

# Regularity and Rigidity for Nonlocal Curvatures in Conformal Geometry

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**Dedicated to Professors Sun-Yung Alice Chang and Paul C.-P. Yang on their 70th birthdays**

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**Abstract.** In this paper, we will explore the geometric effects of conformally covariant operators and the induced nonlinear curvature equations in certain nonlocal nature. Mainly, we will prove some regularity and rigidity results for the distributional solutions to those equations.

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## 1 Introduction

In conformal geometry, a primary goal is to understand both analytic and geometric properties of the conformally covariant operators. Specifically, let  $(M^n, g_0)$  be a Riemannian manifold of dimension  $n$  and equipped with a conformal structure  $[g_0]$ . For example, in the case of second order differential operators, the most classical conformally covariant operator is the conformal Laplacian. For  $n \geq 3$ , it is defined as

$$P_{g_0} = L_{g_0} = -\Delta_{g_0} + \frac{(n-2)}{4(n-1)} R_{g_0}, \quad (1.1)$$

where  $R_{g_0} = (\frac{n-2}{4(n-1)})^{-1} L_{g_0}(1)$  is the scalar curvature of the metric  $g_0$ . In the context of *conformally compact Einstein manifolds*, geometric scattering theory gives a much more general manner to study the conformally covariant operators. That is, for each  $\gamma \in (0, n)$ ,

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there is a well-defined conformally covariant operator  $P_{2\gamma}$  of order  $2\gamma$  which is called the fractional GJMS operator, see Definition 1.2 below. Similarly, the scalar function  $Q_{2\gamma} \equiv (\frac{n-2\gamma}{2})^{-1} P_{2\gamma}(1)$  is called the nonlocal  $Q$  curvature of order  $2\gamma$ . In particular, when  $\gamma=2$ , the operator  $P_4$  is called the Paneitz operator. Correspondingly,  $Q_4$  is called Branson's  $Q$  curvature which naturally arises from the Chern-Gauss-Bonnet integral on a 4-manifold and hence deeply related to the geometry and topology of the underlying manifold. There are a lot of fundamental works in this direction (see, e.g., [1, 3, 6, 18–21, 24] and the references therein).

To start with, we introduce some background materials of the conformally compact Einstein manifolds and the precise definition of the nonlocal curvature  $Q_{2\gamma}$ .

**Definition 1.1.** Given a pair of smooth manifolds  $(X^{n+1}, M^n)$  with  $M^n \equiv \partial X^{n+1}$ , we say a complete Einstein metric  $g_+$  on  $X^{n+1}$  is *conformally compact Einstein* if it satisfies

$$\text{Ric}_{g_+} \equiv -ng_+ \quad (1.2)$$

and there is a conformal metric  $\bar{g} = u^2 g_+$  which smoothly extends to the boundary  $M^n$  with a restriction

$$h_0 \equiv u^2 g_+|_{M^n}. \quad (1.3)$$

The conformal manifold  $(M^n, [h_0])$  is called the *conformal infinity*.

Establishing effective connections between the conformal structure of the conformal infinity  $(M^n, [h_0])$  and the Riemannian structure of the Einstein filling-in  $(X^{n+1}, g_+)$  is always a central topic of conformal geometry and the theory of AdS/CFT correspondence. A crucial tool in understanding the above structure is the conformally covariant operators defined on a conformally compact Einstein manifold. Due to Graham-Zworski [16] and Chang-González [5], there are a family of conformally covariant operators called the *fractional GJMS operators* (see [2, 5]). Specifically, in the context of Definition 1.1, we define the operator as below.

**Definition 1.2** (Fractional GJMS operator). Given any real number  $\gamma \in (0, \frac{n}{2})$ ,

$$P_{2\gamma}[g_+, h_0] \equiv 2^{2\gamma} \cdot \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} \cdot S(\frac{n}{2} + \gamma), \quad (1.4)$$

where  $S$  is the scattering operator which is essentially a Dirichlet-to-Neumann operator (we refer the reader to [5] and [16] for the more detailed definition of the scattering operator).

In this case, let  $\hat{h} = v^{\frac{4}{n-2\gamma}} h_0$  and let  $\hat{P}_{2\gamma}$  be the fractional GJMS operator with respect to  $\hat{h}$ , then

$$\hat{P}_{2\gamma}(u) = v^{-\frac{n+2\gamma}{n-2\gamma}} P_{2\gamma}(uv), \quad (1.5)$$

for any  $u \in C^\infty(M^n)$ . Let  $P_{2\gamma}$  be in (1.4) with  $\gamma \in (0, \frac{n}{2})$ , then the function

$$Q_{2\gamma} \equiv \left(\frac{n-2\gamma}{2}\right)^{-1} P_{2\gamma}(1) \quad (1.6)$$

is defined as the *nonlocal Q curvature of order  $2\gamma$* . Note that in the critical case,  $\gamma = \frac{n}{2}$ , the above definition also gives a well-defined conformal invariant. We define

$$Q_n \equiv \lim_{\gamma \rightarrow n/2} \left(\frac{n-2\gamma}{2}\right)^{-1} P_{2\gamma}(1). \quad (1.7)$$

Moreover, it follows from the computations in [16] that the following conformal covariance property always holds in the *critical* case: if  $\hat{h} = e^{2u}h_0$  then,

$$e^{nu} \hat{Q}_n = Q_n + P_n(u). \quad (1.8)$$

The above relations are essentially given by the meromorphic property of the scattering operator, see [16] for more details. In the literature, there are extensive studies in the analytic aspect regarding the nonlocal operators and curvatures in recent years. However, in the literature, so far the geometric properties and applications have not been well and enough explored even in the simplest case. Conformally flat manifolds are the most natural objects such that the nonlocal operators can play their effective roles.

In a recent paper [28], the second author has obtained some topological obstructions for the manifolds admitting conformally flat metrics with positive *nonlocal* curvatures. Moreover, in the lower dimensional case, the second author proved some topological rigidity theorems for the conformally flat manifolds  $(M^n, h)$  with positive nonlocal curvatures. In our current paper, we will further explore the metric aspect of the nonlocal curvatures. More precisely, we will prove some isometric rigidity theorems under the assumption that the nonlocal curvature of the *critical order*  $Q_n$  is a positive constant.

## 1.1 Main results

In this section, we will present our main theorems. We start with a regularity and rigidity theorem for the distributional solutions to the critical order Yamabe equation in  $\mathbb{R}^n$ .

**Theorem 1.1.** *Given any integer  $n \geq 2$ , let  $u$  be a solution to*

$$(-\Delta)^{n/2} u = (n-1)! e^{nu(x)}, \quad x \in \mathbb{R}^n \quad (1.9)$$

*in the distributional sense which satisfies*

$$\int_{\mathbb{R}^n} e^{nu} dx < \infty, \quad (1.10)$$

$$|u(x)| = o(|x|^2) \text{ as } |x| \rightarrow \infty. \quad (1.11)$$

Then there exists some  $\lambda > 0$  and  $x_0 \in \mathbb{R}^n$  such that

$$u(x) \equiv \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}. \quad (1.12)$$

In particular,  $u$  is smooth in  $\mathbb{R}^n$ .

**Remark 1.1.** It is well known that there are examples of solution to (1.9) which are not of the form

$$u(x) \equiv \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}. \quad (1.13)$$

When  $n = 4$ , Chang and the first author proved in [4] that there are non-standard radial solutions satisfying

$$\int_{\mathbb{R}^n} e^{4u(x)} dx < \text{Vol}(\mathbb{S}^4), \quad (1.14)$$

with the asymptotic behavior

$$u(x) = O(|x|^2). \quad (1.15)$$

Such counterexamples tell us that the growth rate condition in Theorem 1.1 is optimal.

A similar classification result was first obtained by Chen-Li in [13] in the case  $n = 2$  under a weaker condition

$$\int_{\mathbb{R}^n} e^{2u(x)} dx < \infty. \quad (1.16)$$

Chang and Yang proved the rigidity result in [9] for higher dimensions under stronger decaying condition

$$u(x) = \log \frac{2}{1 + |x|^2} + w(\zeta(x)) \quad (1.17)$$

for some smooth function  $w$  defined on  $\mathbb{S}^n$ . In [23], for dimension  $n = 4$ , Lin weakened condition (1.17) to

$$\int_{\mathbb{R}^4} e^{4u(x)} dx < \infty \text{ and } u(x) = o(|x|^2) \text{ for } |x| \text{ large} \quad (1.18)$$

and classified the solutions. In [26], Wei and Xu generalized Lin's result to all even dimensions. Xu obtained in [27] the rigidity result in all dimensions under a stronger regularity assumption  $u \in C^\infty(\mathbb{R}^n)$ . As a comparison, in Theorem 1.1, we only need to assume that equation (1.9) holds in the distributional sense.

Next we consider the geometry of the nonlocal curvature  $Q_3$  in the case  $n = 3$ , where the curvature  $Q_3$  is deeply connected to the topology of the underlying conformally compact Einstein manifold  $(X^4, M^3, g_+)$  via the following version of the Chern-Gauss-Bonnet formula. In our specific context of conformally compact Einstein manifold  $(X^4, M^3, g_+)$ , it is a combined result due to [14] and [8] that the following generalized Chern-Gauss-Bonnet formula holds,

$$8\pi^2 \chi(X^4) = \int_{X^4} |W|^2 d\text{vol}_{g_+} + 2 \int_{M^3} Q_3(h_0, g_+) d\text{vol}_{h_0}, \quad (1.19)$$

where  $(M^3, [h_0])$  is the conformal infinity. In a special context that the Einstein filling-in is specified as a *hyperbolic manifold* with constant sectional curvature  $-1$ , the second author proved in [28] a conformal sphere theorem regarding positive  $Q_3$ .

**Theorem 1.2** (Zhang, 2018). *Let  $(M^3, g)$  be a closed locally conformally flat manifold with  $R_g > 0$  and  $Q_3 > 0$ , then  $(M^3, g)$  is conformal to  $S^3$  or  $\mathbb{R}P^3$ .*

We also refer the reader to [28] for more geometric and topological rigidity theorems on the nonlocal curvatures. Based on the conformal sphere theorem, we can obtain an isometric sphere theorem by assuming the nonlocal curvature  $Q_3$  is a positive constant.

**Theorem 1.3.** *Let  $(M^3, g)$  be a closed locally conformally flat manifold with*

$$R_g > 0, \quad Q_3 \equiv 2, \quad (1.20)$$

*then  $(M^3, g)$  is isometric to the round sphere  $S^3$  or the projective space  $\mathbb{R}P^3$  with constant sectional curvature  $\sec_{g_1} \equiv 1$ .*

*Proof of Theorem 1.3.* Since we have assumed

$$R_g > 0, \quad Q_3 \equiv 2 > 0, \quad (1.21)$$

by Theorem 1.2,  $(M^3, g)$  is conformal to  $S^3$  or  $\mathbb{R}P^3$ . In particular, applying the stereographic projection, one can write

$$g = e^{2u} |dx|^2 \quad (1.22)$$

such that

$$\int_{\mathbb{R}^3} e^{3u(x)} dx < \infty \quad (1.23)$$

and  $u(x) \rightarrow -2\log|x|$  as  $|x| \rightarrow +\infty$ . Moreover, the following curvature equation holds globally in  $\mathbb{R}^3$ ,

$$(-\Delta)^{\frac{3}{2}} u = 2e^{3u}. \quad (1.24)$$

Theorem 1.1 implies that  $g$  is isometric to the standard metric on  $S^3$  or  $\mathbb{R}P^3$  with constant sectional curvature  $+1$ .  $\square$

We should emphasize that in the nonlocal cases of the above isometric rigidity theorems, we need to specify the Poincaré-Einstein filling-in which are hyperbolic manifolds in our context. Otherwise, constant  $Q_n$  does not imply the sphere theorems. A standard example is the following AdS-Schwarzschild space.

**Example 1.1** (The AdS-Schwarzschild space). Let  $(M^3, h_0) \equiv (S^1 \times S^2, dt^2 \oplus g_1)$ , there are a family of Einstein metrics on the topological product  $\mathbb{R}^2 \times S^2$  called the AdS-Schwarzschild metrics, which has the explicit expression

$$g_m = \frac{dr^2}{V_m(r)} + V_m(r) dt^2 + r^2 g_1, \quad m > 0, \quad (1.25)$$

where  $V_m(r) \equiv 1 + r^2 - \frac{2m}{r^2}$  and  $g_1$  is the round metric on  $S^2$  with  $\sec_{g_1} \equiv +1$ . In the above, the positive constant  $m > 0$  is called *the ADM mass* of the AdS-Schwarzschild space. Then one can check that  $\text{Ric}_{g_m} \equiv -3g_m$  and  $(M^3, [h_0])$  is the conformal infinity of the AdS-Schwarzschild space  $(\mathbb{R}^2 \times S^2, g_m)$ .

We denote by  $r_h > 0$  the *horizon* of the AdS-Schwarzschild space which is the positive root of  $V_m$ . Then one can compute that the nonlocal curvature  $Q_3$  is a positive constant when  $r_h \in (0, 1)$ . See [7] for the detailed computations. We also notice that, when the ADM mass  $m = 0$ , the metric  $g_m$  degenerates to a hyperbolic metric on the topological product  $S^1 \times \mathbb{R}^3 \cong \mathbb{H}^4 / \mathbb{Z}$  with  $\sec_{g_m} \equiv -1$ . In this case, one can also compute  $Q_3 \equiv 0$ . See [28] for more details in this case.

## 1.2 The analysis of the fractional GJMS operators

In this section we will illustrate the analytic part of the paper, which is the technical foundation in proving the above sphere theorems.

To start with, let  $(X_+^{n+1}, g_+)$  be a conformally compact Einstein manifold with a conformal infinity  $(M^n, [h_0])$ . If we make the conformal change  $\hat{h} = e^{2u}h_0$ , correspondingly the fractional GJMS operator of order  $n$  yields the covariance equation,

$$e^{nu}\hat{Q}_n = Q_n + P_n(u), \quad (1.26)$$

where  $\hat{Q}_n$  is the nonlocal curvature with respect to the conformal metric  $\hat{h}$ . Now we consider conformal covariant equation (1.26) in the model case. That is, let  $(X^{n+1}, g_+) \equiv (\mathbb{H}^{n+1}, g_{-1})$  be the hyperbolic space with curvature  $\equiv -1$  such that the Euclidean space  $\mathbb{R}^n$  is its conformal infinity. In this case, with respect to the Euclidean background metric  $h_0 = |dx|^2$  on  $\mathbb{R}^n$ , we aim to obtain a conformal metric  $h = e^{2u}|dx|^2$  with positive constant curvature  $Q_n$ ,

$$Q_n(e^{2u}|dx|^2) = (n-1)! = Q_n(S^n, g_1). \quad (1.27)$$

Plugging (1.27) into the conformal covariant equation (1.8), we have the nonlocal Yamabe equation

$$(-\Delta)^{\frac{n}{2}}u = Q_n \cdot e^{nu}. \quad (1.28)$$

Based on the above discussion, now the nonlocal Yamabe equation on the universal covering space is given by the following semi-linear equation

$$(-\Delta)^{n/2}u = (n-1)!e^{nu(x)}, \quad x \in \mathbb{R}^n, \quad (1.29)$$

where the dimension  $n \geq 2$  is an arbitrary positive integer. When  $n$  is even,  $(-\Delta)^{n/2}$  is a poly-harmonic operator; while  $n$  is odd, it is a nonlocal fractional operator. For  $n = 2m + 1$ , we define

$$(-\Delta)^{\frac{n}{2}} = (-\Delta)^m \circ (-\Delta)^{1/2} \quad (1.30)$$

with

$$(-\Delta)^{1/2}u(x) = C_n \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+1}} dy, \quad (1.31)$$

where “P.V.” stands for the Cauchy principal value. We say that  $u$  is a solution of (1.29) in the distributional sense if

$$\int_{\mathbb{R}^n} u(x) (-\Delta)^{\frac{n}{2}} \phi(x) dx = \int_{\mathbb{R}^n} (n-1)! e^{nu(x)} \phi(x) dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n). \quad (1.32)$$

In the case when  $n$  is even, first a Pohozaev type identity was obtained and employed to derive the exact asymptotic behavior of the solution  $u$  near infinity, then the method of moving planes was applied to prove the radial symmetry of  $u$ , and finally, by the uniqueness theory of ODEs, the classification of solutions was established [13, 23, 26]. All these are based on local analysis and are no longer at our disposal in the nonlocal setting when the space dimension  $n$  is odd. To our knowledge, there is no corresponding Pohozaev identities except in the case  $u$  is identically zero outside a finite domain (see [25]) and the ODE theory are no longer applicable to radial solutions for fractional equations [15]. To overcome these difficulties, we apply the *method of moving planes and moving spheres in integral forms* to classify solutions and unify the proofs in all dimensions. As is known to the people in this area, when applying the usual method of moving planes, the higher the order of the operator, the more analysis is required. While its counter part in integral forms treat all orders, including fractional orders indiscriminately.

The *method of moving planes in integral forms* was introduced in [12], and since then it has been applied widely to obtain symmetry, monotonicity, non-existence, and classification of solutions for various higher order and fractional order equations. However, the equation here is the so called “critical order” (the order of the operator is the same as the dimension of the space) where the fundamental solution is in the form of logarithm, hence one needs to approach it in an entirely different manner.

### 1.3 Outline of the proof of Theorem 1.1

The proof of our main classification result (Theorem 1.1) is rather involved, so we outline its structure here.

First, we introduce some notions which naturally appear in our proof. When  $n$  is odd, for the higher order fractional Laplacian, there are several equivalent definitions. For instance, when  $n = 3$ , as we mentioned before, one can define

$$(-\Delta)^{3/2}u = (-\Delta)^{1/2} \circ (-\Delta)u, \quad (1.33)$$

or by using a single integral (see [22])

$$(-\Delta)^{3/2}u(x) = C_n \cdot \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y) + \frac{1}{2n} \Delta u(x) |x - y|^2}{|x - y|^{n+3}} dy; \quad (1.34)$$

or via Fourier transform

$$\mathcal{F}[(-\Delta u)^{3/2}u](\xi) = |\xi|^{3/2} \mathcal{F}[u(x)] \quad (1.35)$$

for  $u \in \mathcal{S}$ , the Schwartz space of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^n$ ; and in this space, one can show that all the above three definitions are equivalent. One can extend this operator to a wider space of distributions.

Let

$$\mathcal{L}_{2s} = \left\{ u \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\}. \quad (1.36)$$

Then in this space, one can define  $(-\Delta)^s u$  as a distribution by

$$\langle (-\Delta)^s u(x), \phi \rangle = \int_{\mathbb{R}^n} u(x) (-\Delta)^s \phi(x) dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n). \quad (1.37)$$

In order the integral on the right to converge, one requires  $u \in \mathcal{L}_{2s}$ . Hence, throughout this paper, for odd integer  $n$ , we assume that the solution  $u$  of (1.29) is in  $\mathcal{L}_n$ . For a solution  $u$  of equation (1.29), let

$$\tilde{u} = nu + \log[n(n-1)!], \quad (1.38)$$

then one can easily verify that

$$(-\Delta)^{n/2} \tilde{u} = e^{\tilde{u}(x)}, \quad x \in \mathbb{R}^n. \quad (1.39)$$

Hence for the simplicity of writing, in the sequel, we consider

$$(-\Delta)^{n/2} u = e^{u(x)}, \quad x \in \mathbb{R}^n \quad (1.40)$$

under the condition

$$\int_{\mathbb{R}^n} e^{u(x)} dx < \infty. \quad (1.41)$$

Let  $c_n$  be the constant such that  $c_n \log \frac{1}{|x-y|}$  is the fundamental solution of  $(-\Delta)^{n/2}$  in  $\mathbb{R}^n$ , that is

$$(-\Delta)^{n/2} \left( c_n \log \frac{1}{|x-y|} \right) = \delta(x-y), \quad x, y \in \mathbb{R}^n. \quad (1.42)$$

Let

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy. \quad (1.43)$$

We first consider the case where

$$u(x) = o(|x|^2), \quad \text{for } |x| \text{ large.} \quad (1.44)$$



We only assume that  $u \in \mathcal{L}_n$  be a distributional solution, it is locally integrable, but may not be locally bounded. Without local boundedness assumptions on  $u$ ,  $v(x)$  may not be well defined for every  $x \in \mathbb{R}^n$  and it may even be unbounded locally. We will first prove that it is locally integrable and then in  $\mathcal{L}_n$ . Through a quite complex process involving several new ideas, we prove that

$$u(x) = v(x) + c \quad (1.45)$$

is bounded from above uniformly in  $\mathbb{R}^n$ , and

$$u(x) = -(\alpha + o(1)) \log|x|, \text{ for } |x| \text{ large}, \quad (1.46)$$

where

$$\alpha = c_n \int_{\mathbb{R}^n} e^{u(y)} dy. \quad (1.47)$$

The asymptotic behavior of  $u$  near infinity depends on the value of  $\alpha$ , and to determine it, in the case when  $n$  is even, it was done by using Pohozaev identities (see [23, 26]). However, when  $n$  is odd, due to the nonlocal nature of the fractional operator  $(-\Delta)^{n/2}$ , there are no such identities. To circumvent this nonlocal difficulty, we employ some new ideas. We first apply the *method of moving planes in integral forms* to deduce  $\alpha \geq 2n$ , then use the *method of moving spheres in integral forms* to prove  $\alpha \leq 2n$  and thus arrive at

**Proposition 1.1.** Under condition (1.44), we have that

$$\alpha = 2n. \quad (1.48)$$

With this exact value of  $\alpha$ , we will be able to apply the *method of moving spheres* again to classify all the solutions and establish Theorem 1.1.

In Section 2 and 3, we will focus on the case  $u(x) = o(|x|^2)$ . The asymptotic behavior and regularity of distributional solutions will be derived in Section 2, while the classification of solutions will be established in Section 3. In Section 4, we will consider the case  $u(x) = o(|x|^3)$  and obtained symmetry of solutions.

## 2 Asymptotic behavior of distributional solutions when $u(x) = o(|x|^2)$

In this section, we investigate the asymptotic behavior and the regularity of the distributional solutions and establish.

**Theorem 2.1.** Assume that  $u \in \mathcal{L}_n$  is a distributional solution of equation (1.40) and

$$u(x) = o(|x|^2) \text{ for } |x| \text{ large}. \quad (2.1)$$

Then  $u(x)$  is continuous, differentiable, uniformly bounded from above in  $\mathbb{R}^n$ , and

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\log|x|} \rightarrow -\alpha \equiv c_n \int_{\mathbb{R}^n} e^{u(y)} dy. \quad (2.2)$$

The proof of this theorem is quite complex. It consists of 9 lemmas. Now we give an outline of the proof as follows.

*Outline of the Proof.* Let

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy. \quad (2.3)$$

Without assuming that  $u$  is locally bounded, this  $v(x)$  may not be defined everywhere. We first show that it is locally integrable and in  $\mathcal{L}_n$ , and then it satisfies the equation

$$(-\Delta)^{n/2} v = e^{u(x)}, \quad x \in \mathbb{R}^n \quad (2.4)$$

in the sense of distributions.

Let  $w = u - v$ , then

$$(-\Delta)^{n/2} w(x) = 0, \quad x \in \mathbb{R}^n. \quad (2.5)$$

Applying the Fourier transform to this temper distribution  $w$ , we deduce that  $w(x)$  must be a polynomial.

It is elementary to show that

$$v(x) \geq -c \log|x|, \text{ for } |x| \text{ large.} \quad (2.6)$$

This together with the condition

$$u(x) = o(|x|^2), \text{ for } |x| \text{ large,} \quad (2.7)$$

imply that  $w$  has to be a first degree polynomial, and hence

$$u(x) = v(x) + \sum_{i=1}^n a_i x_i + c. \quad (2.8)$$

Combining the finite volume assumption

$$\int_{\mathbb{R}^n} e^{u(x)} dx < \infty \quad (2.9)$$

and (2.6), we derive that all  $a_i$  must be zero and

$$u(x) = v(x) + c \text{ almost everywhere in } \mathbb{R}^n. \quad (2.10)$$

Based on this, we are able to estimate  $u(x)$  and prove that it is uniformly bounded from above in  $\mathbb{R}^n$ . This is a crucial step involving some interesting ideas. Only after this, can we derive that  $v(x)$  is defined everywhere, continuous, and differentiable, and so does  $u(x)$ .

Under the global bounded-ness (from above) of  $u(x)$ , we derive

$$\lim_{|x| \rightarrow \infty} \frac{v(x)}{\log|x|} \rightarrow -\alpha, \quad (2.11)$$

which apparently also holds for  $u(x)$ . This is a needed behavior later in carrying out the method of moving planes and moving spheres to classify the solutions.  $\square$

In the following, we state the lemmas and present their proofs.

**Lemma 2.1.**  $v(x)$  is in  $\mathcal{L}_n$ , that is,  $v \in L^1_{loc}(\mathbb{R}^n)$ , and

$$\int_{\mathbb{R}^n} \frac{|v(x)|}{(1+|x|)^{2n}} dx < \infty. \quad (2.12)$$

*Proof.* We first prove that  $v(x)$  is locally integrable. For any given  $R > 0$ , we show that

$$\int_{|x| < R} |v(x)| dx < \infty. \quad (2.13)$$

To this end, let us define

$$A_1 \equiv \left\{ y \mid |x-y| \leq \frac{|x|}{2} \right\} \quad \text{and} \quad A_2 \equiv \left\{ y \mid |x-y| > \frac{|x|}{2} \right\}. \quad (2.14)$$

Then it follows that

$$\begin{aligned} & \frac{1}{c_n} \int_{|x| < R} |v(x)| dx \\ & \leq \int_{|x| < R} \int_{A_1} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx + \int_{|x| < R} \int_{A_2} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx \\ & =: I_1 + I_2. \end{aligned}$$

For  $|x| < R$  and  $y \in A_1$ ,

$$|y| \leq |y-x| + |x| \leq \frac{3|x|}{2} \leq \frac{3R}{2}, \quad (2.15)$$

hence

$$\frac{2}{R} \leq \frac{2}{|x|} \leq \frac{|y|+1}{|x|/2} \leq \frac{|y|+1}{|x-y|} \leq \frac{3R/2+1}{|x-y|}. \quad (2.16)$$

Therefore

$$I_1 \leq \int_{|y| \leq 3R/2} \int_{|x| < R} \left| \log \frac{|y|+1}{|x-y|} \right| dx e^{u(y)} dy < \infty. \quad (2.17)$$

For  $|x| < R$  and  $y \in A_2$ ,

$$\frac{|y|+1}{|x-y|} \leq \frac{|x|+|y-x|+1}{|x-y|} \leq \frac{|x|+1}{|x|/2} + 1 \leq 3 + \frac{2}{|x|}, \quad (2.18)$$

while on the other hand

$$\frac{|y|+1}{|x-y|} \geq \frac{|y|+1}{|x|+|y|} \geq \frac{|y|+1}{R+|y|} \geq c_0 > 0. \quad (2.19)$$

It follows that

$$\left| \log \frac{|y|+1}{|x-y|} \right| \leq C + \log(1+|x|), \quad (2.20)$$

and consequently

$$I_2 \leq \int_{|x| < R} [C + \log(|x|+1)] dx \int_{\mathbb{R}^n} e^{u(y)} dy < \infty. \quad (2.21)$$

Combining (2.15), (2.17), and (2.21), we arrive at (2.13).

Now the remainder is to show that

$$\int_{|x| > R} \frac{1}{(1+|x|)^{2n}} \int_{\mathbb{R}^n} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx < \infty. \quad (2.22)$$

For each  $x \in \mathbb{R}^n$ , let

$$\begin{aligned} A_1 &= \{y \mid |x-y| < 1\}, & A_2 &= \left\{y \mid 1 \leq |x-y| \leq \frac{|x|}{2}\right\}, \\ A_3 &= \left\{y \mid |x-y| > \frac{|x|}{2}\right\}. \end{aligned} \quad (2.23)$$

We have that

$$\begin{aligned} & \int_{|x| > R} \frac{1}{(1+|x|)^{2n}} \int_{A_1} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx \\ & \leq \int_{\mathbb{R}^n} e^{u(y)} \left[ \int_{|y-x| < 1} \frac{\log(|y|+1) + |\log|x-y||}{(1+|x|)^{2n}} dx \right] dy \\ & \leq C \int_{\mathbb{R}^n} \frac{\log(|y|+1)}{(1+|y|)^{2n}} e^{u(y)} dy + C \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{2n}} e^{u(y)} \left[ \int_{|y-x| < 1} |\log|x-y|| dx \right] dy \\ & < \infty. \end{aligned} \quad (2.24)$$

For  $y \in A_2$ , we have  $|y| \leq \frac{3|x|}{2}$ , and hence

$$\frac{2}{|x|} \leq \frac{|y|+1}{|x|/2} \leq \frac{|y|+1}{|x-y|} \leq \frac{3|x|/2+1}{|x-y|} \leq 3|x|/2+1. \quad (2.25)$$

Consequently,

$$\left| \log \frac{|y|+1}{|x-y|} \right| \leq C + \log(|x|+1). \quad (2.26)$$

It follows that

$$\int_{|x|>R} \frac{1}{(1+|x|)^{2n}} \int_{A_2} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx < \infty. \quad (2.27)$$

Choose  $R \geq 2$ . For  $|x| \geq 2$  and  $|y| \geq 2$ ,

$$|x-y| \leq |x|+|y| \leq |x||y|, \quad (2.28)$$

and hence

$$\frac{|y|+1}{|x-y|} \geq \frac{|y|+1}{|x||y|} \geq \frac{1}{|x|}. \quad (2.29)$$

For  $|y| \leq 2$ ,

$$\frac{|y|+1}{|x-y|} \geq \frac{|y|+1}{|x|+|y|} \geq \frac{1}{|x|+2}. \quad (2.30)$$

On the other hand, for  $|x-y| \geq \frac{|x|}{2}$ ,

$$\frac{|y|+1}{|x-y|} \leq \frac{|x-y|+|x|+1}{|x-y|} \leq 1 + \frac{|x|}{|x-y|} \leq 1 + \frac{|x|}{|x|/2} = 3. \quad (2.31)$$

Combining (2.29)–(2.31), we derive

$$\left| \log \frac{|y|+1}{|x-y|} \right| \leq C + \log(|x|+2), \text{ for } |x| \geq 2 \text{ and } y \in A_3. \quad (2.32)$$

It follows that

$$\int_{|x|>2} \frac{1}{(1+|x|)^{2n}} \int_{A_3} \left| \log \frac{|y|+1}{|x-y|} \right| e^{u(y)} dy dx < \infty. \quad (2.33)$$

Now, (2.22) is a direct consequence of (2.24), (2.27), and (2.33). This completes the proof of the lemma.  $\square$

**Lemma 2.2.**  $v(x)$  satisfies

$$(-\Delta)^{n/2} v(x) = e^{u(x)}, \quad x \in \mathbb{R}^n \quad (2.34)$$

in the sense of distribution.

*Proof.* By the definition of the distributional solutions, it suffice to show that, for any  $\phi \in C_0^\infty(\mathbb{R}^n)$ , it holds

$$\int_{\mathbb{R}^n} \left[ c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy \right] (-\Delta)^{n/2} \phi(x) dx = \int_{\mathbb{R}^n} e^{u(x)} \phi(x) dx. \quad (2.35)$$

Using the fact that

$$(-\Delta)^{n/2} \phi(x) = C.P.V. \int_K \frac{\phi(x) - \phi(y)}{|x-y|^{2n}} dy \sim \frac{1}{(1+|x|)^{2n}} \text{ for } |x| \text{ large}, \quad (2.36)$$

where  $K$  is the (compact) support of  $\phi(x)$ , and by a similar argument as in the proof of the previous lemma, one can exchange the order of integrations on the left hand side of (2.35) to arrive at the right hand side. This completes the proof of the lemma.  $\square$

Based on Lemma 2.2,  $w = u - v$  is  $n/2$ -harmonic in  $\mathbb{R}^n$ , i.e

$$(-\Delta)^{n/2} w(x) = 0, \quad x \in \mathbb{R}^n, \quad (2.37)$$

in the sense of distributions.

**Lemma 2.3.** *If a tempered distribution  $w(x)$  is  $s$ -harmonic in  $\mathbb{R}^n$  for some positive real number  $s$ , then it is a polynomial.*

*Proof.* The proof is similar to that in [10] in which  $0 < s < 1$ , for readers convenience, we provide it here.

Since  $w$  is a tempered distribution, it admits a Fourier transform  $\mathcal{F}(w)$ . We will show that  $\mathcal{F}(w)$  has support at the origin, hence it is a finite combination of the Dirac's delta measure and its derivatives. Therefore  $w$  is a polynomial (see for instance [17]).

In fact, from

$$\langle (-\Delta)^s w, \psi \rangle = \int_{\mathbb{R}^n} w(x) (-\Delta)^s \psi(x) dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^n), \quad (2.38)$$

and the fact that

$$\mathcal{F}((- \Delta)^s \psi)(\xi) = |\xi|^{2s} \mathcal{F}(\psi)(\xi) \quad \text{for } \psi \in \mathcal{S}, \quad (2.39)$$

we observe that  $(-\Delta)^s w = 0$  means that for any  $\psi \in \mathcal{S}$

$$0 = \langle (-\Delta)^s w, \psi \rangle = \int_{\mathbb{R}^n} w(x) (-\Delta)^s \psi(x) dx = \int_{\mathbb{R}^n} w(x) \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(\psi))(x) dx. \quad (2.40)$$

We claim that

$$\langle \mathcal{F}(w), \phi \rangle = 0 \quad \text{for any } \phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}). \quad (2.41)$$

Indeed let  $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ . The function  $\phi(\xi)/|\xi|^{2s}$  belongs to  $C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset \mathcal{S}$ . Therefore there exists  $\psi \in \mathcal{S}$  such that  $\mathcal{F}(\psi)(\xi) = \phi(\xi)/|\xi|^{2s}$ .

Now, since  $w$  is a tempered distribution and from (2.40), we have

$$\begin{aligned} \langle \mathcal{F}(w), \phi \rangle &= \langle \mathcal{F}(w), |\xi|^{2s} \mathcal{F}(\psi) \rangle = \langle w, \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(\psi)) \rangle \\ &= \int_{\mathbb{R}^n} w(x) \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(\psi))(x) dx = 0. \end{aligned} \quad (2.42)$$

This implies that

$$\text{supp}\{\mathcal{F}(w)\} = \{0\}, \quad (2.43)$$

and thus we complete the proof.  $\square$

**Lemma 2.4.** *We have*

$$v(x) \geq -c_1 \log|x| \quad \text{for } |x| \geq 3 \quad (2.44)$$

*for some positive constant  $c_1$ .*

*Proof.* For  $|x| \geq 2$  and  $|y| \geq 2$ ,

$$|x - y| \leq |x| + |y| \leq |x||y|, \quad (2.45)$$

hence

$$\frac{|y|+1}{|x-y|} \geq \frac{|y|+1}{|x||y|} \geq \frac{1}{|x|}. \quad (2.46)$$

Therefore for  $|x| \geq 3$

$$\begin{aligned} v(x) &\geq c_n \int_{|y| \geq 2} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy + c_n \int_{|y| < 2} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy \\ &\geq -\log|x| c_n \int_{|y| \geq 2} e^{u(y)} dy - c_n \int_{|y| < 2} \log(|x-y|) e^{u(y)} dy \\ &\geq -c_1 \log|x|. \end{aligned}$$

This completes the proof of the lemma.  $\square$

**Lemma 2.5.** *Assume that  $u$  is a distributional solution of (1.40) and*

$$u(x) = o(|x|^2) \quad \text{for } |x| \text{ sufficiently large.} \quad (2.47)$$

*Then*

$$u(x) = v(x) + C \quad (2.48)$$

*for some constant  $C$ , where*

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy. \quad (2.49)$$

*Proof.* Since  $u \in \mathcal{L}_n$ ,  $w(x) = u(x) - v(x)$  is a tempered distribution, and hence by Lemma 2.3, it is a polynomial. Condition (2.47) implies that  $w(x)$  is of at most degree one. Then under the finite volume assumption (1.41) and Lemma 2.4,  $w(x)$  has to be a constant. This completes the proof of the lemma.  $\square$

**Lemma 2.6.** Assume that  $u$  is a distributional solution of (1.40) and

$$u(x) = o(|x|^2) \text{ for } |x| \text{ sufficiently large.} \quad (2.50)$$

Then for any point  $x^0 \in \mathbb{R}^n$ , we have

$$\int_{B_1(x^0)} \exp \left[ \frac{\gamma u(x)}{\|e^u\|_{L^1(B_2(x^0))}} \right] dx \leq C(\gamma), \quad (2.51)$$

where  $\gamma$  is any positive number less than  $\frac{n}{c_n}$  and  $C(\gamma)$  is a constant depending on  $\gamma$  but independent of  $x^0$ .

*Proof.* Based on the finite volume assumption (1.41) and by the Jensen's inequality, we have, for every  $x^0 \in \mathbb{R}^n$ ,

$$\exp \left[ \int_{B_1(x^0)} u(x) \frac{dx}{|B_1|} \right] \leq \int_{B_1(x^0)} e^{u(x)} \frac{dx}{|B_1|} \leq C. \quad (2.52)$$

As a consequence, there exists a constant  $A_1$ , such that

$$\int_{B_1(x^0)} u(x) dx \leq A_1 \text{ for all } x^0 \in \mathbb{R}^n. \quad (2.53)$$

Express

$$\begin{aligned} u(x) &= c_n \int_{\mathbb{R}^n} \log \frac{|y - x^0| + 1}{|x - y|} e^{u(y)} dy + c_n \int_{\mathbb{R}^n} \log \frac{|y| + 1}{|y - x^0| + 1} e^{u(y)} dy + C \\ &= c_n \int_{\mathbb{R}^n} \log \frac{|y - x^0| + 1}{|x - y|} e^{u(y)} dy + A(x^0) + C \\ &= c_n \int_{B_2(x^0)} \log \frac{|y - x^0| + 1}{|x - y|} e^{u(y)} dy + I(x, x^0) + A(x^0) + C. \end{aligned} \quad (2.54)$$

Here

$$A(x^0) = c_n \int_{\mathbb{R}^n} \log \frac{|y| + 1}{|y - x^0| + 1} e^{u(y)} dy, \quad (2.55)$$

$$I(x, x^0) = c_n \int_{B_2^C(x^0)} \log \frac{|y - x^0| + 1}{|x - y|} e^{u(y)} dy. \quad (2.56)$$



From condition (2.53),

$$\begin{aligned} A_1 \geq & \int_{B_1(x^0)} c_n \int_{B_2(x^0)} \log \frac{|y-x^0|+1}{|x-y|} e^{u(y)} dy dx \\ & + \int_{B_1(x^0)} I(x, x^0) dx + A(x^0)|B_1| + C|B_1|. \end{aligned} \quad (2.57)$$

Changing the order of integration, one can easily show that there is a positive constant  $C_1$ , such that

$$\left| \int_{B_1(x^0)} c_n \int_{B_2^C(x^0)} \log \frac{|y-x^0|+1}{|x-y|} e^{u(y)} dy \right| \leq C_1, \quad (2.58)$$

and one can also verify that there is a positive constant  $C_2$ , such that

$$|I(x, x^0)| \leq C_2 \quad \text{for all } x^0 \in \mathbb{R}^n, \quad x \in B_1(x^0). \quad (2.59)$$

For  $x \in B_1(x^0)$  and  $y \in B_2(x^0)$ ,

$$\log \frac{|y-x^0|+1}{|x-y|} \geq \log \frac{|y-x^0|+1}{3} \geq -\log 3. \quad (2.60)$$

Consequently, the first integral in (2.57) is uniformly bounded from below. Hence by (2.57), there is a constant  $A$ , such that

$$A(x^0) \leq A \quad \text{for all } x^0 \in \mathbb{R}^n. \quad (2.61)$$

Then it follows from (2.54) that

$$u(x) \leq c_n \int_{B_2(x^0)} \log \frac{|y-x^0|+1}{|x-y|} e^{u(y)} dy + C_2, \quad \forall x \in B_1(x^0), \text{ for all } x^0 \in \mathbb{R}^n. \quad (2.62)$$

Let

$$a = \int_{B_2(x^0)} e^{u(y)} dy. \quad (2.63)$$

Then again by Jensen's inequality, we derive, for any  $b > 0$

$$\begin{aligned} e^{bu(x)} & \leq C_3 \exp \left[ \int_{B_2(x^0)} abc_n \log \frac{|y-x^0|+1}{|x-y|} \frac{e^{u(y)}}{a} dy \right] \\ & \leq C_3 \int_{B_2(x^0)} \left( \frac{|y-x^0|+1}{|x-y|} \right)^{abc_n} \frac{e^{u(y)}}{a} dy. \end{aligned}$$

Consequently, for any positive number  $b < \frac{n}{ac_n}$ , we have

$$\begin{aligned}
 \int_{B_1(x^0)} e^{bu(x)} dx &\leq C_3 \int_{B_1(x^0)} \int_{B_2(x^0)} \left( \frac{|y-x^0|+1}{|x-y|} \right)^{abc_n} \frac{e^{u(y)}}{a} dy dx \\
 &= C_3 \int_{B_2(x^0)} \left[ \int_{B_1(x^0)} \left( \frac{|y-x^0|+1}{|x-y|} \right)^{abc_n} dx \right] \frac{e^{u(y)}}{a} dy \\
 &\leq C_3 \int_{B_2(x^0)} \left[ \int_{B_3(0)} \left( \frac{3}{|z|} \right)^{abc_n} dz \right] \frac{e^{u(y)}}{a} dy \\
 &\leq C_3 \int_{B_2(x^0)} C(b) \frac{e^{u(y)}}{a} dy \\
 &= C_3 C(b).
 \end{aligned} \tag{2.64}$$

Now letting  $\gamma = ab$ , we complete the proof of the lemma.  $\square$

**Lemma 2.7.** Assume that  $u$  is a distributional solution of (1.40) and

$$u(x) = o(|x|^2) \quad \text{for } |x| \text{ sufficiently large.} \tag{2.65}$$

Then  $u(x)$  is bounded from above, that is there exists a constant  $C$ , such that

$$u(x) \leq C \quad \text{for all } x \in \mathbb{R}^n. \tag{2.66}$$

*Proof.* Entirely similar to the proof of (2.62), one can show that, for all  $x^0 \in \mathbb{R}^n$ ,

$$u(x) \leq c_n \int_{B_1(x^0)} \log \frac{|y-x^0|+1}{|x-y|} e^{u(y)} dy + C_2, \quad \forall x \in B_{1/2}(x^0). \tag{2.67}$$

It follows that, for any  $x \in B_{1/2}(x^0)$ ,

$$\begin{aligned}
 u(x) &\leq c_n \int_{B_1(x^0)} \log \frac{1}{|x-y|} e^{u(y)} dy + C_3 \\
 &\leq c_n \left[ \int_{B_1(x^0)} (\log |x-y|)^2 dy \right]^{1/2} \left[ \int_{B_2(x^0)} e^{2u(y)} dy \right]^{1/2} + C_3 \\
 &\leq C_4 \left[ \int_{B_1(x^0)} e^{2u(y)} dy \right]^{1/2} + C_3.
 \end{aligned} \tag{2.68}$$

In Lemma 2.6, choose  $\gamma = \frac{n}{2c_n}$ . Due to the bounded-ness of the total volume, we can replace  $u$  by  $u - c$  if needed, so that

$$\frac{\gamma}{\|e^u\|_{L^1(B_2(x^0))}} > 2. \tag{2.69}$$

Applying (2.68) and Lemma 2.6, for any  $x \in B_{1/2}(x^0)$ , we have that  $u(x) \leq C$ . Since  $x^0$  is arbitrary, we conclude that for every  $x \in \mathbb{R}^n$ ,  $u(x) \leq C$ . This completes the proof of the lemma.  $\square$

**Lemma 2.8.** Assume that  $u$  is a distributional solution of (1.40) with

$$\int_{\mathbb{R}^n} e^{u(x)} dx < \infty \quad (2.70)$$

and  $u(x) = o(|x|^2)$ . Then  $u$  is continuous and differentiable.

Based on Lemma 2.7,  $u$  is bounded from above, then it is elementary to show that

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy \quad (2.71)$$

is continuous and differentiable, and so does  $u(x)$ .

**Lemma 2.9.** As  $|x| \rightarrow \infty$

$$\frac{v(x)}{\log|x|} \rightarrow -\alpha \equiv c_n \int_{\mathbb{R}^n} e^{u(y)} dy. \quad (2.72)$$

Hence

$$\frac{u(x)}{\log|x|} \rightarrow -\alpha. \quad (2.73)$$

*Proof of Lemma 2.9.* Express

$$\frac{v(x)}{\log|x|} + \alpha = c_n \int_{\mathbb{R}^n} \frac{\log|y| - \log|x-y| + \log|x|}{\log|x|} e^{u(y)} dy =: I_1 + I_2 + I_3, \quad (2.74)$$

where  $I_1, I_2$ , and  $I_3$  are integrals in the following 3 regions respectively:

$$\begin{aligned} D_1 &= \{y \mid |y| > R, |x-y| > 1\}, & D_2 &= \{y \mid |y-x| \leq 1\}, \\ D_3 &= \{y \mid |y| \leq R\}, & \text{with } R &\ll |x|. \end{aligned} \quad (2.75)$$

In the following, we use  $C$  to denote various constants that are independent of  $x$ .

i) In  $D_1$ , we have

$$\left| \frac{\log|y| - \log|x-y| + \log|x|}{\log|x|} \right| \leq C. \quad (2.76)$$

To see this, one may consider two cases:  $|y| \leq 2|x|$  and  $|y| > 2|x|$ . Moreover, it follows from (2.76) that

$$I_1 \leq C \int_{|y|>R} e^{u(y)} dy \rightarrow 0, \text{ as } R \rightarrow \infty. \quad (2.77)$$

ii) By Lemma 2.7,  $u$  is uniformly bounded from above in  $\mathbb{R}^n$ , therefore, in  $D_2$  we have

$$\begin{aligned} I_2 &\leq C \int_{|y-x|\leq 1} e^{u(y)} dy - \frac{1}{\log|x|} \int_{|y-x|\leq 1} \log|y-x| e^{u(y)} dy \\ &\leq C \int_{|y-x|\leq 1} e^{u(y)} dy - \frac{C}{\log|x|} \int_{|y-x|\leq 1} \log|x-y| dy \\ &\leq C \int_{|y-x|\leq 1} e^{u(y)} dy - \frac{C}{\log|x|}. \end{aligned}$$

It follows that

$$I_2 \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (2.78)$$

iii) In  $D_3$ , for each fixed  $R$ ,

$$\frac{\log|y| - \log|x-y| + \log|x|}{\log|x|} \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (2.79)$$

Consequently, due to bounded-ness of the total volume,

$$I_3 \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (2.80)$$

Combining (2.77), (2.78), and (2.80), we arrive at

$$\frac{v(x)}{\log|x|} + \alpha \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (2.81)$$

This completes the proof of the lemma.  $\square$

### 3 Classification of solutions in the case $u(x) = o(|x|^2)$

In this section, we will prove the main classification result and complete the proof of Theorem 1.1

Due to the local bounded-ness of  $u$  (see Theorem 2.1) and by Lemma 2.5, we can also write

$$u(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} e^{u(y)} dy + c, \quad (3.1)$$

and we still denote

$$\alpha \equiv c_n \int_{\mathbb{R}^n} e^{u(y)} dy, \quad (3.2)$$

then Theorem 2.1 implies

$$u(x) = -(\alpha + o(1)) \log|x|, \quad \text{for } |x| \text{ large}. \quad (3.3)$$

We will show that  $\alpha = 2n$ , in two stages. First, using the *method of moving planes in integral forms*, we will prove the following inequality.

**Lemma 3.1.** *Let  $\alpha$  be the constant defined in (3.2). Then it holds that*

$$\alpha \geq 2n. \quad (3.4)$$

*Proof.* Suppose that  $\alpha < 2n$ , then we will prove that  $u$  is radially symmetric about any point by the *method of moving planes*, hence it has to be a constant, this would contradict the equation.

In order to apply the *method of moving planes*, we need the solution to decay fast enough near infinity. However at this moment, we do not know the exact value of  $\alpha$ . To circumvent this difficulty, we make a Kelvin transform centered at a given point  $x^o$ . Let

$$\bar{u}_{x^o}(x) = u\left(\frac{x - x^o}{|x - x^o|^2} + x^o\right) - 2n \log|x - x^o|. \quad (3.5)$$

Then we have that  $\bar{u}_{x^o}(x) \sim -2n \log|x|$  near infinity.

We want to show that  $\bar{u}_{x^o}(x)$  is radially symmetric about  $x^o$ , and so does  $u$  itself. Without loss of generality, we may take  $x^o = 0$  and denote  $\bar{u}_0(x)$  simply as  $\bar{u}(x)$ . By a straight forward calculation, one can verify that

$$\bar{u}(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|}{|x - y|} e^{\bar{u}(y)} dy + I - \gamma \log|x|, \quad (3.6)$$

where

$$I = -c_n \int_{\mathbb{R}^n} \log|y| e^{\bar{u}(y)} dy \quad (3.7)$$

is a constant, and  $\gamma = 2n - \alpha > 0$ . Let

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}. \quad (3.8)$$

Let  $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$  be the reflection of  $x$  about the plane  $T_\lambda$ , and

$$w_\lambda(x) = \bar{u}(x^\lambda) - \bar{u}(x) \equiv \bar{u}_\lambda(x) - \bar{u}(x).$$

For any  $x \in \Sigma_\lambda^-$ , through an elementary argument, we derive

$$\begin{aligned} 0 < -w_\lambda(x) &= \frac{c_n}{2} \int_{\Sigma_\lambda} \left( \log|x^\lambda - y|^2 - \log|x - y|^2 \right) \left[ e^{\bar{u}(y)} - e^{\bar{u}_\lambda(y)} \right] dy - \gamma \log \frac{|x|}{|x^\lambda|} \\ &= 2c_n(\lambda - x_1) \int_{\Sigma_\lambda} \frac{\lambda - y_1}{t(\lambda, x, y)} e^{\xi_\lambda(y)} (-w_\lambda(y)) dy - \gamma \log \frac{|x|}{|x^\lambda|} \\ &\leq 2c_n(\lambda - x_1) \int_{\Sigma_\lambda^-} \frac{\lambda - y_1}{|x - y|^2} e^{\bar{u}(y)} |w_\lambda(y)| dy - \gamma \log \frac{|x|}{|x^\lambda|}. \end{aligned} \quad (3.9)$$

Here we have applied the *mean value theorem* to both

$$\log|x^\lambda - y|^2 - \log|x - y|^2 \quad \text{and} \quad e^{\bar{u}(y)} - e^{\bar{u}_\lambda(y)}, \quad (3.10)$$

where  $t(\lambda, x, y)$  is valued between  $|x^\lambda - y|^2$  and  $|x - y|^2$ , while  $\xi_\lambda(y)$  is between  $\bar{u}_\lambda(y)$  and  $\bar{u}(y)$ .

Now we describe the main steps.

*Step 1.* We show that for  $\lambda$  sufficiently negative,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \quad (3.11)$$

by demonstrating that, the set where the inequality is violated,

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\} = \emptyset. \quad (3.12)$$

In this step, we only need the following consequence of inequality (3.9):

$$0 < -w_\lambda(x) \leq 2c_n(\lambda - x_1) \int_{\Sigma_\lambda^-} \frac{\lambda - y_1}{|x - y|^2} e^{\bar{u}(y)} |w_\lambda(y)| dy. \quad (3.13)$$

In a usual process of *the method of moving planes in integral forms*, one simply take the  $L^q$  norm of both side of inequality (3.13) (see for example, [12] and [11]). However, this does not work in our case, because  $w_\lambda(x)$  is not integrable in the whole space. The other difference is the presence of the function  $(\lambda - x_1)$  on the right hand side. To circumvent this difficulty, we first turn (3.13) to

$$\frac{|w_\lambda(x)|}{\lambda - x_1} e^{\frac{\bar{u}(x)}{3}} \leq C \int_{\Sigma_\lambda^-} \frac{\lambda - y_1}{|x - y|^2} e^{\bar{u}(y)} |w_\lambda(y)| dy, \quad (3.14)$$

due to the upper bounded-ness of  $\bar{u}$  in  $\Sigma_\lambda$  (this bound may become large as  $\lambda$  comes closer to 0, however, for all  $\lambda \leq -c_0 < 0$ , the bound is uniform). Applying first the Hardy-Littlewood-Sobolev and then Holder inequalities to (3.14), we derive

$$\begin{aligned} & \left\| \frac{w_\lambda(x)}{\lambda - x_1} e^{\frac{\bar{u}(x)}{3}} \right\|_{L^q(\Sigma_\lambda^-)} \\ & \leq \|(\lambda - x_1) w_\lambda(x) e^{\bar{u}(x)}\|_{L^{\frac{nq}{n+(n-2)q}}(\Sigma_\lambda^-)} \\ & \leq C \|(\lambda - x_1)^2 e^{\frac{2\bar{u}(x)}{3}}\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^-)} \left\| \frac{w_\lambda(x)}{\lambda - x_1} e^{\frac{\bar{u}(x)}{3}} \right\|_{L^q(\Sigma_\lambda^-)}. \end{aligned} \quad (3.15)$$

Here  $q$  is chosen close to  $n$ , so that the integral in

$$\left\| \frac{w_\lambda(x)}{\lambda - x_1} e^{\frac{\bar{u}(x)}{3}} \right\|_{L^q(\Sigma_\lambda^-)} \quad (3.16)$$

converges. Since

$$\bar{u}(y) \sim -2n \log |y|, \quad (3.17)$$

for  $\lambda$  sufficiently negative, we can make

$$C \left\| (\lambda - x_1)^2 e^{\frac{2\bar{u}(x)}{3}} \right\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^-)} < 1. \quad (3.18)$$

It follows from (3.15) that

$$\left\| \frac{w_\lambda(x)}{\lambda - x_1} e^{\frac{\bar{u}(x)}{3}} \right\|_{L^q(\Sigma_\lambda^-)} = 0, \quad (3.19)$$

hence,  $\Sigma_\lambda^-$  must be empty, therefore

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \text{ for all sufficiently negative } \lambda. \quad (3.20)$$

*Step 2.* Inequality (3.20) provides a starting point to move the plane  $T_\lambda$ , from which we move the plane toward the right to its limiting position as long as (3.20) holds. Let

$$\lambda_o = \sup \{ \lambda < 0 \mid w_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda \}. \quad (3.21)$$

We prove that  $\lambda_o = 0$ . Otherwise, if  $\lambda_o < 0$ , then from

$$w_{\lambda_o}(x) = c_n \int_{\Sigma_{\lambda_o}} \log \frac{|x^\lambda - y|}{|x - y|} \left[ e^{\bar{u}_{\lambda_o}(y)} - e^{\bar{u}(y)} \right] dy + \gamma \log \frac{|x|}{|x^{\lambda_o}|}, \quad (3.22)$$

one can see that

$$w_{\lambda_o}(x) > 0, \quad \forall x \in \Sigma_{\lambda_o}. \quad (3.23)$$

Based on this, using (3.15), and going through a similar argument as in *Step 2* of the proof for Proposition 4.2, we can deduce that, there exists  $\epsilon > 0$ , such that

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \text{ for all } \lambda_o \leq \lambda < \lambda_o + \epsilon. \quad (3.24)$$

This contradicts the definition of  $\lambda_o$ . Therefore, we must have  $\lambda_o = 0$  and thus complete the proof of Lemma 3.1.  $\square$

**Remark 3.1.** Note in (3.20), when we choose  $\lambda \leq \frac{1}{2}\lambda_o$ , the constant  $C$  depends on  $\lambda_o$  only. Hence we can choose  $R$  sufficiently large and then  $\delta$  and  $\epsilon$  sufficiently small, such that

$$C \left\| (\lambda - x_1)^2 e^{\frac{2\bar{u}(x)}{3}} \right\|_{L^{\frac{n}{n-2}}(\Sigma_\lambda^-)} < 1. \quad (3.25)$$

Then we apply the *method of moving spheres in integral forms* to prove the converse direction.

**Lemma 3.2.** *Let  $\alpha$  be the constant defined in (3.2). Then it holds that*

$$\alpha \leq 2n. \quad (3.26)$$

*Proof.* We will use the *method of moving spheres in integral forms* to rule out the possibility that  $\alpha > 2n$ .

Given any point  $x^0 \in \mathbb{R}^n$ , consider the ball  $B_\lambda(x^0)$  of radius  $\lambda$  centered at  $x^0$ . For a point  $x \in B_\lambda(x^0)$ , its inversion point about the sphere  $S_\lambda(x^0) \equiv \partial B_\lambda(x^0)$  is  $\frac{\lambda^2 x}{|x|^2}$ .

We will compare the value of  $u(x)$  and its Kelvin transform with respect to the sphere  $S_\lambda(x^0)$

$$u_{\lambda, x^0}(x) \equiv u\left(\frac{\lambda^2(x-x^0)}{|x-x^0|^2} + x^0\right) + 2n \log \frac{\lambda}{|x-x^0|}. \quad (3.27)$$

In the case  $\alpha > 2n$ , we will show that

$$u_{\lambda, x^0}(x) \leq u(x), \quad \forall x \in B_\lambda(x^0), \quad \text{for all } \lambda > 0. \quad (3.28)$$

To this end, we first establish inequality (3.28) for sufficiently large  $\lambda$ , then we move (more precisely, shrink) the sphere  $S_\lambda(x^0)$  as long as (3.28) holds to its limiting position. The condition  $\alpha > 2n$  enable us to shrink the sphere all the way to its center and thus arrive at (3.28) for all positive real value  $\lambda$ .

Based on (3.28), due to the arbitrariness of the center  $x^0$ , and through an elementary calculus argument, we will deduce that  $\nabla u(x) = 0$  at any point  $x \in \mathbb{R}^n$ , and hence  $u$  must be constant. This would contradicts the equation.

Without loss of generality, we may take  $x^0 = 0$  and write  $u_{\lambda, 0}(x)$  as  $u_\lambda(x)$ . In order that  $u$  and  $u_\lambda$  take the same form, we express

$$u(x) = c_n \int_{\mathbb{R}^n} \log \frac{\sqrt{|y|}}{|x-y|} e^{u(y)} dy + c. \quad (3.29)$$

Then by a straight forward calculation, one can verify that

$$u_\lambda(x) = c_n \int_{\mathbb{R}^n} \log \frac{\sqrt{|y|}}{|x-y|} e^{u_\lambda(y)} dy - \gamma \log \frac{\lambda}{|x|} + c, \quad (3.30)$$

with  $\gamma = \alpha - 2n > 0$ . Consider

$$w_\lambda(x) = u_\lambda(x) - u(x), \quad x \in B_\lambda \equiv B_\lambda(0). \quad (3.31)$$

Then by a long, but straight forward computation, we derive

$$w_\lambda(x) = \frac{c_n}{2\lambda^2} \int_{B_\lambda} \frac{(\lambda^2 - |y|^2)(\lambda^2 - |x|^2)}{t(\lambda, x, y)} \left[ e^{u_\lambda(y)} - e^{u(y)} \right] dy - \gamma \log \frac{\lambda}{|x|}, \quad (3.32)$$



where

$$\left| \frac{x|y|}{\lambda} - \frac{\lambda y}{|y|} \right|^2 \geq t(\lambda, x, y) \geq |x - y|^2. \quad (3.33)$$

*Step 1.* We show that for  $\lambda$  sufficiently large,

$$w_\lambda(x) \leq 0, \quad \forall x \in B_\lambda. \quad (3.34)$$

Let  $B_\lambda^+ = \{x \in B_\lambda \mid w_\lambda(x) > 0\}$ . We argue that it must be empty. In fact, for any  $x \in B_\lambda^+$ , we have that

$$\begin{aligned} 0 < w_\lambda(x) &= \frac{c_n}{2\lambda^2} \int_{B_\lambda} \frac{(\lambda^2 - |y|^2)(\lambda^2 - |x|^2)}{t(\lambda, x, y)} e^{\tilde{c}_\lambda(y)} w_\lambda(y) dy \\ &\leq \frac{c_n \lambda^2}{2} \int_{B_\lambda^+} \frac{1}{|x - y|^2} e^{u_\lambda(y)} w_\lambda(y) dy. \end{aligned} \quad (3.35)$$

Applying the Hardy-Littlewood-Sobolev and Holder inequalities, we arrive at, for some  $q > 0$ ,

$$\begin{aligned} \|w_\lambda\|_{L^q(B_\lambda^+)} &\leq C\lambda^2 \|e^{u_\lambda} w_\lambda\|_{L^{\frac{nq}{n+(n-2)q}}(B_\lambda^+)} \leq C\lambda^2 \|e^{u_\lambda}\|_{L^{\frac{n}{n-2}}(B_\lambda^+)} \|w_\lambda\|_{L^q(B_\lambda^+)} \\ &= \frac{C}{\lambda^2} \|e^{u(x)} |x|^4\|_{L^{\frac{n}{n-2}}(B_\lambda^c)} \|w_\lambda\|_{L^q(B_\lambda^+)}. \end{aligned} \quad (3.36)$$

Taking into account of the asymptotic behavior

$$u(x) \sim -\alpha \log|x| \quad \text{with } \alpha > 2n, \quad (3.37)$$

one can see that, for  $\lambda$  sufficiently large, both  $\frac{1}{\lambda^2}$  and  $\|e^{u(x)} |x|^4\|_{L^{\frac{n}{n-2}}(B_\lambda^c)}$  can be arbitrarily small, hence it follows from (3.36) that

$$\|w_\lambda\|_{L^q(B_\lambda^+)} = 0, \quad (3.38)$$

and this implies the empty-ness of  $B_\lambda^+$ . Therefore (3.34) holds for sufficiently large  $\lambda$ .

*Step 2.* Now we decrease  $\lambda$  (shrink the sphere  $S_\lambda(0)$ ) to its limit as long as inequality (3.34) holds. Let

$$\lambda_o = \inf\{\lambda \mid w_\mu \leq 0, x \in B_\mu, \mu \geq \lambda\}. \quad (3.39)$$

We prove that  $\lambda_o = 0$ . Suppose in the contrary,  $\lambda_o > 0$ , we argue that one can shrink the sphere further (just a tiny bit) while maintaining inequality (3.34). This would contradict the definition of  $\lambda_o$ .

From (4.6), due to the presence of  $-\gamma \log \frac{\lambda_o}{|x|}$ , we see that

$$w_{\lambda_o}(x) < 0, \quad \forall x \in B_{\lambda_o} \setminus \{0\}. \quad (3.40)$$

In a small neighborhood of 0,  $w_{\lambda_o}$  is negatively bounded away from 0, which can be seen from,

$$w_{\lambda_o}(x) \sim -\alpha \log \lambda_o - \gamma \log \frac{\lambda_o}{|x|} - u(x) \quad \text{for } |x| \ll \lambda_o, \quad (3.41)$$

the right hand side of which approaches  $-\infty$  as  $|x| \rightarrow 0$ . Hence by (3.40), for any  $\delta > 0$ , there exists  $c_\delta > 0$ , such that

$$w_{\lambda_o}(x) \leq -c_\delta, \quad \forall x \in B_{\lambda_o - \delta}. \quad (3.42)$$

Consequently, due to the continuity of  $w_\lambda$  with respect to  $\lambda$ , there exists an  $\epsilon > 0$ , such that

$$\text{for any } \lambda \in (\lambda_o - \epsilon, \lambda_o], \quad w_\lambda(x) \leq 0, \quad \forall x \in B_{\lambda_o - \delta}.$$

Now  $B_\lambda^+$  is confined in the (spherical) narrow region  $B_\lambda \setminus B_{\lambda_o - \delta}$ . Again in

$$\|w_\lambda\|_{L^q(B_\lambda^+)} \leq C\lambda^2 \|e^{u_\lambda}\|_{L^{\frac{n}{n-2}}(B_\lambda^+)} \|w_\lambda\|_{L^q(B_\lambda^+)},$$

letting  $\delta$  be sufficiently small, we have

$$\|w_\lambda\|_{L^q(B_\lambda^+)} \leq \frac{1}{2} \|w_\lambda\|_{L^q(B_\lambda^+)}.$$

Hence  $B_\lambda^+$  must be empty, and therefore

$$w_\lambda(x) \leq 0, \quad \forall x \in B_\lambda, \quad \lambda \in (\lambda_o - \delta, \lambda_o].$$

This contradicts the definition of  $\lambda_o$ .

*Step 3.* Now, we have shown that

$$u_{\lambda, x^o}(x) \leq u(x), \quad \forall x \in B_\lambda(x^o), \quad \text{for all } \lambda > 0. \quad (3.43)$$

From here we will derive that for any  $y \in \mathbb{R}^n$ ,

$$\nabla u(y) = 0.$$

Take  $\lambda = |y|$  so that  $y$  lies on the boundary of  $B_\lambda(0)$ . From (3.43)  $x^o = 0$ ,

$$u\left(\frac{\lambda^2 x}{|x|^2}\right) - u(x) \leq 0, \quad \forall x \in B_\lambda(0). \quad (3.44)$$

Let  $\nu$  be the outward normal of  $\partial B_\lambda(0)$ , we show  $\frac{\partial u}{\partial \nu}(y) \leq 0$ . Without loss of generality, we may assume  $y = (\lambda, 0, \dots, 0)$  and hence  $\nu = (1, 0, \dots, 0)$ . For  $x = (x_1, 0, \dots, 0)$ ,

$$\frac{\lambda^2 x}{|x|^2} = \left( \frac{\lambda^2}{x_1}, 0, \dots, 0 \right). \quad (3.45)$$

By the mean value theorem

$$u\left(\frac{\lambda^2 x}{|x|^2}\right) - u(x) = \frac{\partial u}{\partial \nu}(\xi) \left( \frac{\lambda^2}{x_1} - x_1 \right), \quad (3.46)$$

where  $\xi$  is some point on the line segment linking  $x$  and  $\frac{\lambda^2 x}{|x|^2}$ . Taking into account that  $0 < x_1 < \lambda$  and by (3.44), we derive

$$\frac{\partial u}{\partial \nu}(\xi) \leq 0. \quad (3.47)$$

Let  $x_1 \rightarrow \lambda$ , then  $\xi \rightarrow y$  and we arrive at

$$\frac{\partial u}{\partial \nu}(y) \leq 0. \quad (3.48)$$

Now fixed this  $y$  and rotate the center  $x^0$  around  $y$  with  $|x^0 - y| = \lambda$ , we can similarly show that for any unit vector  $\nu = \frac{y - x^0}{|y - x^0|}$ , we have

$$\frac{\partial u}{\partial \nu}(y) \leq 0. \quad (3.49)$$

This implies that

$$\nabla u(y) = 0, \quad (3.50)$$

and since  $y$  is any point in  $\mathbb{R}^n$ ,  $u$  must be constant. This contradicts equation (1.40) and completes the proof of Lemma 3.2.  $\square$

Combining Lemmas 3.1 and 3.2, we obtain Proposition 1.1. Then we will use the *method of moving spheres* to classify the solutions, which completes the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By Lemmas 3.1 and 3.2, we have  $\alpha = 2n$ , and hence (3.32) becomes

$$w_\lambda(x) = \frac{c_n}{2\lambda^2} \int_{B_\lambda} \frac{(\lambda^2 - |y|^2)(\lambda^2 - |x|^2)}{t(\lambda, x, y)} \cdot [e^{u_\lambda(y)} - e^{u(y)}] dy. \quad (3.51)$$

Similar to the proof of Lemma 3.2, we can still show that, for  $\lambda$  sufficiently large,

$$w_\lambda(x) \leq 0, \quad \forall x \in B_\lambda. \quad (3.52)$$

However, without the presence of the term  $-\gamma \log \frac{\lambda}{|x|}$ , one cannot shrink the sphere all the way to its center, and this can be seen from asymptotic behavior of  $w_\lambda$  near 0 for each fixed  $\lambda$ :

$$w_\lambda(x) = u\left(\frac{\lambda^2 x}{|x|^2}\right) + 2n \log \frac{\lambda}{|x|} - u(x) \sim -2n \log \lambda - u(x), \quad (3.53)$$

which approaches  $-2n \log \lambda - u(0)$  as  $x \rightarrow 0$ . This quantity is obviously positive for sufficiently small  $\lambda$ . Therefore, in this case, we must have  $\lambda_0 > 0$ .

So far, we have shown that, for each  $y \in \mathbb{R}^n$ , there is a  $\lambda_y > 0$ , such that

$$u(x) = u\left(\frac{\lambda_y^2(x-y)}{|x-y|^2} + y\right) + 2n \log \frac{\lambda_y}{|x-y|}, \quad (3.54)$$

and, in particular, for  $y=0$ ,

$$u(x) = u\left(\frac{\lambda_0^2 x}{|x|^2}\right) + 2n \log \frac{\lambda_0}{|x|}. \quad (3.55)$$

In (3.54), fix  $y$  and let  $|x| \rightarrow \infty$ ,

$$u(y) + 2n \log \lambda_y = \lim_{|x| \rightarrow \infty} (u(x) + 2n \log |x-y|) = 0. \quad (3.56)$$

Similarly, (3.55) yields

$$u(0) + 2n \log \lambda_0 = 0. \quad (3.57)$$

For  $|x|$  large, via Taylor expansion at  $y$  in (3.54), and taking into account of (3.56), we have

$$u(x) = \nabla u(y) \cdot \frac{\lambda_y^2(x-y)}{|x-y|^2} - 2n \log |x-y| + o\left(\frac{1}{|x|}\right). \quad (3.58)$$

Similarly, (3.55) and (3.57) lead to

$$u(x) = \nabla u(0) \cdot \frac{\lambda_0^2 x}{|x|^2} - 2n \log |x| + o\left(\frac{1}{|x|}\right). \quad (3.59)$$

Combining (3.58) and (3.59), we deduce

$$\nabla u(y) \cdot \frac{\lambda_y^2(x-y)}{|x-y|^2} = \nabla u(0) \cdot \frac{\lambda_0^2 x}{|x|^2} - 2ny \cdot \frac{x}{|x|^2} + o\left(\frac{1}{|x|}\right). \quad (3.60)$$

Equating the coefficients of the order  $\frac{1}{|x|}$ , we arrive at

$$\lambda_y^2 \nabla u(y) = \lambda_0^2 \nabla u(0) - 2ny. \quad (3.61)$$

By virtue of (3.56), if we write  $f(y) = e^{-\frac{u(y)}{n}}$ , then it becomes

$$\nabla f(y) = \nabla f(0) + 2y. \quad (3.62)$$

Integrating yields

$$f(y) = a + |y - y^0|^2. \quad (3.63)$$

Since  $f$  is positive, the constant  $a$  must be positive, hence we can write

$$f(y) = \lambda^2 + |y - y^0|^2. \quad (3.64)$$

Therefore

$$u(y) = n \log \frac{1}{\lambda^2 + |y - y^0|^2}. \quad (3.65)$$

Noticing that, for any constant  $c$ ,  $u + c$  also satisfies identity (3.54), we obtain

$$u(y) = n \log \frac{1}{\lambda^2 + |y - y^0|^2} + c. \quad (3.66)$$

Finally, taking into account of the total volume:

$$c_n \int_{\mathbb{R}^n} e^{u(y)} dy = 2n, \quad (3.67)$$

we are able to determine constant  $c$  and arrive at the conclusion of Theorem 1.1.  $\square$

## 4 Symmetry of solutions in the case $o(|x|^3)$

This section is to study the symmetry of the solutions  $u$  satisfying the asymptotic growth rate  $u = o(|x|^3)$ .

To begin with, we have the following rigidity result.

**Proposition 4.1.** Assume that

$$u(x) = o(|x|^3), \quad \text{for } |x| \text{ large}, \quad (4.1)$$

is a distributional solution to

$$(-\Delta)^{n/2} u = e^{u(x)}, \quad x \in \mathbb{R}^n. \quad (4.2)$$

Then it holds that

$$u(x) = v(x) + \sum_{i=1}^n a_i (x_i - x_i^0)^2 + c \quad (4.3)$$

for some constants  $a_i \leq 0$  and  $c \in \mathbb{R}$ , where

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy. \quad (4.4)$$

*Proof.* Based on Lemmas 2.3 and 2.4 and under the assumption

$$u(x) = o(|x|^3), \quad |x| \rightarrow +\infty, \quad (4.5)$$

$w$  must be a second degree polynomial. After rotating the coordinates and completing the square, we have

$$u(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy + \sum_{i=1}^n a_i (x_i - x_i^o)^2 + c. \quad (4.6)$$

Now, due to the bounded-ness of the total volume and Lemma 2.4, all  $a_i$  must be non-positive. This completes the proof of Proposition 4.1.  $\square$

In the following, we will apply the *method of moving planes in integral forms* to obtain the symmetry of solutions.

**Proposition 4.2.** Assume that  $u$  is locally bounded. Under the growth assumption  $u = o(|x|^3)$  as  $|x| \rightarrow +\infty$ , if  $a_i < 0$ , then  $u(x)$  is symmetric and monotone decreasing in  $x_i$  about the center  $x_i = x_i^o$ .

Before carrying out the method of moving planes, we need the regularity and asymptotic behavior of the solutions as stated in the following.

**Proposition 4.3.** Assume that  $u \in \mathcal{L}_n$  is a distributional solution of Eq. (1.40) and

$$u(x) = o(|x|^3), \quad |x| \rightarrow +\infty. \quad (4.7)$$

Further assume that  $u$  is locally bounded from above. Then  $u(x)$  is continuous, differentiable, uniformly bounded from above in  $\mathbb{R}^n$ , and

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\log|x|} \rightarrow -\alpha \equiv c_n \int_{\mathbb{R}^n} e^{u(y)} dy. \quad (4.8)$$

The proof is similar to some parts of the proof of Theorem 2.1, and it is much simpler in the current case. Under the assumption that  $u$  be locally bounded from above, one can derive immediately that

$$\int_{\mathbb{R}^n} \log \frac{|y|+1}{|x-y|} e^{u(y)} dy \quad (4.9)$$

is well defined for all  $x \in \mathbb{R}^n$ , continuous, and differentiable, so does  $u(x)$ . Now by the finite total volume condition, we can also express

$$u(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} e^{u(y)} dy + \sum_{i=1}^n a_i (x_i - x_i^o)^2 + c. \quad (4.10)$$

The proof of the uniform upper bound of  $u(x)$  is similar to the proof of Lemma 2.6 and 2.7, and the asymptotic behavior comes directly from Lemma 2.9. Notice that due to a technical difficulty (see the proof of Lemma 2.6), we are not able to obtain the uniform upper bound for  $u(x)$  without assuming the local upper bound for  $u(x)$ .

*Proof of Proposition 4.2.* In the following, we let

$$v(x) = c_n \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} e^{u(y)} dy. \quad (4.11)$$

Without loss of generality, we may assume that

$$a_1 = -1 \text{ and } x_1^0 = 0, \quad (4.12)$$

then

$$u(x) = v(x) - x_1^2 + P(x') \quad (4.13)$$

where  $P(x')$  is a polynomial of  $x' = (x_2, \dots, x_n)$ .

We will apply the method of moving planes in integral forms to show that  $u(x)$  is symmetric in  $x_1$  variable about the center  $x_1 = 0$ , and at mean time, we derive that it is monotone decreasing from the center.

Let

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\} \quad \text{and} \quad T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}. \quad (4.14)$$

Let  $x^\lambda \equiv (2\lambda - x_1, x_2, \dots, x_n)$  be the reflection of  $x$  about the plane  $T_\lambda$ , and

$$w_\lambda(x) = u_\lambda(x) - u(x) \equiv u(x^\lambda) - u(x). \quad (4.15)$$

Then it holds that

$$\begin{aligned} w_\lambda(x) &= v(x^\lambda) - v(x) + 4|\lambda|(\lambda - x_1) \\ &= 2c_n(x_1 - \lambda) \int_{\mathbb{R}^n} \frac{\lambda - y_1}{t(\lambda, x, y)} e^{u(y)} dy + 4|\lambda|(\lambda - x_1). \end{aligned} \quad (4.16)$$

Here we have applied the *mean value theorem* to

$$\log|x-y|^2 - \log|x^\lambda-y|^2 \quad (4.17)$$

and  $t(\lambda, x, y)$  is valued between  $|x-y|^2$  and  $|x^\lambda-y|^2$ .

*Step 1.* We show that for sufficiently negative  $\lambda$ ,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda. \quad (4.18)$$

Let  $\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$ , the set where the inequality (4.18) is violated. We prove that  $\Sigma_\lambda^-$  must be empty. For any  $x \in \Sigma_\lambda^-$ , by (4.16), we have

$$\begin{aligned} 0 < -w_\lambda(x) &= \frac{c_n}{2} \int_{\Sigma_\lambda} \left( \log|x^\lambda - y|^2 - \log|x - y|^2 \right) \left[ e^{u(y)} - e^{u_\lambda(y)} \right] dy - 4|\lambda|(\lambda - x_1) \\ &= 2c_n(\lambda - x_1) \int_{\Sigma_\lambda} \frac{\lambda - y_1}{t(\lambda, x, y)} e^{\xi_\lambda(y)} (-w_\lambda(y)) dy - 4|\lambda|(\lambda - x_1) \\ &= 4(\lambda - x_1) \left[ \frac{c_n}{2} \int_{\Sigma_\lambda^-} \frac{\lambda - y_1}{|x - y|^2} e^{u(y)} |w_\lambda(y)| dy - |\lambda| \right]. \end{aligned} \quad (4.19)$$

Here we have applied the *mean value theorem* to both

$$\log|x^\lambda - y|^2 - \log|x - y|^2 \quad (4.20)$$

and

$$e^{u(y)} - e^{u_\lambda(y)}; \quad (4.21)$$

$t(\lambda, x, y)$  is valued between  $|x^\lambda - y|^2$  and  $|x - y|^2$ , while  $\xi_\lambda(y)$  is between  $u_\lambda(y)$  and  $u(y)$ . Since

$$v(y) = -\alpha \log|y| + o(1) \quad \text{for } |y| \text{ large}, \quad (4.22)$$

for each fixed  $\lambda$ ,

$$v(y) - v(y^\lambda) \rightarrow 0, \quad \text{as } |y| \rightarrow \infty, \quad (4.23)$$

hence, for  $y \in \Sigma_\lambda^-$ ,

$$|w_\lambda(y)| = v(y) - v(y^\lambda) - 4|\lambda|(\lambda - y_1) \leq v(y) - v(y^\lambda) \rightarrow 0, \quad \text{as } |y| \rightarrow \infty. \quad (4.24)$$

Therefore the integral

$$\frac{c_n}{2} \int_{\Sigma_\lambda^-} \frac{\lambda - y_1}{|x - y|^2} e^{u(y)} |w_\lambda(y)| dy \quad (4.25)$$

is bounded, and this bound will not increase as  $\lambda$  becomes more negative.

It follows that for sufficiently negative  $\lambda$ , the right hand side of (4.19) is negative, which implies that  $\Sigma_\lambda^-$  is empty.

*Step 2.* Step 1 provides a starting point to move the plane  $T_\lambda$ . Now we move the plane toward the right as long as inequality (4.18) holds to its limiting position. More precisely, let

$$\lambda_o = \sup\{\lambda \leq 0 \mid w_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda\}, \quad (4.26)$$



and we will show that  $\lambda_o = 0$  using the contradiction argument.

Suppose  $\lambda_o < 0$ , we show that the plane can be moved a little bit to the right while inequality (4.18) is still valid, which would contradict the definition of  $\lambda_o$ .

From the expression

$$w_{\lambda_o}(x) = c_n \int_{\Sigma_\lambda} \left( \log|x^{\lambda_o} - y| - \log|x - y| \right) \left[ e^{u_{\lambda_o}(y)} - e^{u(y)} \right] dy + x_1^2 - (x^{\lambda_o})_1^2, \quad (4.27)$$

we see that

$$w_{\lambda_o}(x) > 0 \quad \forall x \in \Sigma_{\lambda_o}. \quad (4.28)$$

Because the integral is non-negative due to the definition of  $\lambda_o$  and  $x_1^2 - (x^{\lambda_o})_1^2 > 0$ .

We will derive that, for  $\lambda$  greater and sufficiently close to  $\lambda_o$ , the right hand side of (4.19) is negative.

a) First choose  $R$  sufficiently large, such that for all  $\lambda$  near  $\lambda_o$ ,

$$\frac{c_n}{2} \int_{\Sigma_\lambda^- \cap B_R^C(0)} \frac{\lambda - y_1}{|x - y|^2} e^{u(y)} |w_\lambda(y)| dy \leq \frac{|\lambda_o|}{3}, \quad (4.29)$$

where  $B_R^C(0)$  is the complement of  $B_R(0)$ .

b) By virtue of (4.28), for each  $\delta > 0$ , there is  $c_o > 0$ , such that

$$w_{\lambda_o}(x) \geq c_o, \quad \forall x \in \Sigma_{\lambda_o - \delta} \cap B_R(0). \quad (4.30)$$

Consequently, by the continuity of  $w_\lambda(x)$  with respect to  $\lambda$ , there exists an  $\epsilon > 0$ , such that

$$\text{for all } \lambda \in [\lambda_o, \lambda_o + \epsilon), \quad w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_o - \delta} \cap B_R(0). \quad (4.31)$$

This implies that  $\Sigma_\lambda^-$  is contained in the union of  $B_R^C(0)$  and a narrow region

$$N \equiv (\Sigma_\lambda \setminus \Sigma_{\lambda_o - \delta}) \cap B_R(0). \quad (4.32)$$

Choose  $\delta$  and  $\epsilon$  small, such that the measure of the narrow region is small enough to ensure that

$$\frac{c_n}{2} \int_N \frac{\lambda - y_1}{|x - y|^2} e^{u(y)} |w_\lambda(y)| dy \leq \frac{|\lambda_o|}{3}. \quad (4.33)$$

Now it follows from (4.19), (4.29), and (4.33) that, for  $x \in \Sigma_\lambda^-$ ,

$$0 < -w_\lambda(x) \leq -4(\lambda - x_1) \frac{|\lambda_o|}{3} < 0, \quad (4.34)$$

a contradiction. Therefore

$$\Sigma_\lambda^- = \emptyset, \quad \forall \lambda \in [\lambda_o, \lambda_o + \epsilon), \quad (4.35)$$

that is,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \forall \lambda \in [\lambda_o, \lambda_o + \epsilon). \quad (4.36)$$

This contradicts the definition of  $\lambda_o$ . Therefore, we must have  $\lambda_o = 0$ . So far, we have shown that

$$u(-x_1, x') \leq u(x_1, x'), \quad \forall x_1 > 0. \quad (4.37)$$

By using an entirely similar argument and starting moving the plane from near  $x_1 = +\infty$  to the left to its limiting position, we can show that

$$u(-x_1, x') \geq u(x_1, x'), \quad \forall x_1 > 0. \quad (4.38)$$

Combining (4.37) and (4.38), we conclude that  $u$  is symmetric about the origin with respect to  $x_1$ , and thus completes the proof of Proposition 4.2.  $\square$

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