Dynamic Active Average Consensus

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***Abstract*—We propose a continuous-time dynamic active average consensus algorithm in which the agents can alternate between active and passive modes depending on their ability to access to their reference input. The objective is to enable all the agents, both active and passive, to track the average of the reference inputs of the active agents. The algorithm is modeled as a switched linear system whose convergence properties are carefully studied considering the agents' piece-wise access to the reference signals. We also study the discrete-time implementation of this algorithm. Next, we show how a containment control problem in which a group of followers should track the convex hull of a set of observed leaders, can be cast as an active average consensus problem, and solved efficiently by our proposed dynamic active average consensus algorithm. Numerical examples demonstrate our results.**

***Index Terms*—Multi-agent coordination, average consensus, containment control, switched systems.**

I. INTRODUCTION

WE PROPOSE a distributed solution for the dynamic active average consensus problem and study its use in solving a distributed containment control problem. In dynamic active average consensus problem, at any time, only a subset of the agents are active, meaning that only a subset of agents collects measurements. The objective then is to enable all the agents, both active and passive, to obtain the average of the collected measurements without knowing the set of active agents. The well-known average consensus problem, extensively studied in the literature for both static [1] and dynamic [2] reference signals, is in fact a special case of this problem with all the agents being active at all times.

The active average consensus problem can be viewed as a *weighted average consensus problem* [3], in which the weights are 1 for active agents and 0 for passive agents. However, the solutions for weighted average consensus (see, e.g., [3]–[5]) use the notation of the 'equivalent' Laplacian matrix, which is the multiplication of the inverse of the weight matrix and the Laplacian matrix. Therefore, the weights should be nonzero, and thus these solutions cannot solve the active average consensus problem. Solutions specifically addressing the active average consensus problem are proposed in [6]–[8], but, they require both the reference signals and their derivatives to be bounded to guarantee bounded error tracking. References [6] and [7] also assume that the active and passive roles of the agents are fixed and agents cannot alternative between modes. On the other hand, [8] allows the agents to change mode but requires the change of the modes to be done smoothly.

In this letter, we propose a continuous-time solution for dynamic active average consensus over connected graphs that requires only the rate of the change of the reference signals to be bounded. Also, the agents can switch between active and passive modes instantaneously, as long as a dwell time exists between the switching incidences. Abrupt switching is usually the case for practical problems where agents are observing dynamic activities that can enter or leave the observation zone of the agents and thus change the agents' role from active to passive or vice versa in a non-smooth fashion. We model our algorithm as a switched linear system and study its convergence properties carefully by taking into account the piece-wise access of the agents to the reference signals. Our study employs the concept of distributional derivatives [9] to model the derivative of piece-wise continuous functions and characterize the transient error at the switching times.

Our next contribution in this letter is studying the discrete-time implementation of our proposed dynamic active average consensus algorithm and using it to solve a containment control problem where a group of followers should track the convex hull of a set of leaders that they observe. We show that the average of the geometric centers of the observed leaders at each active agent is a point in the convex hull of the leaders. Thus, the containment problem can be formulated and solved as an active average consensus problem, see Fig. 1. Continuous-time solutions for containment problems can be found in [10]–[12]. But, the requirement for continuous inter-agent information sharing can be of concern for practical problems where agents communication bandwidth is limited. Discrete-time containment control solutions where agents communicate with each other in a finite rate are given in [13], [14]. To provide perfect tracking, [10]–[14] assume that the leaders are static or if they are dynamic they either follow a certain dynamics that is known to the followers or the leaders' motions have to be coordinated with the followers. In this letter, we consider a tracking problem where the states of the leaders are only measured online, and we make no assumption about the dynamics of the leaders except that the changes of the states of the leaders are bounded. This relaxation however, as known in dynamic consensus literature, is attained by trading off perfect tracking, as the

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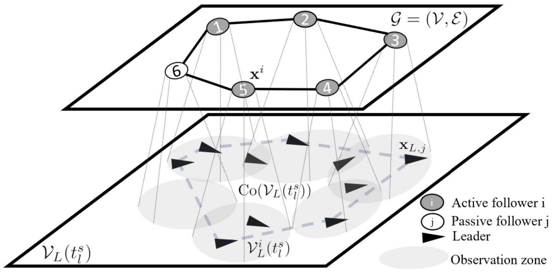

Fig. 1. A containment control scenario where a set of six followers should track the convex hull of a set of the dynamic leaders that they observe: Followers $\mathcal{V}_a = \{1, \dots, 5\}$ are active agents that each observes a subset of the leaders, while follower 6 is the passive agent that should still follow the convex hull of the leaders despite having no measurement.

online time-varying information takes some time to propagate through the network [2]. A preliminary version of our work appeared in [15]. There, we used two parallel conventional dynamic average consensus algorithms, one to generate the sum of the measurements divided by the size of the network and the other to obtain the sum of the active agents divided by the size of the network. Then, the average of the active measurements is obtained from dividing the output of the first algorithm by that of the second one. This letter is offering a computationally more efficient algorithm, which has a lower communication complexity and avoids zero-crossing problem observed in our initial work [15] for its approach to solve dynamic active average consensus problem.

II. Notations and Preliminaries

We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the set of real, positive real, non-negative real, integer, positive integer, and non-negative integer, respectively. For $\mathbf{s} \in \mathbb{R}^d$, $\|\mathbf{s}\| = \sqrt{\mathbf{s}^\top \mathbf{s}}$ denotes the standard Euclidean norm. We let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) denote the vector of n ones (resp. n zeros), and $\mathbf{I}_n$ denote the $n \times n$ identity matrix. When clear from the context, we do not specify the matrix dimensions. $\mathrm{H}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$ is the *Heaviside step* function. $\boldsymbol{\delta}(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$ such that $\int_{-\infty}^{\infty} \boldsymbol{\delta}(t)\mathrm{d}t = 1$ is the *Dirac Delta* function. In a network of N agents, the aggregate vector of local variables $p^i \in \mathbb{R}$, $i \in \{1, \dots, N\}$, is denoted by $\mathbf{p} = (p^1, \dots, p^N)^\top \in \mathbb{R}^N$.

Consider the piece-wise continuous function

$$\mathbf{v}(t) = \begin{cases} \mathbf{v}_0(t), & t_0 \leq t < t_1, \\ \mathbf{v}_1(t), & t_1 \leq t < t_2, \\ \vdots & \\ \mathbf{v}_{\bar{k}}(t), & t_{\bar{k}} \leq t \end{cases}, \tag{1}$$

where $\mathbf{v}_i \in \mathcal{C}^1, i \in \{1, \dots, \bar{k}\}$. Using the Heaviside step function, (1) reads as $\mathbf{v}(t) = \mathbf{v}_0 + \sum_{k=1}^{\bar{k}} (\mathbf{v}_k - \mathbf{v}_{k-1}) \mathrm{H}(t - t_k)$. Then, following [9], the distributional derivative of $\mathbf{v}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \dot{\bar{\mathbf{v}}} + \sum_{k=1}^{\bar{k}} (\mathbf{v}(t_k^+) - \mathbf{v}(t_k^-))\, \boldsymbol{\delta}(t - t_k), \tag{2}$$

where $\dot{\bar{\mathbf{v}}} = \dot{\mathbf{v}}_0 + \sum_{k=1}^{\bar{k}} (\dot{\mathbf{v}}_k - \dot{\mathbf{v}}_{k-1}) \mathrm{H}(t - t_k)$ or equivalently

$$\dot{\bar{\mathbf{v}}} = \begin{cases} \dot{\mathbf{v}}_0(t), & t_0 \leq t < t_1, \\ \dot{\mathbf{v}}_1(t), & t_1 \leq t < t_2, \\ \vdots & \\ \dot{\mathbf{v}}_{\bar{k}}(t), & t_{\bar{k}} \leq t. \end{cases}$$

We assume that the piece-wise continuous signals are right-continuous, i.e., $\mathbf{v}(t_k) = \mathbf{v}(t_k^+)$. Hereafter, we use the notation $\dot{\bar{\cdot}}$ to represent $\dot{\bar{\mathbf{v}}}(t) = \begin{cases} \dot{\mathbf{v}}(t) & t \neq t_k \\ \dot{\mathbf{v}}(t_k^+) & t = t_k \end{cases}$.

An undirected *graph* is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*, and $\mathsf{A} \in \mathbb{R}^{N \times N}$ is a *adjacency* matrix such that $\mathsf{a}_{ij} = \mathsf{a}_{ji} > 0$ if $(i, j) \in \mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. An edge (i, j) from i to j means that agents i and j can communicate. A *connected graph* is an undirected graph in which for every pair of nodes there is a path connecting them. The *degree* of a node i is $\mathsf{d}^i = \Sigma_{j=1}^N \mathsf{a}_{ij}$. The *Laplacian* matrix is $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D} = \mathrm{Diag}(\mathsf{d}^1, \dots, \mathsf{d}^N) \in \mathbb{R}^{N \times N}$. For connected graphs, $\mathbf{L}\mathbf{1}_N = \mathbf{0}$ and $\mathbf{1}_N^T \mathbf{L} = \mathbf{0}$. Moreover, $\mathbf{L}$ has one eigenvalue $\lambda_1 = 0$, and the rest of the eigenvalues $\{\lambda_i\}_{i=2}^N$ are positive. $\mathbf{T} = [\mathfrak{r} \quad \mathfrak{R}] \in \mathbb{R}^{N \times N}$ is an orthonormal matrix, where $\mathfrak{r} = \frac{1}{\sqrt{N}} \mathbf{1}_N$ and $\mathfrak{R} \in \mathbb{R}^{N \times (N-1)}$ is any matrix that makes $\mathbf{T}^\top \mathbf{T} = \mathbf{T}\mathbf{T}^\top = \mathbf{I}$. For a connected graph, $\mathbf{T}^\top \mathbf{L} \mathbf{T} = [\begin{smallmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^+ \end{smallmatrix}]$, where $\mathbf{L}^+ = \mathfrak{R}^\top \mathbf{L} \mathfrak{R}$, is a positive definite matrix with eigenvalues $\{\lambda_i\}_{i=2}^N \subset \mathbb{R}_{>0}$.

Lemma 1: Suppose the nonzero matrix $\mathbf{E} \in \mathbb{R}^{N \times N}$ is a diagonal matrix whose diagonal elements are either 0 or 1, and $\mathbf{L}$ is the Laplacian matrix of a connected graph. Then, $-(\mathbf{E} + \mathbf{L})$ is Hurwitz.

Proof: Consider the system $\dot{\mathbf{x}} = -(\mathbf{E} + \mathbf{L})\mathbf{x}$. Now consider Lyapnov function $V = \frac{1}{2}\mathbf{x}^\top \mathbf{x}$. Then, $\dot{V} = -\mathbf{x}^\top \mathbf{E} \mathbf{x} - \mathbf{x}^\top \mathbf{L} \mathbf{x} \leq 0$, because $-\mathbf{x}^\top \mathbf{E} \mathbf{x} \leq 0$ and $-\mathbf{x}^\top \mathbf{L} \mathbf{x} \leq 0$. However, $\dot{V} \equiv 0$ happens when $-\mathbf{x}^\top \mathbf{E} \mathbf{x} = 0$ and $-\mathbf{x}^\top \mathbf{L} \mathbf{x} = 0$. But, since $-\mathbf{x}^\top \mathbf{L} \mathbf{x} = 0$ if and only if $\mathbf{x} = \alpha \mathbf{1}$, $\alpha \in \mathbb{R}$ then $\dot{V} \equiv 0$ if $\mathbf{x} \equiv \mathbf{0}$. Therefore, invoking [16, Th. 4.11], we conclude that the system $\dot{\mathbf{x}} = -(\mathbf{E} + \mathbf{L})\mathbf{x}$ is uniformly exponentially stable. Thus, $-(\mathbf{E} + \mathbf{L})$ is Hurwitz. ■

III. Problem Definition

Consider a network of N single integrator agents $\dot{x}^i = u^i$, $i \in \mathcal{V}$, interacting over a connected undirected graph $\mathcal{G}$. Suppose each agent $i \in \mathcal{V}$ has access to a measurable locally essentially bounded reference signal $\mathsf{r}^i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ in a possibly intermittent fashion. For every agent $i \in \mathcal{V}$, we let $\eta^i(t)$ be the indicator function for the agent $i \in \mathcal{V}$, which returns 1 if agent i is active and has access to $\mathsf{r}^i(t)$ at time $t \in \mathbb{R}_{\geq 0}$, and 0 otherwise. Let $\mathcal{V}_\mathrm{a}(t) \subset \mathcal{V}$ be the set of active agents at time $t \in \mathbb{R}_{\geq 0}$, i.e., $\mathcal{V}_\mathrm{a}(t) = \{i \in \mathcal{V} \,|\, \eta^i(t) = 1\}$. In what follows, we assume that $\mathcal{V}_\mathrm{a}(t) \neq \emptyset$ for all $t \in \mathbb{R}_{\geq 0}$ and $|\mathcal{V}_\mathrm{a}(t)|$ is a piece-wise constant function of time. We refer to an agent in $\mathcal{V} \backslash \mathcal{V}_\mathrm{a}(t)$ as the passive agent at time t.

Problem 1 (Active Average Consensus Problem): The active average consensus problem over $\mathcal{G}$ is defined as designing a distributed control input u^i such that the agreement state $x^i(t) \in \mathbb{R}$ of every agent $i \in \mathcal{V}$ tracks

$$\mathsf{avg}^\mathrm{a}(t) = \frac{\sum_{i \in \mathcal{V}_\mathrm{a}(t)} \mathsf{r}^i(t)}{|\mathcal{V}_\mathrm{a}(t)|} = \frac{\sum_{i=1}^N \eta^i(t) \mathsf{r}^i(t)}{\sum_{i=1}^N \eta^i(t)}.$$

In what follows, we first propose a distributed continuous-time algorithm to solve Problem 1. Then, we present a discrete-time implementation of this active average consensus algorithm in which the agents sample the reference inputs with a rate of $1/\delta_s$ in a zero-order fashion. Lastly, we show how a containment problem can be cast as dynamic active average consensus problem and solved using our proposed algorithm.

IV. Continuous-Time Dynamic Active Average Consensus

Our solution to solve Problem 1 over a connected undirected graph $\mathcal{G}$ is

$$\dot{x}^i(t) = -\eta^i(t)(x^i(t) - \mathsf{r}^i(t)) - \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i(t) - x^j(t)) - \sum_{j=1}^{N} \mathsf{a}_{ij}(v^i(t) - v^j(t)) + \eta^i(t)\dot{\mathsf{r}}^i(t), \quad (3a)$$

$$\dot{v}^i(t) = \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i(t) - x^j(t)), \quad (3b)$$

with $x^i(0), v^i(0) \in \mathbb{R}$, $i \in \mathcal{V}$. Here, $v^i(t) \in \mathbb{R}$ is an internal state that acts as an integral action. Next, we study the convergence properties of (3) by modeling it as a switched system and analyzing the collective response of the agents. In what follows, we let $\mathbf{E}(t) = \text{Diag}(\eta^1(t), \dots, \eta^N(t))$. $\mathbf{E}(t)$ can be considered as switching in the class of non-zero diagonal matrices $\{\mathbf{E}_p\}_{p\in\mathcal{P}}$, $\mathcal{P} = \{1, \dots, 2^N - 1\}$, whose elements are either 1 or 0. That is $\mathbf{E}(t) = \mathbf{E}_{\sigma(t)} \neq \mathbf{0}$ with the switching signal $\sigma(t) : \mathbb{R}_{\geq 0} \to \mathcal{P}$. We let $N_\sigma(0, t)$ denote the number of switchings of $\sigma(t)$ on the interval $[0, t)$. In our problem of interest, the following common assumption for switch linear systems holds [17], [18].

Assumption 1: There exist some $N_0 \in \mathbb{Z}_{\geq 0}$ and $\tau_\mathrm{D} \in \mathbb{R}_{>0}$ such that, $N_\sigma(0, t) \leq N_0 + \frac{t}{\tau_\mathrm{D}}$, $t \in \mathbb{R}_{>0}$, where τ_D is called the average dwell time and N_0 is the chatter bound.

We let $\mathbf{avg}^{\mathrm{a}} = \mathrm{avg}^{\mathrm{a}}\mathbf{1}$, $\Delta\mathbf{avg}^{\mathrm{a}}_k = \mathbf{avg}^{\mathrm{a}}(t_k^+) - \mathbf{avg}^{\mathrm{a}}(t_k^-)$, $\mathbf{w}(t) = \mathbf{E}_{\sigma(t)}(\mathbf{r}(t) - \mathbf{avg}^{\mathrm{a}}(t))$, and $\Delta\mathbf{w}_k = \mathbf{w}(t_k^+) - \mathbf{w}(t_k^-)$, where t_k, $k \in \mathbb{Z}_{\geq 0}$ is the kth switching time of the switching signal $\sigma(t)$. Throughout this letter we assume $t_0 = 0$. Lastly, given a time $t \in \mathbb{R}_{\geq 0}$, $\bar{k} \in \mathbb{Z}_{\geq 0}$ is the largest integer such that $t_{\bar{k}} \leq t$.

For convenience in the correctness analysis of algorithm (3), we use the change of variables $\bar{\mathbf{e}} = \mathbf{T}^\top(\mathbf{x} - \mathbf{avg}^{\mathrm{a}})$, $\mathbf{q} = [q_1 \;\; \mathbf{q}_{2:N}^\top]^\top = \mathbf{T}^\top(\mathbf{L}\mathbf{v} - \mathbf{w})$ to write the equivalent compact form of (3) as

$$\dot{q}_1 = 0, \quad (4a)$$

$$\begin{bmatrix} \dot{\bar{\mathbf{e}}} \\ \dot{\mathbf{q}}_{2:N} \end{bmatrix} = \overline{\mathbf{A}}_{\sigma(t)} \begin{bmatrix} \bar{\mathbf{e}} \\ \mathbf{q}_{2:N} \end{bmatrix} + \overline{\mathbf{B}} \begin{bmatrix} \mathbf{E}\,\dot{\mathbf{r}} - \dot{\mathbf{avg}}^{\mathrm{a}} \\ -\dot{\mathbf{w}} \end{bmatrix} - \overline{\mathbf{B}} \sum_{k=1}^{\bar{k}} \begin{bmatrix} \Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k \end{bmatrix} \delta(t - t_k). \quad (4b)$$

where $\overline{\mathbf{A}}_{\sigma(t)} = \begin{bmatrix} -\mathbf{T}^\top(\mathbf{E}_{\sigma(t)}+\mathbf{L})\mathbf{T} & -\begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{N-1} \end{bmatrix} \\ [\mathbf{0} \;\; \mathbf{L}^+\mathbf{L}^+] & \mathbf{0} \end{bmatrix}$ and $\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{T}^\top & \mathbf{0} \\ \mathbf{0} & \mathfrak{R}^\top \end{bmatrix}$. Here, we used the facts that $\mathbf{r}^\top\dot{\mathbf{w}} = 0$ and $\mathbf{r}^\top\Delta\mathbf{w}_k = 0$. Also, we used $\mathfrak{R}\mathfrak{R}^\top\mathbf{L} = \mathbf{L}$ to write $\mathfrak{R}^\top\mathbf{L}\mathbf{L}\mathfrak{R} = \mathbf{L}^+\mathbf{L}^+$. Lastly, note that since $\mathbf{avg}^{\mathrm{a}}$ and $\mathbf{w}$ are piece-wise continuous functions, we used (2) to compute their derivatives that appear in $\dot{\bar{\mathbf{e}}}$ and $\dot{\mathbf{q}}$. Using standard results for linear time-varying systems we can write

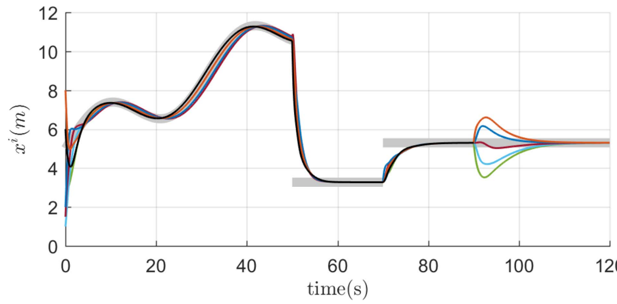

Fig. 2. A network of 6 agents with a ring interaction topology executes the active average consensus algorithm (3). In time interval $t \in [0, 50)$, the observing agents $\mathcal{V}_\mathrm{a}(t) = \{1, 2, 4, 6\}$ all have dynamic inputs. The observing agents at $t \in [50, 70)$ and $t \in [70, 120]$ are, respectively, $\mathcal{V}_\mathrm{a}(t) = \{2,3,5,6\}$ $\mathcal{V}_\mathrm{a}(t) = \{3, 6\}$ and their observations are static signals. Agent 1 (black line) leaves the network at $t = 90$. The gray thick line represents $\mathrm{avg}^{\mathrm{a}}(t)$. The agents can track the dynamic $\mathrm{avg}^{\mathrm{a}}(t)$ with bounded error in $t \in [0, 50)$, while their tracking error is close to zero for the rest of the time as the reference signals are constant after $t = 50$. The transient tracking error at time $t = 70$ is due to switching of some of agents to the passive mode. This error is captured by the second term in the right-hand side of (8). Lastly, agent 1's leaving causes perturbations at $t = 90$ but the network still converge to $\mathrm{avg}^{\mathrm{a}}(t)$.

$$\begin{bmatrix} \bar{\mathbf{e}}(t) \\ \mathbf{q}_{2:N}(t) \end{bmatrix} = \Phi(t, 0)\begin{bmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0) \end{bmatrix} + \int_0^t \Phi(t, \tau)\overline{\mathbf{B}}\left(\begin{bmatrix} \mathbf{E}\,\dot{\mathbf{r}} - \dot{\mathbf{avg}}^{\mathrm{a}} \\ -\dot{\mathbf{w}} \end{bmatrix} - \sum_{k=1}^{\bar{k}} \begin{bmatrix} \Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k \end{bmatrix} \delta(\tau - t_k)\right) \mathrm{d}\tau, \quad (5)$$

where $\Phi(t, \tau)$ is the transition matrix of linear system (4b). The next result shows that the internal dynamics of (4b) is uniformly exponentially stable. Therefore, there always exists κ_s, λ_s such that

$$\|\Phi(t, \tau)\| \leq \kappa_s \mathrm{e}^{-\lambda_s(t-\tau)}, \quad t \geq \tau \geq 0. \quad (6)$$

Lemma 2: Let $\mathcal{G}$ be a connected undirected graph. Then, every subsystem matrix $\overline{\mathbf{A}}_p$, $p \in \mathcal{P}$ of (4b) is Hurwitz. Furthermore, under Assumption 1 the internal dynamics of (4b) is uniformly exponentially stable, i.e., (6) holds.

Proof: Consider the radially unbounded quadratic Lyapunov function $V = \frac{1}{2}\mathbf{q}_{2:N}^\top(\mathbf{L}^+\mathbf{L}^+)^{-1}\mathbf{q}_{2:N} + \frac{1}{2}\bar{\mathbf{e}}^\top\bar{\mathbf{e}}$ (a common Lyapunov function for all the subsystems $\overline{\mathbf{A}}_{p\in\mathcal{P}}$ of the switched system $\overline{\mathbf{A}}_{\sigma(t)}$). Here, note that since $\mathbf{L}^+ > 0$, then $\mathbf{L}^+\mathbf{L}^+ > 0$. The Lie derivative of V along the trajectories of internal dynamics of (4b) is

$$\dot{V} = -\bar{\mathbf{e}}^\top\mathbf{T}^\top(\mathbf{E}_p + \mathbf{L})\mathbf{T}\,\bar{\mathbf{e}} \leq 0, \quad p \in \mathcal{P}. \quad (7)$$

To establish negative semi-definiteness of $\dot{V}$, we invoke Lemma 1. So far we have established that V is a weak Lyapunov function. Next, we use the LaSalle invariant principle and [19, Th. 4] to establish exponential stability of the internal dynamics of (4b). Let $\mathcal{S}_p = \{(\bar{\mathbf{e}}, \mathbf{q}_{2:N}) \in \mathbb{R}^N \times \mathbb{R}^{N-1} | \dot{V} \equiv 0\}$ for all $p \in \mathcal{P}$. Given (7), we then have $\mathcal{S}_p = \{(\bar{\mathbf{e}}, \mathbf{q}_{2:N}) \in \mathbb{R}^N \times \mathbb{R}^{N-1} | \bar{\mathbf{e}} = 0\}$, for all $p \in \mathcal{P}$. Then, it is straightforward to observe that the trajectories of the internal dynamics of (4b) that belong to $\mathcal{S}_{p\in\mathcal{P}}$, should also satisfy $\mathbf{q}_{2:N} \equiv \mathbf{0}$. Therefore, the largest invariant set of the internal dynamics of (4b) in $\mathcal{S}_{p\in\mathcal{P}}$ is the origin. Thus, using [16, Th. 4.4] all the subsystems $\overline{\mathbf{A}}_{p\in\mathcal{P}}$ of the switched system $\overline{\mathbf{A}}_{\sigma(t)}$ are globally asymptotically stable. Moreover, because the all subsystems of the switched system share the common weak quadratic Lyapunov function and the largest invariant set of $\mathcal{S}_{p\in\mathcal{P}}$ contains only the origin, given Assumption 1, by virtue of [19, Th. 4] the internal dynamics of (4b), which is a switched system, is uniformly exponentially stable. Here, we note that according to [20, Th. 2.1] the origin being the largest invariant set of $\mathcal{S}_p$, for all $p \in \mathcal{P}$, ensures that the observability condition in [19, Th. 4] is satisfied. ■

Given (5) and (6), we can characterize the tracking performance of active average consensus algorithm (3) as follows.

Theorem 1: Let $\mathcal{G}$ be a connected undirected graph and suppose Assumption 1 holds. Then, starting from any $x^i(0), v^i(0) \in \mathbb{R}$, $i \in \mathcal{V}$ the trajectories of dynamic active average consensus algorithm (3) satisfy

$$|x^i(t) - \mathsf{avg}^{\mathrm{a}}(t)| \leq \kappa_s \,\mathrm{e}^{-\lambda_s t}\left\|\begin{bmatrix} \mathbf{x}(0)-\mathbf{avg}^{\mathrm{a}}(0) \\ \mathbf{L}\mathbf{v}(0)-\mathbf{w}(0)\end{bmatrix}\right\| + \kappa_s \sum_{k=1}^{\bar{k}} \mathrm{e}^{-\lambda_s(t-t_k)}\left\|\begin{bmatrix} \Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k\end{bmatrix}\right\| \mathrm{H}(t-t_k) + \frac{\kappa_s}{\lambda_s}\sup_{0\leq\tau\leq t}\left\|\begin{bmatrix} \mathbf{E}_{\sigma(\tau)}\,\dot{\mathbf{r}}(\tau)-\dot{\mathbf{avg}}^{\mathrm{a}}(\tau) \\ -\dot{\mathbf{w}}(\tau)\end{bmatrix}\right\|. \quad (8)$$

Proof: We note that $\|\overline{\mathbf{B}}\| \leq 1$. Then, given (5) and (6), we can write

$$\left\|\begin{bmatrix} \bar{\mathbf{e}}(t) \\ \mathbf{q}_{2:N}(t)\end{bmatrix}\right\| \leq \kappa_s\,\mathrm{e}^{-\lambda_s t}\left\|\begin{bmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0)\end{bmatrix}\right\| + \kappa_s\int_0^t \mathrm{e}^{-\lambda_s(t-\tau)}\left\|\begin{bmatrix} \mathbf{E}\,\dot{\mathbf{r}}-\dot{\mathbf{avg}}^{\mathrm{a}} \\ -\dot{\mathbf{w}}\end{bmatrix}\right\|\mathrm{d}\tau + \kappa_s\sum_{k=1}^{\bar{k}}\int_0^t \mathrm{e}^{-\lambda_s(t-\tau)}\left\|\begin{bmatrix} \Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k\end{bmatrix}\boldsymbol{\delta}(\tau-t_k)\right\|\mathrm{d}\tau.$$

Then, the Hölder inequality is used to bound the second term of the right hand side to arrive at

$$\left\|\begin{bmatrix} \bar{\mathbf{e}}(t) \\ \mathbf{q}_{2:N}(t)\end{bmatrix}\right\| \leq \kappa_s\,\mathrm{e}^{-\lambda_s t}\left\|\begin{bmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0)\end{bmatrix}\right\| + \frac{\kappa_s}{\lambda_s}\sup_{0\leq\tau\leq t}\left\|\begin{bmatrix} \mathbf{E}_{\sigma(\tau)}\,\dot{\mathbf{r}}(\tau)-\dot{\mathbf{avg}}^{\mathrm{a}}(\tau) \\ -\dot{\mathbf{w}}(\tau)\end{bmatrix}\right\| + \kappa_s\sum_{k=1}^{\bar{k}}\int_0^t \mathrm{e}^{-\lambda_s(t-\tau)}\left\|\begin{bmatrix} \Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k\end{bmatrix}\boldsymbol{\delta}(\tau-t_k)\right\|\mathrm{d}\tau.$$

Consequently, with integration by parts, the last term is equivalent to $\kappa_s\sum_{k=1}^{\bar{k}}\mathrm{e}^{-\lambda_s(t-t_k)}\|[\begin{smallmatrix}\Delta\mathbf{avg}^{\mathrm{a}}_k \\ \Delta\mathbf{w}_k\end{smallmatrix}]\|\,\mathrm{H}(t-t_k)$. Then, since $\mathbf{T}$ is an orthonormal matrix, we have $\|[\begin{smallmatrix}\bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0)\end{smallmatrix}]\| = \|[\begin{smallmatrix}\mathbf{x}(0)-\mathbf{avg}^{\mathrm{a}}(0) \\ \mathbf{L}\mathbf{v}(0)-\mathbf{w}(0)\end{smallmatrix}]\|$ and $\|\mathbf{x}-\mathbf{avg}^{\mathrm{a}}\| = \|\bar{\mathbf{e}}\|$. Finally, (8) is derived along with the relation $|x^i - \mathsf{avg}^{\mathrm{a}}| \leq \|[\,\bar{\mathbf{e}}^\top \quad \mathbf{q}_{2:N}^\top\,]^\top\|$. ■

We note that the first summand of the tracking error bound (8) is the transient response, which vanishes over time. The second summand is due to the agents alternating between active and passive sets. If the average dwell time τ_D is large, this error also disappears after a while. The third summand can result in a steady-state error. This error that is expected in dynamic average consensus algorithms, as tracking an arbitrarily fast average signal with zero error is not feasible unless agents have some priori information about the dynamics generating the signals [2]. However, the size of this error is proportional to the rate of change of the signals and can be limited by limiting the rate. We recall that to provide bounded tracking, previous work in [6]–[8] require both the reference input signals and their rate of change to be bounded. If the local reference signals are static and the agents do not switch, the agents exponentially converge to $\mathsf{avg}^{\mathrm{a}}$ without steady-state error. Lastly, algorithm (3) does not require specific initialization. In other words, the convergence property of algorithm (3) uniformly holds for any initialization. Therefore, as long as the graph stays connected, agents can leave and join the network without effecting the convergence guarantees. Figure 2 demonstrates the performance of algorithm (3) in a numerical example.

V. Discrete-Time Dynamic Active Average Consensus

We consider a scenario where active agents sample their reference inputs at sampling times $t^s_l = l\delta_s \in \mathbb{R}_{\geq 0}$, $l \in \mathbb{Z}_{\geq 0}$, $\delta_s \in \mathbb{R}_{>0}$. The agents can communicate at discrete-times $t^c_k = k\delta_c \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$, $\delta_c \in \mathbb{R}_{>0}$. The objective of every agent $i \in \mathcal{V}$ is to track $\mathsf{avg}^{\mathrm{a}}(k)$ (where k is the shorthand for t^c_k). To solve the active average consensus problem under this scenario, we propose that every agent $i \in \mathcal{V}$ implements

$$x^i(k) = z^i(k) + \eta^i(k)\mathsf{r}^i(k), \quad (9a)$$

$$z^i(k+1) = z^i(k) - \delta_c\eta^i(k)(x^i(k) - \mathsf{r}^i(k)) - \delta_c\sum_{j=1}^{N}\mathsf{a}_{ij}(x^i(k) - x^j(k)) - \delta_c\sum_{j=1}^{N}\mathsf{a}_{ij}(v^i(k) - v^j(k)), \quad (9b)$$

$$v^i(k+1) = v^i(k) + \delta_c\sum_{j=1}^{N}\mathsf{a}_{ij}(x^i(k) - x^j(k)). \quad (9c)$$

which is an Euler discretized implementation of the active average algorithm (3) with stepszie δ_c. Here, we assume that if $\delta_s \neq \delta_c$, the agents perform a zero-order hold sampling, so that $\mathsf{r}^i(k) = \mathsf{r}^i(\bar{l})$, $i \in \mathcal{V}$, where $\bar{l}$ is the latest sampling time step such that $t^s_{\bar{l}} \leq t^c_k$. We let $\sigma(k) : \mathbb{Z}_{\geq 0} \to \mathcal{P} = \{1, \ldots, 2^N - 1\}$ be the switching signal of $\mathbf{E}(k)$, i.e., $\mathbf{E}(k) = \mathbf{E}_{\sigma(k)}$. Then, we implement the same change of variable as for the continuous-time algorithm (3) to write the compact form of (9) as

$$q_1(k+1) = q_1(k), \quad (10a)$$

$$\begin{bmatrix} \bar{\mathbf{e}}(k+1) \\ \mathbf{q}_{2:N}(k+1)\end{bmatrix} = (\mathbf{I} + \delta_c\overline{\mathbf{A}}_\sigma)\begin{bmatrix} \bar{\mathbf{e}}(k) \\ \mathbf{q}_{2:N}(k)\end{bmatrix} + \overline{\mathbf{B}}\begin{bmatrix} \Delta\mathbf{E}\mathbf{r}(k)-\Delta\mathbf{avg}^{\mathrm{a}}(k) \\ -\Delta\mathbf{w}(k)\end{bmatrix}, \quad (10b)$$

where $\overline{\mathbf{A}}_\sigma$ and $\overline{\mathbf{B}}$ are defined in (4b), $\Delta\mathbf{E}\mathbf{r}(k) = \Delta\mathbf{E}(k+1)\mathbf{r}(k+1) - \Delta\mathbf{E}(k)\mathbf{r}(k)$, $\Delta\mathbf{avg}^{\mathrm{a}}(k) = \mathbf{avg}^{\mathrm{a}}(k+1) - \mathbf{avg}^{\mathrm{a}}(k)$, and $\Delta\mathbf{w}(k) = \mathbf{w}(k+1) - \mathbf{w}(k)$. Then, given $|x^i - \mathsf{avg}^{\mathrm{a}}| \leq \|[\,\bar{\mathbf{e}}^\top \quad \mathbf{q}_{2:N}^\top\,]^\top\|$, the tracking performance of (9) can be understood by studying the convergence properties of (10b). The first result below shows that with a proper choice for δ_c every subsystem $(\mathbf{I} + \delta_c\overline{\mathbf{A}}_p)$, $p \in \mathcal{P}$ is Schur. However, this is not enough to guarantee that the internal dynamics of (10b) is exponentially stable. To provide such guarantee, following [21, Corollary 1], we impose the following standard assumption.

Assumption 2: The average dwell time τ_D of the switching signal $\sigma(k)$ satisfies $\tau_D \geq \tau_D^*$, where τ_D^* is a stable average dwell time of the switched system (10b).

Note that τ_D^* of the switched system (10b) can be computed using the methods introduced in [21], [22].

Lemma 3: Let $\mathcal{G}$ be a connected undirected graph. Then, every subsystem matrix $(\mathbf{I} + \delta_c \bar{\mathbf{A}}_p)$, $p \in \mathcal{P}$ of (10b) is Schur provided $\delta_c \in (0, \bar{d})$, where $\bar{d} = \min\{\{-2\frac{\text{Re}(\mu_{i,p})}{|\mu_{i,p}|^2}\}_{i=1}^{2N-1}\}_{p\in\mathcal{P}}$ and $\{\mu_{i,p}\}_{i=1}^{2N-1}$ are the set of eigenvalues of $\bar{\mathbf{A}}_p$. Furthermore, under Assumption 2 the internal dynamics of (10b) is uniformly exponentially stable, i.e., there always exists $\kappa_d \in \mathbb{R}_{>0}$ and $\omega_d \in (0, 1)$, such that, the state transition matrix $\Phi(k, j)$ of (10b) satisfies

$$\|\Phi(k, j)\| \leq \kappa_d\, \omega_d^{(k-j)}, \quad k \geq j \geq 0, k, j \in \mathbb{Z}_{\geq 0}. \quad (11)$$

Proof: Lemma 2 ensures that every $\bar{\mathbf{A}}_p$, $p \in \mathcal{P}$ is a Hurwitz matrix. Then, it follows from [2, Lemma S1] that $(\mathbf{I} + \delta_{c,p}\bar{\mathbf{A}}_p)$, $p \in \mathcal{P}$ is Schur if $\delta_{c,p} \in (0, \bar{d}_p)$, where $\bar{d}_p = \min\{-2\frac{\text{Re}(\mu_{i,p})}{|\mu_{i,p}|^2}\}_{i=1}^{2N-1}$. As a result, $(\mathbf{I} + \delta_c\bar{\mathbf{A}}_p)$, $p \in \mathcal{P}$ is Schur if $\delta_c \in (0, \bar{d})$, where $\bar{d} = \min\{\bar{d}_p\}_{p\in\mathcal{P}}$. Then, given Assumption 2, it follows from [21, Corollary 1] that the zero input dynamics of switched system (10b) is uniformly exponentially stable. ■

The next result characterizes the tracking performance of (9).

Theorem 2: Let $\mathcal{G}$ be a connected undirected graph and suppose Assumption 2 holds. Then, for any $\delta_c \in (0, \bar{d})$, starting from any $x^i(0), v^i(0) \in \mathbb{R}$, $i \in \mathcal{V}$, the trajectories of dynamic active average consensus algorithm (9) satisfy

$$|x^i(k) - \mathsf{avg}^{\mathrm{a}}(k)| \leq \kappa_d\omega_d^k \left\| \begin{bmatrix} \mathbf{x}(0) - \mathbf{avg}^{\mathrm{a}}(0) \\ \mathbf{L}\mathbf{v}(0) - \mathbf{w}(0) \end{bmatrix} \right\| + \frac{\kappa_d(1-\omega_d^k)}{1-\omega_d} \sup_{0 \leq l \leq k-1} \left\| \begin{bmatrix} \Delta\mathbf{Er}(l) - \Delta\mathbf{avg}^{\mathrm{a}}(l) \\ -\Delta\mathbf{w}(l) \end{bmatrix} \right\|. \quad (12)$$

Proof: Using standard results for linear systems, trajectories of (10b) are given by

$$\begin{bmatrix} \bar{\mathbf{e}}(k) \\ \mathbf{q}_{2:N}(k) \end{bmatrix} = \Phi(k, 0)\begin{bmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0) \end{bmatrix} + \sum_{j=0}^{k-1} \Phi(k, j+1)\bar{\mathbf{B}}\begin{bmatrix} \Delta\mathbf{Er}(j) - \Delta\mathsf{avg}^{\mathrm{a}}(j) \\ -\Delta\mathbf{w}(j) \end{bmatrix}.$$

Then, given that $\|\bar{\mathbf{B}}\| \leq 1$ and (11) we can write

$$\left\| \begin{bmatrix} \bar{\mathbf{e}}(k) \\ \mathbf{q}_{2:N}(k) \end{bmatrix} \right\| \leq \kappa_d\, \omega_d^k \left\| \begin{bmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0) \end{bmatrix} \right\| + \kappa_d \sum_{j=0}^{k-1} \omega_d^j \sup_{0 \leq l \leq k-1} \left\| \begin{bmatrix} \Delta\mathbf{Er}(l) - \Delta\mathbf{avg}^{\mathrm{a}}(l) \\ -\Delta\mathbf{w}(l) \end{bmatrix} \right\|.$$

By the sum of geometric sequence, $\kappa_d \sum_{j=0}^{k-1} \omega_d^j = \frac{\kappa_d(1-\omega_d^k)}{1-\omega_d}$. Then, given that $\|[\begin{smallmatrix} \bar{\mathbf{e}}(0) \\ \mathbf{q}_{2:N}(0) \end{smallmatrix}]\| = \|[\begin{smallmatrix} \mathbf{x}(0) - \mathbf{avg}^{\mathrm{a}}(0) \\ \mathbf{L}\mathbf{v}(0) - \mathbf{w}(0) \end{smallmatrix}]\|$ and $|x^i - \mathsf{avg}^{\mathrm{a}}| \leq \|[\bar{\mathbf{e}}^\top \quad \mathbf{q}_{2:N}^\top]^\top\|$, tracking error (12) is established. ■

VI. Distributed Containment Control via Dynamic Active Average Consensus Modeling

In this section, we use the discrete-time dynamic active average consensus algorithm to solve a containment control problem. Consider a group of M (M can change with time) mobile leaders that are moving with a bounded velocity on a $\mathbb{R}^2$ or $\mathbb{R}^3$ space. $\mathbf{x}_{L,j}(t)$ represents the position vector of leader $j \in \{1, \ldots, M\}$ at time $t \in \mathbb{R}_{\geq 0}$. A set of networked follower agents $\mathcal{V} = \{1, \ldots, N\}$ interacting over a connected graph $\mathcal{G}$ monitors the leaders. The agents can communicate at discrete-times $t_k^c = k\delta_c \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$, $\delta_c \in \mathbb{R}_{>0}$. The agents sample the leaders at sampling times $t_l^s = l\delta_s \in \mathbb{R}_{\geq 0}$, $l \in \mathbb{Z}_{\geq 0}$, $\delta_s \in \mathbb{R}_{>0}$. We let $\mathcal{V}_L^i(t_l^s)$ be the set of leaders observed by agent $i \in \mathcal{V}$ at sampling time t_l^s. Between each sampling time, agent $i \in \mathcal{V}$ uses $\mathbf{x}_{L,j}(t) = \mathbf{x}_{L,j}(t_l^s)$ and $\mathcal{V}_L^i(t) = \mathcal{V}_L^i(t_l^s)$, $t \in [t_l^s, t_{l+1}^s)$, $l \in \mathbb{Z}_{\geq 0}$, $j \in \mathcal{V}_L^i(t_l^s)$. At every sampling time $t_l^s \in \mathbb{R}_{\geq 0}$, we let $\mathcal{V}_L(t_l^s)$ be the set of the mobile leaders that are observed jointly by the agents $\mathcal{V}$, i.e., $\mathcal{V}_L(t_l^s) = \cup_{i=1}^N \mathcal{V}_L^i(t_l^s)$ (see Fig. 1). We let $\mathcal{V}_a(t_l^s) \subset \mathcal{V}$ be the set of the active agents that observe at least one leader at t_l^s, $k \in \mathbb{Z}_{\geq 0}$; we assume that $\mathcal{V}_{\mathrm{a}}(t_l^s) \neq \emptyset$. In what follows, the objective is to design a distributed control that enables each follower $i \in \mathcal{V}$ to derive its local state $\mathbf{x}^i$ to asymptotically track $\text{Co}(\mathcal{V}_L(t_l^s))$, the convex hull of the set of the location of the observed leaders $\mathcal{V}_L(t_l^s)$, with a bounded error $e \geq 0$. To simplify notation, we wrote $\text{Co}(\{\mathbf{x}_{L,j}(t)\}_{j\in\mathcal{V}_L(t)})$ as $\text{Co}(\mathcal{V}_L(t))$. We state the objective of the containment control as $\|\mathbf{x}^i(t_k^c) - \bar{\mathbf{x}}_L(t_k^c)\| \leq e$, $i \in \mathcal{V}$, where $\bar{\mathbf{x}}_L(t_k^c) \in \text{Co}(\mathcal{V}_L(t_k^c))$. The agents have no knowledge about the motion model of the leaders. Since followers observe the dynamic leaders collaboratively, the tracking error e is expected as the measurement of each active follower needs time to propagate through the network to the rest of the followers.

Our solution builds on the key observation that we make below about the convex hull of a set of points $\{\mathbf{x}_i\}_{i=1}^m$ in an Euclidean space.

Lemma 4: Consider a set of points $\{\mathbf{x}_i\}_{i=1}^m$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $\mathcal{S}_j \neq \emptyset$, $j \in \{1, \ldots, s\}$, be a subset of $\{1, \ldots, m\}$. Let $\bar{\mathbf{x}}_j = \frac{\sum_{k\in\mathcal{S}_j}\mathbf{x}_k}{|\mathcal{S}_j|}$, $j \in \{1, \ldots, s\}$. Then, the point $\bar{\mathbf{x}} = \frac{\sum_{i=1}^s \bar{\mathbf{x}}_i}{s}$ is a point in $\text{Co}(\{\mathbf{x}_j\}_{j=1}^m)$.

Proof: It is straightforward to confirm that $\bar{\mathbf{x}}_j \in \text{Co}(\{\mathbf{x}_i\}_{i\in\mathcal{S}_j})$, $j \in \{1, \ldots, s\}$ and $\bar{\mathbf{x}} \in \text{Co}(\{\bar{\mathbf{x}}_i\}_{i=1}^s)$ (recall the definition of the convex hull). Moreover, since $\text{Co}(\{\mathbf{x}_k\}_{k=1}^m)$ is a convex set, we note that $\text{Co}(\{\mathbf{x}_i\}_{i\in\mathcal{S}_j}) \subset \text{Co}(\{\mathbf{x}_k\}_{k=1}^m)$, $j \in \{1, \ldots, s\}$. Thus, for $i \in \{1, \ldots, s\}$, $\bar{\mathbf{x}}_i \in \text{Co}(\{\mathbf{x}_j\}_{j=1}^m)$, and $\text{Co}(\{\bar{\mathbf{x}}_i\}_{i=1}^s) \subset \text{Co}(\{\mathbf{x}_i\}_{i=1}^m)$. As a result, $\bar{\mathbf{x}} \in \text{Co}(\{\mathbf{x}_j\}_{j=1}^m)$. ■

With the right notation at hand, and the observation made in Lemma 4, we are now ready to present in the Lemma below our solution for the containment problem stated above.

Lemma 5: In a containment control problem, let the interaction topology $\mathcal{G}$ of the followers be a connected graph and suppose that the agents communicate at $t_k^c = k\delta_c \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 0}$. Assume that at each sampling time $t_l^s = l\delta_s \in \mathbb{R}_{\geq 0}$, $l \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{V}_{\mathrm{a}}(t_l^s) \neq \emptyset$, and the followers are observing the leaders in a zero-order hold fashion, i.e., $\mathbf{x}_{L,j}(t) = \mathbf{x}_{L,j}(t_l^s)$, $j \in \mathcal{V}_L^i(t_l^s)$ and $i \in \mathcal{V}_{\mathrm{a}}(t_l^s)$ for $t \in [t_l^s, t_{l+1}^s)$. Let

$$\mathsf{r}^i(t_l^s) = \begin{cases} \frac{\sum_{j\in\mathcal{V}_L^i(t_l^s)}\mathbf{x}_{L,j}(t_l^s)}{|\mathcal{V}_L^i(t_l^s)|}, & i \in \mathcal{V}_{\mathrm{a}}(t_l^s), \\ \mathbf{0}, & i \in \mathcal{V}\backslash\mathcal{V}_{\mathrm{a}}(t_l^s). \end{cases} \quad (13)$$

Then, $\bar{\mathbf{x}}_L(t_l^s) = \frac{\sum_{i\in\mathcal{V}_{\mathrm{a}}(t_l^s)}\mathsf{r}^i(t_l^s)}{|\mathcal{V}_{\mathrm{a}}(t_l^s)|}$ is a point in the convex hull of the leaders $\text{Co}(\mathcal{V}_L(t_l^s))$. Moreover, assume $\|\mathbf{x}_{L,j}(t_{l+1}^s) - \mathbf{x}_{L,j}(t_l^s)\|$, $j \in \{1, \ldots, M\}$, is bounded. If the followers implement active average consensus algorithm (9) with inputs (13),

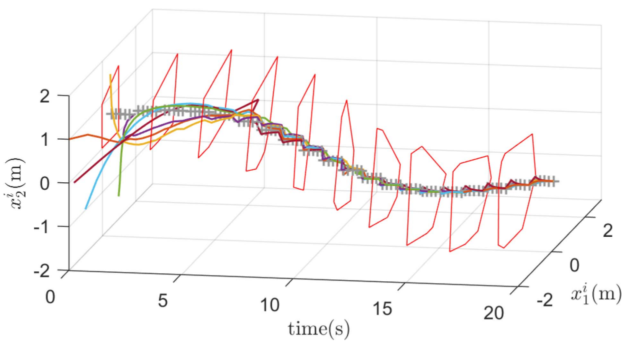

Fig. 3. The containment tracking performance of the follower agents while implementing the distributed algorithm (9): the solid curves show the trajectory of $\mathbf{x}^i$ vs. time, while "+" show the location of $\bar{\mathbf{x}}_L(t_k^c)$ of the leaders. The red polygons indicate the convex hull formed by the moving leaders.

and $\eta^i(t) = 1$ if $i \in \mathcal{V}_a(t_l^s)$, otherwise, $\eta^i(t) = 0$ for $t \in [t_l^s, t_{l+1}^s)$, then the tracking error $\|\mathbf{x}^i(t_k^c) - \bar{\mathbf{x}}_L(t_k^c)\|$ is bounded.

Proof: $\bar{\mathbf{x}}_L(t_l^s) \in \mathrm{Co}(\mathcal{V}_L(t_l^s))$ is true by virtue of Lemma 4. The boundedness of the tracking error $\|\mathbf{x}^i(t_k^c)-\bar{\mathbf{x}}_L(t_k^c)\|$ follows from the guarantees that Theorem 2 provides. ■

Our solution in Lemma 5 applies to scenarios like in Fig. 1 where the observation sets of the followers have overlap. It is interesting to note that in case of overlapping observations, $\bar{\mathbf{x}}_L$ is not the centroid of the leaders. Next, note that by virtue of Theorem 2, if the leaders are static or move towards a static configuration, the algorithm convergences exactly to $\bar{\mathbf{x}}_L$. Otherwise, to ensure that the followers stay in the convex hull while tracking $\bar{\mathbf{x}}_L$ with some error, we may have to require that the convex hull of the leaders should be sufficiently large.

For demonstration, consider a case that 6 followers with a ring interaction graph aim to follow the convex hull of 10 leaders in a two dimensional space. The followers observe the leaders at 1 Hz according to the scenario described below where the set of active followers changes at $t_l^s = 5$ and $t_l^s = 10$ seconds:

- $0 \leq t_l^s < 5$: $\mathcal{V}_L^1(t_l^s) = \{1, 4, 6, 8\}$, $\mathcal{V}_L^2(t_l^s) = \{2, 4, 7, 8, 10\}$, $\mathcal{V}_L^3(t_l^s) = \{3, 4, 5, 9\}$, $\mathcal{V}_L^4(t_l^s) = \emptyset$, $\mathcal{V}_L^5(t_l^s) = \{1, 3, 9\}$ and $\mathcal{V}_L^6(t_l^s) = \emptyset$,
- $5 \leq t_l^s < 10$: $\mathcal{V}_L^1(t_l^s) = \{3, 5, 6, 8\}$, $\mathcal{V}_L^2(t_l^s) = \{1, 2, 7, 9, 10\}$, $\mathcal{V}_L^3(t_l^s) = \{3, 4, 5, 9\}$, $\mathcal{V}_L^4(t_l^s) = \emptyset$, $\mathcal{V}_L^5(t_l^s) = \{1, 3, 9\}$ and $\mathcal{V}_L^6(t_l^s) = \{2, 5, 7, 9\}$,
- $10 \leq t_l^s \leq 20$: $\mathcal{V}_L^1(t_l^s) = \{1, 2, 5, 8\}$, $\mathcal{V}_L^2(t_l^s) = \{2, 3, 6, 7, 10\}$, $\mathcal{V}_L^3(t_l^s) = \{3, 4, 5, 9\}$, $\mathcal{V}_L^4(t_l^s) = \{3, 10\}$, $\mathcal{V}_L^5(t_l^s) = \{1, 3, 9\}$ and $\mathcal{V}_L^6(t_l^s) = \{2, 5, 7, 9\}$.

The communication frequency of the followers is 5 Hz. Figure 3 shows that the proposed distributed containment control of Lemma 5 results in a bounded tracking of the convex hull of the observed leaders. The interested reader can also find an application study of use of our solution in Lemma 5 in solving containment control for a group of unicycle followers with continuous-time dynamics in our preliminary work [15]. There, the algorithm in Lemma 5 is used as an observer to generate the tracking points for the followers.

VII. Conclusion

We proposed a dynamic active average consensus algorithm that makes both active and passive agents track the average of the collected reference signals. The stability and tracking performance were analyzed in both continuous- and discrete-time implementations. We also showed that a containment control can be formulated as an active average consensus problem and solved using our proposed discrete-time algorithm.

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[2] S. S. Kia, B. V. Scoy, J. Cortés, R. A. Freeman, K. M. Lynch, and S. Martínez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms," *IEEE Control Syst. Mag.*, vol. 39, no. 3, pp. 40–72, Jun. 2019.

[3] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[4] J. Cortés, "Analysis and design of distributed algorithms for X-consensus," in *Proc. 45th IEEE Conf. Decis. Control*, San Diego, CA, USA, 2006, pp. 3363–3368.

[5] Y. Shang, "Finite-time weighted average consensus and generalized consensus over a subset," *IEEE Access*, vol. 4, pp. 2615–2620, 2016.

[6] T. Yucelen and J. D. Peterson, "Active-passive networked multiagent systems," in *Proc. 53rd IEEE Conf. Decis. Control*, Los Angeles, CA, USA, 2014, pp. 6939–6944.

[7] J. D. Peterson, T. Yucelen, G. Chowdhary, and S. Kannan, "Exploitation of heterogeneity in distributed sensing: An active-passive networked multiagent systems approach," in *Proc. Amer. Control Conf.*, Chicago, IL, USA, 2015, pp. 4112–4117.

[8] J. D. Peterson, T. Yucelen, J. Sarangapani, and E. L. Pasiliao, "Active-passive dynamic consensus filters with reduced information exchange and time-varying agent roles," *IEEE Trans. Control Syst. Technol.*, vol. 28, no. 3, pp. 844–856, May 2020.

[9] R. P. Kanwal, *Distributional Derivatives of Functions With Jump Discontinuities*, Boston, MA, USA: Birkhäuser Boston, 1998, pp. 99–137.

[10] M. Ji, G. Ferrari-Trecate, M. Egerstedt, and A. Buffa, "Containment control in mobile networks," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1972–1975, Sep. 2008.

[11] X. Wang, S. Li, and P. Shi, "Distributed finite-time containment control for double-integrator multiagent systems," *IEEE Trans. Cybern.*, vol. 44, no. 9, pp. 1518–1528, Sep. 2014.

[12] H. Liu, G. Xie, and L. Wang, "Containment of linear multi-agent systems under general interaction topologies," *Syst. Control Lett.*, vol. 61, no. 4, pp. 528–534, 2012.

[13] L. Galbusera, G. Ferrari-Trecate, and R. Scattolini, "A hybrid model predictive control scheme for containment and distributed sensing in multi-agent systems," *Syst. Control Lett.*, vol. 62, no. 5, pp. 413–419, 2013.

[14] Z. Kan, J. M. Shea, and W. E. Dixon, "Leader–follower containment control over directed random graphs," *Automatica*, vol. 66, pp. 56–62, Apr. 2016.

[15] Y.-F. Chung and S. S. Kia, "Distributed dynamic containment control over a strongly connected and weight-balanced digraph," *IFAC PapersOnLine*, vol. 52, no. 20, pp. 25–30, 2019.

[16] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[17] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[18] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conf. Decis. Control*, vol. 3, Phoenix, AZ, USA, 1999, pp. 2655–2660.

[19] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle," *IEEE Trans. Autom. Control*, vol. 49, no. 4, pp. 470–482, Apr. 2004.

[20] Z. Artstein, "Stability, observability and invariance," *J. Differ. Equ.*, vol. 44, no. 2, pp. 224–248, 1982.

[21] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Qualitative analysis of discrete-time switched systems," in *Proc. IEEE Amer. Control Conf.*, vol. 3. Anchorage, AK, USA, 2002, pp. 1880–1885.

[22] J. C. Geromel and P. Colaneri, "Stability and stabilization of discrete time switched systems," *Int. J. Control*, vol. 79, no. 7, pp. 719–728, 2006.