INSPIRE grant DMS-1344235, NSF grant DMS-1839077, a Simons Foundation Fellowship, and Simons Collaborations in MPS grant 563916. This project is based upon work supported by NSF grant DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the 2017 spring semester.

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