Quantum Hall Network Models as Floquet Topological Insulators

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Network models for equilibrium integer quantum Hall (IQH) transitions are described by unitary scattering matrices that can also be viewed as representing nonequilibrium Floquet systems. The resulting Floquet bands have zero Chern number, and are instead characterized by a chiral Floquet winding number. This begs the question, How can a model without Chern number describe IQH systems? We resolve this puzzle by showing that nonzero Chern number is recovered from the network model via the energy dependence of network model scattering parameters. This relationship shows that, despite their topologically distinct origins, IQH and chiral Floquet topology-changing transitions share identical universal scaling properties.

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Disorder and localization often play a vital role in stabilizing topological matter. In particular, these features are essential for the experimental observation of quantized Hall conductance in a variety of experimental systems such as GaAs quantum wells [1,2], graphene [3], and magnetically doped topological insulator thin films [4]. These systems all allow tuning between Hall plateaus with different quantized conductance via changing gate voltages or external magnetic fields, resulting in a topological-phase transition. This transition is marked by a jump in the Chern number of the occupied electron states and a change in the number of chiral edge states circulating around the perimeter of the sample. It is accompanied by a divergence in the localization length at a critical value of the chemical potential.

Disorder and localization can also stabilize driven systems against drive-induced heating, giving access to new regimes of quantum coherent dynamics. Recent theoretical advances have shown that time-periodically driven (Floquet) systems can exhibit new types of nonequilibrium topological phases with inherently dynamical properties that could not arise in the ground state of static (time-independent) Hamiltonians [5–7]. Striking examples include chiral Floquet (CF) phases [6], which exhibit chiral edge states, despite having only topologically trivial bulk bands.

Although CF edge states are reminiscent of those in integer quantum Hall (IQH) systems, there are important differences. Crucially, the number of CF edge states for a given Floquet operator is the same at all values of the (compact) quasienergy. By contrast, the number of edge states for a (time-independent) IQH Hamiltonian is a function of (noncompact) energy, and is given by the net Chern number of all bulk states at lower energies. As a corollary, CF phases differ from IQH phases in that they do not exhibit charge pumping through bulk states upon adiabatic insertion of magnetic flux, and hence have vanishing Hall conductance and Chern number. Instead, for noninteracting systems, the unitary time-evolution operators for CF phases are characterized by an integer-valued winding number χ, the chiral unitary invariant [6].

In this Letter, we explore a relation between these two distinct types of topological phenomena via the network model introduced by Chalker and Coddington [8] to describe the scattering dynamics of electrons near the quantum Hall plateau transition. The Chalker-Coddington network (CCN) model is defined by a unitary matrix that acts on a wave function sampled at discrete spatial lattice points in a continuum IQH system. This unitary can also be interpreted as the Floquet operator of a time-periodically driven system [9], as previously discussed in the context of photonic networks where it was found that these network models could realize both CF phases and Chern bands for appropriate network geometries and parameters [10–12]. We construct a periodically time-dependent Hamiltonian, acting on the same lattice, whose Floquet operator coincides with the CCN unitary, and demonstrate that this Floquet system hosts a CF phase, and find vanishing Chern number for all bands at all fixed network-model parameters.

This correspondence uncovers a puzzle: IQH systems are characterized by a Chern number, but the CF system has Chern number zero. We resolve this puzzle by showing that the Chern number of an IQH system is recovered by properly accounting for energy dependence of scattering phases and tunneling amplitudes in the model. This relationship provides a fresh perspective on IQH systems, building on the fact that the CF Floquet operator is characterized by the chiral unitary invariant. Since the CCN is represented by the same unitary matrix as a CF system, values of the chiral unitary invariant can be used to label IQH phases within the CCN description.

These observations show that the network model equally describes the fixed-energy scattering behavior of a wave packet in an IQH system and the full dynamics of a CF system. In particular, this implies that the topology-changing phase transitions for disordered, noninteracting IQH and CF phases share the same universal scaling properties, in agreement with recent field-theoretic analysis [13] relating σ models for CF and IQH phases.

Network model definition.—Consider a zero-temperature noninteracting electron gas moving in a uniform magnetic field and a disordered potential, with the lowest Landau level partially occupied on average. If the potential is smooth on the scale of the magnetic length and has fluctuations smaller in amplitude than the cyclotron energy, the system is divided into spatial regions around potential minima where the Landau level is locally fully occupied and regions around maxima where it is empty. Chiral edge states at the chemical potential circulate along the boundaries between these occupied and empty regions. Tunneling between distinct edge segments occurs near saddle points in the potential where their spatial separation is small.

The network model [8] describes a simplified version of this picture, in which the potential is chosen so that occupied and empty regions form alternate plaquettes of a regular square lattice. Edge states propagate on directed links of this lattice, meeting at nodes that correspond to potential saddle points. Consider an eigenstate $\psi(r)$ of the usual single-particle Hamiltonian [14] for this continuum problem. In the CCN this wave function is sampled at a single point r_i on each link and is represented by a current amplitude ψ_i. Amplitudes on incoming and outgoing links at a node are related by a scattering matrix, so that (referring to Fig. 1)

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}e^{i\varphi_3} & 0\\ 0 & e^{i\varphi_4}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \quad (1)$$

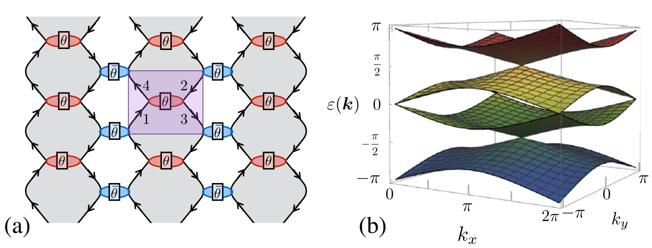

FIG. 1. Network model and Floquet band structure. (a) Schematic of Chalker-Coddington network model. (b) Floquet bands $\varepsilon(\boldsymbol{k})$ of the clean network model evolution operator at criticality, $\theta = \pi/4$.

Here, θ parametrizes tunneling while φ_3 and φ_4 are Aharonov-Bohm phases. A similar S matrix with $\theta \to \bar{\theta} \equiv \pi/2 - \theta$ describes scattering at the $\bar{\theta}$ nodes. Disorder is modeled by taking φ_i to be an independent random variable on each link i, uniform in $[0, 2\pi)$. The CCN may be formulated either for an open system consisting of a sample connected to leads [8] or for a closed system [15]. In a closed system of N links, the model is specified by an $N \times N$ unitary matrix U, which is composed of 2×2 submatrices, each having the form of Eq. (1).

For an open system, the model defines a scattering problem between asymptotic states supported in the leads. Stationary states for this scattering problem exist at all energies within the Landau level, and IQH physics may be extracted by relating the scattering matrix to conductances using the Landauer-Büttiker formula [16]. For a closed system, stationary states exist only at discrete energies and the model defines an eigenvalue problem. A standard way to characterize the IQH for a closed system is via the Chern numbers of eigenstates, which are at the focus of our discussion.

The amplitudes ψ_i of a stationary state satisfy (1) at every node. As a result, in a closed system these amplitudes form components of an eigenvector of the unitary matrix U, with an eigenvalue of unity. To locate the discrete energies at which one of the eigenvalues of U is unity, it is necessary to consider the dependence on energy of the parameters θ and φ_i [17]. From the shape of equipotentials near a saddle point, one sees that θ increases from 0 to $\pi/2$ as energy is increased across the disorder-broadened Landau level, as illustrated in Fig. 2. The accumulated link phase around a plaquette is (2π times) the number of flux quanta passing through this plaquette. Randomness in φ_i arises from small random variations in the area of plaquettes, and energy dependence of φ_i arises due to the change in area of a plaquette with a change in the chemical potential.

The behavior of the model is simplest at the extreme limits $\theta = 0$ and $\theta = \pi/2$. In the first case, the system consists only of isolated plaquettes enclosing occupied regions. In the second case, it is made up of isolated plaquettes enclosing empty regions, together with a chiral edge state at a boundary where the system meets an external empty region. Detailed numerical studies show that the edge state is present for all $\theta > \pi/4$.

Floquet perspective.—The unitary matrix U for fixed θ and φ_i may be viewed as the evolution operator for one time step of a Floquet problem in which the N-component wave function $\Psi(t)$ has the stroboscopic evolution

$$\Psi(t+1) = U\Psi(t). \quad (2)$$

It can be shown that dynamics of $\Psi(t)$ in this Floquet problem matches the dynamics of a wave packet with small energy dispersion in the IQH system, under the

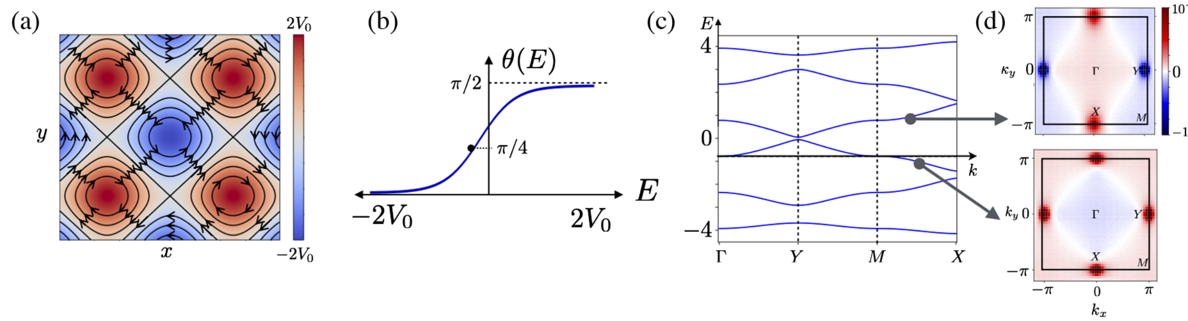

FIG. 2. Chern bands from the scattering network. (a) Semiclassical orbits for an electron in a periodic potential and uniform magnetic field and (b) corresponding energy dependence of scattering parameters (schematic). (c) Partial set of energy bands from the scattering network with $\theta(E) = (\pi/4)(\tanh\{[(E + \pi/4)/4\pi]\} + 1)$ and $\varphi(E) = E$. The horizontal axis is placed at the critical energy $E = E_c$ for which $\theta(E_c) = \pi/4$. (d) Berry curvature for two bands near $E = 0$ for which θ respectively crosses (lower) and does not cross (upper) $\pi/4$, giving total Chern number 1 or 0. All other bands have Chern number zero.

approximations that (i) all φ_i vary with energy linearly and at the same rate and (ii) the energy dependence of θ may be neglected.

Writing the eigenvalues of U as $e^{-i\varepsilon}$, the phases ε are referred to in this context as quasienergies, located within a compact Brillouin zone $\varepsilon \in (-\pi, \pi]$. The fact that quasienergies lie on a circle rather than an open line changes the topological classification of Floquet dynamics compared to that of gapped ground states of static Hamiltonians [5–7].

Noninteracting Floquet bands in systems with conserved particle number are exhaustively classified by two integer-valued topological invariants: (1) the Chern number C_n, defined separately for each Floquet band n, and (2) the chiral unitary index, $\chi(U)$, defined for the full Floquet unitary, which characterizes the number of chiral edge states that wrap around the quasienergy Brillouin zone [6].

The Chern numbers for the Floquet bands of U are identically zero [10,11]. This can be verified by inspection at the trivial and topological limiting points, $\theta = 0$ and $\theta = \pi/2$. At these points, the bulk motion consists of short loops around the elementary plaquettes of the network model (Fig. 1), indicating that the Floquet bands admit a strictly localized Wannier basis of orbitals, each having support only on the four links of a single plaquette. For a system on a torus such a localized Wannier basis implies vanishing Chern number [18], immediately implying that each individual eigenstate has zero Chern number. This conclusion extends to all values of $\theta \neq \pi/4$, since Chern number may only change at a delocalization transition, which in this model occurs only at $\theta = \pi/4$. The regions either side of this critical point inherit the vanishing Chern number of their limiting points, at $\theta = 0$, $\pi/2$, respectively.

This argument, valid for arbitrary disorder, can be directly verified for the clean version of the model by explicit computation of the Floquet band structure. Introducing crystalline momenta $\boldsymbol{k}$, the matrix U is block diagonal with blocks of the form

$$U(\theta, \boldsymbol{k}) = \begin{pmatrix} 0 & 0 & \sin\theta e^{-ik_x} & \cos\theta e^{-ik_y} \\ 0 & 0 & -\cos\theta e^{ik_y} & \sin\theta e^{ik_x} \\ \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \end{pmatrix}. \tag{3}$$

We denote the eigenvalues of $U(\theta, \boldsymbol{k})$ by $e^{-i\varepsilon_n(\theta,\boldsymbol{k})}$ for $n \in \{0, 1, 2, 3\}$. At $\theta = 0$ and $\theta = \pi/2$, ε_n are independent of $\boldsymbol{k}$, taking the values $\varepsilon_n = \pi/4 + \pi n/2$. As θ deviates from these extreme values, the bands disperse, until they touch at $\theta = \pi/4$ in a sequence of Dirac points at wave vectors $\boldsymbol{k} = (0, 0)$ or $\boldsymbol{k} = (\pi, \pi)$ (Fig. 1). Tuning away from $\theta = \pi/4$, the Dirac points develop a mass gap with an alternating sign mass for Dirac points separated in quasienergy by $\Delta\varepsilon = \pi/2$. Equivalently, viewed in the three-dimensional parameter space $(\theta, \boldsymbol{k})$, these degenerate points form monopole sources of Berry flux, with overall canceling monopole charge, resulting in vanishing net Chern number for all θ.

Chiral Floquet invariant.—Viewing the network model as a lattice Floquet system, since U has only topologically trivial bulk bands, any nontrivial topological behavior must emerge from a nontrivial chiral unitary invariant, $\chi \neq 0$. As a first step, we compare behavior at $\theta = 0$ and $\theta = \pi/2$ for a system with open boundaries, where χ can be computed simply by counting the number of chiral edge states wrapping the quasienergy Brillouin zone [6,10]. There is no edge state in the first case, and one in the second case. The eigenvectors of U corresponding to the edge state can be given explicitly. Let integer j label links in order along the boundary, and for simplicity set all $\varphi_j = 0$. Then the vector with $\psi_j = e^{ikj}$ on edge links and $\psi_i = 0$ on all other links is an (unnormalized) eigenvector of U with quasienergy $\varepsilon_{\text{edge}} = k$. This mode indeed wraps the quasienergy Brillouin zone as k varies, and is manifestly absent in the other phase of the model ($\theta \approx 0$), demonstrating that

$$\chi[U_{\mathrm{CCN}}(\theta)] = \begin{cases} 0 & 0 \le \theta < \pi/4 \\ 1 & \pi/4 < \theta \le \pi/2, \end{cases} \tag{4}$$

and hence that the network model describes a Floquet chiral unitary index changing phase transition.

Although inspection of the stroboscopic edge motion is sufficient to give the value of bulk invariant χ via the bulk-boundary correspondence established in Ref. [6], it is also reassuring to compute χ directly from the network model in the bulk by considering a system with periodic boundary conditions. Here, we face an obstacle: to compute χ from bulk behavior alone, it is not sufficient to examine stroboscopic times (for which the bulk motion is always trivial when all $C_n = 0$). Instead, one must examine the micromotion within a single period. The network model as formulated is blind to this micromotion, and additional choices are needed to define it (though the final result will be independent of these choices).

Specifically, we seek a continuous path in the space of unitary matrices, from the identity to U, that is generated by a local, time-dependent Hamiltonian $H(t)$ acting on the Hilbert space of the network model lattice. That is, we require $U(t) = \mathcal{T} e^{-i\int_0^t H(t_1)dt_1}$ (where $\mathcal{T}$ denotes time ordering) such that $U(T) = U$, the CCN unitary. This relation does not uniquely fix $H(t)$. However, if we demand that $H(t)$ is spatially local, any such choice of $H(t)$ will produce the same value of χ. In Ref. [19], we construct a specific example via a coupled-wire perspective, and explicitly verify Eq. (4) via evaluation of the bulk chiral Floquet winding number [6]. The completeness of the Floquet classification for lattice models [6,21,22] also implies that any such local generating $H(t)$ must be explicitly time dependent (in contrast to the static Hamiltonian for continuum Landau levels).

Recovery of Chern bands.—These observations naturally raise the question, How can a model with zero Chern number describe the quantum Hall transition? We now show that a nonzero Chern number for the Landau band of the continuum Hamiltonian is correctly recovered from the eigenvectors of U when the network model parameters are allowed to vary with energy in a realistic manner.

For clarity, we use the clean model in this discussion although the results are general. Without disorder, the link phases obey $\varphi_i = \varphi(E)$ for all i, where $\varphi(E)$ is a monotonic function of energy E with an increment Φ across the energy width of the disorder-broadened Landau level. If there are many magnetic flux quanta per unit cell (the natural regime for the network model), then $\Phi \gg 1$. Including $\varphi(E)$, the quasienergies of U are $\varepsilon_n(\boldsymbol{k},\theta) - \varphi(E)$. Eigenenergies of the continuum problem form bands defined by

$$\varphi(E) = \varepsilon_n(\boldsymbol{k},\theta) + 2\pi m \tag{5}$$

for integer m. For large Φ there are many such bands.

Recalling that $\theta \equiv \theta(E)$ is a (slowly varying) function of E, the solution to Eq. (5) for each m, n defines a surface $\theta_{m,n}(\boldsymbol{k})$ in the space $(\theta,\boldsymbol{k})$ that was introduced following Eq. (3). For all but one of these surfaces, both of the oppositely charged $(\theta,\boldsymbol{k})$-space monopole sources of Berry flux lie on the same side of the surface. The net Berry flux through these surfaces is therefore zero, implying zero Chern number for the associated band of eigenstates in the continuum problem. However, there is one exceptional pair (m,n) for which $\theta_{m,n}(\boldsymbol{k}) < \pi/4$ at $\boldsymbol{k} = (0,0)$, but $\theta_{m,n}(\boldsymbol{k}) > \pi/4$ at $\boldsymbol{k} = (\pi,\pi)$ (or vice versa, depending on the parity of n). For this exceptional pair, oppositely charged monopoles lie on opposite sides of the surface, which therefore has a full unit of Berry flux passing though it. The associated band of eigenstates hence has unit Chern number, and summing over all bands we recover unit Chern number for the Landau level. A numerical demonstration for a particular energy dependence is shown in Fig. 2.

Discussion.—These two lines of analysis show that the network model equally describes both chiral Floquet topological insulators and quantum Hall phases and transitions. Specifically, these results establish a precise equivalence of the dynamics of a wave packet with near-constant energy in these two settings. Since the critical properties at topological phase transitions in both classes arise from delocalization at a fixed critical energy, this implies that these topologically distinct phenomena share the same critical properties at their (disordered) topological-phase transitions. We emphasize that, despite sharing critical scaling properties, the CF and IQH systems are sharply distinct. Beyond their distinct bulk-topology and edge-state structure, they exhibit qualitatively different dynamics of spatially localized wave packets, which remain indefinitely localized in strongly disordered CF phases [23], but instead spread subdiffusively in IQH systems due to overlap with critically extended states [24].

In the absence of interactions, the equilibrium classification and the Floquet classification have related structure [7]: for each equilibrium class with phases classified by group structure $G \in \{\mathbb{Z}, \mathbb{Z}_2\}$, there is an additional set of purely dynamical Floquet phases also having the same classification G. Our results raise the question, Are the critical properties of topological-phase transitions equivalent for all of these equilibrium-Floquet phase pairs? We conjecture that the answer is affirmative. For example, closely related $2D$ network models can be used to establish a similar relation between topological phase transitions in equilibrium and Floquet chiral superconductors with spin-rotation symmetry (spin-quantum Hall effect, class C) whose critical properties correspond to percolation [25–27]. Moreover, analogous $1D$ scattering network constructions can be obtained by compactifying $2D$ examples to extend these results to all $1D$ classes [28], leading us to conjecture a general equivalence of universal scaling structure of noninteracting equilibrium and Floquet topological phase transitions.

The network model construction is special to noninteracting systems with elastic scattering. In interacting settings, the conjectured static-Floquet topological quantum criticality correspondence likely continues to hold for interacting MBL systems in $1d$ where phase transitions are characterized by RG flow to infinite randomness and have equivalent interacting and noninteracting scaling properties [29]. By this route the $1d$ Floquet model studied in Refs. [30,31] is related to an Ising model, which is known to have a network model representation [32]. However, the correspondence will presumably fail for interacting $2d$ systems, since in that context CF phases are governed by rational-fractional-valued topological invariants with a completely distinct structure from Chern number [21,22], and because the noninteracting CF-to-trivial critical point will broaden into an intervening thermal phase upon including interactions [33].

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