Covariance of the extended holonomy

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It has been pointed out that the holonomy of generic extended loops is not gauge covariant. We show how to define a family of extended loops for which previous criticism does not apply. We also give sufficient conditions that extended loops must satisfy in order to yield covariant holonomies. This makes a quantum representation for Yang–Mills theories and gravity based on extended loops viable.

I. INTRODUCTION

The use of loop based variables to study gauge theories can be traced all the way back to Faraday. In the context of Yang–Mills theory holonomies have been widely used to analyze the quantization, both in the continuum and in the lattice in gauge invariant terms. They have also been used to base a complete quantum representation in terms of loops, the loop representation, again both for Yang–Mills theories and gravity (see [1] and references therein). Broadly viewed, the holonomies can be seen as a way of providing test functions against which to smear the fields of a theory. Because loops are one dimensional objects, the resulting smearings tend to be distributional in nature. This creates regularization problems when operators act on the holonomies. In the context of gravity for instance, where one wishes to have quantum states that are invariant under diffeomorphisms, this has led to considering functions of thickened or framed loops [2]. In the Yang–Mills context, the definition of the inner product in the loop representation would require summations over families of loops that are not well defined. To deal with these issues the concept of extended loops and the ensuing extended holonomies was introduced [3, 4]. The idea is to construct smearing functions that share some of the properties of the smearings provided by loops, but that are more general and have three-dimensional support. Maps from the gauge connection to the elements of a group formally similar to the ones used to define the non-Abelian holonomy can be constructed in terms of those smeared functions to construct an extended holonomy.

It was shown [5], however, that in spite of the formal analogy with the case of loops, some convergence issues appeared in the expansion which rendered the extended holonomy to be non-gauge covariant. A potential solution was suggested [6] to this problem by considering certain subsets of extended holonomies. However, the solution was not entirely satisfactory since the proposed subsets were ad-hoc in nature. In spite of these difficulties the techniques attracted some attention in the mathematical and particle physics literature [7]. Here we would like to overcome those limitations by providing a generic definition of the families of extended loops that yield properly covariant holonomies.

The structure of this paper is as follows. In the next section we will review the concept of multitangents in ordinary holonomies. In section 3 we discuss extended holonomies. In section 4 we will propose the construction of extended loops that yield covariant extended holonomies. Section 5 proposes an explicit construction of extended loops leading to covariant holonomies. We end with a discussion.

II. ORDINARY HOLONOMIES AND MULTITANGENTS

The holonomy (whose trace is the *Wilson loop*) of a connection one-form $A(x) = A_a(x)dx^a$ is given by its path ordered exponential along a loop[1]. It can be rewritten as,

$$U_A(\gamma) = 1 + \sum_{n=1}^{\infty} (-i)^n \int d^3x_1 \cdots d^3x_n A_{a_1}(x_1) \cdots A_{a_n}(x_n) T^{a_1 \cdots a_n}(x_1, \ldots, x_n, \gamma) \tag{2.1}$$

where γ is a loop with a base point o which we take as its origin and the loop dependent multitangents T are given by,

$$\begin{aligned} T^{a_1 \cdots a_n}(x_1, \ldots, x_n, \gamma) &= \int_\gamma dy_n^{a_n} \int_0^{y_n} dy_{n-1}^{a_{n-1}} \cdots \int_0^{y_2} dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) \\ &= \oint_\gamma dy_n^{a_n} \cdots \oint_\gamma dy_1^{a_1} \delta(x_n - y_n) \cdots \delta(x_1 - y_1) \Theta_\gamma(0, y_1, \ldots, y_n) \end{aligned} \tag{2.2}$$

with the multi Heaviside function $\Theta_\gamma(0, y_1, \ldots, y_n)$ ordering the points along the loop starting at the origin and the Dirac deltas are three dimensional ones. These relations define the multitangents of "rank" n. It will be convenient to introduce the notation

$$T^{\mu_1 \cdots \mu_n}(\gamma) = T^{a_1 x_1 \cdots a_n x_n}(\gamma) = T^{a_1 \cdots a_n}(x_1, \ldots, x_n, \gamma) \, , \tag{2.3}$$

with $\mu_i \equiv (a_i x_i)$, which suggests better the role played by the x variables under diffeomorphisms [3, 4].

The multitangents satisfy a set of *algebraic identities*, which follow directly from properties of the Heaviside function,

$$T^{\underline{\mu_1 \cdots \mu_k} \mu_{k+1} \cdots \mu_n} \equiv \sum_{P_k} T^{P_k(\mu_1 \cdots \mu_n)} = T^{\mu_1 \cdots \mu_k} \, T^{\mu_{k+1} \cdots \mu_n}, \tag{2.4}$$

with the summation over all the permutations P_k of the first k of the μ variables which preserve the ordering of the $\mu_1, \ldots, \mu_k$ and the $\mu_{k+1}, \ldots, \mu_n$ among themselves.

They also satisfy a *differential constraint*,

$$\frac{\partial}{\partial x_i^{a_i}} T^{a_1 x_1 \cdots a_i x_i \ldots a_n x_n} = \big(\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1}) \big) T^{a_1 x_1 \cdots a_{i-1} x_{i-1} \; a_{i+1} x_{i+1} \cdots a_n x_n}, \tag{2.5}$$

with x_0 and x_{n+1} equal to the origin of the loop o.

The holonomy is gauge covariant, that is, if we consider a gauge transformation U,

$$A_a \to A'_a = U A_a U^\dagger - (\partial_a U) \, U^\dagger, \tag{2.6}$$

[1] It should be noted that in this context a "loop" is an equivalence class of curves that differ by retracings called "trees". All curves in the class yield the same holonomy and have the same multitangents.

we have that,

$$U_{A'}(\gamma) = U_o U_A(\gamma) U_o^\dagger \tag{2.7}$$

Notice that the gauge transformation is a function of point and U_o is the gauge transformation evaluated at the origin of the loop. The covariance of the holonomy follows from the path ordered nature along the loop of the holonomy. It is customary to take the loop origin at infinity and the small gauge transformations (the ones connected to the identity) as the identity at infinity. However, we will not make assumptions about this in this paper.

A gauge field can be viewed as stemming from a representation of the group of loops in a Lie group G. Every representation defines a connection up to gauge transformations that leads to expansions for the holonomy that are convergent (for a detailed discussion see [8]).

The multitangents transform as multivector densities under the subgroup of coordinate transformations that leaves the base point o fixed. That is, if one has a transformation,

$$x^a \longrightarrow x'^a = D^a(x) \tag{2.8}$$

then

$$T^{a_1 x'_1 \cdots a_n x'_n}(D\gamma) = \frac{\partial x'^{a_1}_1}{\partial x^{b_1}_1} \cdots \frac{\partial x'^{a_n}_n}{\partial x^{b_n}_n} \frac{1}{J(x_1)} \cdots \frac{1}{J(x_n)} T^{b_1 x_1 \cdots b_n x_n}(\gamma)\,. \tag{2.9}$$

where J is the Jacobian of the transformation. We will call objects whose components transform in this way *multitensors*.

III. EXTENDED LOOPS AND HOLONOMIES

Given a multitensor $E = (E^0, E^{\mu_1}, \cdots, E^{\mu_1 \cdots \mu_n}, \cdots)$, where E^0 is a real constant that in what follows we take equal to one, we can define an extended holonomy,

$$U_A(E) = 1 + \sum_{n=1}^{\infty} \int (-i)^n \, d^3x_1 \cdots d^3x_n A_{\mu_1} \cdots A_{\mu_n} E^{\mu_1 \cdots \mu_n}\,. \tag{3.1}$$

with E satisfying the differential and algebraic constraints. The multitensors E are a generalization of the multitangents. They have a product,

$$(E_1 \times E_2)^{\mu_1 \ldots \mu_n} = \sum_{i=0}^{n} E_1^{\mu_1 \ldots \mu_i} E_2^{\mu_{i+1} \ldots \mu_n}, \tag{3.2}$$

that is related to the product of loops, which form a group [1],

$$T(\gamma_1 \circ \gamma_2) = T(\gamma_1) \times T(\gamma_2)\,. \tag{3.3}$$

The product is associative and satisfies the differential constraint.

It is convenient to rewrite the generalized holonomy as,

$$U_A(E) = \sum_{n=0}^{\infty} A_{\mu_1} \cdots A_{\mu_n} E^{\mu_1 \cdots \mu_n}, \tag{3.4}$$

where from now on $A_a(x) = -iA_a(x)$ and implicit in the sum are the integrals along space, with the term with $n = 0$ (E with zero components, which we will call E^0) equal to one. It is also useful to define the term containing n powers of the connection as,

$$U_A^{(n)}(E) = A_{\mu_1} \cdots A_{\mu_n} E^{\mu_1 \cdots \mu_n}. \tag{3.5}$$

Let us address the issue of gauge invariance. Consider an infinitesimal gauge transformation,

$$A^g = A + dg + [A, g] \tag{3.6}$$

the terms in the sum transform as,

$$U^{(n)}_{A^g} = U^{(n)}_A + \left[U^{(n-1)}_A(E), g_o\right] + f^{(n)}_{(A,g)}(E) - f^{(n-1)}_{(A,g)}(E), \tag{3.7}$$

where g_o stands for the infinitesimal local gauge transformation evaluated at the origin of the loop and,

$$f^{(n)}_{(A,g)}(E) = \sum_k E^{\mu_1\ldots\mu_n} A_{\mu_1} \ldots A_{\mu_{k-1}} [A, g]_{\mu_k} A_{\mu_{k+1}} \ldots A_{\mu_n}, \tag{3.8}$$

and therefore,

$$\sum_{n=0}^{N} U^{(n)}_{A^g}(E) = \sum_{n=0}^{N} U^{(n)}_A(E) + \left[\sum_{n=0}^{N} U^{(n)}_A(E), g_o\right] - \left[U^{(N)}_A(E), g_o\right] + f^{(N)}_{(A,g)}(E). \tag{3.9}$$

If $U_A(E)$ converges, $U^{(N)}_A(E) \to 0$ for $N \to \infty$, the 2nd term corresponds to the infinitesimal gauge transformation at the origin, and the 3rd term vanishes. Therefore, the extended holonomy is covariant if and only if $f^{(N)}_{(A,g)}(E) = 0$ for $N \to \infty$. It is not obvious that this holds for all E and A. In a nutshell, this was one of the points of the criticism in [5]. The possibility of having extended loops for which $f_{(A,g)}$ is non-vanishing allows in principle [5] to find counterexamples of extended loops that do not lead to gauge covariant holonomies. In what follows we will show how to construct explicitly extended loops that lead to gauge covariant holonomies and we will establish sufficient conditions that ensure the covariance.

IV. CONSTRUCTING EXTENDED INVARIANTS

Let us see that one can define extended loops that lead to gauge covariant extended holonomies. The starting point of the construction is the invariance for the case of loops, equation (2.7), and the observation made in [6]: one needs to restrict the type of extended loops considered, as we shall see in detail. We will confine the discussion to $SU(N)$ but it can be extended to other Lie groups. Let us define, as we did with the extended loop a multitangent $T(\gamma)$ with $T^0(\gamma) = 1$ and introduce its product with a "multiconnection" (a product of connections),

$$T(\gamma) \cdot A \equiv U_A(\gamma) = \sum_{n=0}^{\infty} T^{\mu_1\ldots\mu_n} A_{\mu_1} \ldots A_{\mu_n}, \tag{4.1}$$

and

$$T(\gamma) \cdot A_g = U_{A_g}(\gamma) = U_g \left(T(\gamma) \cdot A\right) U_g^\dagger, \tag{4.2}$$

where A_g is the gauge transformed A connection.

A first example of an extended loop could be given by the real power of a loop. Integer powers of $T(\gamma)$ are trivially defined,

$$\left(T(\gamma)\right)^n = T(\gamma) \times T(\gamma) \times \cdots \times T(\gamma), = T(\gamma \circ \gamma \circ \cdots \circ \gamma), \tag{4.3}$$

and $(T(\gamma))^n \cdot A = U_A(\gamma)^n$ is also gauge covariant.

Let us consider non-integer powers $T(\gamma)^\lambda$ with λ real resulting in $T(\gamma)^\lambda \cdot A = U_A(\gamma)^\lambda$. This is non trivial because in general the real power of a complex number (recall that the connections are complex) is not uniquely defined.

Indeed since $z = \exp(2\pi n i)\, z$, the quantity $(\exp(2\pi n i) z)^\beta$ is multi-valued. A real power can be uniquely defined via the principal part of the logarithm, Log,

$$\mathrm{Log}\,(z) \equiv \ln(|z|) + i \mathrm{Arg}\,(z)\,, \tag{4.4}$$

with $-\pi \leq \mathrm{Arg}\,(z) \leq \pi$.

For $z = 1 + u$ with $|u| < 1$,

$$\mathrm{Log}(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} u^n \tag{4.5}$$

If one defines $\tilde{z} = \exp(\mathrm{Log}(z))$ the phase of $\tilde{z}$ is between $\exp(-i\pi)$ and $\exp(i\pi)$ and $\tilde{z}^\beta$ is well defined,

$$\tilde{z}^\beta = \exp(\beta \mathrm{Log}(z)). \tag{4.6}$$

It will be useful in what follows to introduce $\tilde{z}^\beta$ in terms of an analytic extension as follows. We define

$$\tilde{z}(a)^\beta \equiv \exp\left(\beta \mathrm{Log}\,(1 + a(z-1))\right) = \exp\left(\beta \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(a(z-1)\right)^n\right), \tag{4.7}$$

which exists $\forall a < 1/(|z-1|)$ real. Therefore one can write,

$$\tilde{z}^\beta = \mathrm{A.E.}\left(\tilde{z}(a)^\beta\right)|_{a=1}. \tag{4.8}$$

With A.E. means the analytic extension of the expression given in (4.7). Note that even though the series has a finite radius of convergence, the analytic extension of the exponent is well defined and leads to the principal value of the logarithm, the exponential also has a well defined extension. The idea is to repeat this argument for the mappings from the connections to the Lie group induced by the multitangents $T(\gamma)$.

Let us define

$$T(a,\gamma) \equiv (1-a)\mathcal{I} + aT(\gamma), \tag{4.9}$$

with $\mathcal{I}$ a multitensor that has as only non-vanishing component $\mathcal{I}^0 = 1$, that is, $\mathcal{I}^{\mu_1\cdots\mu_n} = 0$ for $n \neq 0$. To put it another way, $T^0(a,\gamma) = 1$ and $T^{\mu_1\cdots\mu_n}(a,\gamma) = aT^{\mu_1\cdots\mu_n}(\gamma)$. We therefore have that,

$$T(a,\gamma)\cdot A = (1-a)I + aU_A(\gamma) \equiv U_A(a,\gamma), \tag{4.10}$$

with I the identity in the group, and for $a = 1$ we have that $T(1,\gamma)\cdot A = U_A(1,\gamma) = U_A(\gamma)$. Notice however, that $U(a,\gamma)$ is not an element of the group $SU(N)$ if $a \neq 1$.

In order to define the family of extended loops that will lead to a covariant holonomy, let us consider the formal expansion of the logarithm of T,

$$F(a,\gamma) \equiv \mathrm{Log}\left(T(a,\gamma)\right) = \sum_{i=1}^{\infty} \frac{(-1)}{i} \left(\mathcal{I} - T(a,\gamma)\right)^i, \tag{4.11}$$

(the $i-th$ power is computed with the multitensor product introduced before) and therefore,

$$F(a,\gamma)\cdot A=\sum_{i=1}^{\infty}\frac{(-1)}{i}\left(I-U_A(a,\gamma)\right)^i, \tag{4.12}$$

and we shall see that the series converges for a sufficiently small. Let $\|M\|$ be the Frobenius norm of the matrix M defined as,

$$\|M\|\equiv\sqrt{\mathrm{Tr}\left(M^\dagger M\right)}=\sqrt{\sum_i|\lambda_i(M)|^2} \tag{4.13}$$

with λ_i the eigenvalues, for diagonalizable matrices. An important property of the Frobenius norm is that it is submultiplicative $\|AB\|\leq\|A\|\|B\|$.

Let us consider the norm of $I-U_A(a,\gamma)$. If we have that

$$\|I-U_A(a,\gamma)\|<1, \tag{4.14}$$

given the submultiplicative property of the norm, this immediately implies the expansion of the logarithm converges and therefore $F(a,\gamma)$ is well defined. Evaluating,

$$\begin{aligned}\|I-U_A(a,\gamma)\| &= \|aI-aU_A(\gamma)\|=\sqrt{\mathrm{Tr}\left(\left(aI-aU_A^\dagger(\gamma)\right)\left(aI-aU_A(\gamma)\right)\right)}\\ &= \sqrt{\mathrm{Tr}\left(2a^2I-a^2\left(U_A^\dagger(\gamma)+U_A(\gamma)\right)\right)}\leq 2a\sqrt{N},\end{aligned} \tag{4.15}$$

which can be seen given the generic form of a unitary transformation in $SU(N)$. Therefore $F(a,\gamma)\cdot A$ is well defined if $a<1/(2\sqrt{N})$.

Let us proceed to verify that the extended holonomy transforms appropriately. Given that $\ln U_A(a,\gamma)=F(a,\gamma)\cdot A$ and taking into account that,

$$U_{A_g}(a,\gamma)=U_o(1-a)U_o^\dagger+aU_oU_A(\gamma)U_o^\dagger=U_oU_A(a,\gamma)U_o^\dagger, \tag{4.16}$$

where U_o is the gauge transformation at the loop origin, and that,

$$U_A(a,\gamma)=\exp\left(F(a,\gamma)\cdot A\right), \tag{4.17}$$

it follows from (4.12) that

$$F(a,\gamma)\cdot A_g=U_oF(a,\gamma)\cdot AU_o^\dagger. \tag{4.18}$$

Equality (4.17) holds for $a<1/(2\sqrt{N})$ and can be extended analytically to the value $a=1$. The resulting series in the exponent of (4.17) converges for $a<1/(2\sqrt{N})$, and can be analytically extended in the same way that for a complex number x the analytic extension leads to $\ln(|x|)+i\mathrm{Arg}(x)$ with $\mathrm{Arg}(x)$ is the phase of x in the interval $[-\pi,\pi]$. Summarizing, the analytic extension of (4.17) to a=1 is perfectly well defined.

Evaluating at $a=1$ we have,

$$U_A(1,\gamma)=U_A(\gamma), \tag{4.19}$$

and the evaluation for $a = 1$ of the exponential of F yields a distribution in the space of multitensors that coincides with T,

$$T(\gamma) = T(1, \gamma) = \text{A.E.}\left(\exp\left(F(a, \gamma)\right)\right)|_{a=1}, \tag{4.20}$$

where in this case A.E. means that, when considered as a mapping from the space $\mathcal{G}$ of connections fields to $SU(N)$, $T(\gamma)$ and the exponential have identical action. This is analogous to when one defines a distribution as (non-existent) limit of functions, that exists as a linear map from test functions to the real numbers. Note that $F(a, \gamma)|_{a=1} \cdot A$ belongs in the algebra $su(N)$ and is well defined and its exponential is unitary.

The analytic extension allows various generalizations of the concept of loop that constitute extended loops for which the holonomy transforms covariantly.

For instance, starting from (4.17) and using (4.18) we can define a gauge covariant real power of a holonomy,

$$U_{A_g}(a, \gamma)^\lambda = \exp\left(\lambda F(a, \gamma) \cdot A_g\right) = \exp\left(U_o \lambda F(a, \gamma) \cdot A U_o^\dagger\right) = U_o U_A^\lambda(a, \gamma) U_o^\dagger, \tag{4.21}$$

and $U_A(a = 1, \gamma)^\lambda$ is the real λ-th power of a holonomy associated with

$$T^\lambda(\gamma) = \text{A.E.} \exp\left(\lambda F(a, \gamma)\right)|_{a=1}, \tag{4.22}$$

of the mapping $\mathcal{G} \to SU(N)$, and is an example of an extended loop that leads to a covariant holonomy. Here, the limit taken after evaluating the contraction with a gauge connection converges to a group element, just as the limit of a family of functions converge to a Dirac delta only if the limit is taken after acting on a function. There are many examples of covariant extensions of loops. The technique presented obviously includes ordinary loops. Real powers of loops are clearly invertible and form a group, the associated holonomies are unitary and gauge covariant.

The covariant extensions stem from observing that the analytic extension,

$$F_A(\gamma) \equiv F(a, \gamma) \cdot A|_{a=1}, \tag{4.23}$$

belongs to the $SU(N)$ algebra (see appendix) and is gauge covariant for all loops γ. That implies that,

$$F_A(\gamma_1) + F_A(\gamma_2), \lambda F_A(\gamma), \left[F_A(\gamma_1), F_A(\gamma_2)\right], \tag{4.24}$$

lead, through exponentiation, to elements of the group that are gauge invariant and define an extended loop algebra with their corresponding extended holonomies.

The idea that this algebra allows to define smoothed loops can be confirmed considering a bi-parametric family $\gamma(\alpha, \beta)$ of loops. The quantity,

$$\exp\left(\int d\alpha d\beta \lambda(\alpha, \beta)\left[F(a, \gamma(\alpha, \beta)) \cdot A\right]\right)\Bigg|_{a=1} = U_A(\gamma(\alpha, \beta), \lambda(\alpha, \beta)) \tag{4.25}$$

where $\lambda(\alpha, \beta)$ a suitable functional coefficient for each member of the family is an example of smoothed loop. The analytic extensions of $F(a, \gamma).A|_{a=1}$ and its exponential are well defined and allow to define a generating set of the space of extended loops. Therefore, in the same way that we may define distributions as limits of functions when considered as linear applications acting on certain space of functions, one can define a family of extended loops $E(\lambda F_\gamma)$ as

$$T^\lambda(\gamma) = \text{A.E.} \exp\left(\lambda F(a, \gamma)\right)|_{a=1} \equiv E(\lambda F_\gamma), \tag{4.26}$$

and construct,

$$E(\lambda F_{\gamma_1} + \mu F_{\gamma_2}) = \exp\left(\lambda F(a,\gamma_1) + \mu F(a,\gamma_2)\right)|_{a=1}, \tag{4.27}$$

or $E(\int d\alpha d\beta \lambda(\alpha,\beta) F_{\gamma(\alpha,\beta)})$, or $E\left([F_{\gamma_1}, F_{\gamma_2}]\right)$ and E of multiple commutators defined analogously. All of them take the form,

$$E = \left(E^0 = 1, E^{\mu_1}, E^{\mu_1,\mu_2}, \ldots, E^{\mu_1\cdots\mu_n}\right), \tag{4.28}$$

but the $E^{\mu_1\cdots\mu_n}$ satisfy additional conditions to the differential and algebraic constraints: they are exponentials of F's, as constructed above. The components of E lead to a series that converges when $a = 1$ to unitary and gauge covariant transformations and are examples of extensions that satisfy equations (3.3) and (3.4) of [5].

Notice that the F's satisfy the differential constraint and a simpler version of the algebraic constraint given by $F(a,\gamma)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}\Big|_{a=1} = 0$. This constraint is a key ingredient in the construction of elements of the Lie group algebra [4].

To prove that the constraint is satisfied, let us consider the continuous binomial expansion

$$T^\lambda(a,\alpha) = (\mathcal{I}(1-a) + aT(\alpha))^\lambda = \sum_{m=0}^{\infty} \binom{\lambda}{m} (1-a)^m a^{\lambda-m} T^{\lambda-m}(\alpha), \tag{4.29}$$

with α a loop. Given that the multitangents satisfy the algebraic constraint

$$T^{\lambda-m}(\alpha)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n} = T^{\lambda-m}(\alpha)^{\mu_1\cdots\mu_k} T^{\lambda-m}(\alpha)^{\mu_{k+1}\cdots\mu_n}, \tag{4.30}$$

differentiating the product in $T^\lambda(a,\alpha)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}$,

$$\begin{aligned}
\frac{d}{d\lambda} T^\lambda(a,\alpha)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n} &= \sum_{m=0}^{\infty} (1-a)^m \frac{d}{d\lambda}\left(\binom{\lambda}{m} a^{\lambda-m}\right) T^{\lambda-m}(\alpha)^{\mu_1\cdots\mu_k} T^{\lambda-m}(\alpha)^{\mu_{k+1}\cdots\mu_n} \\
&+ \sum_{m=0}^{\infty} \binom{\lambda}{m} (1-a)^m a^{\lambda-m} \frac{d}{d\lambda} T^{\lambda-m}(\alpha)^{\mu_1\cdots\mu_k} T^{\lambda-m}(\alpha)^{\mu_{k+1}\cdots\mu_n} \\
&+ \sum_{m=0}^{\infty} \binom{\lambda}{m} (1-a)^m a^{\lambda-m} T^{\lambda-m}(\alpha)^{\mu_1\cdots\mu_k} \frac{d}{d\lambda} T^{\lambda-m}(\alpha)^{\mu_{k+1}\cdots\mu_n},
\end{aligned} \tag{4.31}$$

where $T^{-m}(\alpha) \equiv T^m(\bar{\alpha})$ [4]. Evaluating the derivative at $\lambda = 0$, we obtain the algebraic constraint of the generators of $T^\lambda(a,\alpha)$,

$$\begin{aligned}
F(a,\alpha)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n} &= \frac{d}{d\lambda} T^\lambda(a,\alpha)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}\Big|_{\lambda=0} \\
&= \sum_{m=1}^{\infty} (\frac{1}{a} - 1)^m \frac{(-1)^{m-1}}{m} (1 + \gamma(m-1)) T^m(\bar{\alpha})^{\mu_1\cdots\mu_k} T^m(\bar{\alpha})^{\mu_{k+1}\cdots\mu_n} \\
&+ F(\alpha)^{\mu_1\cdots\mu_k} \mathcal{I}^{\mu_{k+1}\cdots\mu_n} + \mathcal{I}^{\mu_1\cdots\mu_k} F(\alpha)^{\mu_{k+1}\cdots\mu_n},
\end{aligned} \tag{4.32}$$

where γ is the Euler–Mascheroni constant. Finally, for $1 < k < n$ and $a \to 1$, $F(a,\gamma)^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}\Big|_{a=1} = 0$ since $\mathcal{I}^{\mu_1\cdots\mu_n} = 0$ for $n \geq 1$. This ensures that the product $F \cdot A$ is in the algebra (see appendix). One can also demonstrate that the exponential of any quantity satisfying the homogeneous algebraic constraint produces a multitensor that satisfies the algebraic constraint.

V. AN EXPLICIT CHARACTERIZATION OF EXTENDED LOOPS LEADING TO COVARIANT HOLONOMIES

Up to now we have followed a constructive process to identify extended loops, either considering real powers of a loop or more general constructions, always starting from ordinary loops. However, it is convenient to have a notion of extended loops that lead to covariant holonomies, independent of their construction procedure .

The non covariance of the extended holonomy discussed in [5] was based on a two fold argument. One of them can be solved by regularization: gauge transformations of extended holonomies based on real powers of loop holonomies may appear to be non invariant due to the appearance of different branches of the exponentiation of a function by a real parameter. The complex power function is a multi-valued function The principal branch of the function is obtained by replacing $\ln(z)$ with the principal branch of the logarithm. If one adds to this observation that ordinary loops lead to gauge invariant holonomies then this leads to extensions that also yield gauge invariant holonomies as we showed in previous sections.

The second problem regards the difficulty to prove that the gauge transformation of the holonomy is correct in the limit where the holonomy includes contributions with an infinite number of nodes, that is, connection fields contracted with the extended loop (3.9). That is, the last term in (4.12) vanish when $i \to \infty$. It is known that holonomies constructed in terms of loops transform correctly and one can prove it as in [9], by partitioning the loop γ into a collection of infinitesimal straight segments $S_{z_i}^{z_{i+1}}$ that form a polygonal that, as you increase the number of segments approaches the curve . For a large number $N+1$ of segments

$$U_A(\gamma) = \mathcal{P}\exp\left\{\int_\gamma A_a(x)dx^a\right\} \simeq \Big(1 + (x_{N+1}^{a_N} - x_N^{a_N})A_{a_N}(x_N) + O((x_N - x_{N-1})^2)\Big)\ldots \times\Big(1 + (x_1^{a_0} - x_0^{a_0})A_{a_0}(x_0) + O((x_1 - x_0)^2)\Big), \qquad (5.1)$$

and we note that in the limit $N \to \infty$ the right hand side reproduces the left hand side. We can identify the loop holonomy as an ordered product of infinitesimal open path parallel transports

$$U_A(S_{x_i}^{x_{i+1}}) = 1 + (x_{i+1}^{a_i} - x_i^{a_i})A_{a_i}(x_i) + O((x_{i+1} - x_i)^2) \qquad (5.2)$$

where it has been assumed that the distance between one point and the next is infinitesimal. It can easily be proven that arbitrary gauge transformations act as

$$U_{A^g}[S_{x_i}^{x_{i+1}}] = U_g(x_i)U_A[S_{x_i}^{x_{i+1}}]U_g^\dagger(x_{i+1}) + O(x_{i+1} - x_i)^2,$$

with g the element of the group associated with the gauge transformation, and we immediately get in the limit where the infinitesimal intervals go to zero

$$U_{A^g}[\gamma] = U_g(x_0)U_A[\gamma]U_g^\dagger(x_0),$$

where we have used $x_{N+1} = x_0$.

To understand how these results extend to the case of extended loops, let us rewrite the above expression in terms of multitangents. To this aim, it is convenient to partition space into cubes and consider the set of cubes that are intersected by the path. We consider a cubic lattice characterized by a lattice size l, and substitute the curve inside each cell by a straight line, entering through y_i and exiting through y_{i+1}, and composing the straight lines into an $N+1$ sided polygonal. Substituting the tangent vector $dy^a \to y^a_{i+1} - y^a_i = \epsilon_i \hat{u}^a_i$, with ϵ_i of order l and $\hat{u}^a_i$ unit vectors, we get,

$$
\begin{aligned}
T^{a_1\ldots a_n}(x_1,\ldots,x_n,\gamma) &= \int_\gamma dy_n^{a_n} \int_0^{y_n} dy_{n-1}^{a_{n-1}} \cdots \int_0^{y_2} dy_1^{a_1} \delta(x_n - y_n)\cdots\delta(x_1-y_1) \\
\to T_N^{a_1x_1\ldots a_nx_n}(\gamma) &= \sum_{\substack{i_1 < i_2 \ldots < i_n \\ 0 \le i_j \le N}} \epsilon_{i_1}\cdots\epsilon_{i_n} \hat{u}^{a_1}_{i_1}\cdots\hat{u}^{a_n}_{i_n} \delta(y_{i_1} - x_1)\cdots\delta(y_{i_n} - x_n), \quad (5.3)
\end{aligned}
$$

where $y^a_{i_j} = \sum_{k=0}^{i_j - 1} \epsilon_k \hat{u}^a + y^a_0$. Notice that this expression is valid for $n \le N+1$ since the inequalities in the sum cannot be satisfied if $n > N+1$, in that case $T_N^{a_1x_1\ldots a_nx_n} = 0$.

Taking into account (5.1) we have that,

$$
U_A(\gamma) = \lim_{N\to\infty} \left(1 + T_N^{\mu_1} A_{\mu_1} + \cdots + T_N^{\mu_1\ldots\mu_p} A_{\mu_1}\cdots A_{\mu_p} + \cdots + T_N^{\mu_1\ldots\mu_{N+1}} A_{\mu_1}\cdots A_{\mu_{N+1}}\right) \quad (5.4)
$$

and the term,

$$
T_N^{\mu_1\ldots\mu_N} A_{\mu_1}\cdots A_{\mu_N} = O\left(\left(\tfrac{1}{N}\right)^N\right), \quad (5.5)
$$

and this is true for any finite connection A since $T_N^{\mu_1\ldots\mu_N}$ will have N factors of order $l = O(L/N)$. In particular for both A and its gauge transformed A_g. Let us note that the polygonal multitangents T_N satisfy the algebraic constraint and the differential constraint up to higher order terms,

$$
\begin{aligned}
\partial_{x_i^a} T_N^{a_1x_1,\ldots,a_ix_i,\ldots,a_nx_n} &= \Theta(N-n)\left[\delta(x_i - x_{i+1}) - \delta(x_i - x_{i-1})\right] T_N^{a_1x_1,\ldots,a_{i-1}x_{i-1},a_{i+1}x_{i+1},\ldots,a_nx_n} \\
&\quad + O(\frac{1}{N^2}). \quad (5.6)
\end{aligned}
$$

where $\Theta(N-n)$ is 1 for $N \ge n$.

Since expression (5.4) is equivalent to (5.1) which is manifestly gauge invariant in the limit $N \to \infty$, (5.4) is too. This can be verified directly observing that the remainder term of $f^{(N)}_{(A,\Lambda)}(E)$ of (3.9) is $O((1/N))$ and vanishes in the limit $N \to \infty$ due to (5.5) and (5.6).

In what concerns the algebraic constraint

$$
T_N^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+1}\cdots\mu_n} = T_N^{\mu_1\cdots\mu_k}\, T_N^{\mu_{k+1}\cdots\mu_n}, \quad (5.7)
$$

Notice, however, that the polygonal multitangents do not form a group because $T_N \times T_M = T_{N+M+1}$. However the inverse of T_N is a T_N and the polygonal loops for arbitrary N do form a group.

So the multitangent,

$$
T = \lim_{N\to\infty} T_N, \quad (5.8)
$$

where T are the vectors associated with the multitangent in the notation of (4.28) and,

$$
\lim_{N\to\infty} T_N^{\mu_1\ldots\mu_N} = 0. \quad (5.9)
$$

These properties of the multitangents can be directly generalized to extended loops giving a sufficient criterion for the latter to lead to a covariant holonomy: the extended loop must be given by a limit $E = \lim_{N\to\infty} E_N$ with the E_N's satisfying a differential constraint (5.6) and a condition like (5.5) that implies limit like (5.9).

It is immediate to show that the explicit constructions presented in previous sections satisfy these conditions for a suitable definition of the limits involved. Let us see this explicitly for the case of a real power λ of a loop. Let us define a polygonal approximation to the algebra element $F(a,\gamma)$,

$$F(m,a,\gamma) = \sum_{i=1}^{m} \frac{-1}{i} \left(\mathcal{I} - T_{m^3}(a,\gamma)\right)^i , \tag{5.10}$$

with the extended loop,

$$E(m,a,\gamma^\lambda) = \sum_{j=0}^{m} \frac{(\lambda F(m,a,\gamma))^j}{j!}. \tag{5.11}$$

The mutitangents $T_{m^3}(a,\gamma)$ need to include polygonals with m^3 segments in order to ensure in the limit $m \to \infty$ the gauge invariance of the sums in the two previous equations. That is, the covariance of the holonomy $T_{m^3} \cdot A$ can be checked up to order $O(\frac{1}{m^3})$, for $F(m)$ the error in the covariance of $F(m) \cdot A$ goes as $O(\frac{1}{m^2})$, and for $E(m)$ the error in the covariance of $E(m) \cdot A$ is $O(\frac{1}{m})$. Thus, the extended loops introduced in section 4 satisfy the sufficient conditions given here.

VI. CONCLUSIONS

We have shown how to generate large families of extended loops that yield covariant holonomies. They include ordinary loops and real powers of them, among others. The real powers of loops constitute a group with associated holonomies that are unitary and gauge invariant. The center of the idea is to construct extended loops using the expansion of the logarithm of multitensors. Through an analytic extension they can be shown to yield covariant holonomies. We have also given sufficient conditions for extended loops that lead to covariant holonomies and showed that the extended loops obtained from the previous construction satisfy them.

This opens the possibility of using the ensuing extended loops to create extended loop representations of interest for the non perturbative quantization of Yang–Mills theories and potentially gravity. In the case of Yang–Mills theories the use of extended loops could have advantages over the use of ordinary loops when one wishes to define the inner product and the closure relations, and for the renormalization of non-perturbative Schrödinger-like equations.

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Appendix

Let us show explicitly that $F(a,\gamma)\cdot A$ with F satisfying the homogeneous algebraic identity $F^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+1}\cdots\mu_n} = 0$ is in the algebra. To do that we use a technique developed in [3, 4]. We define the matrix of delta functions,

$$\delta^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = \delta_{n,m}\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_n}_{\nu_n} \tag{7.1}$$

and the vector (using the same notation as in (4.28)),

$$\delta_{\nu_1\cdots\nu_i} = (\,0,\ \delta^{\mu_1}{}_{\nu_1\cdots\nu_i},\ \cdots,\delta^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_i},\ \cdots)\ . \tag{7.2}$$

We define then a projector from generic multitangents to those satisfying the homogeneous algebraic constraint, given by

$$\Omega^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \equiv \frac{\delta_{n,m}}{m}[[\cdots[\delta_{\nu_1},\delta_{\nu_2}],\cdots],\delta_{\nu_n}]^{\mu_1\cdots\mu_n}\theta(m-1)\,+\,\delta_{m,1}\delta^{\mu_1}_{\nu_1} \tag{7.3}$$

where

$$[\delta_{\nu_1},\delta_{\nu_2}]^{\mu_1\mu_2} = (\delta_{\nu_1}\delta_{\nu_2}-\delta_{\nu_2}\delta_{\nu_1})^{\mu_1\mu_2} = \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}-\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}, \tag{7.4}$$

and the θ function means that the terms is non-vanishing for $m>1$. From the above definition,

$$\begin{aligned}\Omega^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+1}\cdots\mu_{n+1}}{}_{\nu_1\ldots\nu_{n+1}} &= \frac{n}{n+1}\Big(\Omega^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+1}\cdots\mu_n}{}_{\nu_1\ldots\nu_n}\delta^{\mu_{n+1}}_{\nu_{n+1}}\\ &+\ \Omega^{\underline{\mu_1\cdots\mu_{k-1}}\,\mu_{k+1}\cdots\mu_{n+1}}{}_{\nu_1\ldots\nu_n}\delta^{\mu_k}_{\nu_{n+1}}\\ &-\delta^{\mu_1}_{\nu_{n+1}}\Omega^{\underline{\mu_2\cdots\mu_k}\,\mu_{k+1}\cdots\mu_{n+1}}{}_{\nu_1\ldots\nu_n}\\ &-\ \delta^{\mu_{k+1}}_{\nu_{n+1}}\Omega^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+2}\cdots\mu_{n+1}}{}_{\nu_1\ldots\nu_n}\Big)\,,\end{aligned} \tag{7.5}$$

we immediately have that if,

$$\Omega^{\underline{\mu_1\cdots\mu_l}\,\mu_{l+1}\cdots\mu_n}{}_{\nu_1\ldots\nu_n} = 0,\quad \forall l/1\le l<n, \tag{7.6}$$

then,

$$\Omega^{\underline{\mu_1\cdots\mu_k}\,\mu_{k+1}\cdots\mu_{n+1}}{}_{\nu_1\ldots\nu_{n+1}} = 0,\quad \forall k/1\le l<n+1. \tag{7.7}$$

Given that Ω is a projector, one has that

$$F\cdot A = (\Omega\cdot F)\cdot A = F\cdot(\Omega\cdot A)\,, \tag{7.8}$$

and $\Omega\cdot A$ is in the algebra and $F\cdot A$ is too. This can be seen in the following way,

$$(\Omega_{\nu_1\ldots\nu_n}\cdot A)_{\nu_1\ldots\nu_n} = \Omega^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_n}A_{\mu_1}\cdots A_{\mu_n} \tag{7.9}$$

$$= \frac{1}{n}\Big(\Omega^{\mu_1\ldots\mu_{n-1}}{}_{\nu_1\ldots\nu_{n-1}}A_{\mu_1}\cdots A_{\mu_{n-1}}\cdots A_{\mu_{n-1}}A_{\nu_n} -A_{\nu_n}\Omega^{\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_{n-1}}A_{\mu_2}\cdots A_{\mu_n}\Big) \tag{7.10}$$

$$= \frac{1}{n}\left[\Omega_{\nu_1\ldots\nu_{n-1}}\cdot A, A_{\nu_n}\right] \tag{7.11}$$

$$\vdots$$

$$= \frac{1}{n!}\left[\ldots\left[A_{\nu_1},A_{\nu_2}\right],\ldots,A_{\nu_n}\right], \tag{7.12}$$

as can be seen inductively and therefore if A_ν is in the algebra, so is $\Omega_{\nu_1\ldots\nu_n}\cdot A$.

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