

DECOMPOSITIONS OF GROTHENDIECK POLYNOMIALS

OLIVER PECHENIK AND DOMINIC SEARLES

ABSTRACT. We investigate the longstanding problem of finding a combinatorial rule for the Schubert structure constants in the K -theory of flag varieties (in type A). The Grothendieck polynomials of A. Lascoux–M.-P. Schützenberger (1982) serve as polynomial representatives for K -theoretic Schubert classes; however no positive rule for their multiplication is known in general. We contribute a new basis for polynomials (in n variables) which we call *glide polynomials*, and give a positive combinatorial formula for the expansion of a Grothendieck polynomial in this basis. We then provide a positive combinatorial Littlewood-Richardson rule for expanding a product of Grothendieck polynomials in the glide basis. Our techniques easily extend to the β -Grothendieck polynomials of S. Fomin–A. Kirillov (1994), representing classes in connective K -theory, and we state our results in this more general context.

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1. INTRODUCTION

Let $X = \text{Flags}(\mathbb{C}^n)$ be the parameter space of complete flags

$$\mathbb{C}^0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n.$$

The space X is a smooth projective complex variety and carries an action of $\text{GL}_n(\mathbb{C})$ induced from the standard action of $\text{GL}_n(\mathbb{C})$ on \mathbb{C}^n . There are then restricted actions by the **Borel subgroup** B of invertible lower triangular matrices and the **maximal torus** T of invertible diagonal matrices. The T -fixed points of X are the flags $F_\bullet^{(w)}$ defined by $F_k^{(w)} = \langle e_{w(1)}, e_{w(2)}, \dots, e_{w(k)} \rangle$, where e_i is the i th standard basis vector and $w \in S_n$ is a permutation. Hence the T -fixed points are naturally indexed by the permutations w in the symmetric group S_n . The B -orbits

$$X_w^\circ := B \cdot F_\bullet^{(w)}$$

give a cell decomposition of X and each contains a unique T -fixed point. The closure of X_w° is the **Schubert variety** X_w .

Since the structure sheaf \mathcal{O}_{X_w} of a Schubert variety has a resolution by locally free sheaves

$$0 \rightarrow V_k \rightarrow V_{k-1} \rightarrow \cdots \rightarrow V_0 \rightarrow \mathcal{O}_{X_w} \rightarrow 0,$$

one may thereby define classes

$$[\mathcal{O}_{X_w}] := \sum_{i=0}^k (-1)^i [V_i]$$

in the Grothendieck ring $K(X)$ of algebraic vector bundles over X . Indeed the set $\{[\mathcal{O}_{X_w}]\}_{w \in S_n}$ of these K -theoretic Schubert classes is an additive basis for $K(X)$. Hence the product structure of $K(X)$ (given by tensor product of vector bundles) is encoded in the structure coefficients $C_{u,v}^w$ appearing in

$$[\mathcal{O}_{X_u}] \cdot [\mathcal{O}_{X_v}] = \sum_{w \in S_n} C_{u,v}^w [\mathcal{O}_{X_w}].$$

It was conjectured by A. Buch [Buc02] and proved by M. Brion [Bri02] that the signs of these coefficients are determined simply by the codimensions of the Schubert varieties in X . More precisely, $(-1)^{\ell(w)-\ell(u)-\ell(v)} C_{u,v}^w \geq 0$, where $\ell(w) = \text{codim}_X(X_w)$ (or equivalently the Coxeter length of w).

Since the numbers $(-1)^{\ell(w)-\ell(u)-\ell(v)} C_{u,v}^w$ are nonnegative integers, one might hope for a combinatorial rule expressing them as the cardinality of some explicit set of combinatorial objects. Giving such a rule remains a major, long-standing problem in algebraic combinatorics.

Most important of the available combinatorial tools are the *Grothendieck polynomials* \mathfrak{G}_w (introduced by A. Lascoux–M.-P. Schützenberger [LS82]), which are polynomial representatives for the K -theoretic Schubert classes in $K(X)$, in the sense that

$$\mathfrak{G}_u \cdot \mathfrak{G}_v = \sum_{w \in S_n} C_{u,v}^w \mathfrak{G}_w,$$

with the same structure coefficients as before (cf. [LS82, FL94]). Indeed more general β -Grothendieck polynomials \mathfrak{K}_w (introduced by S. Fomin–A. Kirillov [FK94]) play the analogous role with respect to the richer *connective K-theory* of X [Hud14], which, as shown by P. Bressler–S. Evens [BE90], is the most general complex-oriented generalized cohomology theory in which the standard method of constructing Schubert classes is well-defined.

In this paper, we contribute the following towards this problem. We use the philosophy of [AS17] to introduce the *glide polynomials* \mathcal{G}_a , which refine the β -Grothendieck polynomials and form a new basis (indexed by weak compositions) of polynomials in n variables. A β -Grothendieck polynomial can be expressed as a generating function for certain combinatorial objects called *pipe dreams* [FK94, KM05]. We define a *quasi-Yamanouchi* condition on pipe dreams and use it to give a *positive* formula for the basis change from β -Grothendieck to glide polynomials:

$$\mathfrak{K}_w = \sum_{Q \in \text{QPD}(w)} \beta^{\text{ex}(Q)} \mathcal{G}_{\text{wt}(Q)},$$

where the sum is over quasi-Yamanouchi pipe dreams for the permutation w .

We define the *glide product* \sqcup_{gen} on weak compositions, which is manifestly positive and extends T. Lam–P. Pylyavskyy’s [LP07] K -theoretic analogue of the *shuffle product* of S. Eilenberg–S. Mac Lane [EML53]. Using the glide product, we prove that the basis of glide polynomials has positive structure constants by providing an explicit positive combinatorial Littlewood-Richardson rule:

$$\mathcal{G}_a \mathcal{G}_b = \sum_c \beta^{|c|-|a|-|b|} g_{a,b}^c \mathcal{G}_c,$$

where $g_{a,b}^c$ denotes the multiplicity of c in the glide product $a \sqcup_{\text{gen}} b$.

Using our Littlewood-Richardson rule for the structure constants of the glide basis and our rule for the glide expansion of a β -Grothendieck polynomial, we then give an explicit positive combinatorial Littlewood-Richardson rule for expanding a product of β -Grothendieck polynomials in the glide basis.

The glide basis has close connections to other important families of polynomials. With β specialized to zero, the glide polynomials $\mathcal{G}_a^{(0)}$ are precisely the *fundamental slide polynomials*, which were introduced in [AS17] and play an analogous role in decomposing Schubert polynomials. Moreover, with β specialized to 1, the stable limits of the glide polynomials $\mathcal{G}_a^{(1)}$ are precisely the *multi-fundamental quasisymmetric functions* of T. Lam–P. Pylyavskyy [LP07], a basis of the ring of quasisymmetric functions. The product structure on glide polynomials extends that of both the fundamental slide polynomials and the multi-fundamental quasisymmetric functions; in particular, the glide product \sqcup_{gen} on weak compositions is a common generalization of the *slide product* of [AS17] and the *multi-shuffle product* of [LP07].

This paper is organized as follows. In Section 2, we first recall the Grothendieck and β -Grothendieck polynomials. We then introduce the basis of glide polynomials and give our positive combinatorial rule for expressing β -Grothendieck polynomials in the glide basis. Finally, we show that specializing the glide polynomials to $\beta = 0$

yields the fundamental slide polynomials of [AS17]. In Section 3, we show the stable limits of glide polynomials (specialized to $\beta = 1$) are the multi-fundamental quasisymmetric functions of [LP07]. Moreover, the glide polynomials refining symmetric β -Grothendieck polynomials (i.e., those representing classes in Grassmannians) are a new basis of quasisymmetric polynomials and can be seen as (connective) K -theoretic analogues of I. Gessel's *fundamental quasisymmetric polynomials* [Ges84]. We give a positive combinatorial formula for expressing symmetric β -Grothendieck polynomials in this basis, compacting the set-valued tableau formula of A. Buch [Buc02]. In Section 4, we introduce the glide product and use it to present our positive combinatorial Littlewood-Richardson rules for the structure constants of the glide basis and for the expansion of a product of β -Grothendieck polynomials in the glide basis.

2. GROTHENDIECK AND GLIDE POLYNOMIALS

Here, we recall the Grothendieck polynomials of A. Lascoux–M.-P. Schützenberger [LS82] and the more general β -Grothendieck polynomials of S. Fomin–A. Kirillov [FK94]. We then introduce the glide polynomials as certain refinements.

2.1. Grothendieck polynomials. While the original definition of Grothendieck polynomials was in terms of divided difference operators, we will follow a more concretely combinatorial description based on work of various authors [BJS93, BB93, FK94, KM05]. Indeed, we will describe first the more general β -Grothendieck polynomials introduced by S. Fomin–A. Kirillov [FK94].

The β -Grothendieck polynomials naturally represent Schubert classes in the *connective K -theory* of X [Hud14] and specialize to the ordinary Grothendieck polynomials at $\beta = -1$. They moreover specialize at $\beta = 0$ to the *Schubert polynomials*, representing the Schubert classes in the Chow ring of X . While our interest is primarily in these two specializations, we will write most of our theorems for general β as a convenient way to describe both theories simultaneously. We find that using general β requires little extra complication beyond considering the $\beta = -1$ case.

We now turn to defining β -Grothendieck polynomials. A **pipe dream** P is a tiling of the fourth quadrant of the plane by **crossing pipes** $+$ and **turning pipes** \curvearrowright that uses finitely-many crossing pipes. The lines of P , traveling from the y -axis to the x -axis, are called **pipes**. We number the pipes by the absolute value of the y -coordinate of their left endpoint. In the case that no two pipes of P cross each other more than once, we say P is **reduced**. For any pipe dream P , its **reduction** $\text{reduct}(P)$ is the reduced pipe dream obtained by replacing all but the southwestmost $+$ between each pair of pipes with \curvearrowright . Note that if P is reduced, then $\text{reduct}(P) = P$.

The **permutation** of a reduced pipe dream P is the permutation given by the x -coordinates of the right endpoints of the pipes, while the permutation of a nonreduced pipe dream is the permutation of its reduction. The **excess** $\text{ex}(P)$ of a pipe dream P is the number of $+$'s in P minus the number of $+$'s in $\text{reduct}(P)$. Let $\text{PD}(w)$ denote the set of all pipe dreams for the permutation w , and let $\text{PD}_e(w)$ denote the subset of pipe dreams with excess e , so that $\text{PD}_0(w)$ denotes the subset of reduced pipe dreams. The **weight** $\text{wt}(P)$ of a pipe dream P is the **weak composition** (i.e.,

finite sequence of nonnegative integers) (a_1, a_2, \dots) , where a_i records the number of $+$'s in the i th row of P (from the top).

Example 2.1. The pipe dream

$$P = \begin{array}{cccc} & 1 & 2 & 3 & 4 \\ 1 & \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & \nearrow & \nearrow & \nearrow & \nearrow \\ 4 & \nearrow & \nearrow & \nearrow & \nearrow \end{array}$$

is not reduced since pipes 3 and 4 cross twice. Its reduction is the reduced pipe dream

$$\text{reduct}(P) = \begin{array}{cccc} & 1 & 2 & 3 & 4 \\ 1 & \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & \nearrow & \nearrow & \nearrow & \nearrow \\ 4 & \nearrow & \nearrow & \nearrow & \nearrow \end{array}$$

obtained by removing the second crossing between those pipes. Since $\text{reduct}(P) \in \text{PD}_0(1432)$, we have $P \in \text{PD}_1(1432) \subset \text{PD}(1432)$. The weight of P is the weak composition $(2, 1, 1)$, while the weight of $\text{reduct}(P)$ is $(2, 0, 1)$. \diamond

For $w \in S_n$, the **β -Grothendieck polynomial** $\mathfrak{K}_w^{(\beta)}$ is the following generating function for pipe dreams of w :

$$\mathfrak{K}_w^{(\beta)} := \sum_{P \in \text{PD}(w)} \beta^{\text{ex}(P)} \mathbf{x}^{\text{wt}(P)},$$

where $\mathbf{x}^a := x_1^{a_1} x_2^{a_2} \dots$. Here we treat β as a formal parameter. Two specializations of $\mathfrak{K}_w^{(\beta)}$ are particularly significant: For $\beta = -1$, the **Grothendieck polynomials**

$$\mathfrak{G}_w := \mathfrak{K}_w^{(-1)}$$

represent the Schubert classes in $K(X)$, while for $\beta = 0$, the **Schubert polynomials**

$$\mathfrak{S}_w := \mathfrak{K}_w^{(0)} = \sum_{P \in \text{PD}_0(w)} \mathbf{x}^{\text{wt}(P)}$$

represent the Schubert classes in the Chow ring of X ; this reflects the fact that Chow rings are isomorphic to the associated graded algebras of K -theory rings (at least after tensoring with \mathbb{Q}). Henceforth, we optionally drop β from the notation, unless it is specialized to a particular value.

2.2. Glide polynomials. Given a weak composition a , the **flattening** of a is the (strong) composition $\text{flat}(a)$ obtained by deleting all zero terms from a . For example, $\text{flat}(0102) = 12$.

Given weak compositions a and b of length n , say that b **dominates** a , denoted by $b \geq a$, if

$$b_1 + \dots + b_i \geq a_1 + \dots + a_i$$

for all $i = 1, \dots, n$. For example, $0120 \geq 0111$. Note that this partial ordering on weak compositions extends the usual dominance order on partitions.

Define a **weak komposition** to be a weak composition where the positive integers may be colored arbitrarily black or red. The **excess** $\text{ex}(a)$ of a weak komposition a is the number of red entries in a .

Definition 2.2. Let a be a weak composition with ℓ nonzero entries, in positions $n_1 < \dots < n_\ell$. The weak komposition b is a **glide** of a if there exist positions $i_1 < \dots < i_\ell$ such that for each $j \in \{1, \dots, \ell\}$ we have

$$(G.1) \quad b_{i_{j-1}+1} + \dots + b_{i_j} = a_{n_j} + \text{ex}(b_{i_{j-1}+1}, \dots, b_{i_j}) \quad (\text{where } i_0 = 0),$$

$$(G.2) \quad i_j \leq n_j, \text{ and}$$

$$(G.3) \quad \text{the leftmost nonzero entry of } b_{i_{j-1}+1}, \dots, b_{i_j} \text{ is black.}$$

In other words, b is a glide of a if one can decompose b into ℓ contiguous blocks, plus a rightmost (possibly empty) block of zeros, corresponding to the ℓ nonzero entries of a as follows. In the j th block of b , the sum of entries minus the number of red entries is equal to the j th nonzero entry of a ; the rightmost entry of the j th block of b occupies a weakly earlier position than the j th nonzero entry of a ; and the leftmost nonzero entry in each block of b is black.

Example 2.3. Let $a = (0, 1, 0, 0, 0, 3)$. Then $\ell = 2$ with $n_1 = 2$ and $n_2 = 6$. The weak komposition $(1, \textcolor{red}{1}, 0, 2, \textcolor{red}{2}, 0)$ is a glide of a . Here $i_1 = 2$ and $i_2 = 5$; the first block is the first two entries and the second block is the third, fourth and fifth entries. The rightmost block of zeros is the sixth entry.

Another glide of a is $(1, 1, \textcolor{red}{1}, 0, \textcolor{red}{1}, \textcolor{red}{3})$. Here, $i_1 = 1$ and $i_2 = 6$; the first block is the first entry and the second block is the last five entries. The rightmost block of zeros is empty.

On the other hand, the weak komposition $(0, 1, \textcolor{red}{1}, 0, 1, 2)$ is not a glide of a . Condition (G.3) forces the red 1 to be in the same block as the black 1 to its left. That means this first block must contain the third entry of b . Thus $i_1 \geq 3 > 2 = n_1$, violating (G.2). \diamond

Remark 2.4. If a glide b of a weak composition a has no red entries, then b is itself a weak composition. In this case, (G.3) is redundant, while conditions (G.1) and (G.2) say that entries of b form contiguous blocks, such that entries of the j th block occur weakly left of and sum to the j th nonzero entry of a . That is, $b \geq a$ and $\text{flat}(b)$ refines $\text{flat}(a)$.

Definition 2.5. For a weak composition a of length n , the **glide polynomial** $\mathcal{G}_a^{(\beta)} = \mathcal{G}_a^{(\beta)}(x_1, \dots, x_n)$ is

$$\mathcal{G}_a^{(\beta)} = \sum_b \beta^{\text{ex}(b)} x_1^{b_1} \cdots x_n^{b_n},$$

where the sum is over all weak kompositions b that are glides of a . As for $\mathfrak{K}_w^{(\beta)}$, we may drop β from the notation, unless it is specialized to a particular value.

Example 2.6. We have

$$\begin{aligned} \mathcal{G}_{0102} = & \mathbf{x}^{0102} + \mathbf{x}^{1002} + \mathbf{x}^{0120} + \mathbf{x}^{1020} + \mathbf{x}^{1200} + \mathbf{x}^{0111} + \mathbf{x}^{1011} + \mathbf{x}^{1101} + \mathbf{x}^{1110} + \\ & \beta \mathbf{x}^{0112} + \beta \mathbf{x}^{1012} + 2\beta \mathbf{x}^{1102} + 2\beta \mathbf{x}^{1120} + \beta \mathbf{x}^{1021} + \beta \mathbf{x}^{0121} + 3\beta \mathbf{x}^{1111} + \beta \mathbf{x}^{1210} + \beta \mathbf{x}^{1201} + \end{aligned}$$

$$2\beta^2 \mathbf{x}^{1112} + 2\beta^2 \mathbf{x}^{1121} + \beta^2 \mathbf{x}^{1211},$$

where $\mathbf{x}^b = x_1^{b_1} \dots x_n^{b_n}$.

For the term $2\beta \mathbf{x}^{1120}$, the corresponding glides are $(1, \textcolor{red}{1}, 2, 0)$ and $(1, 1, \textcolor{red}{2}, 0)$. \diamond

Let $\text{Poly}_n := \mathbb{Z}[x_1, x_2, \dots, x_n]$ denote the ring of polynomials in n variables.

Theorem 2.7. *The set*

$$\{\beta^k \mathcal{G}_a : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is an additive basis of the free \mathbb{Z} -module $\text{Poly}_n[\beta]$. Hence for any fixed $p \in \mathbb{Z}$,

$$\{\mathcal{G}_a^{(p)} : a \text{ is a weak composition of length } n\}$$

is a basis of Poly_n .

Proof. A monomial m in $\text{Poly}_n[\beta]$ is determined by a pair (k, a) , where $k \in \mathbb{Z}_{\geq 0}$ records the degree of β in m and a is the weak composition of length n that records the degrees of x_1, \dots, x_n in m . Let \mathcal{M} denote the set of such pairs (k, a) .

Define a total order on \mathcal{M} by $(k, a) > (\ell, b)$ if

- a contains strictly more 0's than b ,
- a and b contain equal numbers of 0's and b precedes a in reverse lexicographic order, or
- $a = b$ and $k > \ell$.

Now the $<$ -leading term of \mathcal{G}_a is $\beta^0 \mathbf{x}^a$. Hence if the $<$ -leading term of $p \in \text{Poly}_n[\beta]$ is $c_a \beta^k \mathbf{x}^a$, then the $<$ -leading term of

$$p_2 := p - c_a \beta^k \mathcal{G}_a$$

is $c_b \beta^\ell \mathbf{x}^b$ for some $(\ell, b) < (k, a)$. Then

$$p_3 := p_2 - c_b \beta^\ell \mathcal{G}_b$$

has $<$ -leading term $c_d \beta^m \mathbf{x}^d$ for some $(m, d) < (\ell, b)$, etc. Since there are no infinite strictly $<$ -decreasing sequences in \mathcal{M} , this process terminates with an expansion of p as a finite sum of glide polynomials times powers of β , proving the first sentence of the theorem.

The second sentence of the theorem is immediate from the first. \square

2.3. Expanding β -Grothendieck polynomials in the glide basis. By Theorem 2.7, \mathfrak{K}_w may be uniquely written as a sum of glide polynomials in the form

$$\mathfrak{K}_w = \sum_{(k,a)} c_w^{(k,a)} \beta^k \mathcal{G}_a,$$

where $c_w^{(k,a)} \in \mathbb{Z}$. This subsection is devoted to showing that these coefficients $c_w^{(k,a)}$ are in fact *nonnegative* integers; we show this by giving an explicit positive combinatorial formula for $c_w^{(k,a)}$.

The following two notions extend definitions of S. Assaf–D. Searles [AS17] to non-reduced pipe dreams.

Definition 2.8. For $P \in \text{PD}(w)$, the **destandardization** of P , denoted by $\text{dst}(P)$, is the pipe dream constructed from P as follows. We start with the top row and move down through the rows of P . For each row, say row $i - 1$, with no $+$ in the first column, if every $+$ in row $i - 1$ lies strictly east of every $+$ in row i , then shift every $+$ in row $i - 1$ southwest one position (if the westmost $+$ of row $i - 1$ is immediately northeast of a $+$, then these two crosses merge during the shift).

Example 2.9.

$$P = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & \nearrow & \nearrow & \nearrow & \nearrow \\ 4 & \nearrow & \nearrow & \nearrow & \nearrow \\ 5 & \nearrow & \nearrow & \nearrow & \nearrow \end{array} \quad \text{dst}(P) = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & \nearrow & \nearrow & \nearrow & \nearrow \\ 2 & \nearrow & \nearrow & \nearrow & \nearrow \\ 3 & \nearrow & \nearrow & \nearrow & \nearrow \\ 4 & \nearrow & \nearrow & \nearrow & \nearrow \\ 5 & \nearrow & \nearrow & \nearrow & \nearrow \end{array}.$$

◇

Remark 2.10. In fact, in Definition 2.8 one may equivalently consider the rows in arbitrary order, rather than from top to bottom. An arbitrary order may be less efficient, as one may have to act on an individual row more than once, repeating until there is no row on which one can act.

Definition 2.11. A pipe dream is **quasi-Yamanouchi** if the following is true for the westmost $+$ in every row: Either

- (1) it is in the westmost column, or
- (2) it is weakly west of some $+$ in the row below it.

Let $\text{QPD}(w)$ denote the set of quasi-Yamanouchi pipe dreams for the permutation w and let $\text{QPD}_e(w)$ be the subset of those with excess e .

Example 2.12. The pipe dream $\text{reduct}(P)$ of Example 2.1 is not quasi-Yamanouchi, since the westmost $+$ in the top row is not in the first column and there is no $+$ in the row below. In the pipe dream P of Example 2.1 the westmost $+$ in the top row is weakly west of a $+$ in the second row. However the $+$ in the second row is neither in the first column nor weakly west of a $+$ in the third row. Hence P is not quasi-Yamanouchi either.

A quasi-Yamanouchi pipe dream for 1432 is

$$Q = \begin{array}{ccccc} 1 & 2 & 3 & 4 \\ 1 & \nearrow & \nearrow & \nearrow \\ 2 & \nearrow & \nearrow & \nearrow \\ 3 & \nearrow & \nearrow & \nearrow \\ 4 & \nearrow & \nearrow & \nearrow \end{array}.$$

(The reduction of Q is formed by removing the $+$'s in the top row.)

◇

The **Lehmer code** $L(w)$ of a permutation w is the weak composition whose i th term is the number of indices j for which $i < j$ and $w(i) > w(j)$. For example, $L(146235) = (0, 2, 3, 0, 0, 0)$.

Lemma 2.13. *The destandardization map is well-defined and satisfies the following:*

- (1) for $P \in \text{PD}(w)$, $\text{dst}(P) \in \text{QPD}(w)$;

- (2) for $P \in \text{PD}(w)$, $\text{dst}(P) = P$ if and only if $P \in \text{QPD}(w)$;
- (3) $\text{dst} : \text{PD}(w) \rightarrow \text{QPD}(w)$ is surjective;
- (4) $\text{dst} : \text{PD}(w) \rightarrow \text{QPD}(w)$ is injective if and only if $w_i < w_{i+1}$ for all $i \geq w^{-1}(1)$.

Proof. Observe that if $P \in \text{PD}(w)$, applying destandardization at row i gives another pipe dream for w . The destandardization procedure terminates only when the quasi-Yamanouchi condition is satisfied, proving (1) and (2). Property (3) is immediate from (2).

For property (4), note that for any w , there is a reduced pipe dream $P_{L(w)}$ given by placing $L(w)_i$ '+'s in row i , columns 1 through $L(w)_i$. Suppose w has no descent after the m th position, where $m := w^{-1}(1)$. Then $P_{L(w)}$ has a '+' in row i , column 1 for all $i < m$, and no '+'s in row i for $i \geq m$. It is then immediate from the local moves connecting elements of $\text{PD}_0(w)$ ([BB93]) that every reduced pipe dream for w has a '+' in row i , column 1 for all $i < m$, and no '+'s in row i for $i \geq m$. Thus, the same is true for all $P \in \text{PD}(w)$ and hence $\text{dst}(P) = P$ for all $P \in \text{PD}(w)$. Conversely, if w has a descent after the m th position, then by [AS17, Lemma 3.12(4)], the map $\text{dst} : \text{PD}_0(w) \rightarrow \text{QPD}_0(w)$ is not injective, so certainly the extension $\text{dst} : \text{PD}(w) \rightarrow \text{QPD}(w)$ is not injective. \square

Theorem 2.14. *For any permutation w , we have*

$$\mathfrak{K}_w = \sum_{(k,a)} c_w^{(k,a)} \beta^k \mathcal{G}_a,$$

where $c_w^{(k,a)}$ is the number of quasi-Yamanouchi pipe dreams for w with weight a and excess k .

Proof. By Lemma 2.13, it suffices to show that, for $Q \in \text{QPD}(w)$, we have

$$\mathcal{G}_{\text{wt}(Q)} = \sum_{P \in \text{dst}^{-1}(Q)} \beta^{\text{ex}(P) - \text{ex}(Q)} x^{\text{wt}(P)}.$$

By definition,

$$\mathcal{G}_{\text{wt}(Q)} = \sum_{b \text{ is a glide of } \text{wt}(Q)} \beta^{\text{ex}(b)} x_1^{b_1} \cdots x_n^{b_n}.$$

For a pipe dream P , the **colored weight** of P is the weak komposition $\text{kwt}(P)$ obtained by coloring the i th entry of $\text{wt}(P)$ red if a '+' can merge into the rightmost '+' of the i th row of P during application of dst . It is not hard to see that if $\text{dst}(P) = Q$, then $\text{kwt}(P)$ is a glide of $\text{wt}(Q)$.

Conversely, we claim that given $Q \in \text{QPD}(w)$, for every weak komposition b that is a glide of $\text{wt}(Q)$, there is a unique $P \in \text{PD}(w)$ with $\text{kwt}(P) = b$ such that $\text{dst}(P) = Q$. To construct this P from b and Q , for $j = 1, \dots, n$, if $\text{wt}(Q)_j = b_{i_{j-1}+1} + \cdots + b_{i_j} - \text{ex}(b_{i_{j-1}+1}, \dots, b_{i_j})$, then, from east to west, shift the first $b_{i_{j-1}+1} + \cdots + b_{i_j-1}$ '+'s northeast from row j to row $j-1$ while leaving a copy of the leftmost of these moved '+'s in place if b_{i_j} is red, the first $b_{i_{j-1}+1} + \cdots + b_{i_j-2}$ '+'s northeast from row $j-1$ to row $j-2$ while leaving a copy of the leftmost of these moved '+'s in place if b_{i_j-1} is

red, and so on. This proves existence, and uniqueness follows from the lack of choice at each step. \square

2.4. Fundamental slide polynomials. The fundamental slide basis of Poly_n was introduced by S. Assaf-D. Searles [AS17], who applied it to the study of Schubert polynomials. We say that a composition b **refines** a composition a if a can be obtained by summing consecutive entries of b , e.g., $(1, 1, 2, 1)$ refines $(2, 3)$ but $(1, 2, 1, 1)$ does not. For a weak composition a of length n , define the **fundamental slide polynomial** $\mathfrak{F}_a = \mathfrak{F}_a(x_1, \dots, x_n)$ by

$$\mathfrak{F}_a = \sum_{\substack{b \geq a \\ \text{flat}(b) \text{ refines flat}(a)}} x_1^{b_1} \dots x_n^{b_n}.$$

Example 2.15.

$$\mathfrak{F}_{0102} = \mathbf{x}^{0102} + \mathbf{x}^{1002} + \mathbf{x}^{0120} + \mathbf{x}^{1020} + \mathbf{x}^{1200} + \mathbf{x}^{0111} + \mathbf{x}^{1011} + \mathbf{x}^{1101} + \mathbf{x}^{1110}.$$

Notice that $\mathfrak{F}_{0102} = \mathcal{G}_{0102}^{(0)}$ (see Example 2.6). \diamond

Proposition 2.16. *The fundamental slide polynomials are a specialization of the glide polynomials. More precisely,*

$$\mathfrak{F}_a = \mathcal{G}_a^{(0)}.$$

Proof. If b is a glide of a with excess 0, then all entries of b are black, so b is a weak composition such that $\text{flat}(b)$ refines $\text{flat}(a)$ and b dominates a , as noted in Remark 2.4. Conversely, every such weak composition may be so obtained. \square

Remark 2.17. Setting $\beta = 0$ in Theorem 2.14 recovers [AS17, Theorem 3.13] for the fundamental slide expansion of Schubert polynomials.

3. SYMMETRIC GROTHENDIECK POLYNOMIALS AND QUASISYMMETRIC GLIDE POLYNOMIALS

3.1. Glide expansions of symmetric β -Grothendieck polynomials. When w is a Grassmannian permutation, i.e., w has at most one descent, $\mathfrak{K}_w^{(\beta)}$ is a symmetric polynomial (with coefficients in $\mathbb{Z}[\beta]$). Let n be the index of the rightmost nonzero entry of $L(w)$, or equivalently the position of the unique descent of w . Then the symmetric β -Grothendieck polynomial \mathfrak{K}_w may be written as $\mathfrak{K}_\lambda(x_1, \dots, x_n)$ where λ is the partition given by reading the nonzero entries of $L(w)$ in reverse. We identify the partition λ with its corresponding Young diagram (in English orientation).

A **set-valued tableau** of shape λ is obtained by filling each box of the Young diagram λ with a nonempty set of positive integers, subject to the condition that if a box is filled with a set A , then the smallest number in the box immediately to the right (respectively, immediately below) is at least as large as (respectively, strictly larger than) $\max(A)$. The **weight** $\text{wt}(T)$ of a set-valued tableau T is the weak composition whose i th entry is the number of occurrences of i in T . Let $|T|$ denote the sum of the entries of $\text{wt}(T)$.

In [Buc02, Theorem 3.1], A. Buch expressed the monomial expansion of $\mathfrak{K}_\lambda^{(-1)}$ as a weighted sum of set-valued tableaux; this formula easily extends to the case of general β . Let $SV_n(\lambda)$ denote the collection of all set-valued tableaux of shape λ using labels from $\{1, \dots, n\}$.

Theorem 3.1 ([Buc02]).

$$\mathfrak{K}_\lambda(x_1, \dots, x_n) = \sum_{T \in SV_n(\lambda)} \beta^{|T| - |\lambda|} \mathbf{x}^{\text{wt}(T)},$$

where $|\lambda|$ denotes the number of boxes in λ .

In [AS17, Definition 2.4], S. Assaf–D. Searles gave a condition for a semistandard Young tableau to be *quasi-Yamanouchi*, and used this to express the fundamental slide expansion of a Schur polynomial $s_\lambda(x_1, \dots, x_n)$ in terms of quasi-Yamanouchi tableaux of shape λ . We extend this concept to set-valued tableaux in order to give a tableau formula for the glide expansion of a symmetric β -Grothendieck polynomial.

Definition 3.2. A set-valued tableau T is **quasi-Yamanouchi** if for all $i > 1$, some instance of i in T is weakly left of some $i - 1$ that is not in the same box.

In the case there is only one entry per box, i.e., T is a semistandard Young tableau, Definition 3.2 reduces to the definition of quasi-Yamanouchi tableaux from [AS17, Definition 2.4]. For a weak composition a of length n , let $\text{rev}(a)$ be the weak composition of length n obtained by reversing the entries of a .

Theorem 3.3. For λ any partition, we have

$$\mathfrak{K}_\lambda(x_1, \dots, x_n) = \sum_{T \in QSV_n(\lambda)} \beta^{|T| - |\lambda|} \mathcal{G}_{\text{rev}(\text{wt}(T))}.$$

Proof. Fix n and a partition λ , and let w be the corresponding Grassmannian permutation. Define a map $\phi : SV_n(\lambda) \rightarrow \text{PD}(w)$ as follows. Given $T \in SV_n(\lambda)$, flip T upside-down, and place it in the fourth quadrant so that the boxes of T are placed exactly over the crosses of the pipe dream $P_{L(w)}$ associated to the Lehmer code of w . Then for each label i of T , turn it into a cross and move it $i + r - n - 1$ steps northeast, where r is the index of the row in which the cross starts. This map ϕ is, up to convention, the bijection of [KMY09, Theorem 5.5].

We now show that the restriction of ϕ to $QSV_n(\lambda)$ is a bijection from $QSV_n(\lambda)$ to $\text{QPD}(w)$. Let $T \in SV_n(\lambda)$. Notice that under ϕ , labels i in boxes of T become crosses in row $n + 1 - i$ of $\phi(T)$.

First suppose T is quasi-Yamanouchi. Then for every i , some instance of i is weakly left of some instance of $i - 1$ in T (and in a different box). By semistandardness the box containing this i is strictly below the box containing this $i - 1$. Therefore, the cross corresponding to this i moves weakly fewer steps northeast than the cross corresponding to this $i - 1$, so there is a cross in row $n + 1 - i$ weakly west of a cross in row $n + 2 - i$ in $\phi(T)$, satisfying the quasi-Yamanouchi condition on these two rows. Since i was arbitrary, $\phi(T)$ is therefore quasi-Yamanouchi.

Now suppose T is not quasi-Yamanouchi. Then for some $i > 1$, all the i 's in T are strictly right of all the $i-1$'s, except possibly for a unique box containing both an i and an $i-1$. If a label i in T is to the right of another label i , then by semistandardness the first label is also weakly above the second; hence, the cross of $\phi(T)$ corresponding to this first i is right of the cross of $\phi(T)$ corresponding to the second i . Since moreover there cannot be two instances of i in the same column of T , it is therefore enough to check that the cross of $\phi(T)$ corresponding to the leftmost i in T is strictly east of the cross corresponding to the rightmost $i-1$ in T . If there is a box b of T containing both i and $i-1$, then b contains the leftmost i and rightmost $i-1$. The cross $+$ _{i} of $\phi(T)$ corresponding to this i sits immediately northeast (and thus strictly east) of the cross $+$ _{$i-1$} corresponding to this $i-1$. If there is no such box b , then let b_i denote the box of the leftmost i and let b_{i-1} denote the box of the rightmost $i-1$. By semistandardness, b_i is weakly above b_{i-1} . So the cross $+$ _{i} corresponding to the $i \in b_i$ moves strictly more steps northeast than the cross $+$ _{$i-1$} corresponding to the $i-1 \in b_{i-1}$. Therefore $+$ _{i} is strictly right of $+$ _{$i-1$} in $\phi(T)$ and $\phi(T)$ is not quasi-Yamanouchi.

Since $\phi : \text{SV}_n(\lambda) \rightarrow \text{PD}(w)$ is a bijection and we have just shown $\phi^{-1}(\text{QPD}(w)) = \text{QSV}_n(\lambda)$, it follows that the restriction $\phi|_{\text{QSV}_n(\lambda)} : \text{QSV}_n(\lambda) \rightarrow \text{QPD}(w)$ is well-defined and bijective.

Finally, if $T \in \text{QSV}_n(\lambda)$, then it is clear that $\text{wt}(\phi(T)) = \text{rev}(\text{wt}(T))$. The theorem now follows from Theorem 2.14. \square

Example 3.4. Let $w = 13524$. Then $L(w) = (0, 1, 2, 0, 0)$, $n = 3$ and the partition λ corresponding to w is $(2, 1)$. We have

$$\begin{aligned} \mathfrak{K}_{13524} = \mathfrak{K}_{(2,1)}(x_1, x_2, x_3) = & x^{210} + x^{201} + x^{120} + 2x^{111} + x^{102} + x^{021} + x^{012} \\ & + \beta x^{220} + 3\beta x^{211} + \beta x^{202} + 3\beta x^{121} + 3\beta x^{112} + \beta x^{022} \\ & + 2\beta^2 x^{221} + 2\beta^2 x^{212} + 2\beta^2 x^{122} + \beta^3 x^{222}. \end{aligned}$$

The elements of $\text{QSV}_3(2, 1)$ are

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Rather than summing over the 27 elements of $\text{SV}_3(2, 1)$ to obtain \mathfrak{K}_{13524} , we may use Theorem 3.3 to sum over the 7 elements of $\text{QSV}_3(2, 1)$, obtaining:

$$\mathfrak{K}_{13524} = \mathfrak{K}_{(2,1)}(x_1, x_2, x_3) = \mathcal{G}_{012} + \mathcal{G}_{021} + \beta \mathcal{G}_{022} + \beta \mathcal{G}_{121} + \beta^2 \mathcal{G}_{122} + \beta^2 \mathcal{G}_{221} + \beta^3 \mathcal{G}_{222}. \quad \diamond$$

3.2. Quasisymmetric polynomials and stable limits of glide polynomials. A polynomial $f \in \text{Poly}_n$ is **quasisymmetric** if the coefficient of $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ is equal to the coefficient of $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ for any two strictly increasing sequences $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. These polynomials were introduced by I. Gessel in [Ges84], who used them in the study of P -partitions. We write QSym_n for the subspace of quasisymmetric polynomials in Poly_n . I. Gessel also defined the **fundamental basis** $\{F_a\}$ of QSym_n , indexed by compositions:

$$F_a(x_1, \dots, x_n) = \sum_{\substack{b \text{ is a weak composition} \\ \text{flat}(b) \text{ refines } a}} \mathbf{x}^b.$$

In [LP07], T. Lam–P. Pilyavskyy introduced the *multi-fundamental quasisymmetric functions* (defined below), which form a basis of the ring of quasisymmetric functions (in countably-many variables). The multi-fundamental quasisymmetric functions are a K -theoretic analogue of I. Gessel’s [Ges84] basis of fundamental quasisymmetric functions, and have been further studied in [Pat16].

Let S_1 and S_2 be nonempty subsets of $\mathbb{Z}_{>0}$. Say that $S_1 < S_2$ if $\max(S_1) < \min(S_2)$, and $S_1 \leq S_2$ if $\max(S_1) \leq \min(S_2)$. For a strong composition a , let \tilde{A}_a be the collection of all chains $\sigma = (S_1, \dots, S_{|a|})$ of nonempty subsets of positive integers such $S_i < S_{i+1}$ if there is some k such that $a_1 + \dots + a_k = i$, and $S_i \leq S_{i+1}$ otherwise.

The **multi-fundamental quasisymmetric function** $\tilde{L}_a(\mathbf{x}) = \tilde{L}_a(x_1, x_2, \dots)$ is defined by

$$\tilde{L}_a(\mathbf{x}) = \sum_{\sigma \in \tilde{A}_a} \mathbf{x}^{\text{wt}(\sigma)},$$

where the i th entry of $\text{wt}(\sigma)$ is the number of occurrences of i in σ .

We now show that the multi-fundamental quasisymmetric functions are the stable limits of the glide polynomials (specialized to $\beta = 1$). Let $0^m a$ denote the weak composition obtained by prepending m zeros to a .

Theorem 3.5. *For any weak composition a ,*

$$\lim_{m \rightarrow \infty} \mathcal{G}_{0^m a}^{(1)} = \tilde{L}_{\text{flat}(a)}(\mathbf{x}).$$

Proof. We give a bijection between the glides indexing monomials in $\mathcal{G}_{0^m a}^{(1)}(x_1, \dots, x_m)$ and the chains $\sigma \in \tilde{A}_{\text{flat}(a)}$ indexing monomials in the truncation $\tilde{L}_{\text{flat}(a)}(x_1, \dots, x_m)$.

Let $\sigma \in \tilde{A}_{\text{flat}(a)}$, where σ uses numbers in $\{1, \dots, m\}$ only. Then the corresponding glide b is simply the weight vector $\text{wt}(\sigma)$, with entries colored as follows: If some j appears in the same subset as some $i < j$, then b_j is red. Otherwise it is black. For example, let $a = (0, 0, 0, 0, 3)$, so $\text{flat}(a) = (3)$. If $\sigma = (\{1, 3\}, \{3, 4\}, \{5\})$, then $b = (1, 0, \text{red}, 1, 1)$.

For the reverse direction, let b be a glide of $0^m a$ such that $b_i = 0$ for $i > m$. Then σ partitions the collection of b_1 ones, b_2 twos, etc., into a chain of nonempty subsets of $\mathbb{Z}_{>0}$. Suppose the first nonzero entry of b is b_j . Then the first $b_j - 1$ subsets in σ are all singletons $\{j\}$, and the final j is assigned to the b_j th subset. If the next nonzero entry, say b_k , of b is black, then the b_j th subset is also the singleton $\{j\}$; now continue

the process with b_k . If, on the other hand, b_k is red, then assign a k to the b_j th subset and continue in this manner. For example, let $a = (0, 0, 0, 0, 3)$, so $\text{flat}(a) = (3)$. If $b = (1, 0, 2, \textcolor{red}{1}, \textcolor{red}{1})$, then $\sigma = (\{1\}, \{3\}, \{3, 4, 5\})$, while if $b = (1, 0, \textcolor{red}{2}, 1, \textcolor{red}{1})$, then $\sigma = (\{1, 3\}, \{3\}, \{4, 5\})$.

These maps are clearly mutually inverse. \square

Remark 3.6. Taking stable limits of the glide polynomials for general β yields a β -deformation of the multi-fundamental quasisymmetric functions. In terms of chains of subsets of integers, the power of β counts the number of integers that occur in a subset along with a smaller integer. It is not hard to see that specializing β to 0 recovers Gessel's fundamental quasisymmetric functions.

We say a polynomial in $\text{Poly}_n[\beta]$ is **quasisymmetric** if it lies in $\text{QSym}_n[\beta]$. Define a weak composition a to be **quasiflat** if the nonzero entries of a occur in an interval. In [AS17], it was shown that \mathfrak{F}_a is quasisymmetric in x_1, \dots, x_n if and only if a is quasiflat with last nonzero term in position n , and that moreover in this case $\mathfrak{F}_a = F_{\text{flat}(a)}(x_1, \dots, x_n)$. Since $\mathfrak{F}_a = \mathcal{G}_a^{(0)}$, this immediately implies that $\mathcal{G}_a^{(\beta)}$ is not quasisymmetric if a is not quasiflat.

Using the glide polynomials, we define a family of polynomials $G_a^{(\beta)}$ indexed by strong compositions.

Definition 3.7. Given a strong composition a , let the **quasisymmetric glide** be

$$G_a^{(\beta)}(x_1, \dots, x_n) = \begin{cases} \mathcal{G}_{0^{n-\ell(a)}a}^{(\beta)} & \text{if } \ell(a) \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $G_a^{(\beta)}$ is quasisymmetric, and that indeed $G_a^{(1)}$ is a truncation of \tilde{L}_a , follows immediately from the bijection in the proof of Theorem 3.5 and the fact that no nonzero entry precedes a zero entry in $0^{n-\ell(a)}a$. Nonetheless, we will use combinatorics of glides to give a direct proof of quasisymmetry, proving moreover that G_a expands positively in the basis of fundamental quasisymmetric polynomials. Define $\text{flat}(b)$ for a weak komposition b to be the strong composition given by deleting all 0 entries of b and forgetting the coloring.

Definition 3.8. Let a be a weak composition. A glide b of a is **unsplitt** if

- b has the same number of nonzero black entries as a does and
- no 0 in b is right of a nonzero entry.

Theorem 3.9. For a strong composition a with $\ell(a) \leq n$,

$$G_a(x_1, \dots, x_n) = \sum_b \beta^{\text{ex}(b)} F_{\text{flat}(b)}(x_1, \dots, x_n),$$

where the sum is over the unsplitt glides b of $0^{n-\ell(a)}a$.

Proof. Suppose that c is a glide of $0^{n-\ell(a)}a$. Observe that the following local operations on the weak komposition c produce another glide of $0^{n-\ell(a)}a$:

- (1) replacing the subword $0k$ by $k0$,

- (2) replacing the subword $0k$ by $k0$,
- (3) replacing the subword $0k$ by ij with $i + j = k$,
- (4) replacing the subword $0k$ with ij with $i + j = k$.

Let b be a unsplit glide of $0^{n-\ell(a)}a$. It is clear by repeated application of (1)–(4) that all monomials of $F_{\text{flat}(b)}(x_1, \dots, x_n)$ appear from glides of $0^{n-\ell(a)}a$.

Now suppose b and b' are distinct unsplit glides of $0^{n-\ell(a)}a$. We need to ensure repeated application of (1)–(4) to b and b' yields disjoint sets of glides of $0^{n-\ell(a)}a$. Since (1)–(4) preserve the number of red entries, we may assume that b and b' both have r red entries. For any weak komposition c , let R_c denote the strong composition whose i th entry is the sum of the entries of c that are strictly right of the $(i-1)$ th red entry and weakly left of the i th red entry. Clearly, if d is obtained from c by any of (1)–(4), then $R_c = R_d$. It remains to note that $R_b \neq R_{b'}$.

Finally, suppose c is a glide of $0^{n-\ell(a)}a$. We need to show c can be obtained from some unsplit glide b of $0^{n-\ell(a)}a$ by repeated application of (1)–(4). By definition of glides, there exists a sequence of nonnegative integers $i_1 < \dots < i_\ell$ such that

- $c_{i_{j-1}+1} + \dots + c_{i_j} = (0^{n-\ell(a)}a)_{n_j} + \text{ex}(c_{i_{j-1}+1}, \dots, c_{i_j})$,
- $i_j \leq n_j$, and
- the leftmost nonzero entry of $c_{i_{j-1}+1}, \dots, c_{i_j}$ is black.

In each block $(c_{i_{j-1}+1}, \dots, c_{i_j})$, shift and combine entries to the right as much as possible by the inverses of (1)–(4). Concatenate the results in order into a new weak komposition c' . Then push all entries of c' as far right as possible by the inverses of (1) and (2). The result b is a glide of $0^{n-\ell(a)}a$, since all entries of $0^{n-\ell(a)}a$ are themselves as far right as possible. Since b has exactly one black entry for each block of c , b is a unsplit glide of $0^{n-\ell(a)}a$, and since we obtained b from c by applying only the inverses of (1)–(4), c is associated to the unsplit glide b . \square

Example 3.10. Let $a = (1, 2)$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Then Theorem 3.9 gives that

$$\begin{aligned} G_{(1,2)}(\mathbf{x}) = & F_{(1,2)}(\mathbf{x}) + 2\beta F_{(1,1,2)}(\mathbf{x}) + \beta F_{(1,2,1)}(\mathbf{x}) \\ & + 3\beta^2 F_{(1,1,1,2)}(\mathbf{x}) + 2\beta^2 F_{(1,1,2,1)}(\mathbf{x}) + \beta^2 F_{(1,2,1,1)}(\mathbf{x}), \end{aligned}$$

because the unsplit glides of $(0, 0, 1, 2)$ are

$$\begin{array}{ccccc} (0, 0, 1, 2) & (0, 1, 1, 2) & (0, 1, 1, 2) & (0, 1, 2, 1) & (1, 1, 1, 2) \\ (1, 1, 1, 2) & (1, 1, 1, 2) & (1, 1, 2, 1) & (1, 1, 2, 1) & (1, 2, 1, 1) \end{array} \quad \diamond$$

Corollary 3.11. *The fundamental quasisymmetric polynomials are a specialization of the quasisymmetric glide polynomials. More precisely,*

$$F_a(x_1, \dots, x_n) = G_a^{(0)}(x_1, \dots, x_n).$$

Remark 3.12. Our Theorem 3.9 (at $\beta = 1$) is a finite-variable analogue of [LP07, Theorem 5.12], which is instead expressed in the language of injective order-preserving maps between chain posets.

Corollary 3.13.

$$\mathfrak{K}_\lambda(x_1, \dots, x_n) = \sum_{T \in \text{QSV}_n(\lambda)} \beta^{|T| - |\lambda|} G_{\text{rev}(\text{wt}(T))}(x_1, \dots, x_n).$$

Proof. By definition, if a quasi-Yamanouchi tableau T uses the label $i > 1$, it must also use the label $i - 1$. Hence $\text{wt}(T)$ is a strong composition (up to trailing 0s). The corollary is then immediate from Theorem 3.3. \square

Specializing Corollary 3.13 to $\beta = 0$ recovers [AS17, Theorem 2.7], a rephrasing of I. Gessel's celebrated expression [Ges84] for writing a **Schur polynomial** $s_\lambda := \mathfrak{F}_\lambda^{(0)}$ as a sum of fundamental quasisymmetric polynomials. Specializing instead to $\beta = 1$ essentially gives an alternate formulation of (a special case of) [LP07, Theorem 5.6] about expansions into multi-fundamental quasisymmetric functions.

Theorem 3.14. *The set $\{\beta^k G_a : k \in \mathbb{Z}_{\geq 0}$ and $\ell(a) \leq n\}$ is a basis of the ring of quasisymmetric polynomials $\text{QSym}_n[\beta]$. Hence for any fixed $p \in \mathbb{Z}$, $\{G_a^{(p)} : \ell(a) \leq n\}$ is a basis of QSym_n .*

Proof. The map $a \mapsto 0^{n-\ell(a)}a$ is injective when $\ell(a) \leq n$, and by Theorem 2.7 the polynomials $\{\beta^k G_a\}$ are linearly independent. Hence $\{\beta^k G_a : k \in \mathbb{Z}_{\geq 0}$ and $\ell(a) \leq n\}$ is linearly independent. Since $G_a^{(0)} = F_a$ by Corollary 3.11 and $\{F_a : \ell(a) \leq n\}$ is a basis of QSym , the set $\{\beta^k G_a : k \in \mathbb{Z}_{\geq 0}$ and $\ell(a) \leq n\}$ spans $\text{QSym}_n[\beta]$. Thus it is a basis of $\text{QSym}_n[\beta]$. The second sentence of the theorem is then immediate. \square

Putting together results of this section and the previous, we have the following relationships between bases of QSym_n and Poly_n . Here upward arrows represent a lifting from quasisymmetric polynomials to polynomials, and rightward arrows represent a lift from ordinary cohomology to connective K -theory.

$$\begin{array}{ccc} \{\mathfrak{F}_b\} \subset \text{Poly}_n(\text{AS17}) & \dashrightarrow & \{\mathcal{G}_b\} \subset \text{Poly}_n \\ \uparrow & & \uparrow \\ \{F_a\} \subset \text{QSym}_n(\text{Ges84}) & \dashrightarrow & \{G_a\} \subset \text{QSym}_n(\text{cf. LP07}) \end{array}$$

4. MULTIPLICATION OF GLIDE POLYNOMIALS

By Theorem 2.7, the glide polynomials form a basis of $\text{Poly}_n[\beta]$. Hence the product of two glide polynomials can be written uniquely as a sum of glide polynomials times powers of β . In this section, we show that this sum involves only *positive* coefficients. We give an explicit positive combinatorial formula for these structure constants, extending the rule of S. Assaf–D. Searles [AS17, Theorem 5.11] for the multiplication of fundamental slide polynomials. Our rule also essentially restricts to [LP07, Proposition 5.9] in the quasisymmetric (and $\beta = 1$) case, though we have some additional complexity related to having finitely-many variables.

4.1. The genomic shuffle product. Here we give a reformulation of the *multishuffle product* of [LP07], a K -theoretic generalization of the *shuffle product* of S. Eilenberg–S. Mac Lane [EML53]. This reformulation is necessary for the statement

of our Littlewood-Richardson rule. In this reformulation, we refer to the multishuffle product as the *genomic shuffle product* because of resemblances to the *genomic tableau* theory for (torus-equivariant) K -theoretic Schubert calculus introduced in [PY17a] and further expounded in [PY17b].

First we recall the classical shuffle product of S. Eilenberg–S. Mac Lane. Let $A = A_1 A_2 \dots A_n$ and $B = B_1 \dots B_m$ be words on disjoint alphabets \mathcal{A} and \mathcal{B} , respectively. The **shuffle product** $A \sqcup B$ of A and B is the set of all permutations of the concatenation AB such that the subword on the alphabet \mathcal{A} is A and the subword on the alphabet \mathcal{B} is B .

Example 4.1. The shuffle product of 331 and 62 is the set

$$331 \sqcup 62 = \{62331, 63231, 63321, 63312, 36231, 36321, 36312, 33621, 33612, 33162\}. \diamond$$

Add a superscript to each letter of A so that, if A_h is the j th instance of i in a (counting from left to right), it becomes i^j . Write A^{gen} for this superscripted version of A . Add superscripts to B to obtain B^{gen} in the same way. For an alphabet \mathcal{A} , let \mathcal{A}^{gen} denote the set of symbols i^j , where $i \in \mathcal{A}$ and $j \in \mathbb{Z}_{>0}$. For A a word in \mathcal{A}^{gen} , a **genotype**¹ is given by deleting all superscripts from any subword obtained by deleting all but one instance of each symbol i^j . Let A and B be words in the alphabets \mathcal{A} and \mathcal{B} , respectively. The **genomic shuffle product** $A \sqcup_{\text{gen}} B$ of A and B is the set of all words in the alphabet $(\mathcal{A} \sqcup \mathcal{B})^{\text{gen}}$ such that

- if i^j appears to the left of i^k , then $j \leq k$;
- no two instances of i^j are consecutive;
- every genotype is an element of $A^{\text{gen}} \sqcup B^{\text{gen}}$.

Remark 4.2. The original definition of \sqcup_{gen} by T. Lam–P. Pylyavskyy [LP07] avoids reference to genotypes. We will need this language of genotypes to formulate some extra relations in dominance order that are necessary for describing the more-general structure constants of the glide basis. In the quasisymmetric case, if one works in countably-many variables, one may simplify our Littlewood-Richardson rule to coincide with theirs [LP07, Proposition 5.9].

Example 4.3. The genomic shuffle product $331 \sqcup_{\text{gen}} 62$ is an infinite set of words, but contains finitely many words of any fixed length. It contains the 10 words of length 5 that are in $331 \sqcup 62$ (but with superscripted 1's on every letter, except the second 3 which has a superscripted 2), together with 35 words of length 6, 81 words of length 7, 154 words of length 8, and many longer words. The words of length 6 in

¹See [PY17b] for motivation of this terminology.

$331 \sqcup_{\text{gen}} 62$ are

$$\begin{array}{cccccc}
 6^1 3^1 2^1 3^1 3^2 1^1 & 3^1 6^1 2^1 3^1 3^2 1^1 & 3^1 6^1 3^1 2^1 3^2 1^1 & 3^1 6^1 3^1 3^2 2^1 1^1 & 3^1 6^1 3^1 3^2 1^1 2^1 \\
 6^1 3^1 3^2 2^1 3^2 1^1 & 3^1 6^1 3^2 2^1 3^2 1^1 & 3^1 3^2 6^1 2^1 3^2 1^1 & 3^1 3^2 6^1 3^2 2^1 1^1 & 3^1 3^2 6^1 3^2 1^1 2^1 \\
 6^1 3^1 3^2 1^1 2^1 1^1 & 3^1 6^1 3^2 1^1 2^1 1^1 & 3^1 3^2 6^1 1^1 2^1 1^1 & 3^1 3^2 1^1 6^1 2^1 1^1 & 3^1 3^2 1^1 6^1 1^1 2^1 \\
 3^1 3^2 6^1 1^1 6^1 2^1 & 3^1 6^1 3^2 1^1 6^1 2^1 & 3^1 6^1 3^2 6^1 1^1 2^1 & 3^1 6^1 3^2 6^1 2^1 1^1 & 6^1 3^1 3^2 1^1 6^1 2^1 \\
 6^1 3^1 3^2 6^1 1^1 2^1 & 6^1 3^1 3^2 6^1 2^1 1^1 & 6^1 3^1 6^1 3^2 1^1 2^1 & 6^1 3^1 6^1 3^2 2^1 1^1 & 6^1 3^1 6^1 2^1 3^2 1^1 \\
 3^1 3^2 6^1 2^1 1^1 2^1 & 3^1 6^1 3^2 2^1 1^1 2^1 & 3^1 6^1 2^1 3^2 1^1 2^1 & 3^1 6^1 2^1 3^2 2^1 1^1 & 6^1 3^1 3^2 2^1 1^1 2^1 \\
 6^1 3^1 2^1 3^2 1^1 2^1 & 6^1 3^1 2^1 3^2 2^1 1^1 & 6^1 2^1 3^1 3^2 1^1 2^1 & 6^1 2^1 3^1 3^2 2^1 1^1 & 6^1 2^1 3^1 2^1 3^2 1^1
 \end{array}$$

The two genotypes of $6^1 3^1 2^1 3^1 3^2 1^1$ are 63231 and 62331. \diamond

4.2. The glide product on weak compositions. Let S be a sequence of words in the alphabet \mathcal{A} and let $\mathcal{B} \subseteq \mathcal{A}$ be a subalphabet. Then the **\mathcal{B} -composition** $\text{Comp}_{\mathcal{B}}(S)$ of S is the weak composition whose i th coordinate is the number of letters of \mathcal{B} in the i th word of S . If $\mathcal{B} = \mathcal{A}$, we may drop \mathcal{B} from the notation.

Order the alphabet $\mathbb{Z}_{>0}^{\text{gen}}$ lexicographically; that is, $i^j < k^l$ if either $i < k$ or else $i = k$ and $j < l$. If C is a word in $\mathbb{Z}_{>0}^{\text{gen}}$, its **run structure** $\text{Runs}(C)$ is the sequence of successive maximally increasing runs of the symbols i^j read from left to right. A **genotype** of $\text{Runs}(C)$ is given by deleting all superscripts from a sequence that comes from deleting all but one instance of each symbol i^j in $\text{Runs}(C)$. In particular, a genotype G of $\text{Runs}(C)$ is a sequence of (possibly empty) words in the alphabet $\mathbb{Z}_{>0}$.

Example 4.4. Let $C = 6^1 3^1 6^1 3^2 1^1 2^1$. Then the run structure of C is $\text{Runs}(C) = (6^1, 3^1 6^1, 3^2, 1^1 2^1)$ and so $\text{Comp}(\text{Runs}(C)) = (1, 2, 1, 2)$. There are two genotypes of $\text{Runs}(C)$, namely $G_1 = (6, 3, 3, 12)$ and $G_2 = (\epsilon, 36, 3, 12)$, where ϵ denotes the empty word. If \mathcal{B} denotes the alphabet of even positive integers, then $\text{Comp}_{\mathcal{B}}(G_1) = (1, 0, 0, 1)$ and $\text{Comp}_{\mathcal{B}}(G_2) = (0, 1, 0, 1)$. \diamond

Definition 4.5. Let a, b be weak compositions of length n . Let A and B be the words $A := (2n-1)^{a_1} \dots (3)^{a_{n-1}} (1)^{a_n}$ and $B := (2n)^{b_1} \dots (4)^{b_{n-1}} (2)^{b_n}$. Define the **genomic shuffle set** $\text{GSS}(a, b)$ of a and b by

$\text{GSS}(a, b) := \{C \in A \sqcup_{\text{gen}} B : \text{for every genotype } G \text{ of } \text{Runs}(C), \text{Comp}_{\mathcal{A}}(G) \geq a \text{ and } \text{Comp}_{\mathcal{B}}(G) \geq b\}$,
where \mathcal{A}, \mathcal{B} respectively denote the alphabets of odd and even positive integers.

Example 4.6. Let $a = 021$ and $b = 101$. Then $A = 331$ and $B = 62$. The genomic shuffle product $331 \sqcup_{\text{gen}} 62$ is (partially) described in Example 4.3. We have

$$\text{GSS}(021, 101) = \left\{ \begin{array}{ccc} 6^1 2^1 3^1 3^2 1^1 & 6^1 2^1 3^1 3^2 1^1 2^1 & 3^1 6^1 2^1 3^1 3^2 1^1 2^1 \\ 6^1 3^1 3^2 1^1 2^1 & 3^1 6^1 2^1 3^1 3^2 1^1 & 3^1 3^2 6^1 2^1 3^2 1^1 2^1 \\ 3^1 6^1 2^1 3^2 1^1 & 3^1 6^1 3^1 3^2 1^1 2^1 & 3^1 3^2 6^1 1^1 2^1 1^1 2^1 \\ 3^1 6^1 3^2 1^1 2^1 & 3^1 3^2 6^1 2^1 3^2 1^1 & \\ 3^1 3^2 6^1 2^1 1^1 & 3^1 3^2 6^1 3^2 1^1 2^1 & \\ 3^1 3^2 6^1 1^1 2^1 & 3^1 6^1 2^1 3^2 1^1 2^1 & \\ & 3^1 3^2 6^1 2^1 1^1 2^1 & \\ & 3^1 3^2 6^1 1^1 2^1 1^1 & \end{array} \right\}. \quad \diamond$$

Note that, while $A \sqcup_{\text{gen}} B$ is usually an infinite set, $\text{GSS}(a, b)$ is necessarily finite, since certainly no element of $\text{GSS}(a, b)$ can have length more than $n \cdot (|a| + |b|)$. (We will significantly improve this upper bound in Proposition 4.10.)

Remark 4.7. Although it is convenient to define $\text{GSS}(a, b)$ by a condition on all genotypes, in fact it is sufficient to verify this condition on a particular ‘worst’ genotype. Specifically let \hat{G} be the genotype of $\text{Runs}(C)$ obtained by preserving the rightmost instance of each letter and deleting the others. Then \hat{G} satisfies the desired dominance conditions if and only if every genotype of $\text{Runs}(C)$ does.

Definition 4.8. Let a, b be weak compositions of length n . For $C \in \text{GSS}(a, b)$, let $\text{BumpRuns}(C)$ denote the unique dominance-minimal way to insert words of length 0 into $\text{Runs}(C)$ while preserving $\text{Comp}_A(G) \geq a$ and $\text{Comp}_B(G) \geq b$ for every genotype G of $\text{BumpRuns}(C)$. The **glide product** $a \sqcup_{\text{gen}} b$ of a and b is the *multiset* of weak compositions

$$a \sqcup_{\text{gen}} b := \{\text{Comp}(\text{BumpRuns}(C)) : C \in \text{GSS}(a, b)\}.$$

Theorem 4.9. For weak compositions a and b of length n , we have

$$\mathcal{G}_a \mathcal{G}_b = \sum_c \beta^{|c|-|a|-|b|} g_{a,b}^c \mathcal{G}_c,$$

where $g_{a,b}^c$ denotes the multiplicity of c in the glide product $a \sqcup_{\text{gen}} b$.

Proof. For simplicity, we explicitly prove the theorem for the specialization $\beta = 1$. It is clear that if the theorem is true for $\beta = 1$, then it is true for general β .

Given a word $C \in \text{GSS}(a, b)$, let \bar{C} be the word in the alphabet $\mathbb{Z}_{>0}^{\text{gen}} \cup \{\emptyset\}$ obtained by inserting ‘|’s into C to separate the elements of $\text{BumpRuns}(C)$. For example, let $C = 3^1 3^2 6^1 1^1 2^1$ from $\text{GSS}(021, 101)$ in Example 4.6. Then $\text{Runs}(C) = (3^1 3^2 6^1, 1^1 2^1)$, and $\text{BumpRuns}(C) = (3^1 3^2 6^1, \epsilon, 1^1 2^1)$. Hence $\bar{C} = 3^1 3^2 6^1 || 1^1 2^1$, where the ‘|’s reflect the locations of the commas in $\text{BumpRuns}(C)$.

Let $\text{shift}(\bar{C})$ denote the set of all words that can be formed from \bar{C} by optionally replacing any letter from $\mathbb{Z}_{>0}^{\text{gen}}$ with a nonempty string of copies of that letter, and by moving ‘|’s to the right, such that $i^j < k^\ell$ whenever i^j and k^ℓ are consecutive.

For example, if $\bar{C} = 3^1 3^2 6^1 || 1^1 2^1$, then the elements of $\text{shift}(\bar{C})$ are

$$3^1 3^2 6^1 || 1^1 2^1 \quad 3^1 3^2 6^1 | 1^1 | 2^1 \quad 3^1 3^2 6^1 | 1^1 2^1 | \quad 3^1 3^2 6^1 | 1^1 | 1^1 2^1 \quad 3^1 3^2 6^1 | 1^1 2^1 | 2^1.$$

Define a set

$$\overline{\text{GSS}(a, b)} := \bigcup_{C \in \text{GSS}(a, b)} \text{shift}(\bar{C}).$$

Let $M(a, b)$ denote the set of ordered pairs (a', b') of weak compositions such that a' is a glide of a and b' is a glide of b . Then the elements of $M(a, b)$ obviously correspond to the monomials in the product $\mathcal{G}_a^{(1)} \mathcal{G}_b^{(1)}$. We claim a bijection between $\overline{\text{GSS}(a, b)}$ and $M(a, b)$; in particular, the elements of $\overline{\text{GSS}(a, b)}$ represent the monomials appearing in $\mathcal{G}_a^{(1)} \mathcal{G}_b^{(1)}$.

Given an element $D \in \overline{\text{GSS}(a, b)}$, let $\text{Seq}(D)$ be the sequence of maximal consecutive subwords in the alphabet $\mathbb{Z}_{>0}^{\text{gen}}$. One then recovers an element $(a', b') \in M(a, b)$ by

$$(a', b') = \left(\text{Comp}_{A^{\text{gen}}}(\text{Seq}(D)), \text{Comp}_{B^{\text{gen}}}(\text{Seq}(D)) \right),$$

where we color a'_i (respectively, b'_i) red if and only if the i th element of $\text{Seq}(D)$ contains a letter $i^j \in A^{\text{gen}}$ (respectively, $i^j \in B^{\text{gen}}$) that also appears in a previous element of $\text{Seq}(D)$.

For example, we have

$$\text{Seq}(3^1 3^2 6^1 || 1^1 2^1) = (3^1 3^2 6^1, \epsilon, 1^1 2^1),$$

which maps to $((2, 0, 1), (1, 0, 1))$, while

$$\text{Seq}(3^1 3^2 6^1 | 1^1 | 1^1 2^1) = (3^1 3^2 6^1, 1^1, 1^1 2^1),$$

which maps to $((2, 1, \textcolor{red}{1}), (1, 0, 1))$.

Given an element $(a', b') \in M(a, b)$, create $D \in \overline{\text{GSS}(a, b)}$ as follows. The first run of D is the first a'_1 letters of A^{gen} followed by the first b'_1 letters of B^{gen} , sorted into increasing order, then the second run is the next a'_2 letters of A^{gen} followed by the next b'_2 letters of B^{gen} , sorted into increasing order, etc, with the exception that whenever you see a red entry in a' (respectively b'), the corresponding run of D has a copy of the most-recently placed letter of A^{gen} (respectively B^{gen}).

For example, if $A^{\text{gen}} = 3^1 3^2 1^1$, $B^{\text{gen}} = 6^1 2^1$, then $(a', b') = ((2, 1, 0), (1, 1, \textcolor{red}{1}))$ maps to $3^1 3^2 6^1 | 1^1 2^1 | 2^1$.

It is clear that these two maps are mutually inverse. Hence the elements of $\overline{\text{GSS}(a, b)}$ represent the set of monomials in $\mathcal{G}_a^{(1)} \mathcal{G}_b^{(1)}$. By construction, for any $C \in \text{GSS}(a, b)$, the monomials associated to the elements of $\text{shift}(\overline{C})$ together comprise the glide polynomial $\mathcal{G}_{\text{Comp}(\text{BumpRuns}(C))}^{(1)}$. Continuing the running example of $C = 3^1 3^2 6^1 1^1 2^1$, the monomials corresponding to elements of $\text{shift}(\overline{C})$ are $\mathbf{x}^{302}, \mathbf{x}^{311}, \mathbf{x}^{320}, \mathbf{x}^{312}, \mathbf{x}^{321}$ and their sum is the glide polynomial $\mathcal{G}_{302}^{(1)}$ corresponding to C .

Hence the elements of $\overline{\text{GSS}(a, b)}$ are partitioned by the elements of $\text{GSS}(a, b)$, with the sum of the monomials in each part equal to the appropriate glide polynomial. \square

We can use Theorem 4.9 to better understand $\text{GSS}(a, b)$ and the glide polynomials appearing in the product $\mathcal{G}_a \mathcal{G}_b$. For a weak composition a , let $z(a)$ denote the number of zeros in a that precede a nonzero entry.

Proposition 4.10. *If \mathcal{G}_c appears in the glide expansion of $\mathcal{G}_a \mathcal{G}_b$, then*

$$|c| \leq |a| + |b| + z(a) + z(b).$$

Moreover, if \mathcal{G}_a and \mathcal{G}_b use the same number of variables, then this bound is attained by some glide polynomial \mathcal{G}_d in the glide expansion of $\mathcal{G}_a \mathcal{G}_b$.

Proof. By Theorem 4.9, the length of an element of $\text{GSS}(a, b)$ is the degree of the lowest-degree monomial of the corresponding glide polynomial. This degree is bounded above by the maximum possible degree of a monomial appearing in the product $\mathcal{G}_a \mathcal{G}_b$, i.e., the sum of the highest degrees of monomials in \mathcal{G}_a and \mathcal{G}_b . These highest-degree

monomials arise from glides of a and of b with as many red entries as possible. Since the number of red entries in a glide of a is clearly at most $z(a)$, the greatest possible degree of a glide of a is $|a| + z(a)$. The analogous statement holds for glides of b .

To see the bound is attained, first note that if \mathcal{G}_a and \mathcal{G}_b use the same number of variables then we may suppose that neither a nor b have trailing zeros (by deleting trailing zeros of a and b if necessary). Suppose we have a glide a' of a and a glide b' of b , each with as many red entries as possible. Then both a' and b' must have no zero entries at all. Let $D \in \overline{\text{GSS}(a, b)}$ be the image of (a', b') under the map from $M(a, b)$ to $\overline{\text{GSS}(a, b)}$ given in the proof of Theorem 4.9. We claim that in fact, $D \in \text{GSS}(a, b)$. Suppose for a contradiction that D has two adjacent copies of the same letter; without loss of generality, we have b'_i is black and b'_{i+1} is red, the letters of A^{gen} in the i th run of D are smaller than the letters of B^{gen} in this run, and the letters of A^{gen} in the $(i+1)$ th run of D are larger than the letters of B^{gen} in the $(i+1)$ th run. But this is impossible since a'_i and a'_{i+1} are both nonzero, and letters of A^{gen} decrease from right to left. Therefore D does not have two adjacent copies of the same letter. Moreover, D cannot have a bar between an ascent, since clearly that would require some a'_i or b'_i to be zero. Thus $D \in \text{GSS}(a, b)$, and so \mathcal{G}_d appears in the product $\mathcal{G}_a \mathcal{G}_b$, where $d = \text{Comp}(\text{BumpRuns}(D))$. \square

Proposition 4.11. *If \mathcal{G}_c appears in the glide expansion of $\mathcal{G}_a \mathcal{G}_b$ and $|c| > |a| + |b|$, then there is a glide polynomial \mathcal{G}_d in the glide expansion of $\mathcal{G}_a \mathcal{G}_b$ with $|d| = |c| - 1$.*

Proof. By Theorem 4.9, there is a $C \in \text{GSS}(a, b)$ corresponding to the weak komposition c . Since $|c| > |a| + |b|$, C has at least one letter appearing more than once. Let D be the subword formed from C by deleting the rightmost letter of C that is a repeat. Note that $D \in A \sqcup_{\text{gen}} B$. Since the set of genotypes of D is a subset of the set of genotypes of C and $C \in \text{GSS}(a, b)$, all genotypes of D satisfy the dominance conditions. Thus $D \in \text{GSS}(a, b)$. The corollary follows by taking $d = \text{Comp}(\text{BumpRuns}(D))$. \square

Theorem 4.9 and the positive combinatorial expansion of a Grothendieck polynomial in the glide basis (Theorem 2.14) together yield a positive Littlewood-Richardson rule for the expansion of a product of Grothendieck polynomials in the glide basis. For a permutation w , let $\text{inv}(w)$ denote the number of inversions of w .

Theorem 4.12. *For a weak composition a and permutations u and v , we have*

$$\mathfrak{K}_u \mathfrak{K}_v = \sum_a \beta^{|a| - \text{inv}(u) - \text{inv}(v)} c_{u,v}^a \mathcal{G}_a,$$

where

$$c_{u,v}^a = \sum_{(P,Q) \in \text{QPD}(u) \times \text{QPD}(v)} g_{\text{wt}(P), \text{wt}(Q)}^a.$$

Proof. Immediate from Theorems 2.14 and 4.9. \square

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REFERENCES

- [AS17] Sami Assaf and Dominic Searles, *Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams*, Adv. Math. **306** (2017), 89–122. MR 3581299
- [BB93] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. **2** (1993), no. 4, 257–269. MR 1281474
- [BE90] Paul Bressler and Sam Evens, *The Schubert calculus, braid relations, and generalized cohomology*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 799–811. MR 968883
- [BJS93] Sara C. Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374. MR 1241505
- [Bri02] Michel Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi’s 60th birthday. MR 1958901
- [Buc02] Anders Skovsted Buch, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR 1946917
- [EML53] Samuel Eilenberg and Saunders Mac Lane, *On the groups of $H(\Pi, n)$. I*, Ann. of Math. (2) **58** (1953), 55–106. MR 0056295
- [FK94] Sergey Fomin and Anatol N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique, DIMACS, Piscataway, NJ, 1994, pp. 183–189. MR 2307216
- [FL94] William Fulton and Alain Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), no. 3, 711–729. MR 1309327
- [Ges84] Ira M Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Contemp. Math. **34** (1984), no. 289–301, 101.
- [Hud14] Thomas Hudson, *A Thom-Porteous formula for connective K-theory using algebraic cobordism*, J. K-Theory **14** (2014), no. 2, 343–369. MR 3319705
- [KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318. MR 2180402
- [KMY09] Allen Knutson, Ezra Miller, and Alexander Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. **630** (2009), 1–31. MR 2526784
- [LP07] Thomas Lam and Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmannians*, Int. Math. Res. Not. IMRN (2007), no. 24, Art. ID rnm125, 48. MR 2377012
- [LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR 686357
- [Pat16] Rebecca Patrias, *Antipode formulas for some combinatorial Hopf algebras*, Electron. J. Combin. **23** (2016), no. 4, Paper #P4.30, 1–32. MR 3604788
- [PY17a] Oliver Pechenik and Alexander Yong, *Equivariant K-theory of Grassmannians*, Forum Math. Pi **5** (2017), 1–128. MR 3668467
- [PY17b] ———, *Genomic tableaux*, J. Algebraic Combin. **45** (2017), no. 3, 649–685. MR 3627499

(OP) DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854
E-mail address: `oliver.pechenik@rutgers.edu`

(DS) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES,
CA 90089
E-mail address: `dsearles@usc.edu`