



Efficient random graph matching via degree profiles

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Abstract

Random graph matching refers to recovering the underlying vertex correspondence between two random graphs with correlated edges; a prominent example is when the two random graphs are given by Erdős-Rényi graphs $G(n, \frac{d}{n})$. This can be viewed as an average-case and noisy version of the graph isomorphism problem. Under this model, the maximum likelihood estimator is equivalent to solving the intractable quadratic assignment problem. This work develops an $\tilde{O}(nd^2 + n^2)$ -time algorithm which perfectly recovers the true vertex correspondence with high probability, provided that the average degree is at least $d = \Omega(\log^2 n)$ and the two graphs differ by at most $\delta = O(\log^{-2}(n))$ fraction of edges. For dense graphs and sparse graphs, this can be improved to $\delta = O(\log^{-2/3}(n))$ and $\delta = O(\log^{-2}(d))$ respectively, both in polynomial time. The methodology is based on appropriately chosen distance statistics of the degree profiles (empirical distribution of the degrees of neighbors). Before this work, the best known result achieves $\delta = O(1)$ and $n^{o(1)} \leq d \leq n^c$ for some constant c with an $n^{O(\log n)}$ -time algorithm and $\delta = \tilde{O}((d/n)^4)$ and $d = \tilde{\Omega}(n^{4/5})$ with a polynomial-time algorithm.

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1 Introduction

Graph matching [12,32], also known as network alignment [20], aims at finding a bijective mapping between the vertex sets of two networks so that the number of adjacency disagreements between the two networks is minimized. It reduces to the graph isomorphism problem in the noiseless setting where the two networks can be matched perfectly.

The paradigm of graph matching has found numerous applications across a variety of diverse fields, such as network privacy, computational biology, computer vision, and natural language processing. For instance, it was convincingly demonstrated [41,42] that hidden vertex identities in a network can nevertheless be recovered by matching the anonymized network (such as Netflix) to a secondary network with known vertex identities (such as the Internet Movie Database). In system biology, graph matching is used in discovering protein functions by matching protein-protein interaction networks across different species [27,53]. In computer vision, using graphs to represent images, where vertices are regions in the images and edges encode the adjacency relationships between different regions, graph matching is widely applied in finding similar images [12,50]. In natural language processing, using graphs to represent sentences, where vertices are phrases and edges represent syntactic and semantic relationships, graph matching is used in question answering, machine translation, and information retrieval [24].

Given two graphs with adjacency matrices A and B , the graph matching problem can be viewed as a special case of the *quadratic assignment problem* (QAP) [11,45]: namely,

$$\max_{\Pi} \langle A, \Pi B \Pi^{\top} \rangle, \quad (1)$$

where Π ranges over all $n \times n$ permutation matrices, and $\langle \cdot, \cdot \rangle$ denotes the matrix inner product. QAP is NP-hard in the worst case. Moreover, approximating QAP within a factor of $2^{\log^{1-\epsilon}(n)}$ for $\epsilon > 0$ is also NP-hard [36].

These hardness results, however, are applicable in the worst case, where the observed networks are designed by an adversary. In contrast, the networks in many aforementioned applications can be modeled by random graphs with latent structures; as such, our focus is not in the worst-case instances, but rather in recovering the underlying vertex permutation with high probability in order to reveal the hidden structures.

1.1 Correlated Erdos-Rényi graphs model

Driven by applications in social networks and biology, a recent line of work [4,13–15,17,20,28,30,33,35,46,55] initiated the statistical analysis of graph matching by

assuming that A and B are generated randomly. The simplest such model is the following correlated Erdős-Rényi graph model:

Definition 1 (*Correlated Erdős-Rényi model $\mathcal{G}(n, q; s)$*) Given an integer n and $q, s \in [0, 1]$, let A and B denote the adjacency matrix of two Erdős-Rényi random graphs $\mathcal{G}(n, q)$ on the same vertex set $[n]$. Let $\pi^* : [n] \rightarrow [n]$ denote a latent permutation. We assume that conditional on A , for all $i < j$, $B_{\pi^*(i)\pi^*(j)}$ are independent and distributed as

$$B_{\pi^*(i)\pi^*(j)} \sim \begin{cases} \text{Bern}(s) & \text{if } A_{ij} = 1 \\ \text{Bern}\left(\frac{q(1-s)}{1-q}\right) & \text{if } A_{ij} = 0 \end{cases}, \quad (2)$$

where $\text{Bern}(s)$ denotes a Bernoulli distribution with mean s .

Equivalently, the two graphs can be viewed as edge-subsampled subgraphs of a parent Erdős-Rényi graph $G \sim \mathcal{G}(n, p)$ with $p = q/s$. Let A be the adjacency matrix of a graph obtained by keeping or deleting each edge of G independently with probability s and $\delta \triangleq 1 - s$ respectively. Repeat the sampling process independently and relabel the vertices according to the latent permutation π^* to obtain B .¹ Note that by (2), the parameter s can be viewed as a measure of the edge correlations. Alternatively, $\delta = 1 - s$ can be interpreted as the fraction of edges in A that are substituted in B on average.

Upon observing A and B , the goal is to exactly recover the latent vertex correspondence π^* with probability converging to 1 as $n \rightarrow \infty$. For instance, in network de-anonymization, the parent Erdős-Rényi graph G corresponds to the underlying friendship network of a group of people, A corresponds to a Facebook friendship network of the same group of people with known identities, and B is the Twitter network of the same set of users with identities removed; the task is to de-anonymize the vertex identities in the Twitter network by finding the underlying mapping between the vertex sets of A and B .

In the noiseless case of $s = 1$, graph matching under the $\mathcal{G}(n, q; 1)$ model reduces to the problem of random graph isomorphism for Erdős-Rényi graph $\mathcal{G}(n, q)$. In this case, a celebrated result [54] (see also [8, Chap. 9]) shows that exact recovery of the underlying permutation is information-theoretically possible if and only if $nq \geq \log n + \omega(1)$ for $q \leq 1/2$;² in other words, the symmetry (i.e., the automorphism group) of the graph is trivial with high probability. Recent work [13, 14] has extended this result to the noisy case where $s < 1$, showing that exact recovery is information-theoretically possible if and only if $nqs \geq \log n + \omega(1)$, under the additional assumption that $q \leq O(\log^{-1} n)$ and $q(1-s)^2/s \leq O(\log^{-3}(n))$.³

¹ To ensure the Bernoulli parameter in (2) is well-defined, we need to assume $q(1-s) \leq 1-q$, or equivalently $s \geq 2 - 1/q$. Similarly, to ensure the edge probability in the parent graph $p = q/s \leq 1$, we need to assume $s \geq q$.

² Throughout the paper, we use standard big O notation, e.g., for any sequences $\{a_n\}$ and $\{b_n\}$, $a_n = \Theta(b_n)$ (or $a_n \asymp b_n$) if $1/c \leq a_n/b_n \leq c$ holds for all n for some absolute constant $c > 0$; $a_n = \Omega(b_n)$ and $b_n = O(a_n)$ (or $a_n \gtrsim b_n$ and $b_n \lesssim a_n$) if $a_n/b_n \geq c$. We use big $\tilde{O}$ notation to hide logarithmic factors.

³ Achievability and converse bounds for more general correlated Erdős-Rényi random graph models are also available in [13, 14].

From a computational perspective, in the noiseless case of $s = 1$, linear-time algorithms have been found to attain the recovery threshold of $nq = \log n + \omega(1)$ [7,16]. However, in the noisy case, very little is known about the performance guarantees of graph matching algorithms that run in polynomial time. Recently a quasi-polynomial-time ($n^{O(\log n)}$) algorithm is proposed in [4] which succeeds when $nqs \in [n^{o(1)}, n^{1/153}] \cup [n^{2/3}, n^{1-\epsilon}]$ and $s \geq (\log n)^{-o(1)}$. Another recent work [17] adapts the classical degree-matching algorithms in [3] and [8, Section 3.5] from the noiseless case to the noisy case, and shows that it exactly recovers π^* with high probability, provided that $q \gg \log^{7/5}(n)/n^{1/5}$ and $1-s \ll q^4/\log^6(n)$. This result requires $1-s$, the fraction of edges differed in the two observed graphs, to decay polynomially in q and is thus far from being optimal.

1.2 Main results

In this work, we significantly improve the state of the art of efficient graph matching algorithms in terms of time complexity, noise tolerance, and sparsity. In particular, we give an $\tilde{O}(nd^2 + n^2)$ -time algorithm for exactly recovering the true permutation π^* with high probability under the correlated Erdős-Rényi graph model, when the fraction of differed edges $\delta = 1 - s$ can be as large as $1/\log^2(n)$ and the average degree d can be as low as $\log^2 n$. Furthermore, we obtain two improved polynomial-time algorithms that aim for dense and sparse graphs respectively. These results are summarized as below:

Theorem 1 *Consider the correlated Erdős-Rényi model $\mathcal{G}(n, q; 1 - \delta)$ with $q \leq q_0$ for some sufficiently small constant q_0 . If*

$$nq \gtrsim \log^2 n \quad \text{and} \quad \delta \lesssim \frac{1}{(\log n)^2}, \quad (3)$$

then there exists an $\tilde{O}(nd^2 + n^2)$ -time algorithm (cf. Algorithm 1) that recovers π^ with probability $1 - O(1/n)$.*

Furthermore,

– *if*

$$q = e^{-O((\log n)^{1/3})} \quad \text{and} \quad \delta \lesssim \frac{1}{(\log n)^{2/3}}, \quad (4)$$

then there exists a polynomial-time algorithm (cf. Algorithm 2) that recovers π^ with probability $1 - O(q/\log n)$;*

– *if*

$$\frac{\log n}{n} \lesssim q \leq n^{-\epsilon} \quad \text{and} \quad \delta \lesssim \frac{1}{(\log(nq))^2}, \quad (5)$$

for some constant $\epsilon > 9/10$, then there exists a polynomial-time algorithm (cf. Algorithm 4) that recovers π^ with probability $1 - O(n^{9-10\epsilon})$.*

1.3 Key algorithmic ideas and techniques for analysis

Many existing matching algorithms for random graph isomorphism are *signature-based*: first attach some appropriately chosen signature μ_i to vertex i in A and v_k to vertex k in B , then match each pair based on their similarity, or equivalently, some distance between the signatures. For example, degree matching simply uses the vertex degree as the signature. In addition, spectral method can be viewed as assigning the i th entry in the leading eigenvector(s) of the matrix A (resp. B) as the signature μ_i (resp. v_i). However, these signatures are highly sensitive to noise. Indeed, it can be shown that (cf. Remark 1 in Sect. 2) for degree sorting to yield the exact matching, the minimum spacing between the ordered degrees needs to overcome the effective noise, which entails $\delta = \tilde{O}(q^2)$. For spectral methods, due to the lack of low-rank structure and the vanishing spectral gap of Erdős-Rényi graphs, the eigenstructure is extremely fragile. Indeed, it can be shown via perturbation bounds that even for dense graphs, matching via top eigenvectors requires $\delta = O(n^{-c})$ for some constant c to succeed, which agrees with the numerical experiments in Sect. 5. Therefore, to deal with sparser graphs and smaller edge correlation, we need to find better signatures that are more robust to random perturbation.

Note that in the absence of any label information, we can only compute signatures that are *permutation-invariant*. The main finding of this work is that *degree profiles*, that is, empirical distribution of the degrees of neighbors, can be used as a signature which is significantly more noise-resilient than degrees or eigenvectors. Using a suitable distance between distributions to construct the matching (see the forthcoming Algorithm 1), this allows us to correctly match graphs that differ by almost linear number of edges. Specifically, for each vertex i in A , its degree profile μ_i is defined as the empirical distribution of the degrees of i 's neighbors. Similarly, for each vertex k in B , let v_k denote its degree profile. Then we match vertex i to vertex k which minimizes the total variation (L_1 -distance) between the appropriately discretized versions of μ_i and v_k (into $\text{polylog}(n)$ bins). The intuitive explanation for why this works is the following:

- if $k = \pi^*(i)$, which we call a “true pair”, then they have a large number of common neighbors, whose degrees, thanks to the edge correlations between A and B , are correlated random variables, which tend to lie in the same bin. This leads to a small distance between the degree profiles μ_i and v_k ;
- if $k \neq \pi^*(i)$, which we call a “fake pair”, then μ_i and v_k are empirical distributions consisting mainly independent samples, and their distance is typically large.

Clearly, in reality the situation is significantly more complicated due to various dependencies and the possibility that fake pairs can still have a non-negligible number of common neighbors. Furthermore, since for each vertex there exists a unique match but many more $(n - 1)$ potential mismatches, one needs to carefully control the total variation distance between degree profiles for true pairs and fake pairs as well as their large deviation behavior (their distance being atypically small). Nevertheless, our analysis rigorously justifies the above intuition and shows the distance statistic for true pairs and fake pairs are indeed separated with high probability under the condition (3).

Ideas related to degree profiles have been used for the random graph isomorphism problem. In particular, it is shown in [16,38] that *degree neighborhood* (i.e., the multiset of the degrees of neighbors of each vertex) constitutes a canonical labeling for $G(n, q)$ with high probability provided that $q \gg \frac{\log^2 n}{n}$. In the absence of noise, it suffices to prove that the degree neighborhoods of different vertices are distinct with high probability. However, how to match vertices in the noisy case and by how many edges the two graphs can differ is far less clear. In fact, although degree neighborhood (multiset) contains the same amount of information as degree profile (empirical distribution), for the development of our matching algorithm as well as the analysis, it is crucial to adopt the view of degree profiles as *probability measures*, which enables us to construct a greedy matching based on natural distances between probability distributions. The main observation is that although each degree profile is centered around the same mean (binomial distribution), the stochastic fluctuations are nearly independent for fake pairs and correlated for true pairs. This perspective allows us to leverage insights from empirical process theory to study the large deviation behavior of distances between degree profiles.

For relatively dense graphs with edge probability $q = \exp(-O(\log^{1/3} n))$, we further relax the condition from $\delta \lesssim \log^{-2} n$ to $\delta \lesssim \log^{-2/3} n$ by combining the degree profile matching with vertex degrees in conjunction with the paradigm of seeded graph matching (cf. Algorithm 2). In particular, we show that even if for some vertices the distance statistics between degree profiles of fake pairs can be smaller than that of the true match, with high probability this does not occur for vertices of sufficiently high degrees. Although the matched high-degree vertices occupy only a vanishing fraction of the vertex set, they provide enough initial “seeds” (correctly matched pairs) to match the remaining vertices with high probability under the condition (4). A key challenge in the analysis is to carefully control the dependency between vertex degrees and degree profiles, and to characterize the statistical correlation among vertex degrees. Furthermore, we provide an efficient seeded graph matching subroutine via maximum bipartite matching, which is guaranteed to succeed with $\Omega(\frac{\log n}{q})$ seeds, even if the seed set is chosen adversarially. A different seeded matching algorithm was previously proposed in [4] allowing possibly incorrect seeds and assuming a relaxed condition on the graph sparsity; however, the number of seeds needed in the worst-case is $\Omega(\max\{\frac{\log n}{q}, qn \log n\})$ (see the condition in Lemma 3.21 and before Lemma 3.26 in [4]), which cannot be afforded in the dense regime.

Note that degree profile matching is a *local algorithm* that uses only 2-hop neighborhood information for each vertex. It turns out that for relatively sparse graphs with edge probability $q \leq n^{-\epsilon}$ for a fixed constant $\epsilon > 9/10$, we can further relax the condition from $\delta \lesssim \log^{-2}(n)$ to $\delta \lesssim \log^{-2}(nq)$, using the 3-hop neighborhood information. This is carried out in three steps: for each neighbor j of vertex i in A and each neighbor j' of vertex k in B , we first compute the total variation distance between the degree profiles of j and j' as before, and then threshold the distances to construct a bipartite graph between the neighbors of vertex i and the neighbors of vertex k , and finally define a similarity score W_{ik} as the size of the maximum matching of this bipartite graph (cf. Algorithm 4). We show that these new similarity measures for true pairs and fake pairs are separated with high probability under the condition (5). Finally,

we mention that in the noiseless case, the algorithm of [7] that achieves the optimal threshold for sparse graphs (with average degree $\text{polylog}(n)$) uses as the signature the distance sequence of each vertex, which consists of the number of ℓ -hop neighbors for ℓ from 1 up to $\Theta(\frac{\log n}{\log \log n})$. This significantly improves the performance of degree matching [3]. It remains open whether local algorithms that use larger neighborhood information can further improve the graph matching performance in the noisy case.

1.4 Further related work

Convex relaxation There exists a large body of literature on convex relaxation of the graph matching problem; for a comprehensive discussion we refer the reader to [19]. One popular approach is doubly stochastic relaxation, which entails replacing the objective (1) by minimizing $\|AX - XB\|_F^2$, with $\|\cdot\|_F$ standing for the Frobenius norm, and relaxing the decision variable X from the set of permutation matrices into its convex hull, i.e., all doubly stochastic matrices [1,21]. This leads to a quadratic programming problem which is solvable in polynomial time but still much slower than the degree profile algorithm. Some initial statistical analysis for the correlated Erdős-Rényi graph model was carried out in [34]; however, its performance guarantees remain far from being well-understood.

There exists a conceptual connection between the degree profile matching algorithm and the doubly stochastic relaxation. In graph theory, two graphs are said to be fractionally isomorphic if their adjacency matrices A and B satisfy $AX = XB$ for some doubly stochastic matrix X . A result due to Ramana, Scheinerman, and Ullman (cf. [49, Theorem 6.5.1]) states that a necessary and sufficient condition for fractional isomorphism is that two graphs have identical *iterated degree sequences*; see [49, Sec. 6.4] for a precise definition. In particular, the first term of the iterated degree sequence corresponds to the degree distribution of the graph (i.e. the empirical distribution of the vertex degrees), while the second term is precisely the empirical distribution of degree profiles. In this perspective, our algorithm can be thought as using the leading two terms in the iterated degree sequence to construct the matching. Thus it is to be expected that degree profile matching algorithm outperforms degree matching but not the doubly stochastic relaxation.

Another approach is the semidefinite programming (SDP) relaxation for QAP [56] which is provably tighter than the doubly stochastic relaxation (cf. [29]). However, this entails solving an SDP in the lifted domain of $n^2 \times n^2$ matrices and the computational cost becomes prohibitively high even for moderate n .

Seeded Graph Matching Another recent line of work [22,30,35,46,51,55] in graph matching considers a relaxed version of the problem, where an initial seed set of correctly matched vertex pairs is revealed. This is motivated by the fact that in many practical applications, some side information on the vertex identities is available and has been successfully utilized to match many real-world networks [41,42]. It is shown in [55] that if $nq = \Theta(\log n)$ and the number of seeds is $\Omega(n/(s^2 \log n)^{4/3})$, then a percolation-based graph matching algorithm correctly matches all but $o(n)$ vertices in polynomial time with high probability. Another work [30] shows that if $q < 1/6$, then with at least $24 \log n/(qs^2)$ seeds, one can match all vertices correctly in polynomial

time with high probability. More recently, it is shown in [39] that the information-theoretic limit $nqs \geq \log n + \omega(1)$ in terms of the graph sparsity can be attained in polynomial time, provided that $s = \Theta(1)$ and the number of seeds is $\Omega(n^{3\epsilon})$ in the sparse graph regime ($nq \leq n^\epsilon$ for $\epsilon < 1/6$) and $\Omega(\log n)$ in some dense graph regime.

1.5 Notation and organization

Denote the identity matrix by $\mathbf{I}$. We let $\|X\|_F$ denote the Frobenius norm of a matrix X and $\|x\|_2$ denote the ℓ_2 norm of a vector x . For any positive integer n , let $[n] = \{1, \dots, n\}$. For any set $T \subset [n]$, let $|T|$ denote its cardinality and T^c denote its complement. Let δ_x denote the Dirac measure (point mass) at x . We say a sequence of events $\mathcal{E}_n$ indexed by a positive integer n holds with high probability, if the probability of $\mathcal{E}_n$ converges to 1 as $n \rightarrow +\infty$. Without further specification, all the asymptotics are taken with respect to $n \rightarrow \infty$. All logarithms are natural and we use the convention $0 \log 0 = 0$. For two real numbers a and b , we use $a \vee b = \max\{a, b\}$ (resp. $a \wedge b = \min\{a, b\}$) to denote the maximum (resp. minimum) between a and b . We denote by $\text{Bern}(p)$ the Bernoulli distribution with mean p and $\text{Binom}(n, p)$ the Binomial distribution with n trials and success probability p .

The rest of the paper is organized as follows: In Sect. 2, we provide a self-contained account of the problem of matching two Wigner random matrices. This part is intended as a warm-up for Erdős-Rényi graphs and serves to explain the main intuition behind the degree profile algorithms and the connection to empirical process theory and small ball probability. Section 3 describes the matching algorithms for the correlated Erdős-Rényi model and presents their theoretical guarantees. Specifically, Sect. 3.2 introduces the main algorithm for degree profile matching, with further improvements given in Sects. 3.3 and 3.4 for dense and sparse graphs, respectively. Section 4 provides the proof of correctness, with some auxiliary lemmas deferred to Appendix A. Appendix B contains our seeded graph matching result. Empirical evaluations of various algorithms on both simulated and real graphs are given in Sect. 5.

2 Warm-up: matching Gaussian Wigner matrices

In this section we take a slight detour to consider the Gaussian version of the graph matching problem, which can also be viewed as a statistical model for the QAP problem (1) with correlated Gaussian weights. Although the proofs for correlated Erdős-Rényi graphs do not exactly follow the same program, by studying this simpler model, we aim to convey the main idea behind the degree profile algorithm and sketch how to deduce the theoretical guarantees from results in empirical process theory and small ball probability.

2.1 Correlated Wigner model

Consider two random symmetric matrices A and B' , whose entries $\{(A_{ij}, B'_{ij}) : 1 \leq i \leq j \leq n\}$ are iid correlated standard normal pairs with correlation coefficient ρ ,

i.e., $(A_{ij}, B'_{ij}) \stackrel{\text{i.i.d.}}{\sim} N(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$. In other words, A and B' are two correlated Wigner matrices. Let $\pi^* \in S(n)$ be a permutation on $[n]$ and Π^* be its corresponding $n \times n$ permutation matrix. Let $B = \Pi^* B' (\Pi^*)^\top$. Observing the two matrices A and B , the goal is to estimate the latent permutation π^* correctly with high probability.

Without loss of generality, we assume $\rho > 0$ and let $\rho = \sqrt{1 - \sigma^2}$ for some $0 < \sigma^2 < 1$, and, furthermore, $\Pi^* = \mathbf{I}$. Therefore, we can write $B = \sqrt{1 - \sigma^2}A + \sigma Z$, where A and Z are two independent Wigner matrices.

2.2 Matching via empirical distributions

Next we describe a procedure for matching Wigner matrices as well as an improved version, which serve as the precursors to Algorithms 1 and 2 for Erdős-Rényi graphs.

The main idea is to use the empirical distribution of each row as the signature, and rely on appropriate distance between distributions to construct the matching. Specifically, for each i , define

$$\mu_i = \frac{1}{n} \sum_{j=1}^n \delta_{A_{ij}}$$

which is the empirical distribution of the i th row of A . Similarly, define

$$\nu_k = \frac{1}{n} \sum_{j=1}^n \delta_{B_{kj}}$$

for the B matrix. Marginally, for any i, k , both μ_i and ν_k are the empirical distributions of n standard normal samples. The difference is that if i and k form a true pair, the samples are correlated; otherwise, the samples are independent.⁴ Therefore, assuming the underlying permutation is the identity, (μ_i, ν_k) behave in distribution as two n -point empirical distributions

$$\mu = \frac{1}{n} \sum_{j=1}^n \delta_{X_j}, \quad \nu = \frac{1}{n} \sum_{j=1}^n \delta_{Y_j} \quad (6)$$

according to two cases:

- For “true pairs” ($i = k$), the X and Y samples consist of independent correlated pairs, namely,

$$(X_1, Y_1), \dots, (X_n, Y_n) \stackrel{\text{i.i.d.}}{\sim} N\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right). \quad (7)$$

⁴ To be precise, all but two elements (namely, A_{ik} and B_{ki}) are independent. This can be easily dealt with by excluding those two from the empirical distribution, which, by the triangle inequality, changes the distance statistic by at most $\frac{1}{n}$.

- For “fake pairs” ($i \neq k$), the X and Y sample are independent, namely,

$$(X_1, \dots, X_n, Y_1, \dots, Y_n) \stackrel{\text{i.i.d.}}{\sim} N(0, 1). \quad (8)$$

Therefore, although both empirical distributions have the same marginal distribution, for true pairs the atoms are correlated and the two empirical distributions tend to be closer than the typical distribution for fake pairs. This offers a test to distinguish true and fake pairs.

Now we introduce our procedure. For two probability measures μ and ν , we define their distance via the L_p -distance between their cumulative distribution function (CDF) F and G :

$$d_p(\mu, \nu) \triangleq \|F - G\|_p = \left(\int_{\mathbb{R}} dt |F(t) - G(t)|^p \right)^{1/p}, \quad (9)$$

where $p \in [1, \infty]$ is some fixed constant. e.g.,

- $p = 1$: 1-Wasserstein distance,
- $p = 2$: Cramér-von Mises goodness of fit statistic,
- $p = \infty$: Kolmogorov-Smirnov distance;

the asymptotic performance of the algorithm turns out to not depend on p . For each vertex i , we match it to the vertex k that minimizes the distance statistic $Z_{ik} \triangleq d_p(\mu_i, \nu_k)$. Next we show that when $\sigma \leq \frac{c}{\log n}$ for sufficiently small constant c , this algorithm succeeds with high probability.

To this end, let us recall the central limit theorem of empirical processes (cf. [52]). Let F_n and G_n denote the empirical CDF of X_i 's and Y_i 's, respectively, i.e.,

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i \leq t\}}, \quad G_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{Y_i \leq t\}}.$$

Let Φ denote the standard normal CDF on the real line. Then it is well-known that, as $n \rightarrow \infty$, $\sqrt{n}(F_n - \Phi)$ converges in distribution to a Gaussian process $\{B_t : t \in \mathbb{R}\}$, with covariance function $\text{Cov}(B_s, B_t) = \min\{\Phi(s), \Phi(t)\} - \Phi(s)\Phi(t)$. In fact, B is a time change of the standard Brownian bridge, which is the limiting process if the samples are drawn from the uniform distribution on $[0, 1]$. Similarly, $\sqrt{n}(G_n - \Phi)$ converges in distribution to another Gaussian process B' with the same distribution as B .

Next we analyze the behavior of true pairs. To get a sense of the order of magnitude of the distance statistic, let us consider the special case of $p = 2$ for convenience, for which direct calculation suffices. Define $F(s, t) = \mathbb{P}\{X \leq s, Y \leq t\}$. Note that we can write $Y = \sqrt{1 - \sigma^2}X + \sigma Z$, where $X, Z \stackrel{\text{i.i.d.}}{\sim} N(0, 1)$. Then

$$\begin{aligned}
\mathbb{E}[\|F_n - G_n\|_2^2] &= \int_{\mathbb{R}} \mathbb{E}[(F_n(t) - G_n(t))^2] dt \\
&\stackrel{(a)}{=} \frac{2}{n} \int_{\mathbb{R}} (F(t) - F(t, t)) dt \\
&= \frac{2}{n} \left(\int_{-\infty}^0 (F(t) - F(t, t)) dt + \int_0^{+\infty} ((1 - F(t, t)) - (1 - F(t))) dt \right) \\
&\stackrel{(b)}{=} \frac{2}{n} (\mathbb{E}[\max(X, Y)] - \mathbb{E}[X]) \\
&= \frac{2}{n} \mathbb{E}[\max(X, Y)] \\
&\stackrel{(c)}{=} \frac{2}{n} \frac{1}{\sqrt{\pi}} \sqrt{1 - \rho} = \frac{2}{n} \frac{1}{\sqrt{\pi}} \underbrace{\sqrt{1 - \sqrt{1 - \sigma^2}}}_{\Theta(\sigma)},
\end{aligned} \tag{10}$$

where (a) is due to $\mathbb{E}[(F_n(t) - G_n(t))^2] = \frac{1}{n} \mathbb{E}[(\mathbf{1}_{\{X \leq t\}} - \mathbf{1}_{\{Y \leq t\}})^2] = \mathbb{P}\{X \leq t\} + \mathbb{P}\{Y \leq t\} - 2\mathbb{P}\{X \leq t, Y \leq t\}$; (b) follows because $\mathbb{E}[U] = \int_0^{+\infty} (1 - F_U(u)) du - \int_{-\infty}^0 F_U(u) du$ for any random variable U whenever at least one of the two integrals is finite; (c) follows from directly differentiating the moment generating function of $\max(X, Y)$, see e.g., [40, Eq. (9)]. In fact, one can show that for small σ , for any $1 \leq p \leq \infty$,

$$\|F_n - G_n\|_p = O_P\left(\sqrt{\frac{\sigma}{n}}\right). \tag{11}$$

Indeed, by the central limit theorem for bivariate empirical processes, as $n \rightarrow \infty$, $\sqrt{n}(F_n - \Phi, G_n - \Phi)$ converges in distribution to a Gaussian process (B, B') indexed by $\mathbb{R}$, which satisfies $\text{Cov}(B_t, B'_t) = \mathbb{P}\{X \leq t, Y \leq t\} - \mathbb{P}\{X \leq t\}\mathbb{P}\{Y \leq t\}$, and furthermore $\sqrt{n}\|F_n - G_n\|_p \rightarrow \|B - B'\|_p$ in distribution. Since $\mathbb{E}|B_t - B'_t|^2 = 2(\Phi(t) - \mathbb{P}\{X \leq t, Y \leq t\})$, following the same calculation that leads to (10), we have $\mathbb{E}[\|B - B'\|_2^2] = \Theta(\sigma)$, which corresponds to (11) for $p = 2$.

Next, we turn to the behavior of fake pairs. Since B and B' are independent and since $B - B' \stackrel{\text{law}}{=} \sqrt{2}B$, we expect $\sqrt{n}\|F_n - G_n\|_p \rightarrow \|B - B'\|_p$ (see [5, Theorem 1.1] for the precise statement). In particular, we have

$$\|F_n - G_n\|_p = \Theta_P\left(\frac{1}{\sqrt{n}}\right). \tag{12}$$

Comparing (11) and (12), we see that the typical distance for true pairs is smaller than that of fake pairs by a factor of $\sqrt{\sigma}$. However, since there are $n - 1$ wrong matches for a given vertex, we need to consider the large-deviation behavior of (12). Recall the classical result from the literature of small ball probability; see [31] for an excellent survey. Let B be some Gaussian process e.g. the Brownian bridge defined on $\mathbb{R}$. Then the probability for the process to be contained in a small ball of radius ϵ behaves as (cf. [31, Sec. 4 and 6.2])

$$\mathbb{P}\{\|B\|_p \leq \epsilon\} \leq \exp\left(-\Theta\left(\frac{1}{\epsilon^2}\right)\right) \quad (13)$$

for some constant C . Indeed, one can show that

$$\mathbb{P}\left\{\|F_n - G_n\|_p \leq \sqrt{\frac{\sigma}{n}}\right\} \leq \exp\left(-\Theta\left(\frac{1}{\sigma}\right)\right). \quad (14)$$

Setting this probability to $o(\frac{1}{n^2})$ and applying a union bound, we conclude that the matching algorithm succeeds with high probability if $\sigma \leq \frac{c}{\log n}$ for sufficiently small constant c .

2.3 Improvement with seeded matching

In this subsection we improve the previous matching algorithm with empirical distributions to $\sigma = O((\log n)^{-1/3})$. To this end, we turn to the idea of *seeded matching*. Given a partial permutation that gives the correct matching for a subset of vertices, which we call *seeds*, one can extend it to a full matching by various methods, e.g., by solving a bipartite matching (see Algorithm 3). It turns out for Wigner matrices, it suffices to obtain $\Omega(\log n)$ seeds, which can be found by combining both the distance-based matching and degree thresholding. The same idea applies to Erdős-Rényi graphs, except that for edge density p , the number of seeds needed is $\Omega(\frac{\log n}{p})$, a fact which will be exploited in Sect. 4.2.

To explain the main idea, let $a_i = \frac{1}{\sqrt{n}} \sum_{j=1}^n A_{ij}$ and $b_k = \frac{1}{\sqrt{n}} \sum_{j=1}^n B_{kj}$ be the standardized row sums, which are the counterparts of “degrees” for Gaussian matrices. Consider the set of pairs (i, k) such that both a_i and b_k exceed some threshold ξ .

Then for any fake pair $i \neq k$, by independence, we have

$$\mathbb{P}\{a_i \geq \xi, b_k \geq \xi\} = \mathbb{P}\{a_i \geq \xi\} \mathbb{P}\{b_k \geq \xi\} = Q(\xi)^2,$$

where $Q \triangleq 1 - \Phi$ is the complementary CDF for the standard normal distribution. For true pairs, since we have the representation

$$b_i = \sqrt{1 - \sigma^2} a_i + \sigma z_i, \quad (15)$$

where $a_i, z_i \stackrel{\text{i.i.d.}}{\sim} N(0, 1)$, we have

$$\begin{aligned} \mathbb{P}\{a_i \geq \xi, b_i \geq \xi\} &\geq \mathbb{P}\left\{a_i \geq \frac{\xi}{\sqrt{1 - \sigma^2}}, z_i \geq 0\right\} \geq \frac{1}{2} Q\left(\frac{\xi}{\sqrt{1 - \sigma^2}}\right) \\ &\geq Q(\xi) \exp(-O(\sigma^2 \xi^2)). \end{aligned}$$

Now let us consider the seed set consisting of those high-degree pairs i and k whose empirical distributions satisfy $d_p(\mu_i, \nu_k) \lesssim \sqrt{\frac{\sigma}{n}}$. Thus to create enough seeds, we need

$$nQ(\xi) \exp\left(-O(\sigma^2 \xi^2)\right) \geq \Omega(\log n), \quad (16)$$

and to eliminate all fake pairs we need (in view of the small-ball estimate (14))

$$n^2 Q(\xi)^2 \exp\left(-\Omega\left(\sigma^{-1}\right)\right) = o(1). \quad (17)$$

Choosing $\xi = \Theta(\sqrt{\log n})$ and substituting it into (16), we get that $Q(\xi) = \Omega\left(\frac{\log n}{n}\right) \exp(O(\sigma^2 \log n))$. Substituting this back into (17), we conclude that $\sigma \leq \frac{c}{(\log n)^{1/3}}$ for some small constant c suffices.

We end this section with a few remarks:

Remark 1 (Order statistics) As described in Sect. 1.3, degree matching fails unless the fraction of differed edges is polynomially small. Similarly, for the Gaussian model directly sorting the degrees (row sums) in both matrices fails to yield the correct matching unless $\sigma \leq n^{-c}$ for some constant c . Indeed, sort the row sums a_i 's decreasingly as $a_{(1)} \geq \dots \geq a_{(n)}$ and similarly for $b_{(1)} \geq \dots \geq b_{(n)}$. Thus, degree matching amounts to match the vertices according to the sorted degrees. Since a_i 's are iid standard normal, it is well-known from the extreme value theory [18] that, with high probability, the order statistics behaves approximately as $a_{(i)} \approx \Phi^{-1}(i/n)$ which is approximately $\sqrt{2 \log \frac{n}{i}}$ for $i \leq n/2$ and $-\sqrt{2 \log \frac{n}{n+1-i}}$ for $i \geq n/2$. In particular, $a_{(1)} = a_{\max} \approx \sqrt{2 \log n}$ and $a_{(n)} = a_{\min} \approx -\sqrt{2 \log n}$. Furthermore, the i th spacing of the order statistics is approximately

$$\sqrt{2 \log \frac{n}{i}} - \sqrt{2 \log \frac{n}{i+1}} = \Theta\left(\frac{1}{i \sqrt{\log \frac{n}{i}}}\right) \quad (18)$$

Therefore, and intuitively so, for most of the samples the spacing is as small as $\Theta(\frac{1}{n})$. In view of (15), we can write $b_i = a_i + \Delta_i$, where $\Delta_i = (\sqrt{1 - \sigma^2} - 1)a_i + \sigma z_i$. Thus degree matching succeeds if $|\Delta_i| \leq \min\{|a_{i-1} - a_i|, |a_i - a_{i+1}|\}$ for all i . Since $|z_i| \leq O(\sqrt{\log n})$ and $|a_i| \leq O(\sqrt{\log n})$ for all i with high probability, this shows that degree matching requires very small noise $\sigma = o(\frac{1}{n \sqrt{\log n}})$, which is much worse than degree profiles. Simulation shows that this condition is necessary up to logarithmic factors.

Following the same idea in this subsection, an immediate improvement is to use degree matching to produce enough seeds to initiate the seeded graph matching process. Indeed, this is possible because the spacing of the first few order statistics is much bigger and more robust to noise. More precisely, in order to produce $\Omega(\log n)$ seeds, it suffices to ensure that the *minimum spacing* of the first i order statistics, which is at least $\widehat{\Omega}(\frac{1}{i^2 \sqrt{\log n}})$, far exceeds the noise which is $O(\sigma \sqrt{\log n})$. With $i = \Theta(\log n)$, this translates to $\sigma = o(\frac{1}{(\log n)^4})$, which is comparable to but still worse than the guarantee of degree profiles of $\sigma = O(\frac{1}{\log n})$ as established in Sect. 2.2. More importantly, a fundamental limitation of degree matching is that it fails for sparse graphs, because

the number of seeds needed is $\Omega(\frac{\log n}{q})$ where q is the edge density of the observed graphs (cf. Lemma 18 and [30, Theorem 1]). Following the similar analysis above for binomial distribution, for the correlated Erdős-Rényi graph model $\mathcal{G}(n, q; 1 - \delta)$, it is well-known that (cf. [8, Theorem 3.15]) the minimum of the first i spacing of sorted degrees is $\tilde{\Omega}(\frac{\sqrt{nq}}{i^2})$ with high probability and the degrees of a true pair differ by at most $\tilde{O}(\sqrt{\delta nq})$. Thus, producing $\Omega(\frac{\log n}{q})$ seeds requires the deletion probability to be as small as $\delta = \tilde{o}(q^4)$. This explains the recent result of [17], which shows that degree-matching algorithm with seeded improvement succeeds under some extra conditions.

Remark 2 (From Gaussian matrices to Erdős-Rényi graphs) To extend the matching algorithm based on empirical distributions from Gaussian matrices to Erdős-Rényi graphs, the main difficulty is that Bernoulli random variables are zero-one valued and hence directly implementing the same empirical distribution matching algorithm using adjacency matrices does not work. As mentioned in Sect. 1.3, the idea is to use the degree profile of each vertex, that is, the empirical distribution of the degrees of the neighbors, each of which is binomially distributed and well-approximated by Gaussians. Indeed, the ideas in Sects. 2.2 and 2.3 lead to Algorithms 1 and 2, respectively, for Erdős-Rényi graphs. However, the major technical difficulty is to address the dependency in the degree profiles. In the Gaussian case, each pair of degree profiles follows the simple dichotomy in (7)–(8), behaving as a pair of empirical distributions of correlated (resp. independent) samples for true (resp. fake) pairs. This is no longer the case for Erdős-Rényi graphs. For this reason, the approach for Erdős-Rényi graphs deviates from the program for Gaussian matrices, in that the algorithms in Sect. 4 are based on a quantized version of the total variation distance as opposed to distances between empirical CDFs, and the analysis in Sect. 4 does not explicitly resort to empirical process theory, although it is still guided by similar intuitions.

3 Matching algorithms for correlated Erdős-Rényi graphs

3.1 Preliminary definitions

For each vertex i , define its *open* neighborhood $N_A(i)$ (resp. $N_B(i)$) in graph A (resp. B) as the set of vertices connecting to i by an edge in A (resp. B); define its *closed* neighborhood $N_A[i]$ (resp. $N_B[i]$) in graph A (resp. B) as the union of its open neighborhood in A (resp. B) and $\{i\}$.

Denote the degrees by

$$a_i = |N_A(i)| = \sum_{j \in [n]} A_{ij} \quad (19)$$

$$b_i = |N_B(i)| = \sum_{j \in [n]} B_{ij}. \quad (20)$$

For each i and j , define

$$a_j^{(i)} = \frac{1}{\sqrt{(n - a_i - 1)q(1 - q)}} \sum_{\ell \notin N_A[i]} (A_{\ell j} - q) \quad (21)$$

$$b_j^{(i)} = \frac{1}{\sqrt{(n - b_i - 1)q(1 - q)}} \sum_{\ell \notin N_B[i]} (B_{\ell j} - q), \quad (22)$$

Note that $a_j^{(i)}$ (resp. $b_j^{(i)}$) can be viewed as the standardized version of the “outdegree” of vertex j by excluding i ’s closed neighborhood in A (resp. B).

To each vertex i in A , attach a distribution which is the empirical distribution of the set $\{a_j^{(i)} : j \in N_A(i)\}$:

$$\mu_i \triangleq \frac{1}{a_i} \sum_{j \in N_A(i)} \delta_{a_j^{(i)}}, \quad (23)$$

and the centered version (viewed as a signed measure)

$$\bar{\mu}_i \triangleq \mu_i - \overline{\text{Binom}}(n - a_i - 1, q), \quad (24)$$

where $\overline{\text{Binom}}(k, q)$ denotes the standardized binomial distribution, that is, the law of $\frac{X - kq}{\sqrt{kq(1 - q)}}$ for $X \sim \text{Binom}(k, q)$. The centering in (24) is due to the fact that conditioned on the neighborhood $N_A(i)$, each $a_j^{(i)}$ is distributed as $\overline{\text{Binom}}(n - a_i - 1, q)$ marginally. Similarly, for B we define

$$v_i \triangleq \frac{1}{b_i} \sum_{j \in N_B(i)} \delta_{b_j^{(i)}}. \quad (25)$$

and the centered version

$$\bar{v}_i \triangleq v_i - \overline{\text{Binom}}(n - b_i - 1, q). \quad (26)$$

Intuitively, μ_i is the degree profile for the neighbors of i in A , if the summation in (21) is over all $[n]$. We exclude edges within the neighborhood itself to reduce dependency and simplify the analysis. Note that conditioned on $N_A(i)$, $\{a_j^{(i)} : j \in N_A(i)\}$ are iid as $\overline{\text{Binom}}(n - a_i - 1, q)$; conditioned on $N_B(i)$, $\{b_j^{(i)} : j \in N_B(i)\}$ are iid as $\overline{\text{Binom}}(n - b_i - 1, q)$.

Fix $L \in \mathbb{N}$ to be specified later. Define $I_1, \dots, I_L$ as the uniform partition of $[-1/2, 1/2]$ such that $|I_\ell| = 1/L$. For each i and k , define the following distance statistic:

$$Z_{ik} \triangleq \sum_{\ell \in [L]} |\bar{\mu}_i(I_\ell) - \bar{v}_k(I_\ell)|. \quad (27)$$

In other words,

$$Z_{ik} = d(\bar{\mu}_i, \bar{v}_k) \triangleq \|[\bar{\mu}_i]_L - [\bar{v}_k]_L\|_1, \quad (28)$$

where $[\mu]_L$ denotes the discretized version of μ according to the partition $I_1, \dots, I_L$, with

$$[\mu]_L(\ell) \triangleq \mu(I_\ell), \quad \ell \in [L]. \quad (29)$$

Throughout the rest of the paper, for simplicity we use the parameterization

$$s \triangleq 1 - \sigma^2, \quad \delta \triangleq \sigma^2 \quad (30)$$

to denote the sampling and deletion probability respectively, where σ corresponds to the magnitude of the “effective noise”.

3.2 Matching via degree profiles

We present our first algorithm which matches the vertices in A to vertices in B based on the pairwise distance statistic $\{Z_{ik}\}$ in (27).

Algorithm 1 Graph matching via degree profiles

```

1: Input: Graphs  $A$  and  $B$  on  $n$  vertices, an integer  $L$ .
2: Output: A permutation  $\hat{\pi} \in S_n$ .
3: For each  $i, k \in [n]$ , compute  $Z_{ik}$  in (27).
4: Sort  $\{Z_{ik} : i, k \in [n]\}$  and let  $\mathcal{S}$  be the set of indices of the smallest  $n$  elements.
5: if  $\mathcal{S}$  defines a perfect matching on  $[n]$ , i.e.,  $\mathcal{S} = \{(i, \hat{\pi}(i)) : i \in [n]\}$  for some permutation  $\hat{\pi}$  then
6:   Output  $\hat{\pi}$ ;
7: else
8:   Output error.
9: end if

```

The key intuition underlying Algorithm 1 is as follows:

- For true pairs $k = \pi^*(i)$, we expect i and k to share many (about nqs) “common neighbors” j , in the sense that j is i ’s neighbor in A and $\pi^*(j)$ is k ’s neighbor in B . For each such common neighbor j , its outdegree $a_j^{(i)}$ in A is statistically correlated with the outdegree $b_{\pi^*(j)}^{(k)}$ in B . As a consequence, the two empirical distributions are strongly correlated, leading to a small distance Z_{ik} .
- For wrong pairs $k \neq \pi^*(i)$, we expect i and k share very few (about nq^2) “common neighbors”. Hence, the two empirical distributions μ_i and ν_k are weakly correlated, leading to a large distance Z_{ik} .

Remark 3 (Time complexity) Implementing Algorithm 1 entails three steps. First, we precompute all outdegrees. Assuming the graph is represented as an adjacency list and the list of degrees are given, for each i and each $j \in N_A(i)$, we have $a_j^{(i)} =$

$a_j - 1 - |N_A(i) \cap N_A(j)|$, where a_j is the degree of j and $|N_A(i) \cap N_A(j)|$ is the number of common neighbors, which can be computed in $O(a_i + a_j)$ time. Thus, computing all outdegrees can be done in time that is $\sum_{i \sim j} O(a_i + a_j) = O(\sum_i a_i^2) = O(|E||d_{\max}|)$.⁵ Next, we compute the discretized and centered degree profiles $[\tilde{\mu}_i]_L$ for each i in graph A and $[\tilde{\nu}_k]_L$ for each k in graph B . These are identified as L -dimensional vectors (where $L = \text{polylog}(n)$) and can be done in $\tilde{O}(|E|)$ time. Finally, we compute the distance statistic Z_{ik} in (27) for all pairs i and k and implement greedy matching via sorting. Since Z_{ik} is the ℓ_1 -distance between two L -dimensional vectors, this step can be computed in a total of $\tilde{O}(n^2)$ time. In summary, the total time complexity of Algorithm 1 is at most $\tilde{O}(|E||d_{\max}| + |V|^2)$, which, for Erdős-Rényi graphs under the assumption of Theorem 1, reduces $\tilde{O}(n^3q^2 + n^2)$.

The reason we use outdegrees instead of degrees in Algorithm 1 is a technical one, which aims at reducing the dependency and facilitating the theoretical analysis. In practice we can use degree profiles defined through the usual degrees and empirically the algorithm performs equally well. In this case, the time complexity reduces to $\tilde{O}(n^2)$.

Theorem 2 (Performance guarantee of Algorithm 1) *Let $s = 1 - \sigma^2$ and $q \leq q_0$ for some sufficiently small positive constant q_0 . Assume that*

$$\sigma \leq \frac{\sigma_0}{\log n}, \quad (31)$$

for some sufficiently small absolute constant σ_0 . Set

$$L = L_0 \log n \quad (32)$$

and assume that

$$nq \geq C_0 \log^2 n \quad (33)$$

for some large absolute constants L_0, C_0 . Then with probability $1 - O(1/n)$, Algorithm 1 outputs $\hat{\pi} = \pi^$.*

3.3 Dense graphs: combining with high-degree vertices

For relatively dense graphs, Algorithm 1 can be improved as follows. Recall the notion of seeded graph matching previously mentioned in Sect. 2.3, where a number of correctly matched vertices are given, known as seeds, and the goal is to match the remaining vertices. It turns out that for $G(n, q)$, provided $m = \Omega(\frac{\log n}{q})$ seeds, solving a linear assignment problem (maximum bipartite matching) can successfully match the rest of the vertices with high probability. Note that the condition $\sigma = O((\log n)^{-1})$ in Theorem 2 ensures Algorithm 1 succeeds in one shot, in the sense that with high

⁵ Alternatively, outdegrees can be computed via the number of common neighbors by squaring the adjacency matrix using fast matrix multiplication.

probability the distance statistics are below the threshold for *all* n true pairs and above the threshold for *all* $\binom{n}{2}$ wrong pairs. Thus, we can weaken this condition so that even if the distance statistics for most of the pairs are not correctly separated, those high-degree vertices can provide enough seeds that allow bipartite matching to succeed. This idea leads to the improvement to $\sigma = O((\log n)^{-1/3})$ when the edge density $q = \exp(-O((\log n)^{1/3}))$.

Specifically, fix some thresholds τ and ξ . Consider the collection of pairs of vertices whose degrees are atypically high and the degree profiles are close:

$$\mathcal{S} = \{(i, k) : a_i \geq \tau, b_k \geq \tau + 1, Z_{ik} \leq \xi\}. \quad (34)$$

We show that, with high probability,

1. $\mathcal{S}$ does not contain any fake pairs, i.e., $(i, k) \notin \mathcal{S}$ for any $k \neq \pi^*(i)$.
2. $\mathcal{S}$ contain enough true pairs, i.e., $|\mathcal{S}| = \Omega(\frac{\log n}{q})$.

Finally, we use the matched pairs in $\mathcal{S}$ as seeds to resolve the rest of the matching by linear assignment; this is done in Algorithm 3. The full procedure is given in Algorithm 2.

As for the time complexity, compared to Algorithms 1, 2 has an extra step of computing the maximum matching on an $n \times n$ unweighted bipartite graph, which can be done in either $O(n^3)$ time using Ford–Fulkerson algorithm [23] or $O(n^{2.5})$ time using the Hopcroft–Karp algorithm [25].

Algorithm 2 Combining degree profiles and large-degree vertices

- 1: **Input:** Graph A and B on n vertices; thresholds $\tau, \xi > 0$.
 - 2: **Output:** A permutation $\hat{\pi} \in S_n$.
 - 3: Compute the distance statistic Z_{ik} for each $i, k \in [n]$. Let $\mathcal{S}$ be given in (34).
 - 4: **if** $\mathcal{S}$ defines a matching, i.e., there exists $S \subset [n]$ and an injection $\pi_0 : S \rightarrow [n]$, such that $S = \{(i, \pi_0(i)) : i \in S\}$, **then**
 - 5: Run Algorithm 3 using π_0 as the seeds and output $\hat{\pi}$.
 - 6: **else**
 - 7: output error;
 - 8: **end if**
-

Theorem 3 (Performance guarantee of Algorithm 2) *Assume that $q \leq q_0$ and*

$$\sigma \leq \sigma_0 \min \left\{ \frac{1}{(\log n)^{1/3}}, \frac{1}{\log \frac{\log n}{q}} \right\}, \quad (36)$$

for some small absolute constants q_0, σ_0 . Define

$$\alpha \triangleq \left(\alpha_0 \frac{\log n}{nq} \right)^{\frac{(1-p)s}{1-q}} \quad (37)$$

Algorithm 3 Seeded graph matching

-
- 1: **Input:** Graphs A and B on n vertices; a bijection $\pi_0 : S \rightarrow T$, where $S, T \subset [n]$;
 2: **Output:** A permutation $\hat{\pi} \in S_n$.
 3: For each $i \in S^c$ and each $k \in T^c$, define $n_{ik} = \sum_{j \in S} A_{ij} B_{k\pi_0(j)}$.
 4: Define a bipartite graph with vertex set $S^c \times T^c$ and adjacency matrix H given by $H_{ik} = \mathbf{1}_{\{n_{ik} \geq \kappa\}}$ for each $i \in S^c$ and each $k \in T^c$, where $\kappa = \frac{1}{2}|S|qS$. Find a maximum bipartite matching of H , i.e., a perfect matching $\tilde{\pi}_1$ between S^c and T^c such that $\tilde{\pi}_1 \in \arg \max_{\pi} w(\pi)$, where
-

$$w(\pi) \triangleq \sum_{i \in S^c} H_{i\pi(i)}. \quad (35)$$

- Let π_1 denote a perfect matching on $[n]$ such that $\pi_1|_S = \pi_0$ and $\pi_1|_{S^c} = \tilde{\pi}_1$.
 5: For each $i, k \in [n]$, define $w_{ik} = \sum_{j=1}^n A_{ij} B_{k\pi_1(j)}$.
 6: Sort $\{w_{ik} : i, k \in [n]\}$ and let $\mathcal{T}$ be the set of indices of the largest n elements.
 7: **if** $\mathcal{T}$ defines a perfect matching on $[n]$, i.e., $\mathcal{T} = \{(i, \hat{\pi}(i)) : i \in [n]\}$ for some permutation $\hat{\pi}$ **then**
 8: Output $\hat{\pi}$;
 9: **else**
 10: Output error;
 11: **end if**
-

and

$$L = L_0 \max \left\{ \log^{1/3}(n), \log \frac{\log n}{q} \right\} \quad (38)$$

for some large absolute constants α_0, L_0 . Let

$$\tau \triangleq \min \{0 \leq k \leq n : \mathbb{P} \{\text{Binom}(n-1, q) \geq k\} \leq \alpha\}, \quad (39)$$

and

$$\xi = C \sqrt{\frac{L}{nq}} \quad (40)$$

for some absolute constant C . Assume that

$$nq^2 \geq C_0 \log^2 n \quad (41)$$

for some large absolute constant C_0 . Then with probability $1 - O\left(\frac{q}{\log n}\right)$, Algorithm 2 outputs $\hat{\pi} = \pi^*$.

We briefly explain the choice of parameters and the condition (36) on σ . According to (39), the threshold τ is chosen to be the $(1-\alpha)$ -quantile of a_i , so that $\mathbb{P}\{a_i \geq \tau\} \approx \alpha$. The crucial observation is the following:

- For true pairs $k = \pi^*(i)$, the degrees a_i and b_k are both sampled from the same vertex in the parent graph and are hence positively correlated. Indeed, we have

$$\mathbb{P}\{a_i \geq \tau, b_k \geq \tau + 1\} = \Omega\left(\alpha^{\frac{1-q}{(1-p)^S}}\right), \quad k = \pi^*(i). \quad (42)$$

Here the exponent $\frac{1-q}{(1-p)s}$ is slightly bigger than one:

$$\frac{1-q}{(1-p)s} = 1 + \frac{1-s}{(1-p)s} = 1 + \frac{\sigma^2}{(1-p)s}. \quad (43)$$

- For fake pairs $k \neq \pi^*(i)$, the degrees a_i and b_k are almost independent, and indeed we have

$$\mathbb{P}\{a_i \geq \tau, b_k \geq \tau + 1\} = O\left(\alpha^2\right), \quad k \neq \pi^*(i). \quad (44)$$

Both (42) and (44) will be made precise in Lemma 3.

In order for Algorithm 2 to succeed, on the one hand, we need to ensure the seed set $\mathcal{S}$ in Algorithm 2 contains at least $\Omega\left(\frac{\log n}{q}\right)$ correctly matched pairs. Indeed, under the condition $L = O(1/\sigma)$ and the choice of ξ in (40), we will show that for any true pair (i, k) the distance statistic Z_{ik} is below ξ with high probability. Thus, we have in expectation:

$$\mathbb{E}[|\mathcal{S}|] \stackrel{(42)}{\geq} n\alpha^{\frac{1-q}{(1-p)s}} \stackrel{(37)}{=} \alpha_0 \frac{\log n}{q},$$

and we will show that this holds with high probability as well.

On the other hand, we need to ensure that no fake pair is included in $\mathcal{S}$ with high probability. We will show that for any wrong pair (i, k) , $Z_{ik} \leq \xi$ with probability at most $e^{-\Omega(L)}$ (see Lemma 2). By the union bound, in view of (44), it suffices to guarantee that

$$\begin{aligned} n^2 \alpha^2 \exp(-\Omega(L)) &\stackrel{(37)}{=} n^2 \left(\alpha_0 \frac{\log n}{nq} \right)^{2\frac{(1-p)s}{1-q}} \exp(-\Omega(L)) \\ &\stackrel{(43)}{\leq} \left(\alpha_0 \frac{\log n}{q} \right)^2 \exp\left(\frac{2\sigma^2}{1-q} \log n - \Omega(L) \right) = o(1). \end{aligned} \quad (45)$$

Also, recall that $L = O(1/\sigma)$. Thus, the desired (45) holds provided that $\sigma \lesssim \frac{1}{(\log n)^{1/3}} \wedge \frac{1}{\log \frac{\log n}{q}}$, and q is bounded away from 1, by choosing $L \gtrsim (\log n)^{1/3} \vee \log \frac{\log n}{q}$.

Finally, we mention that since the seed set obtained from Algorithm 1 and degree thresholding depends on the entire graph, the analysis of Algorithm 2 entails a worst-case analysis of the seeded matching subroutine. This is done in Lemma 19 in Appendix B, which guarantees the correctness of Algorithm 3 even for an adversarially chosen seed set.

3.4 Sparse graphs: matching via neighbors' degree profiles

For relatively sparse graphs, we can improve the condition from $\sigma = O(1/\log(n))$ to $\sigma = O(1/\log(nq))$ by comparing neighbors' degree profiles. Next we describe our improved local algorithm, which uses the information of 3-hop neighborhoods.

We start with some basic definitions. The ℓ -hop neighborhood of i in graph G is the subgraph of G induced by the vertices within distance ℓ from i . Let $\tilde{N}_A(i)$ (resp. $\tilde{N}_B(i)$) denote the set of vertices in the 2-hop neighborhood of i in graph A (resp. B). Denote the size of the 2-hop neighborhood of i in graph A and B by respectively

$$\tilde{a}_i = |\tilde{N}_A(i)|, \quad \text{and} \quad \tilde{b}_i = |\tilde{N}_B(i)|.$$

For each vertex i and each vertex ℓ at distance two from i in graph A (resp. B), define $\tilde{a}_\ell^{(i)}$ (resp. $\tilde{b}_\ell^{(i)}$) as

$$\tilde{a}_\ell^{(i)} = \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} \sum_{k \notin \tilde{N}_A(i)} (A_{k\ell} - q), \quad (46)$$

$$\tilde{b}_\ell^{(i)} = \frac{1}{\sqrt{(n - \tilde{b}_i)q(1 - q)}} \sum_{k \notin \tilde{N}_B(i)} (B_{k\ell} - q), \quad (47)$$

Analogous to (21) and (22), $\tilde{a}_\ell^{(i)}$ (resp. $\tilde{b}_\ell^{(i)}$) can also be viewed as the normalized “outdegree” of vertex ℓ , this time with the closed 2-hop neighborhood of i in A (resp. B) excluded.

To each vertex $j \in N_A(i)$, attach the centered empirical distribution of the set $\{\tilde{a}_\ell^{(i)} : \ell \in N_A(j) \setminus N_A[i]\}$:

$$\tilde{\mu}_j^{(i)} \triangleq \frac{1}{|N_A(j) \setminus N_A[i]|} \sum_{\ell \in N_A(j) \setminus N_A[i]} \delta_{\tilde{a}_\ell^{(i)}} - \overline{\text{Binom}}(n - \tilde{a}_i, q). \quad (48)$$

Similarly, to each vertex $j \in N_B(i)$, attach the centered empirical distribution of the set $\{\tilde{b}_\ell^{(i)} : \ell \in N_B(j) \setminus N_B[i]\}$:

$$\tilde{\nu}_j^{(i)} \triangleq \frac{1}{|N_B(j) \setminus N_B[i]|} \sum_{\ell \in N_B(j) \setminus N_B[i]} \delta_{\tilde{b}_\ell^{(i)}} - \overline{\text{Binom}}(n - \tilde{b}_i, q). \quad (49)$$

Analogous to (24) and (26), $\tilde{\mu}_j$ (resp. $\tilde{\nu}_j$) is the centered “outdegree” profile of j , this time defined over only j 's neighbors which are at exactly distance two from i in A (resp. B).

We now introduce a new distance statistic W based on aggregating the original Z statistic in (27) over neighbors. Recall the uniform partition $I_1, \dots, I_L$ of $[-1/2, 1/2]$ such that $|I_\ell| = 1/L$. For each $j \in N_A(i)$ and $j' \in N_B(k)$, define the following

distance statistic:

$$\tilde{Z}_{jj'}^{(ik)} \triangleq \sum_{\ell \in [L]} \left| \tilde{\mu}_j^{(i)}(I_\ell) - \tilde{v}_{j'}^{(k)}(I_\ell) \right|, \quad (50)$$

which is analogous to (27) except that the definition of the outdegrees are modified.

For each $i, k \in [n]$, construct a bipartite graph with vertex set $N_A(i) \times N_B(k)$, whose adjacency matrix $Y^{(ik)}$ is given by

$$Y_{jj'}^{(ik)} = \mathbf{1}_{\{\tilde{Z}_{jj'}^{(ik)} \leq \eta\}}, \quad j \in N_A(i), j' \in N_B(k). \quad (51)$$

Here η is a threshold to be specified later. Define a similarity matrix W , where W_{ik} is the size of a maximum bipartite matching of $Y^{(ik)}$:

$$\begin{aligned} W_{ik} &= \max \left\langle Y^{(ik)}, M \right\rangle \\ \text{s.t. } &\sum_j M_{jj'} \leq 1, \\ &\sum_{j'} M_{jj'} \leq 1, \\ &M_{jj'} \in \{0, 1\}. \end{aligned} \quad (52)$$

Finally, we match vertices in A to vertices in B greedily by sorting the similarities W_{ik} 's. The entire algorithm is summarized in Algorithm 4 below.

Algorithm 4 Graph matching via neighbors' degree profiles

- 1: **Input:** Graphs A and B on n vertices, an integer L , and a threshold $\eta > 0$.
 - 2: **Output:** A permutation $\hat{\pi} \in S_n$.
 - 3: For each $i, k \in [n]$, compute W_{ik} as in (52).
 - 4: Sort $\{W_{ik} : i, k \in [n]\}$ and let $\mathcal{S}$ be the set of indices of the largest n elements.
 - 5: **if** $\mathcal{S}$ defines a perfect matching on $[n]$, i.e., $\mathcal{S} = \{(i, \hat{\pi}(i)) : i \in [n]\}$ for some permutation $\hat{\pi}$ **then**
 - 6: Output $\hat{\pi}$;
 - 7: **else**
 - 8: Output error.
 - 9: **end if**
-

The intuition behind Algorithm 4 is as follows. Even if the $\tilde{Z}$ distance statistics of degree profiles are not correctly separated for all pairs, the new W statistics are guaranteed to be well separated. Indeed, by setting

$$\eta = \eta_0 \sqrt{\frac{L}{nq}} \quad (53)$$

for some sufficiently small absolute constant η_0 , we expect that

- for true pairs $k = \pi^*(i)$, i and k share many (about nq s) “common neighbors” (in the sense that $j \in N_A(i)$ and $\pi^*(j) \in N_B(k)$). Moreover, most of such common neighbors have $\tilde{Z}$ distance smaller than η . As a consequence, W_{ik} is at least $nq/4$ with high probability;
- for fake pairs $k \neq \pi^*(i)$, i and k share very few (about nq^2) “common neighbors”. Moreover, most of the fake pair of vertices $j \in N_A(i)$ and $j' \in N_B(k)$ have $\tilde{Z}$ distance larger than η . As a consequence, when q is small, W_{ik} is smaller than $nq/4$ with high probability.

The performance guarantee of Algorithm 4 is as follows:

Theorem 4 Fix any constant $\epsilon > 9/10$. Suppose

$$C_0 \log n \leq nq \leq n^{1-\epsilon} \quad \text{and} \quad \sigma \leq \frac{\sigma_0}{\log(nq)}$$

for some sufficiently large absolute constant C_0 and some sufficiently small absolute constant σ_0 . Set $L = L_0 \log(nq)$ and η as in (53) for some sufficiently large absolute constant L_0 and some sufficiently small absolute constant η_0 . Then with probability at least $1 - O(n^{9-10\epsilon})$, Algorithm 4 outputs $\hat{\pi} = \pi^*$.

We briefly explain the condition on the graph sparsity in Theorem 4. On the one hand, the analysis of Algorithm 4 requires the graphs to be sufficiently sparse ($nq \leq n^{1-\epsilon}$ for $\epsilon > 9/10$), so that all 2-hop neighborhoods are tangle-free, each containing at most one cycle. On the other hand, Theorem 4 requires the graphs cannot be too sparse (i.e., $nq \gtrsim \log n$) so that each vertex has enough neighbors; this lower bound is information-theoretically necessary for exact recovery [13,14].

4 Analysis

Throughout this section, without loss of generality, we assume the true permutation π^* is the identity.

We introduce a number of events regarding the neighborhoods $N_A(i)$ and $N_B(k)$. Recall that $a_i = |N_A(i)|$ and $b_k = |N_B(k)|$ denote the degrees. Put

$$c_{ik} = |N_A(i) \cap N_B(k)|. \quad (54)$$

First, for each $i \in [n]$, define the events

$$\Gamma_A(i) = \left\{ \frac{1}{2}nq \leq a_i \leq 2nq \right\}, \quad \Gamma_B(i) = \left\{ \frac{1}{2}nq \leq b_i \leq 2nq \right\}, \quad (55)$$

$$\Gamma_{ii} = \left\{ c_{ii} \geq \frac{1}{2}nq \right\}. \quad (56)$$

Second, for each pair of $i, k \in [n]$ with $i \neq k$, define the event

$$\Gamma_{ik} = \left\{ \sqrt{c_{ik}} \leq \sqrt{nq^2} + \sqrt{2 \log n} \right\}. \quad (57)$$

Note that $a_i, b_i \sim \text{Bin}(n-1, q)$. Moreover, $c_{ii} \sim \text{Bin}(n-1, qs)$ which is stochastically larger than $\text{Bin}(n-1, 3q/4)$ under the assumption $s = 1 - \sigma^2 \geq 3/4$; for $i \neq k$, $c_{ik} \sim \text{Bin}(n-2, q^2)$. Thus, it follows from the binomial tail bounds (165) and (168) that

$$\mathbb{P}\{\Gamma_A^c(i)\}, \mathbb{P}\{\Gamma_B^c(i)\}, \mathbb{P}\{\Gamma_{ii}^c\} \leq e^{-\Omega(nq)} \leq n^{-3}, \quad \forall i \in [n], \quad (58)$$

$$\mathbb{P}\{\Gamma_{ik}^c\} \leq n^{-3}, \quad \forall i \neq k \in [n], \quad (59)$$

where we use the assumption that $nq \geq C \log n$ for a sufficiently large constant C .

Third, given any $\Delta > 0$, for each pair of $i, k \in [n]$, define the event

$$\Theta_{ik} \triangleq \{|a_i - b_k| \leq 4\sqrt{nq}\Delta\}. \quad (60)$$

In view of the binomial tail bounds (167) and (168), we have that

$$\mathbb{P}\{\sqrt{nq} - \sqrt{\Delta} \leq \sqrt{a_i} \leq \sqrt{nq} + \sqrt{\Delta}\} \geq 1 - 2e^{-\Delta}$$

and similarly for b_k . Thus it follows from the union bound that

$$\mathbb{P}\{\Theta_{ik}\} \geq \mathbb{P}\{\sqrt{nq} - \sqrt{\Delta} \leq \sqrt{a_i}, \sqrt{b_k} \leq \sqrt{nq} + \sqrt{\Delta}\} \geq 1 - 4e^{-\Delta}. \quad (61)$$

Lastly, for each $i \in [n]$, define the event

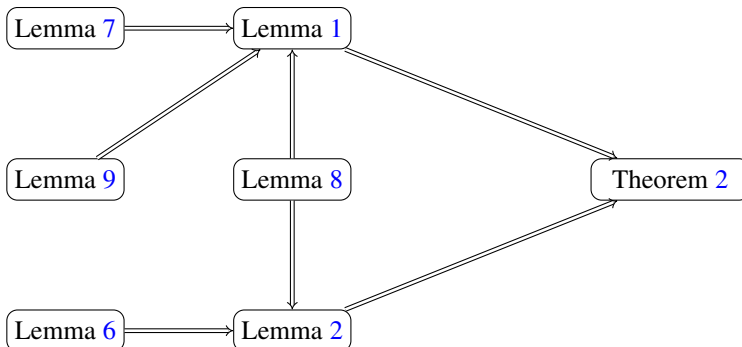
$$\Theta_i = \{\max\{\sqrt{a_i - c_{ii}}, \sqrt{b_i - c_{ii}}\} \leq \sqrt{nq(1-s)} + \sqrt{\Delta}\}. \quad (62)$$

Since both $a_i - c_{ii}$ and $b_i - c_{ii}$ are distributed as $\text{Binom}(n-1, q(1-s))$, it follows from the binomial tail bound (168) and the union bound that

$$\mathbb{P}\{\Theta_i\} \geq 1 - 2e^{-\Delta}. \quad (63)$$

4.1 Proof of Theorem 2

The proof of Theorem 2 is structured as follows:



We start with the following results on separating the maximum distance among true pairs $\max_{i \in [n]} Z_{ii}$ and the minimum distance among wrong pairs $\min_{i \neq k \in [n]} Z_{ik}$:

Lemma 1 (True pairs) *Assume that $\sigma \leq \frac{1}{2}$, $q \leq q_0 \leq \frac{1}{8}$, $nq \geq C \max\{\log n, L^2, \Delta\}$ for some sufficiently large constant C , and*

$$4L\sqrt{nq\Delta} \leq n. \quad (64)$$

There exist absolute constants τ_1, τ_2 such that for each $i \in [n]$,

$$\mathbb{P}\{Z_{ii} \geq \xi_{\text{true}} \mid N_A(i), N_B(i)\} \mathbf{1}_{\{\Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_{ii}\}} \leq O(e^{-\Delta/2}), \quad (65)$$

where

$$\xi_{\text{true}} \triangleq L\sqrt{\frac{2\beta}{nq}} + \tau_2\sqrt{\frac{\Delta}{nq}} + \tau_2\sigma\sqrt{\frac{L}{nq}} \quad (66)$$

and

$$\beta \triangleq \tau_2 \left(\sigma + \sqrt{\frac{\Delta}{n}} + \frac{1}{\sqrt{nq}} + e^{-\Delta} \right) + \frac{1}{L} \exp \left(-\tau_1 \min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right). \quad (67)$$

Lemma 2 (Fake pairs) *Assume that $\sigma \leq \frac{1}{2}$, $q \leq q_0$ for some sufficiently small constant q_0 , $nq \geq C \max\{\log n, L^2, \Delta\}$, and $L \geq L_0$ for some sufficiently large constant L_0 . Then there exist universal constants c_1, c_2, c_3 , such that for each distinct pair $i \neq k$ in $[n]$,*

$$\mathbb{P}\{Z_{ik} \leq \xi_{\text{fake}} \mid N_A(i), N_B(k)\} \mathbf{1}_{\{\Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}\}} \leq O(e^{-\Delta/2}), \quad (68)$$

where

$$\xi_{\text{fake}} \triangleq c_1\sqrt{\frac{L}{nq}} - c_2\sqrt{\frac{\Delta}{nq}}. \quad (69)$$

Note that the conclusions of Lemma 1 and 2 are stated in a conditional form conditioned on the neighborhoods $N_A(i)$ and $N_B(k)$. This is for the purpose of analyzing Algorithm 2, where we will need to apply these lemmas to high-degree vertices (see proof of Theorem 3).

We now prove Theorem 2:

Proof It suffices to show that with probability $1 - O(1/n)$,

$$\min_{i \neq k \in [n]} Z_{ik} > \max_{i \in [n]} Z_{ii}.$$

Choose

$$\Delta = \left(\frac{c_1}{4 \max\{c_2, \tau_2\}} \right)^2 L, \quad (70)$$

where c_1 , c_2 and τ_2 are the absolute constants given in Lemmas 2 and 1, respectively.

In view of the theorem assumptions $\sigma \leq \sigma_0/\log n$, $L = L_0 \log n$, and $nq \geq C_0 \log^2 n$, we have that β in (67) satisfies

$$\begin{aligned} \beta L &\leq \tau_2 L_0 \left(\sigma_0 + \log n \sqrt{\frac{\Delta}{n}} + \frac{1}{\sqrt{C_0}} + e^{-\Delta} \log n \right) \\ &\quad + \exp \left(-\tau_1 \min \left\{ \frac{1}{\sigma_0^2 L_0^2}, \frac{n}{L_0^2 \Delta \log^2 n}, \frac{\sqrt{C_0}}{L_0} \right\} \right) \\ &\leq \frac{c_1^2}{32}, \end{aligned}$$

provided that $\sigma_0 L_0$ is sufficiently small, and n and $\sqrt{C_0}/L_0$ are sufficiently large. Moreover, $\tau_2 \sigma \leq 1/8$ when σ_0 is sufficiently small. Thus, in view of (66), (69), and (70), we have

$$\xi_{\text{fake}} \geq \frac{3c_1}{4} \sqrt{\frac{L}{nq}} > \frac{5c_1}{8} \sqrt{\frac{L}{nq}} \geq \xi_{\text{true}}. \quad (71)$$

Also, since $L = L_0 \log n$, (64) is satisfied for sufficiently large n . Hence, all the conditions of Lemmas 1 and 2 are fulfilled. Furthermore, for L_0 sufficiently large, we have $e^{-\Delta/2} \leq n^{-3}$.

Applying Lemma 1 and averaging over $N_A(i)$ and $N_B(i)$ over both sides of (65), we get that

$$\mathbb{P} \{ \{Z_{ii} \geq \xi_{\text{true}}\} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii} \} \leq O \left(e^{-\Delta/2} \right).$$

By the union bound, we get that

$$\begin{aligned} \mathbb{P} \left\{ \max_{i \in [n]} Z_{ii} \geq \xi_{\text{true}} \right\} &\leq \sum_{i \in [n]} \left(\mathbb{P} \{ \{Z_{ii} \geq \xi_{\text{true}}\} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii} \} \right. \\ &\quad \left. + \mathbb{P} \{ \Gamma_A^c(i) \} + \mathbb{P} \{ \Gamma_B^c(i) \} + \mathbb{P} \{ \Gamma_{ii}^c \} + \mathbb{P} \{ \Theta_i^c \} + \mathbb{P} \{ \Theta_{ii}^c \} \right) \\ &\leq O \left(n^{-2} \right) + O \left(ne^{-\Delta/2} \right) \leq O \left(n^{-2} \right). \end{aligned} \quad (72)$$

where the second-to-the-last inequality holds due to (58), (61) and (63).

Similarly, for $i \neq k$, applying Lemma 2 and averaging over $N_A(i)$ and $N_B(k)$ over both hand sides of (68), we get that

$$\mathbb{P}\{\{Z_{ik} \leq \xi_{\text{fake}}\} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}\} \leq O\left(e^{-\Delta/2}\right).$$

By the union bound, we get that

$$\begin{aligned} & \mathbb{P}\left\{\min_{i \neq k \in [n]} Z_{ik} \leq \xi_{\text{fake}}\right\} \\ & \leq \sum_{i \neq k} (\mathbb{P}\{\{Z_{ik} \leq \xi_{\text{fake}}\} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}\} \\ & \quad + \mathbb{P}\{\Gamma_A^c(i)\} + \mathbb{P}\{\Gamma_B^c(k)\} + \mathbb{P}\{\Gamma_{ik}^c\} + \mathbb{P}\{\Theta_{ik}^c\}) \\ & \leq O(n^2) \times (e^{-\Delta/2} + n^{-3}) \leq O(n^{-1}), \end{aligned} \quad (73)$$

where the second-to-the-last inequality holds due to (58), (59), and (61).

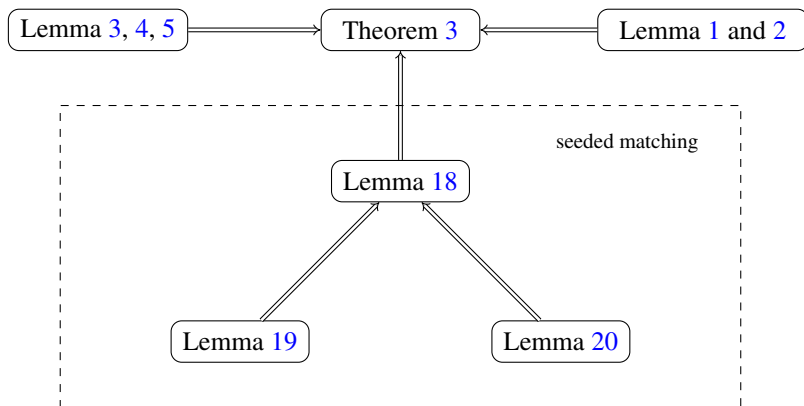
Finally, combining (72) and (73), we conclude that, with probability at least $1 - O(1/n)$,

$$\min_{i \neq k \in [n]} Z_{ik} \geq \xi_{\text{fake}} > \xi_{\text{true}} \geq \max_{i \in [n]} Z_{ii},$$

and hence Algorithm 1 succeeds. $\square$

4.2 Proof of Theorem 3

The proof of Theorem 3 is structured as follows:



We start with a few intermediate lemmas, whose proofs are postponed till Sect. 4.4. Recall that α is defined in (37) as

$$\alpha \triangleq \left(\alpha_0 \frac{\log n}{nq} \right)^{\frac{(1-p)s}{1-q}}$$

and τ is defined in (39) as

$$\tau \triangleq \min \{0 \leq k \leq n : \mathbb{P} \{ \text{Binom}(n-1, q) \geq k \} \leq \alpha \}.$$

Note that $\sigma^2 = 1 - s$ and $p = q/s$.

The first lemma bounds the correlations between the degree of vertex i in graph A and the degree of vertex k in graph B .

Lemma 3 Suppose $q \leq \frac{1}{8}$, $nq \rightarrow +\infty$, $1/(nq) \leq \alpha \leq 1/4$, and $\sigma^2 \log \log(nq) = o(1)$. Then

$$\mathbb{P} \{a_i \geq \tau, b_k \geq \tau + 1\} \begin{cases} \geq \Omega \left(\alpha^{\frac{1-q}{(1-p)s}} \right) & \text{if } i = k \\ \leq \alpha^2 & \text{o.w.} \end{cases} \quad (74)$$

We also need the following two auxiliary lemmas.

Lemma 4 Suppose $q \leq 1/8$, $1/(nq) \leq \alpha \leq \alpha_1$ for a sufficiently small constant $\alpha_1 > 0$, $nq \geq C_0 \Delta^2$, and $\Delta \geq C_0$ for a sufficiently large constant $C_0 > 0$. Let event Θ_{ik} be given in (60) as $\Theta_{ik} \triangleq \{|a_i - b_k| \leq 4\sqrt{nq\Delta}\}$. Then

$$\mathbb{P} \{ \{a_i \geq \tau, b_k \geq \tau + 1\} \cap \Theta_{ik}^c \} \leq O \left(\alpha^{1+\mathbf{1}_{\{i \neq k\}}} e^{-\Delta/2} \right). \quad (75)$$

Lemma 5 Let the event Θ_i be defined in (62). Then

$$\mathbb{P} \{ \{a_i \geq \tau, b_i \geq \tau + 1\} \cap \Theta_i^c \} \leq 2\alpha e^{-\Delta/2} + 2e^{-\Delta/(2\sigma^2)}. \quad (76)$$

Proof (Proof of Theorem 3) Recall that L is given in (38) as $L = L_0 \max \left\{ \log^{1/3}(n), \log \frac{\log n}{q} \right\}$. Choose Δ as per (70): $\Delta = \left(\frac{c_1}{4 \max\{c_2, \tau_2\}} \right)^2 L$ and set $\xi = \frac{3c_1}{4} \sqrt{\frac{L}{nq}}$, where c_1, c_2 are from Lemma 2 and τ_2 are from Lemma 1. Then ξ_{fake} in (69) satisfies $\xi_{\text{fake}} \triangleq c_1 \sqrt{\frac{L}{nq}} - c_2 \sqrt{\frac{\Delta}{nq}} \geq \xi$. Under the condition (36): $\sigma \leq \sigma_0 \min \left\{ \frac{1}{(\log n)^{1/3}}, \frac{1}{\log \frac{\log n}{q}} \right\}$, we have $\sigma L \leq \sigma_0 L_0$. Moreover, under the assumption (41): $nq^2 \geq C_0 \log^2 n$ for some large absolute constant C_0 , we have $nq \geq CL^2$ for a sufficiently large constant C . Thus, β in (67) satisfies $\beta L \leq c_1^2/32$. Moreover, $\tau_2 \sigma \leq \frac{1}{4}$ provided that σ is a sufficiently small constant. Hence, ξ_{true} in (66) satisfies $\xi_{\text{true}} \triangleq L \sqrt{\frac{2\beta}{nq}} + \tau_2 \sigma \sqrt{\frac{L}{nq}} + \tau_2 \sqrt{\frac{\Delta}{nq}} \leq \xi$.

For ease of notation, for each pair of $i, k \in [n]$, denote the event that $\mathcal{D}_{ik} = \{a_i \geq \tau, b_k \geq \tau + 1\}$. Then, for wrong pairs $i \neq k$,

$$\begin{aligned}
 & \mathbb{P}\{a_i \geq \tau, b_k \geq \tau + 1, Z_{ik} \leq \xi\} \\
 &= \mathbb{E}\left[\mathbb{P}\{Z_{ik} \leq \xi \mid N_A(i), N_B(k)\} \mathbf{1}_{\{\mathcal{D}_{ik}\}}\right] \\
 &\leq \mathbb{E}\left[\mathbb{P}\{Z_{ik} \leq \xi \mid N_A(i), N_B(k)\} \mathbf{1}_{\{\mathcal{D}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}\}}\right] \\
 &\quad + \mathbb{P}\{\mathcal{D}_{ik} \cap (\Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik})^c\} \\
 &\stackrel{(a)}{\leq} O\left(e^{-\Delta/2}\right) \mathbb{P}\{\mathcal{D}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}\} + \mathbb{P}\{\mathcal{D}_{ik} \cap \Gamma_A^c(i)\} \\
 &\quad + \mathbb{P}\{\mathcal{D}_{ik} \cap \Gamma_B^c(k)\} + \mathbb{P}\{\mathcal{D}_{ik} \cap \Gamma_{ik}^c\} + \mathbb{P}\{\mathcal{D}_{ik} \cap \Theta_{ik}^c\} \\
 &\leq O\left(e^{-\Delta/2}\right) \mathbb{P}\{\mathcal{D}_{ik}\} + \mathbb{P}\{\Gamma_A^c(i)\} + \mathbb{P}\{\Gamma_B^c(k)\} + \mathbb{P}\{\Gamma_{ik}^c\} + \mathbb{P}\{\Theta_{ik}^c\} \\
 &\stackrel{(b)}{\leq} O\left(\alpha^2 e^{-\Delta/2}\right) + O\left(n^{-3}\right),
 \end{aligned}$$

where (a) is due to Lemma 2 and $\xi_{\text{fake}} \geq \xi$; (b) is due to Lemma 3, Lemma 4, (58), and (59). Therefore, it follows from the union bound that

$$\begin{aligned}
 \mathbb{P}\{\exists(i, k) \in \mathcal{S} : i \neq k\} &\leq \sum_{i \neq k} \mathbb{P}\{a_i \geq \tau, b_k \geq \tau + 1, Z_{ik} \leq \xi\} \\
 &\leq O\left(n^2\right) \alpha^2 \exp(-\Delta/2) + O\left(n^{-1}\right) \\
 &\stackrel{(a)}{\leq} O\left(\alpha_0 \frac{\log n}{q}\right)^2 \exp\left(\frac{2\sigma^2}{1-q} \log n - \Omega(L)\right) + O\left(n^{-1}\right) \\
 &\stackrel{(b)}{=} O\left(e^{-\Omega(L)}\right) + O\left(n^{-1}\right),
 \end{aligned}$$

where (a) was previously explained in (45); (b) is due to the condition (36) on σ and the choice of L in (38).

For true pairs, let

$$T = \sum_{i \in [n]} \mathbf{1}_{\{a_i \geq \tau, b_i \geq \tau + 1, Z_{ii} \leq \xi\}}.$$

To show that $T = \Omega\left(\alpha_0 \frac{\log n}{q}\right)$ with high probability, we compute its first and second moment. Since Z_{ii} and the degrees a_i, b_i are dependent, one needs to be careful with respect to conditioning. Note that

$$\begin{aligned}
 & \mathbb{P}\{a_i \geq \tau, b_i \geq \tau + 1, Z_{ii} \leq \xi\} \\
 &= \mathbb{E}\left[\mathbb{P}\{Z_{ii} \leq \xi \mid N_A(i), N_B(i)\} \mathbf{1}_{\{\mathcal{D}_{ii}\}}\right] \\
 &\geq \mathbb{E}\left[\mathbb{P}\{Z_{ii} \leq \xi \mid N_A(i), N_B(i)\} \mathbf{1}_{\{\mathcal{D}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}\}}\right] \\
 &\geq \left(1 - O\left(e^{-\Delta/2}\right)\right) \mathbb{P}\{\mathcal{D}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}\}, \quad (77)
 \end{aligned}$$

where the last inequality holds due to Lemma 1 and $\xi_{\text{true}} \leq \xi$.

By Lemma 3,

$$t \triangleq \mathbb{P}\{a_i \geq \tau, b_i \geq \tau + 1\} = \mathbb{P}\{\mathcal{D}_{ii}\} \geq \Omega\left(\alpha^{\frac{1-q}{(1-p)s}}\right) \stackrel{(37)}{=} \Omega\left(\alpha_0 \frac{\log n}{nq}\right). \quad (78)$$

Combining Lemma 4 and Lemma 5 together with the union bound, we get that

$$\mathbb{P}\{\mathcal{D}_{ii} \cap (\Theta_i \cap \Theta_{ii})^c\} \leq O\left(\alpha e^{-\Delta/2} + e^{-\Delta/(2\sigma^2)}\right) \quad (79)$$

Combining the last two displayed equations yields that

$$\begin{aligned} & \mathbb{P}\{\mathcal{D}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}\} \\ & \geq \mathbb{P}\{\mathcal{D}_{ii}\} - \mathbb{P}\{\mathcal{D}_{ii} \cap (\Theta_i \cap \Theta_{ii})^c\} - \mathbb{P}\{\Gamma_A^c(i)\} - \mathbb{P}\{\Gamma_B^c(i)\} - \mathbb{P}\{\Gamma_{ii}\} \\ & \geq t - O\left(\alpha e^{-\Delta/2} + e^{-\Delta/(2\sigma^2)}\right) - 3n^{-3}, \end{aligned}$$

where in the last inequality we used $\mathbb{P}\{\Gamma_A^c(i)\}, \mathbb{P}\{\Gamma_B^c(i)\}, \mathbb{P}\{\Gamma_{ii}\} \leq 1/n^3$ by (58).

In view of the definition of α given in (37), we get that

$$\begin{aligned} \alpha e^{-\Delta/2} &= \left(\alpha_0 \frac{\log n}{nq}\right)^{\frac{(1-p)s}{1-q}} e^{-\Delta/2} \\ &\stackrel{(a)}{\leq} \alpha_0 \frac{\log n}{nq} \exp\left(\frac{\sigma^2}{1-q} \log n - \frac{\Delta}{2}\right) \\ &\stackrel{(b)}{=} \alpha_0 \frac{\log n}{nq} \exp\left(\frac{\sigma^2}{1-q} \log n - \Omega(L)\right) \\ &\stackrel{(c)}{=} O(t) e^{-\Omega(L)}, \end{aligned}$$

where (a) is by (43); (b) is due to $\Delta = \Omega(L)$ by our choice of Δ ; (c) holds because of (78) and the facts that $\sigma \leq \sigma_0/\log^{1/3}(n)$ in view of condition (36) and $L \geq L_0 \log^{1/3}(n)$ in view of (38).

Furthermore, by our choice of Δ and the theorem assumptions, $\Delta/\sigma^2 \geq 6 \log n$ by letting L_0/σ_0^2 sufficiently large. Combining this fact with the last two displayed equations, we get that

$$\mathbb{P}\{\mathcal{D}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}\} \geq t \left(1 - O\left(e^{-\Omega(L)}\right)\right) - 4n^{-3}. \quad (80)$$

By (77) and (80), we get that

$$\begin{aligned} \mathbb{E}[T] &\geq nt \left(1 - O\left(e^{-\Delta/2}\right)\right) \left(1 - O\left(e^{-\Omega(L)}\right)\right) - O(n^{-2}) \\ &= nt \left(1 - O\left(e^{-\Omega(L)}\right)\right) - O(n^{-2}), \end{aligned} \quad (81)$$

where the last equality holds because $\Delta = \Theta(L)$.

Next we estimate the second moment of T :

$$\begin{aligned}\mathbb{E}[T^2] &\leq \sum_{i,j} \mathbb{P}\{a_i \geq \tau, b_i \geq \tau + 1, a_j \geq \tau, b_j \geq \tau + 1\} \\ &= nt + \sum_{i \neq j} \mathbb{P}\{a_i \geq \tau, b_i \geq \tau + 1, a_j \geq \tau, b_j \geq \tau + 1\}.\end{aligned}$$

We will show that for $i \neq j$,

$$\mathbb{P}\{a_i \geq \tau, b_i \geq \tau + 1, a_j \geq \tau, b_j \geq \tau + 1\} \leq t^2 \left(1 + e^{-\Omega(L)}\right). \quad (82)$$

It then follows that

$$\mathbb{E}[T^2] \leq nt + n^2 t^2 \left(1 + e^{-\Omega(L)}\right). \quad (83)$$

Combining (81) and (83), we get that

$$\text{var}(T) = \mathbb{E}[T^2] - (\mathbb{E}[T])^2 \leq O\left(n^2 t^2 e^{-\Omega(L)} + nt\right)$$

and hence by Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}\left\{T \geq \frac{1}{2}nt\right\} &\leq \frac{\text{var}(T)}{(\mathbb{E}[T] - nt/2)^2} = O\left(e^{-\Omega(L)} + \frac{1}{nt}\right) \\ &= O\left(e^{-\Omega(L)} + \frac{q}{\log n}\right) = O\left(\frac{q}{\log n}\right),\end{aligned}$$

where the last two equalities holds because $nt = \Omega(\log n/q)$ and $L \geq L_0 \log \frac{\log n}{q}$ in view of (38). Therefore, the set $\mathcal{S}$ defines a partial matching with $|\mathcal{S}| = T \geq nt/2$ with probability $1 - O(q/\log n)$. Finally, the success of Algorithm 2 follows from applying the seeded graph matching result Lemma 18 given in Appendix B.

It remains to prove (82). Fix $i \neq j$. Recall that $\mathcal{D}_{ii}$ is the event that $a_i \geq \tau$ and $b_i \geq \tau + 1$. Also, let g_i denote the degree of vertex i in the parent graph. Abusing notation slightly, we let k denote the realization of g_i in the remainder of the proof. Then

$$\mathbb{P}\{\mathcal{D}_{ii} \cap \mathcal{D}_{jj}\} = \sum_{k,k'} \mathbb{P}\{g_i = k, g_j = k'\} \mathbb{P}\{\mathcal{D}_{ii} \mid g_i = k\} \mathbb{P}\{\mathcal{D}_{jj} \mid g_j = k'\}$$

and

$$\begin{aligned}\mathbb{P}\{g_i = k, g_j = k'\} &= p \cdot \mathbb{P}\{\text{Binom}(n-2, p) = k-1\} \mathbb{P}\{\text{Binom}(n-2, p) = k'-1\} \\ &\quad + (1-p) \mathbb{P}\{\text{Binom}(n-2, p) = k\} \mathbb{P}\{\text{Binom}(n-2, p) = k'\}.\end{aligned}$$

For ease of notation, we write $c_k \triangleq \mathbb{P}\{\text{Binom}(n-2, p) = k\}$. Then

$$\begin{aligned} & \mathbb{P}\{g_i = k, g_j = k'\} - \mathbb{P}\{g_i = k\}\mathbb{P}\{g_j = k'\} \\ &= pc_{k-1}c_{k'-1} + (1-p)c_k c_{k'} - (pc_{k-1} + (1-p)c_k)(pc_{k'-1} + (1-p)c_{k'}) \\ &= p(1-p)(c_{k-1} - c_k)(c_{k'-1} - c_{k'}). \end{aligned}$$

By definition,

$$\frac{c_{k-1} - c_k}{c_{k-1}} = \left(1 - \frac{(n-k-1)p}{k(1-p)}\right) = \frac{k - (n-1)p}{k(1-p)}$$

and

$$\frac{c_{k-1} - c_k}{c_k} = \left(\frac{k(1-p)}{(n-k-1)p} - 1\right) = \frac{k - (n-1)p}{(n-k-1)p}.$$

We let

$$\eta \triangleq \frac{\sqrt{3} \log(np)}{\sqrt{np}}$$

and $I \triangleq [(1-\eta)(n-1)p, (1+\eta)(n-1)p]$. Then for all $k \in I$, we have

$$\frac{|c_{k-1} - c_k|}{\min\{c_{k-1}, c_k\}} \leq \frac{\eta}{\min\{(1-\eta)(1-p), 1 - (1+\eta)p\}} \leq \frac{2\eta}{1-\eta},$$

where the last equality holds due to $p \leq 1/2$. Thus, for all $k, k' \in I$, we have

$$\mathbb{P}\{g_i = k, g_j = k'\} \leq \left(1 + \frac{4\eta^2}{(1-\eta)^2}\right) \mathbb{P}\{g_i = k\} \mathbb{P}\{g_j = k'\}.$$

Moreover, by Chernoff's bound given in (165),

$$\mathbb{P}\{g_i \notin I\} \leq 2 \exp(-\eta^2 np/3) = 2 \exp(-\log^2(np)).$$

Therefore,

$$\begin{aligned} & \mathbb{P}\{\mathcal{D}_{ii} \cap \mathcal{D}_{jj}\} \\ & \leq \mathbb{P}\{g_i \notin I\} + \mathbb{P}\{g_j \notin I\} + \sum_{k, k' \in I} \mathbb{P}\{g_i = k, g_j = k'\} \\ & \quad \mathbb{P}\{\mathcal{D}_{ii} \mid g_i = k'\} \mathbb{P}\{\mathcal{D}_{jj} \mid g_j = k'\} \\ & \leq 4 \exp(-\log^2(np)) + \left(1 + \frac{4\eta^2}{(1-\eta)^2}\right) \end{aligned}$$

$$\begin{aligned}
& \times \sum_{k,k'} \mathbb{P}\{g_i = k\} \mathbb{P}\{g_j = k'\} \mathbb{P}\{\mathcal{D}_{ii} \mid g_i = k\} \mathbb{P}\{\mathcal{D}_{jj} \mid g_j = k'\} \\
& = 4 \exp\left(-\log^2(np)\right) + \left(1 + \frac{4\eta^2}{(1-\eta)^2}\right) \mathbb{P}\{\mathcal{D}_{ii}\} \mathbb{P}\{\mathcal{D}_{jj}\} = \left(1 + e^{-\Omega(L)}\right) t^2,
\end{aligned}$$

where the last equality holds due to $\mathbb{P}\{\mathcal{D}_{ii}\} = t = \Omega(\log(n)/(nq))$ and $\eta^2 + \frac{1}{t^2} \exp(-\log^2(np)) = \exp(-\Omega(L))$ under the assumptions of Theorem 3.

4.3 Proof of Lemma 1 and Lemma 2

Note that for both the case of $i = k$ and $i \neq k$, the empirical distribution μ_i and ν_k will both involve correlated samples arising from common neighbors. So we start by decomposing the empirical distribution according to the common neighbors. Fix i, k . Recall that $c_{ik} = |N_A(i) \cap N_B(k)|$. Then

$$\mu_i = \frac{c_{ik}}{a_i} \left(\frac{1}{c_{ik}} \sum_{j \in N_A(i) \cap N_B(k)} \delta_{a_j^{(i)}} \right) + \left(1 - \frac{c_{ik}}{a_i} \right) \left(\frac{1}{a_i - c_{ik}} \sum_{j \in N_A(i) \setminus N_B(k)} \delta_{a_j^{(i)}} \right), \quad (84)$$

$$\nu_k = \frac{c_{ik}}{b_k} \left(\frac{1}{c_{ik}} \sum_{j \in N_A(i) \cap N_B(k)} \delta_{b_j^{(k)}} \right) + \left(1 - \frac{c_{ik}}{b_k} \right) \left(\frac{1}{b_k - c_{ik}} \sum_{j \in N_B(k) \setminus N_A(i)} \delta_{b_j^{(k)}} \right). \quad (85)$$

As a consequence, the centered empirical distribution can be rewritten as

$$\bar{\mu}_i = \rho P + (1 - \rho) P' \quad (86)$$

$$\bar{\nu}_k = \rho' Q + (1 - \rho') Q' \quad (87)$$

where

$$\rho \triangleq \frac{c_{ik}}{a_i}, \quad \rho' \triangleq \frac{c_{ik}}{b_k}$$

and

$$\begin{aligned}
P & \triangleq \frac{1}{c_{ik}} \sum_{j \in N_A(i) \cap N_B(k)} \delta_{a_j^{(i)}} - \nu, & P' & \triangleq \frac{1}{a_i - c_{ik}} \sum_{j \in N_A(i) \setminus N_B(k)} \delta_{a_j^{(i)}} - \nu, \\
Q & \triangleq \frac{1}{c_{ik}} \sum_{j \in N_A(i) \cap N_B(k)} \delta_{b_j^{(k)}} - \nu', & Q' & \triangleq \frac{1}{b_k - c_{ik}} \sum_{j \in N_B(k) \setminus N_A(i)} \delta_{b_j^{(k)}} - \nu',
\end{aligned}$$

and $\nu = \overline{\text{Binom}}(n - a_i - 1, q)$ and $\nu' = \overline{\text{Binom}}(n - b_k - 1, q)$. Note that if $c_{ik} = 0$, we set $P = Q = \text{Bin}(n - 1, q)$ by default.

The following lemmas are the key ingredients of the proof:

Lemma 6 (Independent two samples) *Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_{m'}$ be two independent sequence of real-valued random variables, where X_i 's are independently distributed as v_i and Y_i 's are independently distributed as v'_i . Assume that for some m_0 ,*

$$\kappa_1 \leq \frac{m}{m_0}, \frac{m'}{m_0} \leq \kappa_2$$

for some absolute constants $\kappa_1, \kappa_2 > 0$.

Suppose the partition $I_1, \dots, I_L$ is chosen so that there exists a set $J_0 \subset [m]$ with $|J_0| \geq m/4$ such that for all $i \in J_0$ and for all $\ell \in [L]$,

$$\frac{c_1}{L} \leq v_i(I_\ell) \leq \frac{c_2}{L} \quad (88)$$

for some absolute constants $c_1, c_2 \in (0, 1]$.

Given any two distributions v and v' on the real line, define $\pi = \frac{1}{m} \sum_{i=1}^m \delta_{X_i} - v$ and $\pi' = \frac{1}{m'} \sum_{i=1}^{m'} \delta_{Y_i} - v'$. Assume that $m_0 \geq CL$ and $L \geq L_0$ for some sufficiently large constants C, L_0 . Then for any $\Delta > 0$,

$$d(\pi, \pi') \geq \alpha_1 \sqrt{\frac{L}{m_0}} - \alpha_2 \sqrt{\frac{\Delta}{m_0}} \quad (89)$$

with probability at least $1 - e^{-\Delta}$, where d is the pseudo-distance defined in (28) with respect to the partition $I_1, \dots, I_L$, and α_1, α_2 are absolute constants.

Lemma 7 (Correlated two samples) *Let $(X_1, Y_1), \dots, (X_m, Y_m)$ be iid so that $X_i \sim v$ and $Y_i \sim v'$. Let $\pi = \frac{1}{m} \sum_{i=1}^m \delta_{X_i} - v$ and $\pi' = \frac{1}{m} \sum_{i=1}^m \delta_{Y_i} - v'$. Assume that for any $\ell \in [L]$,*

$$\mathbb{P}\{X_1 \in I_\ell, Y_1 \notin I_\ell\} + \mathbb{P}\{X_1 \notin I_\ell, Y_1 \in I_\ell\} \leq \beta. \quad (90)$$

Then for any $\Delta > 0$,

$$d(\pi, \pi') \leq L \sqrt{\frac{\beta}{m}} + c_3 \sqrt{\frac{\Delta}{m}} \quad (91)$$

with probability at least $1 - e^{-\Delta}$, where β is defined in (67) and c_3 is an absolute constant.

Remark 4 In Lemma 6, the samples X_i 's and Y_i 's need not be identically distributed, and v and v' can be arbitrary so that π and π' need not be centered (which is the case when we apply Lemma 6 for proving Lemmas 2 and 17). This is because Lemma 6 aims to lower bound the distance and centering tends to make the distance smaller. However, in Lemma 7 which bounds the distance from above, the samples are required to be iid and the empirical distributions must be correctly centered.

Lemma 8 (Concentration of total variation) *Let $X_1, \dots, X_m$ be drawn independently from a discrete distribution v supported on k elements. Then the empirical distribution $v_m = \frac{1}{m} \sum_{i=1}^m \delta_{X_i}$ satisfies that for any $\Delta > 0$,*

$$\mathbb{P} \left\{ \|v - v_m\|_1 \geq \sqrt{\frac{k}{m}} + \sqrt{\frac{\Delta}{m}} \right\} \leq e^{-\Delta/2}.$$

In order to apply Lemma 7, we need to quantify the correlation and upper bound the probability β in (90). This is given by the following (elementary but extremely tedious) lemma:

Lemma 9 *Assume that $\sigma \leq 1/2$, $q \leq \frac{1}{8}$, $nq \geq C \max\{L^2, \Delta\}$, and (64) holds, i.e., $4L\sqrt{nq}\Delta \leq n$. Then for any $j \in N_A(i) \cap N_B(i)$ and any interval $I \subset [-1/2, 1/2]$ with $|I| = 1/L$,*

$$\begin{aligned} & \left(\mathbb{P} \left\{ a_j^{(i)} \in I, b_j^{(i)} \notin I \mid N_A(i), N_B(i) \right\} + \mathbb{P} \left\{ a_j^{(i)} \notin I, b_j^{(i)} \in I \mid N_A(i), N_B(i) \right\} \right) \\ & \quad \mathbf{1}_{\{\Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}\}} \\ & \lesssim \sigma + \sqrt{\frac{\Delta}{n}} + \frac{1}{\sqrt{nq}} + \frac{1}{L} \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right) + e^{-\Delta}. \end{aligned} \quad (92)$$

Remark 5 Note that for the right hand side of (92) to be much smaller than $1/L$, it suffices to have $L \ll \min\{1/\sigma, \sqrt{n/\Delta}, \sqrt{nq}\}$ and $\Delta \gg \log L$.

4.3.1 Proof of Lemma 1

Proof (Proof of Lemma 1) Fix $i \in [n]$. Throughout the proof, we condition on the neighborhoods $N_A(i)$ and $N_B(i)$ such that $\Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}$ holds.

Recall the pseudo-distance d defined in (28), namely,

$$d(\mu, v) = \|[\mu]_L - [v]_L\|_1 \quad (93)$$

where $[\mu]_L$ is the discretized version of μ , defined in (29), according to the uniform partition $I_1, \dots, I_L$ of $[-1/2, 1/2]$ such that $|I_\ell| = 1/L$. Using the decomposition in (86)–(87) and the triangle inequality for the total variation distance, we have

$$\begin{aligned} Z_{ii} &= d(\rho P + (1 - \rho)P', \rho Q + (1 - \rho')Q' + (\rho' - \rho)Q) \\ &\leq \underbrace{d(P, Q)}_{(I)} + \underbrace{(1 - \rho)\| [P']_L \|_1 + (1 - \rho')\| [Q']_L \|_1}_{(II)} + \underbrace{|\rho - \rho'|}_{(III)}, \end{aligned} \quad (94)$$

where $\rho = \frac{c_{ii}}{a_i}$ and $\rho' = \frac{c_{ii}}{b_i}$.

For (I), in view of the assumption (64): $4L\sqrt{nq\Delta} \leq n$, Lemma 9 yields that for any $j \in N_A(i) \cap N_B(i)$ and any interval $I \subset [-1/2, 1/2]$ with $|I| = 1/L$,

$$\begin{aligned} & \mathbb{P} \left\{ a_j^{(i)} \in I, b_j^{(i)} \notin I \mid N_A(i), N_B(i) \right\} + \mathbb{P} \left\{ a_j^{(i)} \notin I, b_j^{(i)} \in I \mid N_A(i), N_B(i) \right\} \\ & \leq \beta \triangleq O(\sigma) + O \left(\sqrt{\frac{\Delta}{n}} + e^{-\Delta} \right) + \frac{1}{L} \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right). \end{aligned}$$

We apply Lemma 7 with $\{X_j\}_{j=1}^m$ given by $\{a_j^{(i)}\}_{j \in N_A(i) \cap N_B(i)}$, $\{Y_j\}_{j=1}^m$ given by $\{b_j^{(i)}\}_{j \in N_A(i) \cap N_B(i)}$, and $m = c_{ii} = |N_A(i) \cap N_B(i)|$. Recall that $a_j^{(i)}$ is a function of $\{A_{j\ell}\}_{\ell \in N_A^c[i]}$ and $b_{j'}^{(i)}$ is a function of $\{B_{j'\ell'}\}_{\ell' \in N_B^c[i]}$. For any $j \neq j' \in N_A(i) \cap N_B(i)$, it holds that $\{j, \ell\} \neq \{j', \ell'\}$. Hence, $(a_j^{(i)}, b_{j'}^{(i)})$'s are independently and identically distributed across different $j \in N_A(i) \cap N_B(i)$. Therefore, Lemma 7 yields that with probability at least $1 - e^{-\Delta}$,

$$d(P, Q) \leq L \sqrt{\frac{\beta}{c_{ii}}} + c_3 \sqrt{\frac{\Delta}{c_{ii}}} \leq L \sqrt{\frac{2\beta}{nq}} + c_3 \sqrt{\frac{2\Delta}{nq}}, \quad (95)$$

where $c_3 > 0$ is some absolute constant given in Lemma 7, and the last inequality holds due to $c_{ii} \geq nq/2$ by (56).

For (II), applying Lemma 8 with $k = L$ implies that $\| [P']_L \|_1 \leq \sqrt{\frac{L}{a_i - c_{ii}}} + \sqrt{\frac{\Delta}{a_i - c_{ii}}}$ and $\| [Q']_L \|_1 \leq \sqrt{\frac{L}{b_i - c_{ii}}} + \sqrt{\frac{\Delta}{b_i - c_{ii}}}$, each with probability at least $1 - e^{-\Delta/2}$. Therefore, by the union bound, with probability at least $1 - 2e^{-\Delta/2}$,

$$\begin{aligned} & (1 - \rho) \| [P']_L \|_1 + (1 - \rho') \| [Q']_L \|_1 \\ & \leq \frac{1}{a_i} (\sqrt{L} + \sqrt{\Delta}) \sqrt{a_i - c_{ii}} + \frac{1}{b_i} (\sqrt{L} + \sqrt{\Delta}) \sqrt{b_i - c_{ii}} \\ & \leq \frac{4}{nq} (\sqrt{L} + \sqrt{\Delta}) \left(\sqrt{nq\sigma^2} + \sqrt{\Delta} \right), \end{aligned} \quad (96)$$

where the last inequality holds due to $a_i, b_i \geq nq/2$ and $\sqrt{a_i - c_{ii}}, \sqrt{b_i - c_{ii}} \leq \sqrt{nq\sigma^2} + \sqrt{\Delta}$ on the event (55) and (62), respectively.

Finally, for (III),

$$|\rho - \rho'| = \frac{c_{ii}|a_i - b_i|}{a_i b_i} \leq \frac{|a_i - b_i|}{a_i} \leq 8 \sqrt{\frac{\Delta}{nq}}, \quad (97)$$

where the last inequality holds due to $|a_i - b_i| \leq 4\sqrt{nq\Delta}$ by (60).

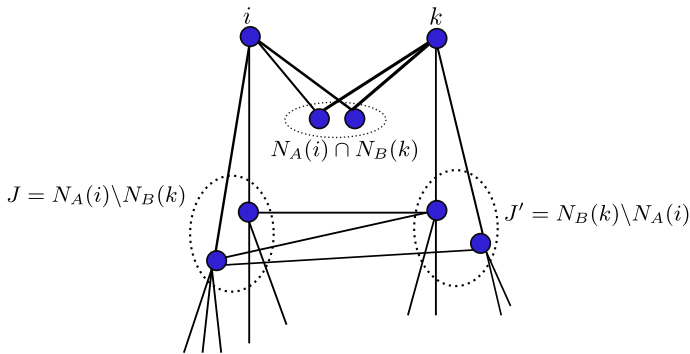


Fig. 1 Conditioned on the edge set $E_A(J, J')$ and $E_B(J, J')$, the outdegrees are independent

Combining (94) with (95), (96), (97), we get that with probability at least $1 - 3e^{-\Delta/2}$,

$$\begin{aligned} Z_{ii} &\leq L\sqrt{\frac{2\beta}{nq}} + c_3\sqrt{\frac{2\Delta}{nq}} + \frac{4}{nq} \left(\sqrt{L} + \sqrt{\Delta} \right) \left(\sqrt{nq\sigma^2} + \sqrt{\Delta} \right) + 8\sqrt{\frac{\Delta}{nq}} \\ &\leq L\sqrt{\frac{2\beta}{nq}} + \tau_2\sigma\sqrt{\frac{L}{nq}} + \tau_2\sqrt{\frac{\Delta}{nq}} \end{aligned}$$

for some absolute constant $\tau_2 > 0$, where the last inequality holds due to the assumption that $nq \geq C \max\{L^2, \Delta\}$ for some sufficiently large constant C . Thus we arrive at the desired (65). $\square$

4.3.2 Proof of Lemma 2

Proof (Proof of Lemma 2) Fix $i \neq k$. We proceed as in the proof of Lemma 1 and condition on the neighborhoods $N_A(i)$ and $N_B(k)$ such that $\Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \cap \Theta_{ik}$ holds.

By the triangle inequality for the total variation distance, we have

$$\begin{aligned} Z_{ik} &= d(\rho P + (1 - \rho)P', \rho Q + (1 - \rho)Q' + (\rho - \rho')(Q' - Q)) \\ &\geq \underbrace{(1 - \rho)d(P', Q')}_{\text{(I)}} - \underbrace{\rho d(P, Q)}_{\text{(II)}} - \underbrace{2|\rho - \rho'|}_{\text{(III)}}. \end{aligned} \quad (98)$$

where $\rho = \frac{c_{ik}}{a_i}$ and $\rho' = \frac{c_{ik}}{b_k}$.

For (I), note that $a_i, b_k \geq nq/2$ by (55), and $c_{ik} \leq nq/4$ for all $i \neq k$ by (57) and the assumptions that $nq \geq C \log n$ and $q \leq q_0$. Thus

$$\rho, \rho' \leq 1/2. \quad (99)$$

Let

$$J = N_A(i) \setminus N_B(k), \quad J' = N_B(k) \setminus N_A(i).$$

To analyze $d(P', Q')$, we aim to apply Lemma 6 with $m = |J|$, $m' = |J'|$, $m_0 = nq$, $\{X_j\}_{j=1}^m$ given by $\{a_j^{(i)}\}_{j \in J}$, and $\{Y_j\}_{j=1}^{m'}$ given by $\{b_j^{(k)}\}_{j \in J'}$. However, Lemma 6 is not directly applicable because the outdegrees are not independent due to the edges between nodes in J and J' (cf. Fig. 1). Indeed, note that $a_j^{(i)}$'s are independent across j , and $b_{j'}^{(k)}$'s are independent across j' , but $a_j^{(i)}$ and $b_{j'}^{(k)}$ are dependent, because $A_{jj'}$ contributes to the outdegree $a_j^{(i)}$, $B_{jj'}$ contributes to the outdegree $b_{j'}^{(k)}$, and $A_{jj'}$ are correlated with $B_{jj'}$. To deal with this dependency issue, define $E_A(J, J')$ as the set of edges between vertices in J and vertices in J' in A and let $e_A(J, J') = |E_A(J, J')|$. Similarly, define $E_B(J, J')$ and $e_B(J, J')$. Conditioned on the edge sets $E_A(J, J')$ and $E_B(J, J')$, the outdegrees $\{a_j^{(i)} : j \in J\}$ and $\{b_{j'}^{(k)} : j' \in J'\}$ are mutually independent (although not identically distributed as binomials). Indeed, let $\ell = |J \setminus \{k\}|$ and $\ell' = |J' \setminus \{i\}|$. Then

$$a_j^{(i)} = \frac{1}{\sqrt{(n - a_i - 1)q(1 - q)}} \left[e_A(j, N_A^c[i] \setminus J') - (n - a_i - 1 - \ell')q \right. \\ \left. + e_A(j, J' \setminus \{i\}) - \ell'q \right]$$

and

$$b_{j'}^{(k)} = \frac{1}{\sqrt{(n - b_k - 1)q(1 - q)}} \left[e_B(j', N_B^c[k] \setminus J) - (n - b_k - 1 - \ell)q \right. \\ \left. + e_B(j', J \setminus \{k\}) - \ell q \right]$$

Note that $\{e_A(j, N_A^c[i] \setminus J')\}_{j \in J}$ are independent from $\{e_B(j', N_B^c[k] \setminus J)\}_{j' \in J'}$.

For each $j \in J \setminus \{k\}$, define the indicator random variable

$$\mathcal{X}(j) = \mathbf{1}_{\{|e_A(j, J' \setminus \{i\}) - \ell'q| \leq \sqrt{nq(1-q)/2}\}}.$$

Let

$$J_0 = \{j \in J \setminus \{k\} : \mathcal{X}(j) = 1\} \quad (100)$$

Define the event

$$\mathcal{H} \triangleq \{|J_0| \geq m/4\}.$$

Note that for each $j \in J \setminus \{k\}$, $e_A(j, J' \setminus \{i\}) \sim \text{Binom}(\ell', q)$. Hence, by Chebyshev's inequality,

$$\mathbb{P}\{\mathcal{X}(j) = 1\} \geq 1 - \frac{2\ell'}{n} \geq 1/2,$$

where the last inequality holds because $\ell' \leq 2nq$ on the event $\Gamma_B(k)$ and $q \leq 1/8$. Moreover, $e_A(j, J' \setminus \{i\})$ are independent across $j \in J \setminus \{k\}$. Hence, $\sum_{j \in J} \mathcal{X}_A(j)$ is

stochastically lower bounded by $\text{Binom}(m-1, 1/2)$. It follows from the binomial tail bound (165) that

$$\mathbb{P}\{\mathcal{H}\} = \mathbb{P}\{|J_0| \geq m/4\} \geq 1 - e^{-m/32}.$$

We first condition on $(E_A(J, J'), E_B(J, J'))$ such that the event $\mathcal{H}$ holds and then apply Lemma 6. In view of (99), $m \geq a_i/2$ and $m' \geq b_k/2$ and thus

$$\frac{1}{4} \leq \frac{m}{m_0}, \frac{m'}{m_0} \leq 2.$$

Moreover, $nq \geq CL$ and $L \geq L_0$ by assumption. It remains to check the condition (88) in Lemma 6.

Let I denote any subinterval of $[-1/2, 1/2]$ with length $1/L$. Let

$$u_j = \frac{1}{\sqrt{(n-a_i-1-\ell')q(1-q)}} [e_A(j, N_A^c[i] \setminus J') - (n-a_i-1-\ell')q]$$

and

$$v_j = \frac{1}{\sqrt{(n-a_i-1)q(1-q)}} [e_A(j, J' \setminus \{i\}) - \ell'q].$$

Let

$$\alpha_j = \sqrt{\frac{n-a_i-1-\ell'}{n-a_i-1}}.$$

Then $a_j^{(i)} = \alpha_j u_j + v_j$. It follows that

$$\mathbb{P}\{a_j^{(i)} \in I\} = \mathbb{P}\left\{u_j \in \frac{I - v_j}{\alpha_j}\right\}.$$

Next we fix $j \in J_0$. Note that on event $\Gamma_A(i) \cap \Gamma_B(k)$, $a_i, \ell' \leq 2nq$. By the assumptions $q \leq 1/8$ and $n \geq 4$,

$$1 \geq \alpha_j \geq \sqrt{\frac{n-4nq-1}{n-2nq-1}} \geq \sqrt{\frac{1}{2}},$$

and, by the definition of J_0 ,

$$|v_j| \leq \frac{\sqrt{nq(1-q)/2}}{\sqrt{(n-2nq-1)q(1-q)}} \leq 1.$$

Hence, $(I - v_j)/\alpha_j \subset [-3, 3]$. It follows that

$$\frac{1}{\sqrt{2\pi}L}e^{-1/18} \leq \mathbb{P}\left\{N(0, 1) \in \frac{I - v_j}{\alpha_j}\right\} \leq \frac{1}{\sqrt{\pi}L}.$$

Note that $u_j \sim \overline{\text{Binom}}(n - a_i - 1 - \ell', q)$. By the Berry-Esseen theorem [47, Theorem 5.5], we have

$$\frac{1}{\sqrt{2\pi}L}e^{-1/18} - \frac{O(1)}{\sqrt{nq(1-q)}} \leq \mathbb{P}\left\{u_j \in \frac{I - v_j}{\alpha_j}\right\} \leq \frac{1}{\sqrt{\pi}L} + \frac{O(1)}{\sqrt{nq(1-q)}}.$$

In view of the assumption $nq \geq CL^2$ for a sufficiently large constant C , we have for all $j \in J_0$ and all $\ell \in [L]$,

$$\frac{c_1}{L} \leq \mathbb{P}\left\{a_j^{(i)} \in I\right\} \leq \frac{c_2}{L}.$$

for two absolute constants $c_1, c_2 \in [0, 1]$. Finally, recall that we have conditioned on $E(J, J')$ such that event $\mathcal{H}$ holds. Hence, $|J_0| \geq m/4$. Thus, condition (88) in Lemma 6 is satisfied.

In conclusions, the assumptions of Lemma 6 are all satisfied. Then it follows from Lemma 6 that

$$\mathbb{P}\left\{d(P', Q') \geq \alpha_1\sqrt{\frac{L}{nq}} - \alpha_2\sqrt{\frac{\Delta}{nq}} \mid E_A(J, J'), E_B(J, J')\right\} \geq (1 - e^{-\Delta}) \mathbf{1}_{\mathcal{H}}, \quad (101)$$

where α_1 and α_2 are absolute constants given in Lemma 6. Taking the expectation of $(E_A(J, J'), E_B(J, J'))$ over the both hand sides of the last display, we get that

$$\begin{aligned} \mathbb{P}\left\{d(P', Q') \geq \alpha_1\sqrt{\frac{L}{nq}} - \alpha_2\sqrt{\frac{\Delta}{nq}}\right\} &\geq (1 - e^{-\Delta}) \mathbb{P}\{\mathcal{H}\} \\ &\geq (1 - e^{-\Delta}) (1 - e^{-m/32}) \geq 1 - 2e^{-\Delta}, \end{aligned} \quad (102)$$

where the last inequality holds due to $m \geq nq/4 \geq C\Delta/4$ for a sufficiently large constant C .

For (II), Lemma 8 implies that $\|[P]_L\|_1 \leq \sqrt{\frac{L}{c_{ik}}} + \sqrt{\frac{\Delta}{c_{ik}}}$ holds with probability at least $1 - e^{-\Delta/2}$; similarly for $\|[Q]_L\|_1$. Thus, by the triangle inequality and union bound, with probability at least $1 - 2e^{-\Delta/2}$,

$$d(P, Q) \leq \|[P]_L\|_1 + \|[Q]_L\|_1 \leq 2\sqrt{\frac{L}{c_{ik}}} + 2\sqrt{\frac{\Delta}{c_{ik}}}.$$

Therefore,

$$\rho \cdot d(P, Q) \leq \frac{c_{ik}}{a_i} \left(2\sqrt{\frac{L}{c_{ik}}} + 2\sqrt{\frac{\Delta}{c_{ik}}} \right) \leq \frac{4}{nq} (\sqrt{L} + \sqrt{\Delta}) (\sqrt{nq^2} + \sqrt{2 \log n}) \quad (103)$$

where the last inequality holds due to $a_i \geq \frac{1}{2}nq$ and $\sqrt{c_{ik}} \leq \sqrt{nq^2} + \sqrt{2 \log n}$ on the event (55) and (57) respectively.

For (III),

$$|\rho - \rho'| = \frac{c_{ik}|a_i - b_k|}{a_i b_k} \leq \frac{|a_i - b_k|}{a_i} \leq 8\sqrt{\frac{\Delta}{nq}}, \quad (104)$$

where the last inequality holds due to (60).

Combining (98) with (99)–(104), we have that with probability at least $1 - 3e^{-\Delta/2}$,

$$\begin{aligned} Z_{ik} &\geq \frac{\alpha_1}{2} \sqrt{\frac{L}{nq}} - \alpha_2 \sqrt{\frac{\Delta}{nq}} - \frac{4}{nq} (\sqrt{L} + \sqrt{\Delta}) (\sqrt{nq^2} + \sqrt{2 \log n}) - 8\sqrt{\frac{\Delta}{nq}} \\ &\geq c_1 \sqrt{\frac{L}{nq}} - c_2 \sqrt{\frac{\Delta}{nq}}, \end{aligned}$$

for some absolute constants $c_1, c_2 > 0$, where the last inequality holds by the assumptions that $q \leq q_0$ and $nq \geq C \log n$ for some sufficiently small constant q_0 and sufficiently large constant C . $\square$

4.3.3 Proof of Lemma 6, 7, 8, and 9

Proof (Proof of Lemma 6) Recall from (28) that

$$d(\pi, \pi') = \sum_{\ell \in [L]} |\pi(I_\ell) - \pi'(I_\ell)|.$$

We first show that it suffices to establish

$$\mathbb{E}d(\pi, \pi') \geq c_0 \sqrt{\frac{L}{m_0}}. \quad (105)$$

To prove the concentration inequality (89), note that $d(\pi, \pi')$, as a function of the independent random variables $(X_1, \dots, X_m, Y_1, \dots, Y_{m'})$, satisfies the bounded difference property. Indeed, let

$$d(\pi, \pi') = f(X_1, \dots, X_m, Y_1, \dots, Y_{m'})$$

for some function f . Then for any i and any x_i, x'_i , we have, for some $\ell, \ell' \in [L]$,

$$\begin{aligned} & |f(x_1, \dots, x_i, \dots, x_m, y_1, \dots, y_{m'}) - f(x_1, \dots, x'_i, \dots, x_m, y_1, \dots, y_{m'})| \\ & \leq \left| |\pi(I_\ell) + \frac{1}{m} - \pi'(I_\ell)| + |\pi(I_{\ell'}) - \frac{1}{m} - \pi'(I_{\ell'})| \right. \\ & \quad \left. - |\pi(I_\ell) - \pi'(I_\ell)| - |\pi(I_{\ell'}) - \pi'(I_{\ell'})| \right| \\ & \leq \frac{2}{m}. \end{aligned} \quad (106)$$

Thus, f satisfies the bounded difference property with parameter $\frac{2}{m \wedge m'}$. By McDiarmid's inequality, we have

$$\mathbb{P} \left\{ d(\pi, \pi') \leq \mathbb{E}d(\pi, \pi') - c_1 \sqrt{\frac{\Delta}{m_0}} \right\} \leq e^{-\Delta},$$

where c_1 depends only on κ_1 and κ_2 .

It remains to show (105). For any $\ell \in [L]$,

$$\begin{aligned} \mathbb{E} [|\pi(I_\ell) - \pi'(I_\ell)|] &= \mathbb{E} \left[\left| \frac{1}{m} \sum_{i=1}^m \mathbf{1}_{\{X_i \in I_\ell\}} - \frac{1}{m'} \sum_{i=1}^{m'} \mathbf{1}_{\{Y_i \in I_\ell\}} - v(I_\ell) + v'(I_\ell) \right| \right] \\ &\geq \frac{1}{m} \inf_{x \in \mathbb{R}} \mathbb{E} \left[\left| \sum_{i \in J_0} \mathbf{1}_{\{X_i \in I_\ell\}} - x \right| \right], \end{aligned} \quad (107)$$

where the last inequality holds because X_i 's and Y_i 's are independent.

For $i \in J_0$, define $\alpha_i \triangleq \mathbb{P}\{X_i \in I_\ell\}$ and $\alpha \triangleq 1/L$. It follows from assumption (88) that $c_1\alpha \leq \alpha_i \leq c_2\alpha$ for two absolute constants $c_1, c_2 \in (0, 1]$. Therefore, we can write $\mathbf{1}_{\{X_i \in I_\ell\}} = W_i Z_i$, where $Z_i \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(\alpha)$ and W_i 's are independently distributed as $\text{Bern}(\eta_i)$ where $c_1 \leq \eta_i \leq c_2$. Let $T = \{i \in J_0 : W_i = 1\}$. Then for any $x \in \mathbb{R}$, conditional on T ,

$$\mathbb{E} \left[\left| \sum_{i \in J_0} \mathbf{1}_{\{X_i \in I_\ell\}} - x \right| \mid T \right] = \mathbb{E} \left[\left| \sum_{i \in T} Z_i - x \right| \right] \geq \mathbb{E} [|\text{Binom}(|T|, \alpha) - x_0|], \quad (108)$$

where x_0 is the median of $\text{Binom}(|T|, \alpha)$, which satisfies $|x_0 - |T|\alpha| \leq 1$ [26]. Using the estimate for the mean absolute deviation of binomial distribution (e.g. [6, Theorem 1]), we have

$$\mathbb{E} [|\text{Binom}(|T|, \alpha) - |T|\alpha|] \geq \frac{\sqrt{|T|\alpha(1-\alpha)}}{\sqrt{2}}, \quad \frac{1}{|T|} \leq \alpha \leq 1 - \frac{1}{|T|}.$$

By assumption, $L \geq L_0$ for some large constant L_0 . Thus if $|T| \geq 16L$, then $|T|\alpha(1-\alpha) \geq 8$. Hence, by triangle inequality,

$$\mathbb{E}[|\text{Binom}(|T|, \alpha) - x_0|] \geq \left(\frac{\sqrt{|T|\alpha(1-\alpha)}}{\sqrt{2}} - 1 \right) \mathbf{1}_{\{|T| \geq 16L\}} \geq \frac{\sqrt{|T|\alpha(1-\alpha)}}{2\sqrt{2}} \mathbf{1}_{\{|T| \geq 16L\}}$$

Therefore, combining the last displayed equation with (108), we get that for any $x \in \mathbb{R}$,

$$\mathbb{E} \left[\left| \sum_{i \in J_0} \mathbf{1}_{\{X_i \in I_\ell\}} - x \right| \mid T \right] \geq \frac{\sqrt{|T|\alpha(1-\alpha)}}{2\sqrt{2}} \mathbf{1}_{\{|T| \geq 16L\}}.$$

Taking expectation over T and then infimum over $x \in \mathbb{R}$ on both hand sides of the last displayed equation yields that

$$\inf_{x \in \mathbb{R}} \mathbb{E} \left[\left| \sum_{i \in J_0} \mathbf{1}_{\{X_i \in I_\ell\}} - x \right| \right] \geq \frac{\sqrt{\alpha(1-\alpha)}}{2\sqrt{2}} \mathbb{E} \left[\sqrt{|T|} \mathbf{1}_{\{|T| \geq 16L\}} \right]$$

It remains to bound $\mathbb{E} \left[\sqrt{|T|} \mathbf{1}_{\{|T| \geq 16L\}} \right]$ from the below. By assumption, it holds that $|J_0| \geq m/4$. Further, recall that W_i 's are independently distributed as $\text{Bern}(\eta_i)$ where $c_1 \leq \eta_i \leq c_2$. Hence $|T|$ is stochastically lower bounded by $U \sim \text{Binom}(m/4, c_1)$ and thus

$$\mathbb{E} \left[\sqrt{|T|} \mathbf{1}_{\{|T| \geq 16L\}} \right] \geq \mathbb{E} \left[\sqrt{U} \mathbf{1}_{\{U \geq 16L\}} \right] = \mathbb{E} \left[\sqrt{U} \right] - \mathbb{E} \left[\sqrt{U} \mathbf{1}_{\{U < 16L\}} \right].$$

Note that for any $y > 0$, $\sqrt{y} \geq 1 + (y-1)/2 - (y-1)^2/2$. Plugging $y = U/\mathbb{E}[U]$ and taking expectation, we get that

$$\mathbb{E} \left[\sqrt{U} \right] \geq \sqrt{\mathbb{E}[U]} \left(1 - \frac{\text{var}(U)}{2(\mathbb{E}[U])^2} \right) = \sqrt{mc_1/4} - \frac{1-c_1}{2\sqrt{mc_1/4}}.$$

Moreover,

$$\mathbb{E} \left[\sqrt{U} \mathbf{1}_{\{U < 16L\}} \right] \leq 4\sqrt{L} \mathbb{P}\{U < 16L\} \leq 4\sqrt{L} e^{-\Omega(m)},$$

where the last inequality follows from the Chernoff bound (165) and the fact that $m \geq \kappa_1 m_0 \geq \kappa_1 CL$ for some large constant C . Combining the last four displays, we have that

$$\inf_{x \in \mathbb{R}} \mathbb{E} \left[\left| \sum_{i \in J_0} \mathbf{1}_{\{X_i \in I_\ell\}} - x \right| \right] \geq c_3 \sqrt{\frac{m}{L}},$$

for some absolute constant c_3 . Combining the last display with (107), we get that

$$\mathbb{E}[|\pi(I_\ell) - \pi'(I_\ell)|] \geq c_3 \sqrt{\frac{1}{mL}}.$$

Summing over $\ell \in [L]$ and noting that $m \geq \kappa_1 m_0$ yields (105). $\square$

Proof (Proof of Lemma 7) Similar to the proof of Lemma 6, observe that $d(\pi, \pi')$ is a function of the independent randomness $(X_1, Y_1), \dots, (X_m, Y_m)$ satisfying the bounded difference property with parameter $\frac{4}{m}$. Thus, by McDiarmid's inequality, to show (91), it suffices to show

$$\mathbb{E}d(\pi, \pi') \leq L \sqrt{\frac{\beta}{m}}. \quad (109)$$

Note that

$$\mathbb{E}[d(\pi, \pi')] = \sum_{\ell=1}^L \mathbb{E}[|\pi(I_\ell) - \pi'(I_\ell)|]$$

and

$$\pi(I_\ell) - \pi'(I_\ell) = \frac{1}{m} \sum_{i=1}^m [(\mathbf{1}_{\{X_i \in I_\ell\}} - \mathbf{1}_{\{Y_i \in I_\ell\}}) - (\mathbb{P}\{X_i \in I_\ell\} - \mathbb{P}\{Y_i \in I_\ell\})].$$

For each i ,

$$\mathbf{1}_{\{X_i \in I_\ell\}} - \mathbf{1}_{\{Y_i \in I_\ell\}} = \begin{cases} 1 & \text{w.p. } \mathbb{P}\{X_i \in I_\ell, Y_i \notin I_\ell\} \\ -1 & \text{w.p. } \mathbb{P}\{X_i \notin I_\ell, Y_i \in I_\ell\} \\ 0 & \text{o.w.} \end{cases}$$

Hence,

$$\begin{aligned} & \mathbb{E}[|\pi(I_\ell) - \pi'(I_\ell)|] \\ & \leq \sqrt{\mathbb{E}[(\pi(I_\ell) - \pi'(I_\ell))^2]} \\ & = \frac{1}{m} \sqrt{\mathbb{E}\left[\left(\sum_{i=1}^m (\mathbf{1}_{\{X_i \in I_\ell\}} - \mathbf{1}_{\{Y_i \in I_\ell\}} - \mathbb{P}\{X_i \in I_\ell\} + \mathbb{P}\{Y_i \in I_\ell\})\right)^2\right]} \\ & = \frac{1}{m} \sqrt{\sum_{i=1}^m \mathbb{E}[(\mathbf{1}_{\{X_i \in I_\ell\}} - \mathbf{1}_{\{Y_i \in I_\ell\}} - \mathbb{P}\{X_i \in I_\ell\} + \mathbb{P}\{Y_i \in I_\ell\})^2]} \\ & = \frac{1}{\sqrt{m}} \sqrt{\mathbb{P}\{X_1 \in I_\ell, Y_1 \notin I_\ell\} + \mathbb{P}\{X_1 \notin I_\ell, Y_1 \in I_\ell\} - (\mathbb{P}\{X_1 \in I_\ell, Y_1 \notin I_\ell\} - \mathbb{P}\{X_1 \notin I_\ell, Y_1 \in I_\ell\})^2} \\ & \leq \frac{1}{\sqrt{m}} \sqrt{\mathbb{P}\{X_1 \in I_\ell, Y_1 \notin I_\ell\} + \mathbb{P}\{X_1 \notin I_\ell, Y_1 \in I_\ell\}} \leq \sqrt{\frac{\beta}{m}}. \end{aligned}$$

Summing over $\ell \in [L]$ gives the desired (109). $\square$

Proof (Proof of Lemma 8) Let v be supported on the set $\{a_1, \dots, a_k\}$ with $v_i = v(\{a_i\})$. Then by Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E} \|v - \widehat{v}\|_1 &= \frac{1}{m} \sum_{i=1}^k \mathbb{E} \left| \sum_{j=1}^m (\mathbf{1}_{\{X_j=a_i\}} - v_i) \right| \\ &\leq \frac{1}{m} \sum_{i=1}^k \sqrt{\mathbb{E} \left(\sum_{j=1}^m (\mathbf{1}_{\{X_j=a_i\}} - v_i) \right)^2} \\ &= \frac{1}{m} \sum_{i=1}^k \sqrt{m v_i (1 - v_i)} \leq \sum_{i=1}^k \sqrt{\frac{v_i}{m}} \leq \sqrt{\frac{k}{m}}, \end{aligned}$$

where the last inequality follows from Jensen's inequality. Note that $(X_1, \dots, X_m) \mapsto \|v - \widehat{v}\|_1$ satisfies the bounded difference property with parameter $\frac{2}{m}$. Thus, by McDiarmid's inequality, we have

$$\mathbb{P} \left\{ \|v - \widehat{v}\|_1 \geq \sqrt{\frac{L}{m}} + \sqrt{\frac{\Delta}{m}} \right\} \leq e^{-\frac{2\Delta/m}{m(2/m)^2}} = e^{-\Delta/2}.$$

□

Proof (Proof of Lemma 9) Let us suppress i and j , and abbreviate $a_j^{(i)}$ and $b_j^{(i)}$ as a and b . Throughout the proof, we condition on $N_A[i] = S$ and $N_B[i] = T$ such that event $\Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \cap \Theta_i \cap \Theta_{ii}$ holds, and aim to show that

$$\begin{aligned} &\mathbb{P} \{a \in I, b \notin I\} \\ &\lesssim \sigma + \sqrt{\frac{\Delta}{n}} + \frac{1}{\sqrt{nq}} + \frac{1}{L} \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right) + e^{-\Delta}. \end{aligned} \quad (110)$$

The second probability in (92) follows from the same bound.

Define

$$\zeta = \sqrt{(n - |S|)q(1 - q)} \quad \text{and} \quad \eta = \sqrt{(n - |T|)q(1 - q)}. \quad (111)$$

Recall that on the event Θ ,

$$|S \cup T| \leq |S| + |T| \leq 4nq \leq n/2, \quad (112)$$

where the last inequality holds due to $q \leq 1/8$. Hence,

$$\sqrt{nq}/2 \leq \zeta, \eta \leq \sqrt{nq}. \quad (113)$$

Then we can rewrite a and b as

$$a = \frac{1}{\zeta} \sum_{k \notin S} (\alpha_k g_k - q), \quad (114)$$

$$b = \frac{1}{\eta} \sum_{k \notin T} (\beta_k g_k - q), \quad (115)$$

where g_k 's are iid as $\text{Bern}(p)$ and α_k, β_k 's are iid as $\text{Bern}(s)$. Recall that $\sigma^2 = 1 - s$ and $p = q/s$.

Define

$$E = \{k \notin S : \alpha_k = 1\} \quad \text{and} \quad F = \{k \notin T : \beta_k = 1\}.$$

Then we can decompose a and b as

$$a = \frac{1}{\zeta} (c + x) \quad \text{and} \quad b = \frac{1}{\eta} (c + y),$$

where

$$\begin{aligned} c &= \sum_{k \in E \cap F} (g_k - p) \\ x &= \sum_{k \in E \setminus F} (g_k - p) + p|E| - (n - |S|)q \quad \text{and} \\ y &= \sum_{k \in F \setminus E} (g_k - p) + p|F| - (n - |T|)q. \end{aligned} \quad (116)$$

Conditional on $\{E, F\}$, c, x, y are mutually independent.

We pause to give some intuition behind the remaining argument. Loosely speaking, the quantity c captures the correlation between the outdegrees a and b , while x and y correspond to the fluctuations. A key step of the proof is to relate the event $\{a \in I, b \notin I\}$ to the event that c belongs to an interval of length roughly $|x - y|$. We further show that $|x - y|$ is typically $O(\sqrt{np}\sigma)$. Coupled with the anti-concentration of c (the maximum probability mass of which is at most $O(1/\sqrt{np})$), this shows that c belongs to an interval of length $|x - y|$ with probability at most $O(\sigma)$, giving rise to the first (main) term in the upper bound (110). The complication comes from the fact that we also need to control the large deviation behavior of $|x - y|$, the mismatches between the normalization factors ζ and η , as well as the atypical behavior of E, F .

Returning to the main proof, note that

$$\begin{aligned} E \cap F &= \{k \in S^c \cap T^c : \alpha_k = \beta_k = 1\} \\ E \setminus F &= \{k \in S^c \setminus T^c : \alpha_k = 1\} \cup \{k \in S^c \cap T^c : \alpha_k = 1, \beta_k = 0\} \\ F \setminus E &= \{k \in T^c \setminus S^c : \beta_k = 1\} \cup \{k \in S^c \cap T^c : \alpha_k = 0, \beta_k = 1\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |E \cap F| &\sim \text{Bin}(|S^c \cap T^c|, s^2) \\ |E \setminus F| &\sim \text{Bin}(|S^c \setminus T^c|, s) + \text{Bin}(|S^c \cap T^c|, s(1-s)) \\ |F \setminus E| &\sim \text{Bin}(|T^c \setminus S^c|, s) + \text{Bin}(|S^c \cap T^c|, s(1-s)). \end{aligned}$$

Recall that on event $\Gamma_A(i) \cap \Gamma_B(i) \cap \Theta_i$,

$$\begin{aligned} |S^c \cap T^c| &= n - |S \cup T| \geq n/2 \\ |S^c \setminus T^c| = |T \setminus S| &\leq \left(\sqrt{n(1-s)} + \sqrt{\Delta}\right)^2 \\ |T^c \setminus S^c| = |S \setminus T| &\leq \left(\sqrt{n(1-s)} + \sqrt{\Delta}\right)^2. \end{aligned}$$

Define

$$\tau_1 = \left(\sqrt{n(1-s)} + 2\sqrt{\Delta}\right)^2 + \left(\sqrt{n(1-s)} + \sqrt{\Delta}\right)^2 \quad \text{and} \quad \tau_2 = \left(\sqrt{ns^2/2} - \sqrt{\Delta}\right)^2$$

and the event

$$\mathcal{E} = \{|E \cap F| \geq \tau_2\} \cap \{|E \setminus F| \leq \tau_1\} \cap \{|F \setminus E| \leq \tau_1\}.$$

Then by binomial tail bounds (167) and (168), we have $\mathbb{P}\{\mathcal{E}^c\} \leq e^{-\Delta} + 4e^{-2\Delta} \leq 5e^{-\Delta}$. Moreover, we have that

$$\tau_1 \leq 4n(1-s) + 10\Delta. \quad (117)$$

Also, in view of the assumption $\sigma \leq 1/2$ so that $s \geq 3/4$, we have that

$$\tau_2 \geq \left(\frac{3}{4}\sqrt{n/2} - \sqrt{\Delta}\right)^2 \geq n/4, \quad (118)$$

where the last inequality holds for sufficiently large n due to $nq \geq C\Delta$.

Note that

$$\begin{aligned} &\mathbb{P}\{a \in I, b \notin I\} \\ &= \mathbb{E}_{E,F}[\mathbb{P}\{a \in I, b \notin I \mid E, F\}] \\ &= \mathbb{E}_{E,F}[\mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}}] + \mathbb{E}_{E,F}[\mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}^c}] \\ &\leq \mathbb{E}_{E,F}[\mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}}] + \mathbb{P}\{\mathcal{E}^c\} \\ &\leq \mathbb{E}_{E,F}[\mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}}] + 5e^{-\Delta}. \end{aligned} \quad (119)$$

Hence, it remains to bound $\mathbb{E}_{E,F} [\mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}}]$. Note that

$$\begin{aligned} \mathbb{P}\{a \in I, b \notin I \mid E, F\} \mathbf{1}_{\mathcal{E}} &= \mathbb{P}\{c \in \zeta I - x, c \notin \eta I - y \mid E, F\} \mathbf{1}_{\mathcal{E}} \\ &= \mathbb{E}_{x,y} [\mathbb{P}\{c \in (\zeta I - x) \setminus (\eta I - y) \mid E, F, x, y\} \mathbf{1}_{\mathcal{E}}] \end{aligned}$$

Next consider the following two cases by assuming $I = [l, r]$ with $-1/2 \leq l \leq r \leq 1/2$.

- Case 1: Either $\zeta r - x \leq \eta l - y$ or $\eta r - y \leq \zeta l - x$. In this case, we have $(\zeta I - x) \cap (\eta I - y) = \emptyset$. Thus, we have

$$\mathbb{P}\{c \in (\zeta I - x) \setminus (\eta I - y) \mid S, T, E, F, x, y\} \stackrel{(a)}{\lesssim} \frac{1}{\sqrt{\tau_2 p}} \left(\frac{\zeta}{L} + 1 \right) \lesssim \frac{1}{L},$$

where (a) holds because the maximum probability mass of c is $\Theta(1/\sqrt{|E \cap F|p})$, and the number of integral points in $\zeta I - x$ is at most $\zeta/L + 1$; the last inequality holds because $\tau_2 \geq n/4$ in view of (118), $\zeta \leq \sqrt{nq}$ in view of (113), and $nq \geq CL^2$ for a sufficiently large constant C .

- Case 2: $\zeta r - x \geq \eta l - y$ and $\eta r - y \geq \zeta l - x$. In this case, we have $(\zeta I - x) \cap (\eta I - y) \neq \emptyset$. Moreover,

$$(\zeta I - x) \setminus (\eta I - y) \subset [\zeta l - x, \eta l - y] \cup [\eta r - y, \zeta r - x].$$

Hence,

$$\begin{aligned} |(\zeta I - x) \setminus (\eta I - y)| &\leq |x - y + (\eta - \zeta)l| + |y - x + (\zeta - \eta)r| \\ &\leq 2|x - y| + |\eta - \zeta|, \end{aligned}$$

where the last inequality follows from the triangle inequality and the assumption that $-1/2 \leq l \leq r \leq 1/2$. Thus,

$$\mathbb{P}\{c \in (\zeta I - x) \setminus (\eta I - y) \mid S, T, E, F, x, y\} \lesssim \frac{1}{\sqrt{np}} (2|x - y| + |\eta - \zeta| + 1),$$

where the last step holds because the maximum probability mass of c is $\Theta(1/\sqrt{|E \cap F|p})$, $|E \cap F| \geq \tau_2 \geq n/4$ in view of (118), and the number of integral points in $(\zeta I - x) \setminus (\eta I - y)$ is at most $2|x - y| + |\eta - \zeta| + 1$.

Combining the above two cases, we get that

$$\begin{aligned} &\mathbb{P}\{c \in (\zeta I - x) \setminus (\eta I - y) \mid E, F, x, y\} \mathbf{1}_{\mathcal{E}} \\ &\lesssim \left(\frac{1}{\sqrt{np}} (|x - y| + |\eta - \zeta| + 1) \right. \\ &\quad \left. + \frac{1}{L} \mathbf{1}_{\{x-y \in [\zeta r - \eta l, +\infty) \cup (-\infty, \zeta l - \eta r]\}} \right) \mathbf{1}_{\mathcal{E}}. \end{aligned}$$

Taking expectation of x, y over both hand sides of the last displayed equation, we get that

$$\begin{aligned} \mathbb{P}\{a \in I, b \notin I | E, F\} \mathbf{1}_{\mathcal{E}} &\lesssim \left(\frac{1}{\sqrt{np}} (\mathbb{E}[|x - y| | E, F] + |\eta - \zeta| + 1) \right. \\ &\quad \left. + \frac{1}{L} \mathbb{P}\{x - y \in [\zeta r - \eta l, +\infty) \cup (-\infty, \zeta l - \eta r] | E, F\} \right) \mathbf{1}_{\mathcal{E}}. \end{aligned}$$

Further taking expectation of E, F over both hand sides of the last displayed equation, we get that

$$\mathbb{E}_{E,F} [\mathbb{P}\{a \in I, b \notin I | E, F\} \mathbf{1}_{\mathcal{E}}] \quad (120)$$

$$\lesssim \frac{1}{\sqrt{np}} \mathbb{E}_{E,F} [\mathbb{E}[|x - y| | E, F] \mathbf{1}_{\mathcal{E}}] \quad (121)$$

$$+ \frac{|\eta - \zeta| + 1}{\sqrt{np}} \quad (122)$$

$$+ \frac{1}{L} \mathbb{E}_{E,F} [\mathbb{P}\{x - y \in [\zeta r - \eta l, +\infty) \cup (-\infty, \zeta l - \eta r] | E, F\} \mathbf{1}_{\mathcal{E}}]. \quad (123)$$

Next we upper bound the three terms (121), (122), and (123) separately.

Upper bound (121):

$$\begin{aligned} \mathbb{E}[|x| | E, F] \mathbf{1}_{\mathcal{E}} &\leq \mathbb{E} \left[\sum_{k \in E \setminus F} (g_k - p) | E, F \right] \mathbf{1}_{\mathcal{E}} + |p|E| - (n - |S|)q| \\ &\leq \sqrt{|E \setminus F|p(1-p)} \mathbf{1}_{\mathcal{E}} + \left| p|E| - (n - |S|)q \right| \\ &\leq \sqrt{\tau_1 p(1-p)} + \left| p|E| - (n - |S|)q \right| \\ &\lesssim \sqrt{np(1-s)} + \sqrt{p\Delta} + p \left| |E| - (n - |S|)s \right|, \end{aligned}$$

where the last inequality holds due to $\tau_1 \lesssim n(1-s) + \Delta$ in (117). It follows that

$$\begin{aligned} \mathbb{E}_{E,F} [\mathbb{E}[|x| | E, F] \mathbf{1}_{\mathcal{E}}] &\lesssim \sqrt{np(1-s)} + \sqrt{p\Delta} + p\sqrt{(n - |S|)s(1-s)} \\ &\lesssim \sqrt{np(1-s)} + \sqrt{p\Delta}, \end{aligned}$$

where the first inequality uses the fact that $|E| \sim \text{Binom}(n - |S|, s)$ and hence $\mathbb{E}[||E| - (n - |S|)s|] \leq \sqrt{\mathbb{E}[(|E| - (n - |S|)s)^2]} = \sqrt{(n - |S|)s(1-s)}$. Similarly,

$$\mathbb{E}_{E,F} [\mathbb{E}[|y| | E, F] \mathbf{1}_{\mathcal{E}}] \lesssim \sqrt{np(1-s)} + \sqrt{p\Delta}.$$

Therefore, by triangle inequality,

$$\mathbb{E}_{E,F} [\mathbb{E} [|x - y| \mid E, F] \mathbf{1}_{\mathcal{E}}] \lesssim \sqrt{np(1-s)} + \sqrt{p\Delta}. \quad (124)$$

Upper bound (122): In view of definitions of ζ and η in (111),

$$\begin{aligned} |\eta - \zeta| &= \sqrt{q(1-q)} \left| \sqrt{n - |S|} - \sqrt{n - |T|} \right| \\ &\leq \sqrt{q} \frac{||T| - |S||}{\sqrt{n - |S|} + \sqrt{(n - |T|)}} \\ &\lesssim q\sqrt{\Delta}, \end{aligned} \quad (125)$$

where the last inequality holds because on event $\Gamma_A(i) \cap \Gamma_B(i) \cap \Theta_{ii}$, $|S \cup T| \leq n/2$ and $||T| - |S|| \leq 4\sqrt{nq\Delta}$.

Upper bound (123): It follows from the last displayed equation that

$$\left| \frac{\eta}{\zeta} - 1 \right| \leq \frac{||T \setminus S| - |S \setminus T||}{2(n - |S \cup T|)} \leq \frac{4\sqrt{nq\Delta}}{n} \leq \frac{1}{L},$$

where the last inequality holds by the assumption (64), i.e., $4L\sqrt{nq\Delta} \leq n$. As a consequence,

$$\zeta r - \eta l = \zeta(r - l) + (\zeta - \eta)l \geq \zeta \left(\frac{1}{L} - \frac{1}{2} \left| \frac{\eta}{\zeta} - 1 \right| \right) = \frac{\zeta}{2L}.$$

Similarly, $\eta r - \zeta l \geq \frac{\zeta}{2L}$. Therefore,

$$\begin{aligned} &\mathbb{P} \left\{ x - y \in [\zeta r - \eta l, +\infty) \cup (-\infty, \zeta l - \eta r] \mid E, F \right\} \\ &\leq \mathbb{P} \left\{ |x| \geq \frac{\zeta}{4L} \mid E, F \right\} + \mathbb{P} \left\{ |y| \geq \frac{\zeta}{4L} \mid E, F \right\}. \end{aligned} \quad (126)$$

Recall the definition of x in (116),

$$\begin{aligned} \mathbb{P} \left\{ |x| \geq \frac{\zeta}{4L} \mid E, F \right\} \mathbf{1}_{\mathcal{E}} &\leq \mathbb{P} \left\{ \sum_{k \in E \setminus F} (g_k - p) \geq \frac{\zeta}{8L} \mid E, F \right\} \\ &\quad \mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\{|p|E| - (n - |S|)q \geq \zeta/(8L)\}}. \end{aligned}$$

By Bernstein's inequality,

$$\begin{aligned} \mathbb{P} \left\{ \sum_{k \in E \setminus F} (g_k - p) \geq \frac{\zeta}{8L} \mid E, F \right\} \mathbf{1}_{\mathcal{E}} &\leq \exp \left(-\Omega \left(\min \left\{ \frac{\zeta^2}{|E \setminus F| L^2 p}, \frac{\zeta}{L} \right\} \right) \right) \mathbf{1}_{\mathcal{E}} \\ &\leq \exp \left(-\Omega \left(\min \left\{ \frac{n}{(n(1-s) + \Delta) L^2}, \frac{\sqrt{np}}{L} \right\} \right) \right), \end{aligned}$$

where the last inequality holds because $\zeta \geq \sqrt{nq}/2$ in (113), $s \geq 3/4$, and $|E \setminus F| \leq \tau_1 \lesssim n(1-s) + \Delta$ on the event $\mathcal{E}$ in view of (117). By Bernstein's inequality again,

$$\begin{aligned} \mathbb{P} \{ |p|E| - (n - |S|)q \geq \zeta/(8L) \} &\leq \exp \left(-\Omega \left(\min \left\{ \frac{\zeta^2}{n(1-s)L^2 p^2}, \frac{\zeta}{Lp} \right\} \right) \right) \\ &\leq \exp \left(-\Omega \left(\min \left\{ \frac{1}{(1-s)L^2 p}, \frac{\sqrt{n}}{L\sqrt{p}} \right\} \right) \right), \end{aligned}$$

where the last inequality holds because $\zeta \geq \sqrt{nq}/2$ in (113) and $s \geq 3/4$. Combining the last three displayed equations yields that

$$\mathbb{E}_{E,F} \left[\mathbb{P} \left\{ |x| \geq \frac{\zeta}{4L} \mid E, F \right\} \mathbf{1}_{\mathcal{E}} \right] \leq \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right),$$

where we used $\sigma^2 = 1 - s$. Similarly,

$$\mathbb{E}_{E,F} \left[\mathbb{P} \left\{ |y| \geq \frac{\zeta}{4L} \mid E, F \right\} \mathbf{1}_{\mathcal{E}} \right] \leq \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right).$$

Combining the last two displayed equation with (126), we get that

$$\begin{aligned} &\mathbb{E}_{E,F} [\mathbb{P} \{ x - y \in [\zeta r - \eta l, +\infty) \cup (-\infty, \zeta l - \eta r] \mid E, F \} \mathbf{1}_{\mathcal{E}}] \\ &\leq \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right). \end{aligned} \quad (127)$$

Assembling (119), (120), (124), (125), and (127), we arrive at the desired bound (110):

$$\begin{aligned} &\mathbb{E}_{E,F} [\mathbb{P} \{ a \in I, b \notin I \mid E, F \} \mathbf{1}_{\mathcal{E}}] \\ &\lesssim \sigma + \sqrt{\frac{\Delta}{n}} + \frac{1}{\sqrt{nq}} + \frac{1}{L} \exp \left(-\Omega \left(\min \left\{ \frac{1}{\sigma^2 L^2}, \frac{n}{L^2 \Delta}, \frac{\sqrt{np}}{L} \right\} \right) \right) + e^{-\Delta}. \end{aligned}$$

□

4.4 Proof of Lemma 3, 4, and 5

To prove Lemma 3 and Lemma 4, we need a few auxiliary lemmas.

First, we need the following tight Gaussian approximation results for the binomial distributions [57, Theorem 1]: Let $D(p||q) \triangleq p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}$ denote the Kullback-Leibler divergence between $\text{Bern}(p)$ and $\text{Bern}(q)$.

Lemma 10 Assume that $k \geq nq + 1$. Then

$$h(k) \leq \mathbb{P}\{\text{Binom}(n, q) \geq k\} \leq h(k-1). \quad (128)$$

where

$$h(k) \triangleq Q\left(\sqrt{2nD\left(\frac{k}{n}||q\right)}\right),$$

and $Q(t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the standard normal tail probability.

Also, we need the following bounds on the Kullback-Leibler divergence:

Lemma 11 It holds that

$$D(x||q) \geq \frac{(x-q)^2}{2x(1-q)} \quad \forall 0 < q \leq x \leq 1, \quad (129)$$

$$D(x||q) \leq \frac{(x-q)^2}{2q(1-q)} \quad \forall 0 < q \leq x \leq 1/2. \quad (130)$$

Proof Note that

$$\begin{aligned} \frac{d}{dx} D(x||q) &= \frac{x(1-q)}{q(1-x)}, & \frac{d^2}{dx^2} D(x||q) &= \frac{1}{x(1-x)}, \\ \frac{d^3}{dx^3} D(x||q) &= \frac{1}{(1-x)^2} - \frac{1}{x^2}, \end{aligned}$$

The second-order Taylor expansion of $D(x||q)$ at $x = q$ gives (129) and the third-order Taylor expansion at $x = q$ gives (130). $\square$

Finally, we need the following inequalities relating $Q(tr)$ to $Q(t)^{r^2}$. Note that if we use the approximation $Q(t) \approx e^{-t^2/2}$, these two quantities are equal. The lemma below makes this approximation precise:

Lemma 12 For any $t > 0$ and $r > 0$, we have

$$\frac{tr}{1+(tr)^2} t^{r^2} (\sqrt{2\pi})^{r^2-1} \leq \frac{Q(tr)}{Q(t)^{r^2}} \leq \left(\sqrt{2\pi} \frac{1+t^2}{t} \right)^{r^2-1} \frac{t^2+1}{rt^2}.$$

Proof For the lower bound, using $\frac{x}{1+x^2}\varphi(x) \leq Q(x) \leq \frac{1}{x}\varphi(x)$, where $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$, we have

$$Q(tr) \geq \frac{tr}{1+(tr)^2}\varphi(tr)$$

and

$$(Q(t))^{r^2} \leq \frac{1}{t^{r^2}}\varphi(t)^{r^2}$$

Combining the last two displayed equations, we get that

$$\frac{Q(tr)}{Q(t)^{r^2}} \geq \frac{tr}{1+(tr)^2} t^{r^2} \frac{\varphi(tr)}{\varphi(t)^{r^2}} = \frac{tr}{1+(tr)^2} t^{r^2} \left(\sqrt{2\pi}\right)^{r^2-1}.$$

The upper bound follows similarly from combining $Q(tr) \leq \frac{1}{tr}\varphi(tr)$ and $Q(t) \geq \frac{t}{1+t^2}\varphi(t)$. $\square$

Now we are ready to prove Lemma 3. Recall that $\tau \triangleq \min\{0 \leq k \leq n : \mathbb{P}\{\text{Binom}(n-1, q) \geq k\} \leq \alpha\}$ as defined in (39).

Proof (Proof of Lemma 3) We first prove (74) for $i \neq k$. Let $b'_k = \sum_{j \neq i} B_{jk}$. Then a_i and b'_k are independent. Since $b'_k \leq b_k \leq b'_k + 1$, it follows that

$$\begin{aligned} \mathbb{P}\{a_i \geq \tau, b_k \geq \tau + 1\} &\leq \mathbb{P}\{a_i \geq \tau, b'_k \geq \tau\} \\ &\leq \mathbb{P}\{a_i \geq \tau\} \mathbb{P}\{b'_k \geq \tau\} \\ &\leq \mathbb{P}\{a_i \geq \tau\} \mathbb{P}\{b_k \geq \tau\} \\ &= (\mathbb{P}\{\text{Binom}(n-1, q) \geq \tau\})^2 \leq \alpha^2. \end{aligned} \quad (131)$$

Next we prove (74) for $i = k$. For notational convenience, we abbreviate a_i and b_i as a and b , respectively. Let g denote the degree of vertex i in the parent graph. Abusing notation slightly, we let k denote the realization of g in the remainder of the proof. Then

$$\begin{aligned} \mathbb{P}\{a \geq \tau, b \geq \tau + 1\} &= \sum_{k \geq 0} \mathbb{P}\{a \geq \tau, b \geq \tau + 1, g = k\} \\ &= \sum_{k \geq 0} \mathbb{P}\{g = k\} \mathbb{P}\{a \geq \tau \mid g = k\} \mathbb{P}\{b \geq \tau + 1 \mid g = k\}. \end{aligned} \quad (132)$$

Let

$$k_0 = \left\lceil \frac{\tau + 2}{s} \right\rceil.$$

Since conditional on $g = k$, $a \sim \text{Binom}(k, s)$ and $b \sim \text{Binom}(k, s)$. It follows that for all $k \geq k_0$,

$$\mathbb{P}\{a \geq \tau \mid g = k\} \geq \mathbb{P}\{b \geq \tau + 1 \mid g = k\} \geq \mathbb{P}\{\text{Binom}(k, s) \geq ks - 1\} \geq \frac{1}{2}, \quad (133)$$

where the last inequality holds because the median of $\text{Binom}(k, s)$ is at least $ks - 1$. Combining (132) and (133) yields that

$$\mathbb{P}\{a \geq \tau, b \geq \tau + 1\} \geq \frac{1}{4} \mathbb{P}\{g \geq k_0\} = \frac{1}{4} \mathbb{P}\{\text{Binom}(n - 1, p) \geq k_0\}, \quad (134)$$

where the last equality holds due to $g \sim \text{Binom}(n - 1, p)$ with $p = q/s$.

It remains to prove that $\mathbb{P}\{\text{Binom}(n - 1, p) \geq k_0\} \geq \Omega\left(\alpha^{\frac{1-q}{(1-p)s}}\right)$. By assumption, $\alpha \leq 1/4$ and hence by the Berry-Esseen theorem, $\tau \geq (n - 1)q + 2$ for all n sufficiently large. Thus $k_0 \geq (\tau + 2)/s \geq np + 1$. It follows from Lemma 10 that

$$\mathbb{P}\{\text{Binom}(n - 1, p) \geq k_0\} \geq Q\left(\sqrt{2(n - 1)D(k_0/(n - 1)\|p)}\right). \quad (135)$$

To proceed, we need to bound $D(k_0/(n - 1)\|p)$ from the above. We claim that

$$0 \leq \tau - (n - 1)q + \sqrt{(n - 1)q(1 - q)}Q^{-1}(\alpha) \leq (1 - q)\left(Q^{-1}(\alpha)\right)^2 + 2, \quad (136)$$

where Q^{-1} denote the inverse function of Q function. We defer the proof of (136) to the end.

Note that $Q(x) \leq e^{-x^2/2}$ for $x \geq 0$. Hence,

$$Q^{-1}(\alpha) \leq \sqrt{2 \log \frac{1}{\alpha}} \leq \sqrt{2 \log(nq)},$$

where the last inequality follows due to the assumption $\alpha \geq 1/(nq)$. Thus it follows from (136) that $k_0 \leq (\tau + 3)/s \leq (n - 1)/2$ for sufficiently large n . Hence, by (130),

$$\begin{aligned} \sqrt{2(n - 1)D(k_0/(n - 1)\|p)} &\leq \frac{k_0 - (n - 1)p}{\sqrt{(n - 1)p(1 - p)}} \\ &\leq \frac{Q^{-1}(\alpha)\sqrt{(n - 1)q(1 - q)} + (1 - q)(Q^{-1}(\alpha))^2 + 5}{s\sqrt{(n - 1)p(1 - p)}}, \end{aligned}$$

where the last inequality holds due to $k_0s \leq \tau + 3$ and (136).

Applying the lower bound in Lemma 12 with

$$t \triangleq Q^{-1}(\alpha), \quad r \triangleq \frac{\sqrt{(n - 1)q(1 - q)} + (1 - q)t + 5/t}{s\sqrt{(n - 1)p(1 - p)}},$$

we get that

$$Q\left(\sqrt{2(n-1)D(k_0/(n-1)\|p)}\right) \geq \frac{tr}{1+(tr)^2} t^{r^2} \left(\sqrt{2\pi}\right)^{r^2-1} Q(t)^{r^2}.$$

Note that $Q(t) = Q(Q^{-1}(\alpha)) = \alpha$. Moreover, in view of $\Omega(1) \leq t \leq \sqrt{2\log(nq)}$, we have

$$r = \sqrt{\frac{1-q}{s(1-p)}} + O\left(\sqrt{\frac{\log nq}{nq}}\right).$$

Recall from (43) that $\frac{1-q}{(1-p)s} = 1 + \frac{1-s}{(1-p)s} = 1 + \frac{\sigma^2}{(1-p)s}$. Therefore, we get that

$$\begin{aligned} t^{r^2-1} &\leq \left(\sqrt{2\log nq}\right)^{r^2-1} = \exp\left(\left(\frac{\sigma^2}{s(1-p)} + O\left(\sqrt{\frac{\log nq}{nq}}\right)\right) \left(\log \sqrt{2\log nq}\right)\right) \\ &= 1 - o(1), \end{aligned}$$

where the last inequality holds because by assumptions, $\sigma^2 \log \log(nq) = o(1)$ and $nq \rightarrow \infty$. Moreover,

$$\alpha^{r^2} = \alpha^{\frac{1-q}{s(1-p)} + O\left(\sqrt{\frac{\log nq}{nq}}\right)} \geq \alpha^{\frac{1-q}{s(1-p)}} \exp\left(-O\left(\sqrt{\frac{\log nq}{nq}} \log(nq)\right)\right) = (1 - o(1)) \alpha^{\frac{1-q}{s(1-p)}},$$

where the inequality holds by the assumption $\alpha \geq 1/(nq)$, and the last equality holds due to $nq \rightarrow \infty$. Therefore, we get that

$$Q\left(\sqrt{2(n-1)D(k_0/(n-1)\|p)}\right) \geq (1 - o(1)) \alpha^{\frac{1-q}{(1-p)s}}. \quad (137)$$

Combining (134), (135), and (137) yields that

$$\mathbb{P}\{a \geq \tau, b \geq \tau + 1\} \geq \Omega\left(\alpha^{\frac{1-q}{(1-p)s}}\right),$$

proving (74) for $i = j$.

Finally, we verify the claim (136). By the definition of τ and Lemma 10, we have that

$$\begin{aligned} Q\left(\sqrt{2(n-1)D(\tau/(n-1)\|q)}\right) &\leq \mathbb{P}\{\text{Binom}(n-1, q) \geq \tau\} \\ &\leq \alpha \\ &< \mathbb{P}\{\text{Binom}(n-1, q) \geq \tau - 1\} \\ &\leq Q\left(\sqrt{2(n-1)D((\tau-2)/(n-1)\|q)}\right). \end{aligned}$$

Thus,

$$\sqrt{2(n-1)D((\tau-2)/(n-1)\|q)} \leq Q^{-1}(\alpha) \leq \sqrt{2(n-1)D(\tau/(n-1)\|q)}. \quad (138)$$

In view of (129), $2D(x\|q) \leq t$ for $t \geq 0$ implies

$$x^2 - (2q + t(1-q))x + q^2 \leq 0,$$

which further implies

$$x \leq \frac{2q + t(1-q) + \sqrt{4q(1-q)t + t^2(1-q)^2}}{2} \leq q + \sqrt{q(1-q)t} + t(1-q),$$

where the last inequality holds due to $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$. Therefore, it follows from the lower inequality in (138) that

$$\frac{\tau-2}{n-1} \leq q + \sqrt{q(1-q)} \frac{Q^{-1}(\alpha)}{\sqrt{n-1}} + \frac{(1-q)}{n-1} \left(Q^{-1}(\alpha) \right)^2.$$

Since $q \leq 1/8$ by assumption and $Q^{-1}(\alpha) \leq \sqrt{2 \log(nq)}$, it follows that for sufficiently large n , $\tau/(n-1) \leq 1/2$. Thus, combining the upper inequality in (138) with (130) gives that

$$\tau \geq (n-1)q + \sqrt{(n-1)q(1-q)} Q^{-1}(\alpha).$$

Combining the last two displayed equations yields the desired (136). $\square$

Proof (Proof of Lemma 4) Recall that $\Theta_{ik} = \{|a_i - b_k| \leq 4\sqrt{nq\Delta}\}$. Thus,

$$\begin{aligned} & \{a_i \geq \tau, b_k \geq \tau + 1\} \cap \Theta_{ik}^c \\ & \subset \left\{ a_i \geq \tau, b_k \geq \tau + 4\sqrt{nq\Delta} \right\} \cup \left\{ a_i \geq \tau + 4\sqrt{nq\Delta} + 1, b_k \geq \tau + 1 \right\} \\ & \subset \left\{ a_i \geq \tau, b_k \geq \tau + 4\sqrt{nq\Delta} \right\} \cup \left\{ a_i \geq \tau + 4\sqrt{nq\Delta}, b_k \geq \tau \right\}. \end{aligned}$$

Hence, by the union bound and the symmetry between a_i and b_k , it suffices to prove

$$\mathbb{P} \left\{ a_i \geq \tau, b_k \geq \tau + 4\sqrt{nq\Delta} \right\} \leq O \left(\alpha^{1+\mathbf{1}_{\{i \neq k\}}} e^{-\Delta/2} \right).$$

If $i \neq k$, analogous to the proof of (131), we have that

$$\begin{aligned} \mathbb{P} \left\{ a_i \geq \tau, b_k \geq \tau + 4\sqrt{nq\Delta} \right\} & \leq \mathbb{P} \{a_i \geq \tau\} \mathbb{P} \left\{ b_k \geq \tau + 4\sqrt{nq\Delta} - 1 \right\} \\ & \leq \alpha \mathbb{P} \left\{ b_k \geq \tau + 4\sqrt{nq\Delta} - 1 \right\}. \end{aligned}$$

If $i = k$, then we have that

$$\mathbb{P}\left\{a_i \geq \tau, b_k \geq \tau + 4\sqrt{nq\Delta}\right\} \leq \mathbb{P}\left\{b_k \geq \tau + 4\sqrt{nq\Delta} - 1\right\}.$$

Hence, for both cases, it reduces to proving

$$\mathbb{P}\left\{b_k \geq \tau + 4\sqrt{nq\Delta} - 1\right\} \leq O\left(\alpha e^{-\Delta/2}\right). \quad (139)$$

In view of Lemma 10, we have that

$$\mathbb{P}\left\{b_k \geq \tau + 4\sqrt{nq\Delta} - 1\right\} \leq Q\left(\sqrt{2(n-1)D\left(\frac{\tau + 4\sqrt{nq\Delta} - 2}{n-1}\|q\right)}\right).$$

In view of (136), we have $\tau \geq (n-1)q + \omega t$, where $\omega \triangleq \sqrt{(n-1)q(1-q)}$ and $t \triangleq Q^{-1}(\alpha)$. Let $\eta \triangleq 4\sqrt{nq\Delta} - 2$. Thus,

$$\begin{aligned} \sqrt{2(n-1)D\left(\frac{\tau + \eta}{n-1}\|q\right)} &\geq \sqrt{2(n-1)D\left(q + \frac{\omega t + \eta}{n-1}\|q\right)} \\ &\geq \frac{\omega t + \eta}{\sqrt{((n-1)q + \omega t + \eta)(1-q)}} \\ &\geq \frac{\omega t + \eta}{\sqrt{\omega^2 + \omega t + \eta}}, \end{aligned}$$

where the second inequality follows from (129). Combining the last two displayed equations gives

$$\mathbb{P}\left\{b_k \geq \tau + 4\sqrt{nq\Delta} - 1\right\} \leq Q\left(\frac{\omega t + \eta}{\sqrt{\omega^2 + \omega t + \eta}}\right) = Q(tr), \quad (140)$$

where

$$r \triangleq \frac{\omega + \eta/t}{\sqrt{\omega^2 + \omega t + \eta}}.$$

By the assumption $nq \geq C_0\Delta^2$, $\Delta \geq C_0$, and $t \leq \sqrt{2\log(nq)}$, we have $\eta \leq \omega^2/2$, $\eta \geq 4t^2$, $\eta^2 \geq 4\omega t^3$, and $t \leq \omega/2$. Thus, we get that

$$r^2 \geq \frac{\omega^2 + \eta^2/t^2}{\omega^2 + \omega t + \eta} = 1 + \frac{\eta^2/t^2 - \omega t - \eta}{\omega^2 + \omega t + \eta} \geq 1 + \frac{\eta^2}{4\omega^2 t^2}. \quad (141)$$

In view of the upper bound in Lemma 12, we have

$$Q(tr) \leq \left(\sqrt{2\pi} \frac{1+t^2}{t} \right)^{r^2-1} \frac{t^2+1}{rt^2} Q(t)^{r^2} \leq t^{c_1(r^2-1)} \alpha^{r^2}, \quad (142)$$

for a constant $c_1 > 0$, where the last inequality holds because $r > 1$ by (141) and $t = Q^{-1}(\alpha) \geq Q^{-1}(\alpha_1)$ under the assumption $\alpha \leq \alpha_1$ for a sufficiently small constant α_1 .

Note that

$$r \leq 1 + \frac{\eta}{\omega t} \leq 1 + \frac{c_2}{t} \sqrt{\Delta}$$

for a constant $c_2 > 0$. Therefore,

$$t^{c_1(r^2-1)} \leq t^{c_1(2c_2\sqrt{\Delta}/t + c_2^2\Delta/t^2)} \leq e^{\Delta/2}, \quad (143)$$

where the last inequality holds because $t \geq Q^{-1}(\alpha_1)$ for sufficiently small constant α_1 .

Finally, it remains to bound α^{r^2} . Using (141), we have

$$\alpha^{r^2} \leq \alpha \exp\left(-\frac{\eta^2}{4\omega^2 t^2} \log \frac{1}{\alpha}\right) \leq \alpha \exp\left(-\frac{\eta^2}{8\omega^2}\right) \leq \alpha \exp(-\Delta), \quad (144)$$

where the second inequality holds due to $t^2 \leq 2 \log \frac{1}{\alpha}$ and the last inequality holds because $\eta^2 \geq 8\omega^2 \Delta$.

In conclusion, by combining (140), (142), (143), and (144), we get the desired (139). $\square$

Proof (Proof of Lemma 5) Recall that

$$\Theta_i = \left\{ \max\{\sqrt{a_i - c_{ii}}, \sqrt{b_i - c_{ii}}\} \leq \sqrt{nq(1-s)} + \sqrt{\Delta} \right\}.$$

Thus,

$$\begin{aligned} & \{a_i \geq \tau, b_i \geq \tau + 1\} \cap \Theta_i^c \\ & \subset \left\{ a_i \geq \tau, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & \cup \left\{ b_i \geq \tau, \sqrt{b_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\}. \end{aligned}$$

Hence, by the union bound and the symmetry between a_i and b_i , it suffices to prove

$$\mathbb{P} \left\{ a_i \geq \tau, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \leq \alpha e^{-\Delta/2} + e^{-\Delta/(2\sigma^2)}.$$

Define

$$\bar{\tau} = \left(\sqrt{nq} + \sqrt{\frac{\Delta}{4(1-s)}} \right)^2.$$

Then

$$\begin{aligned} & \left\{ a_i \geq \tau, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & \subset \left\{ \tau \leq a_i \leq \bar{\tau}, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \cup \{a_i \geq \bar{\tau}\} \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{P} \left\{ a_i \geq \tau, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & \leq \mathbb{P} \left\{ \tau \leq a_i \leq \bar{\tau}, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} + \mathbb{P} \{a_i \geq \bar{\tau}\}. \end{aligned}$$

Since conditional on $a_i = k$, $a_i - c_{ii} \sim \text{Binom}(k, 1-s)$, it follows that

$$\begin{aligned} & \mathbb{P} \left\{ \tau \leq a_i \leq \bar{\tau}, \sqrt{a_i - c_{ii}} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & = \sum_{\tau \leq k \leq \bar{\tau}} \mathbb{P} \{a_i = k\} \mathbb{P} \left\{ \sqrt{\text{Binom}(k, 1-s)} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & \leq \mathbb{P} \left\{ \tau \leq a_i \leq \bar{\tau} \right\} \mathbb{P} \left\{ \sqrt{\text{Binom}(\bar{\tau}, 1-s)} > \sqrt{nq(1-s)} + \sqrt{\Delta} \right\} \\ & \leq \alpha \exp \left(-2 \left(\sqrt{nq(1-s)} + \sqrt{\Delta} - \sqrt{\bar{\tau}(1-s)} \right)^2 \right) = \alpha e^{-\Delta/2}, \end{aligned}$$

where the last inequality holds because of the definition of τ in (39) and the binomial tail bound (168). Moreover, since $a_i \sim \text{Binom}(n-1, q)$, it follows from the binomial tail bound (168) that

$$\mathbb{P} \{a_i \geq \bar{\tau}\} \leq \exp \left(-2 \left(\sqrt{\bar{\tau}} - \sqrt{(n-1)q} \right)^2 \right) \leq e^{-\frac{\Delta}{2(1-s)}} = e^{-\Delta/(2\sigma^2)}.$$

Combining the last three displayed equation completes the proof. $\square$

4.5 Proof of Theorem 4

The following classical result about Erdős-Rényi graphs (cf. [9, Lemma 30]) gives an upper bound on the probability that the 2-hop neighborhood of a given vertex i in $G \sim \mathcal{G}(n, p)$ is tangle-free, i.e., containing at most one cycle. This result will be used to control the dependency among outdegrees in analyzing the W similarity defined in (52).

Lemma 13 Consider graph $G \sim \mathcal{G}(n, p)$ with $np \geq C \log n$ for a large constant C . Let $\mathcal{H}$ denote the event that all 2-hop neighborhoods in G are tangle-free. Then

$$\mathbb{P}\{\mathcal{H}\} \geq 1 - n(2np)^8 p^2 - n^{-1}.$$

In particular, when $C \log n \leq np \leq n^{1-\epsilon}$ for $\epsilon > 9/10$, $\mathbb{P}\{\mathcal{H}\} \geq 1 - O(n^{9-10\epsilon})$.

Proof Let $\mathcal{H}_i$ denote the event that the 2-hop neighborhood of the vertex i in G is tangle-free and let $\mathcal{H} = \cap_{i \in [n]} \mathcal{H}_i$. Let $\ell = 2$ throughout the proof. Consider the classical graph branching process to explore the vertices in the ℓ -hop neighborhood of i . See, e.g., [2, Section 11.5] for a reference. Such a branching process discovers a set of edges which form a spanning tree of the ℓ -hop neighborhood of i . Then the ℓ -hop neighborhood of i is tangle-free, provided that the number of edges undiscovered by the branching process is at most one.

Let m denote the size of the ℓ -hop neighborhood of i in graph $G \sim \mathcal{G}(n, p)$. There are at most $\binom{m}{2}$ pairs of two distinct vertices in the ℓ -hop neighborhood of i . Hence, the number of undiscovered edges is stochastically dominated by $\text{Binom}(\binom{m}{2}, p)$. Thus, conditional on the size of the ℓ -hop neighborhood of i being m , the probability of $\mathcal{H}_i^c$, by a union bound, is at most

$$\mathbb{P}\{\text{Binom}(m(m-1)/2, p) \geq 2\} \leq \frac{1}{8} m^4 p^2,$$

Moreover, since $np \geq C \log n$ for a large constant C , the maximum degree in G is at most $2np$ with probability at least $1 - n^{-2}$. Thus, $m \leq (2np)^\ell$ with probability at least $1 - n^{-2}$. Therefore, the unconditional probability

$$\mathbb{P}\{\mathcal{H}_i^c\} \leq \frac{1}{8} (2np)^{4\ell} p^2 + n^{-2} \leq (2np)^{4\ell} p^2 + n^{-2}.$$

The proof is complete by applying a union bound over $i \in [n]$ to the last display. $\square$

Recall that $\tilde{N}_A(i)$ (resp. $\tilde{N}_B(i)$) denote the set of vertices in the 2-hop neighborhood of i in graph A (resp. B). Let $\tilde{G}_A(i)$ (resp. $\tilde{G}_B(i)$) denote the 2-hop neighborhood of i in graph A (resp. B), i.e., the subgraph induced by $\tilde{N}_A(i)$ (resp. $\tilde{N}_B(i)$). For notational simplicity, we use the same notation $a_j^{(i)}$ and $b_j^{(i)}$ as (21) and (22) for unnormalized outdegrees $|N_A(j) \setminus N_A[i]|$ and $|N_B(j) \setminus N_B[i]|$, respectively. Similar to the high-probability events defined in the beginning of Sect. 4, we also need to condition on a number of events regarding the 2-hop neighborhoods of i in A and k in B in analyzing the W statistic.

First, for each $i \in [n]$, define the event $\Gamma_A(i)$ such that the following statements hold simultaneously:

$$\begin{aligned} \frac{nq}{2} &\leq a_i \leq 2nq \\ \frac{nq}{2} &\leq a_j^{(i)} \leq 2nq, \quad \forall j \in N_A(i) \\ \tilde{a}_i &\leq (2nq)^2. \end{aligned}$$

Similarly, define the event $\Gamma_B(i)$ such that the following statements hold simultaneously:

$$\begin{aligned}\frac{nq}{2} &\leq b_i \leq 2nq \\ \frac{nq}{2} &\leq b_j^{(i)} \leq 2nq, \quad \forall j \in N_B(i) \\ \tilde{b}_i &\leq (2nq)^2.\end{aligned}$$

Define the event Γ_{ii} such that the following statements hold simultaneously:

$$\begin{aligned}c_{ii} &\geq \frac{nq}{2} \\ c_j^{(i)} &\geq \frac{nq}{2}, \quad \forall j \in N_A(i) \cap N_B(i) \\ \sqrt{a_i - c_{ii}}, \sqrt{b_i - c_{ii}} &\leq \sqrt{nq(1-s)} + \sqrt{2 \log n},\end{aligned}$$

where

$$c_j^{(i)} \triangleq |(N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])|. \quad (145)$$

Under the assumptions that $nq \geq C \log n$ for some sufficiently large constant C , and $\sigma \leq \sigma_0$ for sufficiently small constants σ_0 , using Chernoff bounds for binomial distributions (165) and the union bound, we have $\mathbb{P}\{\Gamma_A^c(i)\}, \mathbb{P}\{\Gamma_B^c(i)\}, \mathbb{P}\{\Gamma_{ii}^c\} \leq O(n^{-2})$.

Second, for each pair of $i, k \in [n]$ with $i \neq k$, define the event Γ_{ik} such that the following statement holds:

$$\begin{aligned}|N_A(i) \cap N_B(k)| &\leq 2 \\ |N_A(j) \cap \tilde{N}_B(k)| &\leq 2, \quad \forall j \in N_A(i) \setminus N_B[k] \\ |N_B(j) \cap \tilde{N}_A(i)| &\leq 2, \quad \forall j \in N_B(k) \setminus N_A[i].\end{aligned}$$

Lemma 14 *If $1 \leq nq \leq n^{1-\epsilon}$ for $\epsilon > 9/10$, we have $\mathbb{P}\{\Gamma_{ik}^c\} \leq O(n^7 q^{10}) = O(n^{7-10\epsilon})$ for all $i \neq k$.*

Proof Fix $i \neq k$. Note that

$$\begin{aligned}\mathbb{P}\{|N_A(i) \cap N_B(k)| \geq 3\} \\ &= \mathbb{P}\{\exists a, b, c \in [n] : A_{ia} = A_{ib} = A_{ic} = 1, B_{ka} = B_{kb} = B_{kc} = 1\} \\ &\leq \sum_{a,b,c \in [n]} \prod_{j \in \{a,b,c\}} \mathbb{P}\{A_{ij} = 1\} \mathbb{P}\{B_{kj} = 1\} \leq n^3 q^6 \leq n^7 q^{10}.\end{aligned}$$

Next, suppose we are given any $j \in N_A(i) \setminus N_B[k]$ such that $|N_A(j) \cap \tilde{N}_B(k)| \geq 3$. Let a, b, c denote three distinct vertices in $N_A(j) \cap \tilde{N}_B(k)$. For each $j' \in \{a, b, c\}$, let $p(j')$ denote a vertex in $N_B(k) \cap N_B[j']$ (which is non-empty since $j' \in$

$\tilde{N}_B(k)$). Consider the subgraph S of the union graph $A \cup B$ induced by vertices in $\{i, j, k, a, b, c, p(a), p(b), p(c)\}$. Let $V(S)$ denote the set of distinct vertices in S and $v(S) = |V(S)|$. Let $e(S)$ denote the number of edges in S . Note that $v(S) \leq 9$. Also, if we delete the two edges (j, a) and (j, b) , the graph S is still connected; thus $e(S) - v(S) \geq 1$. Therefore, by letting $\mathcal{K}_n$ denote the complete graph on $[n]$ and noting that $A \cup B \sim \mathcal{G}(n, q(2-s))$,

$$\begin{aligned} & \mathbb{P}\{\exists j \in N_A(i) \setminus N_B[k] : |N_A(j) \cap \tilde{N}_B(k)| \geq 3\} \\ & \leq \mathbb{P}\{\exists S \subset A \cup B : v(S) \leq 9, e(S) - v(S) \geq 1\} \\ & \leq \sum_{v \leq 9} \sum_{S \subset \mathcal{K}_n : v(S)=v} \mathbf{1}_{\{(i,k) \in V(S)\}} \mathbf{1}_{\{e(S)-v(S) \geq 1\}} \mathbb{P}\{S \subset A \cup B\} \\ & \leq \sum_{v \leq 9} 2^{\binom{v}{2}} n^{v-2} (2q)^{v+1} \leq O(n^7 q^{10}). \end{aligned}$$

Similarly, we have $\mathbb{P}\{\exists j \in N_B(k) \setminus N_A[i] : |N_B(j) \cap \tilde{N}_A(i)| \geq 3\} \leq O(n^7 q^{10})$ and hence $\mathbb{P}\{\Gamma_{ik}^c\} \leq O(n^7 q^{10})$. $\square$

Third, let $A \cup B$ denote the union graph of A and B . Define

$$\mathcal{H}_{ii} = \{\tilde{G}_{A \cup B}(i) \text{ is tangle-free}\}$$

and

$$\mathcal{H}_{ik} = \{\tilde{G}_A(i) \text{ and } \tilde{G}_B(k) \text{ are both tangle-free}\}.$$

The next two lemmas are the counterparts of Lemma 1 and Lemma 2, which establish the desired separation of the W statistic for true pairs and fake pairs.

Lemma 15 (True pairs) *Assume that $nq \geq C \max\{\log n, L^2\}$, $L \geq L_0$ for some sufficiently large constants C and L_0 , $\sigma \leq \sigma_0/L$ for some sufficiently small constant $\sigma_0 > 0$, and $n^2 q^3 \sqrt{L} \leq c_0$ for some sufficiently small constant $c_0 > 0$. Then*

$$\mathbb{P}\{W_{ii} \leq nq/4 \mid \tilde{G}_A(i), \tilde{G}_B(i), \tilde{G}_{A \cup B}(i)\} \mathbf{1}_{\{\mathcal{H}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii}\}} \leq e^{-\Omega(nq)}. \quad (146)$$

Proof Throughout the proof, we condition on the 2-hop neighborhoods of i in A , B , and $A \cup B$ such that event $\mathcal{H}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii}$ holds.

On the event $\mathcal{H}_{ii}$, there is at most one cycle in the 2-hop neighborhood of i in the union graph $A \cup B$. Hence, there is at most one pair of vertices $j_0 \in N_A(i)$ and $j'_0 \in N_B(i)$ with $j_0 \neq j'_0$ such that in the union graph $A \cup B$,

- (a) either j_0 and j'_0 are adjacent;
- (b) or there exist a neighbor $\ell \neq i$ of j_0 and a neighbor $\ell' \neq i$ of j'_0 , where either $\ell = \ell'$ or ℓ and ℓ' are adjacent.

Then we claim that $\tilde{Z}_{jj}^{(ii)}$ are mutually independent across different j in $N_A(i) \cap N_B(i) \setminus \{j_0, j'_0\}$. Indeed, note that $\tilde{Z}_{jj}^{(ii)}$ is a function of $\{\tilde{a}_\ell^{(i)} : \ell \in N_A(j) \setminus N_A[i]\}$ and

$\{\tilde{b}_\ell^{(i)} : \ell \in N_B(j) \setminus N_B[i]\}$. Fix a pair of $j \neq j' \in N_A(i) \cap N_B(i) \setminus \{j_0, j'_0\}$ and any $\ell \in (N_A(j) \setminus N_A[i]) \cup (N_B(j) \setminus N_B[i])$ and any $\ell' \in (N_A(j') \setminus N_A[i]) \cup (N_B(j') \setminus N_B[i])$. First, we claim $\ell \neq \ell'$, and ℓ, ℓ' are non-adjacent in the union graph $A \cup B$; otherwise, (j, j') is another pair in addition to (j_0, j'_0) satisfying either the condition (a) or (b) mentioned above, violating the tangle-free property. Moreover, since we have excluded i 's closed 2-hop neighborhoods in the definition of outdegree $\tilde{a}_\ell^{(i)}$ and $\tilde{b}_\ell^{(i)}$, it follows that $(\tilde{a}_\ell^{(i)}, \tilde{b}_\ell^{(i)})$ is independent from $(\tilde{a}_{\ell'}^{(i)}, \tilde{b}_{\ell'}^{(i)})$. Thus, $\tilde{Z}_{jj}^{(ii)}$ and $\tilde{Z}_{j'j'}^{(ii)}$ are independent.

By the definition of W similarity in (52), we have

$$W_{ii} \geq \sum_{j \in N_A(i) \cap N_B(i) \setminus \{j_0\}} \mathbf{1}_{\{\tilde{Z}_{jj}^{(ii)} \leq \eta\}},$$

where $\eta = \eta_0 \sqrt{\frac{L}{nq}}$ as defined in (53). We claim that

$$\mathbb{P} \left\{ \tilde{Z}_{jj}^{(ii)} \leq \eta \right\} \geq 1 - e^{-\Omega(L)} \geq \frac{3}{4}, \quad (147)$$

where the last inequality holds due to $L \geq L_0$. Also, on the event Γ_{ii} , $c_{ii} = |N_A(i) \cap N_B(i)| \geq nq/2$. Then it follows from the independence of $\tilde{Z}_{jj}^{(ii)}$ across different $j \in N_A(i) \cap N_B(i) \setminus \{j_0\}$ that

$$W_{ii} \stackrel{s.t.}{\geq} \text{Binom} \left(\frac{nq}{2} - 1, \frac{3}{4} \right).$$

Therefore, by Chernoff's bound (165) for binomials, we get that

$$\mathbb{P} \{W_{ii} \leq nq/4\} \leq e^{-\Omega(nq)}.$$

It remains to verify claim (147). The proof follows the similar argument as the proof of Lemma 1. Specifically, recall that $\tilde{Z}_{jj}^{(ii)} = d \left(\tilde{\mu}_j^{(i)}, \tilde{\nu}_j^{(i)} \right)$, where

$$\tilde{\mu}_j^{(i)} \triangleq \frac{1}{a_j^{(i)}} \sum_{\ell \in N_A(j) \setminus N_A[i]} \delta_{\tilde{a}_\ell^{(i)}} - \overline{\text{Binom}}(n - \tilde{a}_i, q),$$

and

$$\tilde{\nu}_j^{(i)} \triangleq \frac{1}{b_j^{(i)}} \sum_{\ell \in N_B(j) \setminus N_B[i]} \delta_{\tilde{b}_\ell^{(i)}} - \overline{\text{Binom}}(n - \tilde{b}_i, q).$$

Recall that $\tilde{a}_\ell^{(i)}$ (resp. $\tilde{b}_\ell^{(i)}$) are the normalized "outdegree" of vertex ℓ with the closed 2-hop neighborhood of i in A (resp. B) excluded; $\tilde{a}_i$ (resp. $\tilde{b}_i$) are the size the 2-hop neighborhood of i in graph A (resp. B).

Note that for $\ell \in N_A(j) \setminus N_A[i]$,

$$\begin{aligned}\tilde{a}_\ell^{(i)} &= \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} \sum_{k \in \tilde{N}_A(i)^c} (A_{k\ell} - q) \\ &= \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} \left[\sum_{k \in \tilde{N}_A(i)^c \cap \tilde{N}_B(i)^c} A_{k\ell} - (n - \tilde{a}_i)q \right],\end{aligned}$$

where the last equality holds because if $k \in \tilde{N}_B(i)$, then $A_{k\ell} = 0$; otherwise, $\tilde{G}_{A \cup B}(i)$ is not tangle-free. Moreover, note that $\ell \notin N_B(i)$; otherwise $\tilde{G}_{A \cup B}(i)$ is not tangle-free. Therefore, for all $k \in \tilde{N}_A(i)^c \cap \tilde{N}_B(i)^c$, $A_{k\ell} \sim \text{Bern}(q)$. Hence,

$$\tilde{a}_\ell^{(i)} \stackrel{\text{i.i.d.}}{\sim} \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} [\text{Binom}(|\tilde{N}_A(i)^c \cap \tilde{N}_B(i)^c|, q) - (n - \tilde{a}_i)q] \triangleq \mu.$$

Similarly, for $\ell \in N_B(j) \setminus N_B[i]$,

$$\begin{aligned}\tilde{b}_\ell^{(i)} &= \frac{1}{\sqrt{(n - \tilde{b}_i)q(1 - q)}} \sum_{k \in \tilde{N}_B(i)^c} (B_{k\ell} - q) \\ &= \frac{1}{\sqrt{(n - \tilde{b}_i)q(1 - q)}} \left[\sum_{k \in \tilde{N}_B(i)^c \cap \tilde{N}_A(i)^c} B_{k\ell} - (n - \tilde{b}_i)q \right].\end{aligned}$$

Thus,

$$\tilde{b}_\ell^{(i)} \stackrel{\text{i.i.d.}}{\sim} \frac{1}{\sqrt{(n - \tilde{b}_i)q(1 - q)}} [\text{Binom}(|\tilde{N}_A(i)^c \cap \tilde{N}_B(i)^c|, q) - (n - \tilde{b}_i)q] \triangleq \mu'.$$

Analogous to (86) and (87), the centered empirical distribution can be rewritten as

$$\begin{aligned}\tilde{\mu}_j^{(i)} &= \rho P + (1 - \rho)P' + \mu - \nu \\ \tilde{\nu}_j^{(i)} &= \rho' Q + (1 - \rho')Q' + \mu' - \nu',\end{aligned}$$

where

$$\rho \triangleq \frac{c_j^{(i)}}{a_j^{(i)}}, \quad \rho' \triangleq \frac{c_{jj}^{(ii)}}{b_j^{(i)}},$$

and

$$P \triangleq \frac{1}{c_j^{(i)}} \sum_{\ell \in (N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])} \delta_{\tilde{a}_\ell^{(i)}} - \mu,$$

$$\begin{aligned}
P' &\triangleq \frac{1}{a_i^{(j)} - c_j^{(i)}} \sum_{\ell \in (N_A(j) \setminus N_A[i]) \setminus (N_B(j) \setminus N_B[i])} \delta_{a_\ell^{(i)}} - \mu, \\
Q &\triangleq \frac{1}{c_j^{(i)}} \sum_{\ell \in (N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])} \delta_{b_\ell^{(i)}} - \mu', \\
Q' &\triangleq \frac{1}{b_j^{(i)} - c_j^{(i)}} \sum_{\ell \in (N_B(j) \setminus N_B[i]) \setminus (N_A(j) \setminus N_A[i])} \delta_{b_\ell^{(i)}} - \mu',
\end{aligned}$$

and $v = \overline{\text{Binom}}(n - \tilde{a}_i, q)$ and $v' = \overline{\text{Binom}}(n - \tilde{b}_i, q)$. □

Similar to (94), we have that

$$\begin{aligned}
\tilde{Z}_{jj}^{(ii)} &\leq \underbrace{\|[\mu - v]_L\|_1 + \|[\mu' - v']_L\|_1}_{\text{(I)}} + \underbrace{d(P, Q)}_{\text{(II)}} + \underbrace{(1 - \rho)\|P'\|_L + (1 - \rho')\|Q'\|_L}_{\text{(III)}} \\
&\quad + \underbrace{|\rho - \rho'| \times \|Q\|_L}_{\text{(IV)}}. \tag{148}
\end{aligned}$$

For (I), we need the following lemma to control the discrepancy between the distribution μ (resp. μ') and the ideal standardized binomial distribution v (resp. v').

Lemma 16 *Let $m, n \in \mathbb{N}$ with $m \leq n$ and $\eta_1, \dots, \eta_m, q \in [0, 1]$. Suppose $X_i \stackrel{i.i.d.}{\sim} \text{Bern}(q)$ for $1 \leq i \leq n$ and Y_i 's are independently distributed as $\text{Bern}(\eta_i)$ for $1 \leq i \leq m$. Let $S = \sum_{i=1}^n X_i$ and $T = \sum_{i=1}^m Y_i + \sum_{i=m+1}^n X_i$. Let μ_0 and v_0 denote the law of $\frac{S-nq}{\sqrt{nq(1-q)}}$ and $\frac{T-nq}{\sqrt{nq(1-q)}}$, respectively. Assume $m \leq n/2$ and $nq = \Omega(1)$. Then*

$$d(\mu_0, v_0) = \|[\mu_0 - v_0]_L\|_1 \leq O\left(L \frac{\sum_{i=1}^m |\eta_i - q|}{\sqrt{nq}}\right). \tag{149}$$

Proof For $1 \leq i \leq m$, we couple X_i and Y_i as follows. When $\eta_i \leq q$, generate $Y_i \sim \text{Bern}(\eta_i)$, and let $X_i = 1$ if $Y_i = 1$ and $X_i \sim \text{Bern}(q - \eta_i)$ if $Y_i = 0$. When $\eta_i > q$, generate $X_i \sim \text{Bern}(q)$, and let $Y_i = 1$ if $X_i = 1$ and $Y_i \sim \text{Bern}(\eta_i - q)$ if $X_i = 0$. Let $X = \sum_{i=m+1}^n X_i$, $Y = \sum_{i=1}^m Y_i$, and $Z = \sum_{i=1}^m X_i$. Then $S = X + Z$ and $T = X + Y$. Let $\xi = \sqrt{nq(1-q)}$. Then

$$\begin{aligned}
d(\mu_0, v_0) &= \sum_{\ell=1}^L |\mu_0(I_\ell) - v_0(I_\ell)| \\
&= \sum_{\ell=1}^L |\mathbb{P}\{S \in \xi I_\ell + nq\} - \mathbb{P}\{T \in \xi I_\ell + nq\}| \\
&\leq \sum_{\ell=1}^L \max\{\mathbb{P}\{S \in \xi I_\ell + nq, T \notin \xi I_\ell + nq\}, \mathbb{P}\{S \notin \xi I_\ell + nq, T \in \xi I_\ell + nq\}\}.
\end{aligned}$$

It remains to show $\mathbb{P}\{S \in \xi I_\ell + nq, T \notin \xi I_\ell + nq\} \leq O\left(\frac{\sum_{i=1}^m |\eta_i - q|}{\sqrt{nq}}\right)$; the proof for $\mathbb{P}\{S \in \xi I_\ell + nq, T \notin \xi I_\ell + nq\}$ is analogous. Note that

$$\begin{aligned}\mathbb{P}\{S \in \xi I_\ell + nq, T \notin \xi I_\ell + nq\} &= \mathbb{P}\{X \in \xi I_\ell + nq - Z, X \notin \xi I_\ell + nq - Y\} \\ &\leq O\left(\frac{\mathbb{E}[|Y - Z|]}{\sqrt{nq}}\right),\end{aligned}$$

where the last inequality follows analogous to Lemma 9. The conclusion follows since $\mathbb{E}[|Y - Z|] \leq \sum_{i=1}^m \mathbb{E}[|X_i - Y_i|] = (1 - \min\{\eta_i, q\}) \sum_{i=1}^m |\eta_i - q|$ by definition. $\square$

Applying Lemma 16 (with $\eta_i \equiv 0$ and $m \leq \tilde{a}_i + \tilde{b}_i$) and noting that $\tilde{a}_i, \tilde{b}_i \leq (2nq)^2$, we get that

$$\|[\mu - \nu]_L\|_1 + \|[\mu' - \nu']_L\|_1 \leq O\left(\frac{L(nq)^2 q}{\sqrt{nq}}\right). \quad (150)$$

Analogous to Lemma 9, under the assumptions that $\sigma \leq \sigma_0/L$ and $nq \geq CL^2$, we have that for any $\ell \in (N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])$ and any interval $I \subset [-1/2, 1/2]$ with $|I| = 1/L$, conditional on the 2-hop neighborhoods of i in both A and B ,

$$\mathbb{P}\left\{\tilde{a}_\ell^{(i)} \in I, \tilde{b}_\ell^{(i)} \notin I\right\} + \mathbb{P}\left\{\tilde{a}_\ell^{(i)} \notin I, \tilde{b}_\ell^{(i)} \in I\right\} \leq \frac{c_1}{L}$$

for a sufficiently small constant c_1 .

For (II), applying Lemma 7 with

$$\begin{aligned}\{X_j\}_{j=1}^m &= \{\tilde{a}_\ell^{(i)}\}_{\ell \in (N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])}, \\ \{Y_j\}_{j=1}^m &= \{\tilde{b}_\ell^{(i)}\}_{\ell \in (N_A(j) \setminus N_A[i]) \cap (N_B(j) \setminus N_B[i])},\end{aligned}$$

and $m = c_j^{(i)} \geq \frac{nq}{2}$ on the event Γ_{ii} , we get that with probability at least $1 - e^{-\Omega(L)}$,

$$d(P, Q) \leq c_2 \sqrt{\frac{L}{nq}}, \quad (151)$$

for a sufficiently small constant c_2 .

For (III), applying Lemma 8 with $k = L$ implies that $\|P'\|_L \leq 2\sqrt{\frac{L}{a_j^{(i)} - c_j^{(i)}}}$ and $\|Q'\|_L \leq 2\sqrt{\frac{L}{b_j^{(i)} - c_j^{(i)}}}$, each with probability at least $1 - e^{-L/2}$. Therefore, by the union bound, with probability at least $1 - e^{-\Omega(L)}$,

$$\begin{aligned}
(1 - \rho) \| [P']_L \|_1 + (1 - \rho') \| [Q']_L \|_1 &\leq \frac{2}{a_j^{(i)}} \sqrt{L} \sqrt{a_j^{(i)} - c_j^{(i)}} + \frac{2}{b_j^{(i)}} \sqrt{L} \sqrt{b_j^{(i)} - c_j^{(i)}} \\
&\leq \frac{8}{nq} \sqrt{L} \left(\sqrt{nq\sigma^2} + \sqrt{2 \log n} \right), \quad (152)
\end{aligned}$$

where the last inequality holds because on the event $\Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii}$, $a_j^{(i)}, b_j^{(i)} \geq nq/2$,

$$\begin{aligned}
\sqrt{a_j^{(i)} - c_{jj}^{(ii)}} &\leq \sqrt{a_j - c_j + b_i - c_i} \leq \sqrt{a_j - c_j} + \sqrt{b_i - c_i} \\
&\leq 2 \left(\sqrt{nq(1-s)} + \sqrt{2 \log n} \right)
\end{aligned}$$

and similarly for $\sqrt{b_j^{(i)} - c_j^{(i)}}$.

Finally, for (IV), applying Lemma 8 with $k = L$ implies that with probability at least $1 - e^{-L/2}$,

$$\| [Q]_L \|_1 \leq 2 \sqrt{\frac{L}{c_j^{(i)}}} \leq 2\sqrt{2} \sqrt{\frac{L}{nq}},$$

where the last inequality holds due to $c_j^{(i)} \geq nq/2$ on event Γ_{ii} . Moreover,

$$\begin{aligned}
|\rho - \rho'| &\leq \max\{1 - \rho, 1 - \rho'\} \leq \frac{2}{nq} \left(\sqrt{nq(1-s)} + \sqrt{2 \log n} \right)^2 \\
&\leq 4\sigma^2 + 8 \frac{\log n}{nq}.
\end{aligned}$$

Therefore,

$$|\rho - \rho'| \times \| [Q]_L \|_1 \leq 8\sqrt{2} \sqrt{\frac{L}{nq}} \left(\sigma^2 + 2 \frac{\log n}{nq} \right). \quad (153)$$

Assembling (148) with (151), (152), (153), we get that with probability at least $1 - e^{-\Omega(L)}$,

$$\begin{aligned}
\tilde{Z}_{jj}^{(ii)} &\leq c_2 \sqrt{\frac{L}{nq}} + \frac{8}{nq} \sqrt{L} \left(\sqrt{nq\sigma^2} + \sqrt{2 \log n} \right) \\
&\quad + 8\sqrt{2} \sqrt{\frac{L}{nq}} \left(\sigma^2 + 2 \frac{\log n}{nq} \right) + O \left(\frac{L(nq)^2 q}{\sqrt{nq}} \right) \\
&\leq \eta_0 \sqrt{\frac{L}{nq}} = \eta
\end{aligned}$$

for some sufficiently small absolute constant $\eta_0 > 0$, where the last inequality holds due to the assumptions that $nq \geq CL$ for some sufficiently large constant C , $\sigma \leq \sigma_0/L$, and $n^2q^3\sqrt{L} \leq c_0$ for some sufficiently small constant $c_0 > 0$. Thus we arrive at the desired (147). $\square$

Lemma 17 (Fake pairs) *Suppose $L \geq C \log(nq)$, $nq \geq C \max\{\log n, L^2\}$ for some sufficiently large constant C , and $q \leq n^{-\epsilon}$ for $\epsilon > 9/10$. Fix $i \neq k$. Then*

$$\mathbb{P}\{W_{ik} \geq nq/4 \mid \tilde{G}_A(i), \tilde{G}_B(k)\} \mathbf{1}_{\{\mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik}\}} \leq e^{-\Omega(nq)}. \quad (154)$$

Proof Fix a pair of vertices $i \neq k$ and condition on the 2-hop neighborhoods of i in A and k in B such that the event $\mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik}$ holds. Fix a feasible solution M in (52); in other words, M is a bipartite matching (possibly imperfect) between the neighborhoods $N_A(i)$ and $N_B(k)$.

For the ease of notation, let $J = N_A(i) \setminus N_B[k]$ and $J' = N_B(k) \setminus N_A[i]$. Recall the matrix $Y^{(ik)}$ defined in (51). Since M is a matching, it follows that

$$\langle Y^{(ik)}, M \rangle \leq 2|N_A[i] \cap N_B[k]| + \sum_{j \in J, j' \in J'} Y_{jj'}^{(ik)} M_{jj'} \leq \frac{nq}{8} + \sum_{j \in J, j' \in J'} Y_{jj'}^{(ik)} M_{jj'},$$

where the last inequality holds because $|N_A[i] \cap N_B[k]| \leq 4 \leq nq/16$ on the event Γ_{ik} under the assumption that $nq \geq C \log n$.

Note that on the event $\mathcal{H}_{ik}$, there is at most one cycle in the 2-hop neighborhood of i in A , and at most one cycle in the 2-hop neighborhood of k in B .

We next bound $\sum_{j \in J, j' \in J'} Y_{jj'}^{(ik)} M_{jj'}$ using McDiarmid's inequality, where $Y_{jj'}^{(ik)} = \mathbf{1}_{\{\tilde{Z}_{jj'}^{(ik)} \leq \eta\}}$ and $\eta = \eta_0 \sqrt{\frac{L}{nq}}$ as defined in (53). To circumvent the discontinuity of the indicator function, define a piecewise linear function F which decreases linearly from 1 to 0 from η to 2η , so that $\mathbf{1}_{\{x \leq \eta\}} \leq F(x)$ for all x . Furthermore, F is Lipschitz with constant $1/\eta$. Define

$$W' \triangleq \sum_{j \in J, j' \in J'} F(\tilde{Z}_{jj'}^{(ik)}) M_{jj'}. \quad (155)$$

Then we have

$$\langle Y^{(ik)}, M \rangle \leq \frac{nq}{8} + W'. \quad (156)$$

Let $\mathcal{L} = \cup_{j \in J} (N_A(j) \setminus N_A[i])$ and $\mathcal{L}' = \cup_{j' \in J'} (N_B(j') \setminus N_B[k])$. Next we claim that, on the event $\mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k)$, W' , as a function of $\{\tilde{a}_\ell^{(i)}, \tilde{b}_\ell^{(k)} : \ell \in \mathcal{L} \cap \mathcal{L}'\}$, $\{\tilde{a}_\ell^{(i)} : \ell \in \mathcal{L} \setminus \mathcal{L}'\}$, and $\{\tilde{b}_{\ell'}^{(k)} : \ell' \in \mathcal{L}' \setminus \mathcal{L}\}$, satisfies the bounded difference property with constant $O(\frac{1}{nq\eta})$. This is verified by the following reasoning:

- Fix $\ell \in \mathcal{L} \setminus \mathcal{L}'$. We consider the impact of modifying the value of $\tilde{a}_\ell^{(i)}$ on that of W' . On the tangle-free event $\mathcal{H}_{ik}$, there are at most two distinct choices of j such that

$\ell \in N_A(j) \setminus N_A[i]$. Therefore $\tilde{a}_\ell^{(i)}$ appears in the empirical distribution $\tilde{\mu}_j^{(i)}$ for at most two different $j \in N_A(i)$. Furthermore, since $\ell \notin \mathcal{L}'$, $\tilde{a}_\ell^{(i)}$ does not appear in any $\tilde{v}_j^{(k)}$. Recall that any m -observation empirical distribution as a function of each observation satisfies the bounded difference property (with respect to the total variation distance) with constant $O(\frac{1}{m})$ (cf. (106)). On the event $\Gamma_A(i)$, we have $a_j^{(i)} \geq nq/4$. Thus modifying $\tilde{a}_\ell^{(i)}$ can change $\tilde{\mu}_j^{(i)}$ in total variation by at most $O(\frac{1}{nq})$. Furthermore, *crucially*, since M is a matching, for each j there exists at most one j' such that $M_{jj'} \neq 0$ in the double sum (155). Finally, since F is $(1/\eta)$ -Lipschitz continuous by design, we conclude that $\tilde{a}_\ell^{(i)} \mapsto W'$ has the desired bounded difference property with constant $O(\frac{1}{nq\eta})$.

- Entirely analogously, since $b_j^{(k)} \geq nq/4$ on the event $\Gamma_B(k)$, the mappings $\tilde{b}_{\ell'}^{(k)} \mapsto W'$ for any $\ell' \in \mathcal{L}' \setminus \mathcal{L}$ and $(\tilde{a}_\ell^{(i)}, \tilde{b}_{\ell'}^{(k)}) \mapsto W'$ for any $\ell \in \mathcal{L} \cap \mathcal{L}$ all satisfy the bounded difference property with constant $O(\frac{1}{nq\eta})$ on the event $\mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k)$.

Recall that in the definition of outdegree $\tilde{a}_\ell^{(i)}$, we have excluded the 2-hop neighborhood of i in A ; similarly, in the definition of outdegree $\tilde{b}_{\ell'}^{(k)}$, we have excluded the 2-hop neighborhood of k in B . Therefore, we have that

- $\{\tilde{a}_\ell^{(i)}, \tilde{b}_{\ell'}^{(k)}\}$ are independent across different $\ell \in \mathcal{L} \cap \mathcal{L}'$;
- $\{\tilde{a}_\ell^{(i)}\}$ are independent across different $\ell \in \mathcal{L} \setminus \mathcal{L}'$;
- $\{\tilde{b}_{\ell'}^{(k)}\}$ are independent across different $\ell' \in \mathcal{L}' \setminus \mathcal{L}$;
- $\{\tilde{a}_\ell^{(i)}, \tilde{b}_{\ell'}^{(k)} : \ell \in \mathcal{L} \cap \mathcal{L}'\}$ are independent of $\{\tilde{a}_\ell^{(i)} : \ell \in \mathcal{L} \setminus \mathcal{L}', \tilde{b}_{\ell'}^{(k)} : \ell' \in \mathcal{L}' \setminus \mathcal{L}\}$.

However, $\tilde{a}_\ell^{(i)}$ for $\ell \in \mathcal{L} \setminus \mathcal{L}'$ and $\tilde{b}_{\ell'}^{(k)}$ for $\ell' \in \mathcal{L}' \setminus \mathcal{L}$ may be dependent, because $A_{\ell\ell'}$ may contribute to the outdegree $\tilde{a}_\ell^{(i)}$, and $B_{\ell\ell'}$ may contribute to the outdegree $\tilde{b}_{\ell'}^{(k)}$. Fortunately, similar to the reasoning in Fig. 1, conditioned on the edge sets $E_A(\mathcal{L}, \mathcal{L}')$ and $E_B(\mathcal{L}, \mathcal{L}')$, the outdegrees $\{\tilde{a}_\ell^{(i)} : \ell \in \mathcal{L} \setminus \mathcal{L}'\}$ and $\{\tilde{b}_{\ell'}^{(k)} : \ell' \in \mathcal{L}' \setminus \mathcal{L}\}$ are independent, since the definition of the outdegree in (46)–(47) excludes the two-hop neighborhood.

In particular, write $E(\mathcal{L}, \mathcal{L}') = (E_A(\mathcal{L}, \mathcal{L}'), E_B(\mathcal{L}, \mathcal{L}'))$ for simplicity, and let $\mathcal{F}_{ik}$ denote an event (to be specified later) that is measurable with respect to $E(\mathcal{L}, \mathcal{L}')$ and holds with high probability: $\mathbb{P}\{\mathcal{F}_{ik}\} \geq 1 - \exp(-\Omega(nq))$. Conditioned on $E(\mathcal{L}, \mathcal{L}')$ such that the event $\mathcal{F}_{ik}$ holds, applying McDiarmid's inequality and noting that $|\mathcal{L}|, |\mathcal{L}'| \leq (2nq)^2$ on the event $\Gamma_A(i) \cap \Gamma_B(k)$, we get that

$$\mathbb{P}\left\{W' - \mathbb{E}[W' \mid E(\mathcal{L}, \mathcal{L}')] \geq \frac{nq}{16} \mid E(\mathcal{L}, \mathcal{L}')\right\} \leq \exp(-c_1(nq\eta)^2), \quad (157)$$

where c_1 is an absolute constant.

We next compute $\mathbb{E}[W' \mid E(\mathcal{L}, \mathcal{L}')]$. We first claim that for all $j \in J$ and $j' \in J'$,

$$\mathbb{P}\left\{Z_{jj'}^{(ik)} \leq 2\eta \mid E(\mathcal{L}, \mathcal{L}')\right\} \leq e^{-\Omega(L)}. \quad (158)$$

By definition of W' , we have

$$\begin{aligned}\mathbb{E}\left[W' \mid E(\mathcal{L}, \mathcal{L}')\right] &= \sum_{j \in J, j' \in J'} \mathbb{E}\left[F\left(\tilde{Z}_{jj'}^{(ik)}\right) \mid E(\mathcal{L}, \mathcal{L}')\right] M_{jj'} \\ &\leq \sum_{j \in J, j' \in J'} \mathbb{P}\left\{Z_{jj'}^{(ik)} \leq 2\eta \mid E(\mathcal{L}, \mathcal{L}')\right\} \\ &\leq O\left(e^{-\Omega(L)}\right) \sum_{j \in J, j' \in J'} M_{jj'} \\ &\leq O(e^{-\Omega(L)} nq) \leq \frac{nq}{16},\end{aligned}$$

where the first inequality follows by the definition of F ; the second inequality holds due to (158); the third inequality is due to $|J| \leq 2nq$ on the event $\Gamma_A(i)$ and that M is a matching; the last inequality holds due to $L \geq L_0 \log n$. Combining the last displayed equation with (157), we obtain

$$\mathbb{P}\left\{W' \geq nq/8 \mid E(\mathcal{L}, \mathcal{L}')\right\} \mathbf{1}_{\{\mathcal{F}_{ik}\}} \leq \exp\left(-c_1(nq\eta)^2\right).$$

Averaging over the last displayed equation yields that

$$\mathbb{P}\left\{\{W' \geq nq/8\} \cap \mathcal{F}_{ik}\right\} \leq \exp\left(-c_1(nq\eta)^2\right).$$

Combining the last displayed equation with (156), we obtain

$$\mathbb{P}\left\{\left\{\langle Y^{(ik)}, M \rangle \geq nq/4\right\} \cap \mathcal{F}_{ik}\right\} \leq \mathbb{P}\left\{\{W' \geq nq/8\} \cap \mathcal{F}_{ik}\right\} \leq \exp\left(-c_1(nq\eta)^2\right).$$

Finally, applying a union bound over the set of all possible matching M and recalling the definition of similarity W_{ik} in (52), we get that

$$\mathbb{P}\left\{\{W_{ik} \geq nq/4\} \cap \mathcal{F}_{ik}\right\} \leq (2nq)! \times e^{-c_1(nq\eta)^2} \leq e^{-\Omega(nq \log(nq))},$$

where the last inequality holds due to the choice of η in (53) and the assumption that $L \geq L_0 \log(nq)$. Therefore, by a union bound,

$$\mathbb{P}\left\{W_{ik} \geq nq/4\right\} \leq \mathbb{P}\left\{\{W_{ik} \geq nq/4\} \cap \mathcal{F}_{ik}\right\} + \mathbb{P}\left\{\mathcal{F}_{ik}^c\right\} \leq e^{-\Omega(nq)}.$$

It remains to specify the event $\mathcal{F}_{ik}$ and verify the claim (158) when conditioned on $E(\mathcal{L}, \mathcal{L}')$ such that the event $\mathcal{F}_{ik}$ holds. The proof follows a similar argument as in the proof of Lemma 2. Specifically, recall that $\tilde{Z}_{jj'}^{(ik)} = d\left(\tilde{\mu}_j^{(i)}, \tilde{\nu}_{j'}^{(k)}\right)$, where

$$\tilde{\mu}_j^{(i)} \triangleq \frac{1}{a_j^{(i)}} \sum_{\ell \in N_A(j) \setminus N_A[i]} \delta_{a_\ell^{(i)}} - \overline{\text{Binom}}(n - \tilde{a}_i, q),$$

and

$$\tilde{v}_{j'}^{(k)} \triangleq \frac{1}{b_{j'}^{(k)}} \sum_{\ell \in N_B(j') \setminus N_B[k]} \delta_{\tilde{b}_\ell^{(k)}} - \overline{\text{Binom}}(n - \tilde{b}_k, q).$$

Let $v = \overline{\text{Binom}}(n - \tilde{a}_i, q)$ and $v' = \overline{\text{Binom}}(n - \tilde{b}_k, q)$. Observe that for $\ell \in \tilde{N}_B(k)$, $\tilde{a}_\ell^{(i)}$ is no longer distributed as v after conditioning on the 2-hop neighborhood $\tilde{N}_B(k)$, and likewise for $\tilde{b}_\ell^{(k)}$ for $\ell \in \tilde{N}_A(i)$. Therefore, we decompose $\tilde{\mu}_j^{(i)}$ and $\tilde{v}_{j'}^{(k)}$ as

$$\begin{aligned}\tilde{\mu}_j^{(i)} &= \kappa \hat{P} + (1 - \kappa) \tilde{P} \\ \tilde{v}_{j'}^{(k)} &= \kappa' \hat{Q} + (1 - \kappa') \tilde{Q},\end{aligned}$$

where

$$\kappa \triangleq \frac{|\hat{S}|}{a_j^{(i)}}, \quad \kappa' \triangleq \frac{|\hat{T}|}{b_{j'}^{(k)}},$$

and

$$\begin{aligned}\tilde{S} &= (N_A(j) \setminus N_A[i]) \cap \tilde{N}_B(k)^c, & \hat{S} &= (N_A(j) \setminus N_A[i]) \cap \tilde{N}_B(k) \\ \tilde{T} &= (N_B(j') \setminus N_B[k]) \cap \tilde{N}_A(i)^c, & \hat{T} &= (N_B(j') \setminus N_B[k]) \cap \tilde{N}_A(i),\end{aligned}$$

and

$$\begin{aligned}\tilde{P} &\triangleq \frac{1}{|\tilde{S}|} \sum_{\ell \in \tilde{S}} \delta_{\tilde{a}_\ell^{(i)}} - v, & \hat{P} &\triangleq \frac{1}{|\hat{S}|} \sum_{\ell \in \hat{S}} \delta_{\tilde{a}_\ell^{(i)}} - v, \\ \tilde{Q} &\triangleq \frac{1}{|\tilde{T}|} \sum_{\ell \in \tilde{T}} \delta_{\tilde{b}_\ell^{(k)}} - v', & \hat{Q} &\triangleq \frac{1}{|\hat{T}|} \sum_{\ell \in \hat{T}} \delta_{\tilde{b}_\ell^{(k)}} - v' .\end{aligned}$$

Therefore, we have

$$\tilde{Z}_{jj'}^{(ik)} \geq (1 - \kappa) d(\tilde{P}, \tilde{Q}) - \kappa \|[\hat{P}]_L\|_1 - \kappa' \|[\hat{Q}]_L\|_1 - |\kappa - \kappa'| \times \|[\tilde{Q}]_L\|_1. \quad (159)$$

On the event $\Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik}$, we have $a_j^{(i)}, b_{j'}^{(k)} \geq nq/2$, and $|\hat{S}|, |\hat{T}| \leq 2$. Therefore, $\kappa, \kappa' \leq \frac{4}{nq}$. Since $\|[\hat{P}]_L\|_1, \|[\hat{Q}]_L\|_1, \|[\tilde{Q}]_L\|_1 \leq 2$, it follows that

$$\tilde{Z}_{jj'}^{(ik)} \geq \frac{1}{2} d(\tilde{P}, \tilde{Q}) - o\left(\frac{1}{nq}\right). \quad (160)$$

It remains to lower bound $d(\tilde{P}, \tilde{Q})$. Conditioning on $E(\mathcal{L}, \mathcal{L}')$, we aim to apply Lemma 6 with $m = |\tilde{S}|$, $m' = |\tilde{T}|$, $m_0 = nq$, $\{X_\ell\}_{\ell=1}^m = \{\tilde{a}_\ell^{(i)}\}_{\ell \in \tilde{S}}$, and $\{Y_\ell\}_{\ell=1}^{m'} =$

$\{\tilde{b}_\ell^{(k)}\}_{\ell \in \tilde{T}}$. Note that since $\kappa, \kappa' \leq 1/2$ and $nq/2 \leq a_j^{(i)}, b_{j'}^{(k)} \leq 2nq$, it follows that $m, m' = \Theta(m_0)$. Also, as previously argued, after conditioning on $E(\mathcal{L}, \mathcal{L}')$, $\{\tilde{a}_\ell^{(i)}\}_{\ell \in \tilde{S}}$ and $\{\tilde{b}_\ell^{(k)}\}_{\ell \in \tilde{T}}$ are two independent sequence of real-valued random variables. It remains to check the assumption (88) in Lemma 6, that is, there exists a set $\mathcal{L}_0 \subset \tilde{S}$ with $|\mathcal{L}_0| \geq m/4$ and constants $c_1, c_2 \in (0, 1]$ such that $\frac{c_1}{L} \leq \mathbb{P}\{\tilde{a}_\ell^{(i)} \in I\} \leq \frac{c_2}{L}$ for all interval $I \subset [-2, 2]$ of length $1/L$.

To this end, recall that

$$\tilde{a}_\ell^{(i)} = \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} \sum_{u \in \tilde{N}_A(i)^c} (A_{u\ell} - q).$$

For $\ell \in \tilde{S}$:

- If $u \in N_B[k]$, then $B_{u\ell} = 0$; otherwise, $\ell \in \tilde{N}_B(k)$, violating $\ell \in \tilde{S}$. Thus, $A_{u\ell} \sim \text{Bern}(q')$ with $q' = \mathbb{P}\{A_{u\ell} = 1 | B_{u\ell} = 0\} = \frac{q(1-s)}{1-ps} \leq q$;
- If $u \in \mathcal{L}'$, then $A_{u\ell}$ is deterministic when conditioning on $E_A(\mathcal{L}, \mathcal{L}')$;
- If $u \notin \tilde{N}_B(k)$, then $A_{u\ell} \sim \text{Bern}(q)$.

Recall that $e_A(\ell, S)$ denotes the number of edges between vertex ℓ and vertices in S in graph A . Define $\phi = |\mathcal{L}' \setminus \tilde{N}_A(i)|$, $\psi = |N_B[k] \setminus \tilde{N}_A(i)|$, and

$$\mathcal{L}_0 = \left\{ \ell \in \tilde{S} : |e_A(\ell, \mathcal{L}' \setminus \tilde{N}_A(i)) - \phi q| \leq \sqrt{nq(1 - q)/2} \right\}.$$

Define the event

$$\mathcal{F}_{ik} = \{|\mathcal{L}_0| \geq m/4\}, \quad (161)$$

which is measurable with respect to $E_A(\mathcal{L}, \mathcal{L}')$ since $\tilde{S} \subset \mathcal{L}$. Note that for each $\ell \in \mathcal{L}$, $e_A(\ell, \mathcal{L}' \setminus \tilde{N}_A(i)) \sim \text{Binom}(\phi, q)$. Hence, by Chebyshev's inequality,

$$\mathbb{P}\{\ell \in \mathcal{L}_0\} \geq 1 - \frac{2\phi}{n} \geq \frac{1}{2},$$

where the last inequality holds because $\phi \leq |\mathcal{L}'| \leq \tilde{b}_k \leq (2nq)^2$ and $q \leq n^{-\epsilon}$ for $\epsilon > 9/10$. Moreover, $e_A(\ell, \mathcal{L}' \setminus \tilde{N}_A(i))$ are independent across $\ell \in \tilde{S}$. Hence, $|\mathcal{L}_0|$ is stochastically lower bounded by $\text{Binom}(m, 1/2)$. It follows from the binomial tail bound (165) and the fact that $m = \Omega(nq)$ that

$$\mathbb{P}\{\mathcal{F}_{ik}\} = \mathbb{P}\{|\mathcal{L}_0| \geq m/4\} \geq 1 - \exp(-\Omega(nq)).$$

Let

$$u_\ell = \frac{1}{\sqrt{(n - \tilde{a}_i - \phi - \psi)q(1 - q) + \psi q'(1 - q')}} [e_A(\ell, \tilde{N}_A(i)^c \setminus \mathcal{L}') - (n - \tilde{a}_i - \phi - \psi)q - \psi q']$$

and

$$v_\ell = \frac{1}{\sqrt{(n - \tilde{a}_i)q(1 - q)}} [\psi(q' - q) + e_A(\ell, \mathcal{L}' \setminus \tilde{N}_A(i)) - \phi q].$$

Let

$$\alpha_\ell = \sqrt{\frac{(n - \tilde{a}_i - \phi - \psi)q(1 - q) + \psi q'(1 - q')}{(n - \tilde{a}_i)q(1 - q)}}.$$

Then $\tilde{a}_\ell^{(i)} = \alpha_\ell u_\ell + v_\ell$. Note that on event $\Gamma_A(i) \cap \Gamma_B(k)$, $\tilde{a}_i \leq (2nq)^2$ and $\phi, \psi \leq 2nq$. Since $q' \leq q \leq n^{-\epsilon}$, it follows that $1/\sqrt{2} \leq \alpha_\ell \leq 1$. Moreover, $|v_\ell| \leq 1$ for all $\ell \in \mathcal{L}_0$. By the Berry-Esseen theorem, we have

$$\begin{aligned} \mathbb{P}\left\{\tilde{a}_\ell^{(i)} \in I\right\} &= \mathbb{P}\left\{u_\ell \in \frac{I - v_\ell}{\alpha_\ell}\right\} = \mathbb{P}\left\{\mathcal{N}(0, 1) \in \frac{I - v_\ell}{\alpha_\ell}\right\} \\ &\pm \frac{O(1)}{\sqrt{nq}} = \frac{\Theta(1)}{L} \pm \frac{O(1)}{\sqrt{nq}} = \frac{\Theta(1)}{L}, \end{aligned}$$

where the last equality holds due to $nq \geq CL^2$.

Conditioning on $E(\mathcal{L}, \mathcal{L}')$ such that event $\mathcal{F}_{ik}$ holds and applying Lemma 6, we get that

$$\mathbb{P}\left\{d(\tilde{P}, \tilde{Q}) \leq \alpha_1 \sqrt{\frac{L}{nq}} \mid E(\mathcal{L}, \mathcal{L}')\right\} \mathbf{1}_{\{\mathcal{F}_{ik}\}} \leq e^{-\Omega(L)}, \quad (162)$$

where α_1 is some absolute constant.

Combining (160) with (162), we have that conditioned on $E(\mathcal{L}, \mathcal{L}')$ such that event $\mathcal{F}_{ik}$ holds, with probability at least $1 - e^{-\Omega(L)}$,

$$\tilde{Z}_{jj'}^{(ik)} \geq \frac{\alpha_1}{2} \sqrt{\frac{L}{nq}} - O\left(\frac{1}{nq}\right) > 2\eta_0 \sqrt{\frac{L}{nq}} = 2\eta$$

for some sufficiently small constant η_0 , where the last inequality holds due to $nq \geq C \log n$. Thus we arrive at the desired claim (158). $\square$

With Lemmas 15 and 17, we are ready to prove Theorem 4.

Proof (Proof of Theorem 4) Let $\mathcal{H}$ denote the event that all 2-hop neighborhoods in the union graph $A \cup B$ are tangle-free. Under the assumption that $q \leq n^{-\epsilon}$ for $\epsilon > 9/10$ and the fact that the union graph $A \cup B \sim \mathcal{G}(n, ps(2 - s))$, it follows from Lemma 13 that $\mathbb{P}\{\mathcal{H}\} \geq 1 - O(n^{9-10\epsilon})$. Define the event $\mathcal{F} = \mathcal{H} \cap (\cap_i (\Gamma_A(i) \cap \Gamma_B(i))) \cap (\cap_{i,k} \Gamma_{ik})$. It follows that

$$\mathbb{P}\{\mathcal{F}^c\} \leq \mathbb{P}\{\mathcal{H}^c\} + \sum_{i \in [n]} (\mathbb{P}\{\Gamma_A^c(i)\} + \mathbb{P}\{\Gamma_B^c(i)\}) + \sum_{i,k \in [n]} \mathbb{P}\{\Gamma_{ik}^c\} \leq O(n^{9-10\epsilon}).$$

Applying Lemma 15 with $L = C \log(nq)$ and averaging over the 2-hop neighborhoods $\tilde{N}_A(i)$ and $\tilde{N}_B(i)$ and noting that $nq \geq C_0 \log n$ for a large constant C_0 , $q \leq n^{-\epsilon}$ for $\epsilon > 9/10$, and $\sigma \leq \sigma_0/L$ for a sufficiently small constant σ_0 , we get that

$$\mathbb{P} \left\{ \left\{ W_{ii} \leq \frac{nq}{4} \right\} \cap \mathcal{H}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \right\} \leq e^{-\Omega(nq)} \leq n^{-2}.$$

Similarly, for $i \neq k$, applying Lemma 17 with $L = C \log(nq)$ and averaging over the 2-hop neighborhoods $\tilde{N}_A(i)$ and $\tilde{N}_B(k)$, we get that

$$\mathbb{P} \left\{ \left\{ W_{ik} \geq \frac{nq}{4} \right\} \cap \mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \right\} \leq e^{-\Omega(nq)} \leq n^{-3}.$$

By the union bound and the fact that $\mathcal{H} \subset H_{ii}$, we have

$$\begin{aligned} \mathbb{P} \left\{ \left\{ \min_{i \in [n]} W_{ii} \leq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} &\leq \sum_{i \in [n]} \mathbb{P} \left\{ \left\{ W_{ii} \leq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} \\ &\leq \sum_{i \in [n]} \mathbb{P} \left\{ \left\{ W_{ii} \leq \frac{nq}{4} \right\} \cap \mathcal{H}_{ii} \cap \Gamma_A(i) \cap \Gamma_B(i) \cap \Gamma_{ii} \right\} \leq n^{-1}. \end{aligned}$$

Similarly, by the union bound and the fact that $\mathcal{H} \subset H_{ik}$, we have

$$\begin{aligned} \mathbb{P} \left\{ \left\{ \max_{i \neq k} W_{ik} \geq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} &\leq \sum_{i \neq k} \mathbb{P} \left\{ \left\{ W_{ik} \geq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} \\ &\leq \sum_{i \neq k} \mathbb{P} \left\{ \left\{ W_{ik} \geq \frac{nq}{4} \right\} \cap \mathcal{H}_{ik} \cap \Gamma_A(i) \cap \Gamma_B(k) \cap \Gamma_{ik} \right\} \leq n^{-1}. \end{aligned}$$

In conclusion, by the union bound,

$$\begin{aligned} &\mathbb{P} \left\{ \min_{i \in [n]} W_{ii} \leq \max_{i \neq k} W_{ik} \right\} \\ &\leq \mathbb{P} \left\{ \mathcal{F}^c \right\} + \mathbb{P} \left\{ \left\{ \min_{i \in [n]} W_{ii} \leq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} + \mathbb{P} \left\{ \left\{ \max_{i \neq k} W_{ik} \geq \frac{nq}{4} \right\} \cap \mathcal{F} \right\} \\ &\leq O \left(n^{9-10\epsilon} \right). \end{aligned}$$

Thus with probability at least $1 - O \left(n^{9-10\epsilon} \right)$, Algorithm 4 outputs $\hat{\pi} = \pi^*$. $\square$

5 Numerical experiments

In this section, we empirically evaluate the performance of degree profile matching (DP), a quadratic programming relaxation of QAP based on doubly stochasticity (QP), and a spectral relaxation (SP).

The performance metric is defined as follows: for a given estimator $\hat{\pi}$ of the ground-truth permutation π^* , we define its accuracy rate as the fraction of correctly matched

pairs:

$$\text{acc}(\widehat{\pi}) \triangleq \frac{1}{n} \sum_{i \in [n]} \mathbf{1}_{\{\pi^*(i) = \widehat{\pi}(i)\}}. \quad (163)$$

Recall that we use outdegrees instead of degrees in our degree profile matching Algorithm 1 to reduce the dependency and facilitate the theoretical analysis. In all numerical experiments, we simply use degree profiles defined through the usual vertex degrees. Moreover, instead of using the Z distance (28) defined as the total variation distance between discretized degree profiles, we directly use the 1-Wasserstein W_1 -distance between degree profiles; see (9) with $p = 1$. Note that for two empirical distributions with the same sample size, such as μ and ν in (6), one can compute their W_1 -distance by sorting the samples:

$$W_1(\mu, \nu) = \sum_{i=1}^n |X_{(i)} - Y_{(i)}|,$$

where $X_{(1)} \geq \dots \geq X_{(n)}$ and $Y_{(1)} \geq \dots \geq Y_{(n)}$. If the sample sizes are different, as is the case for Erdős-Rényi graphs, it is more convenient to compute the W_1 -distance using either the CDF characterization (6) or the original coupling definition.

For the QP method, note that the optimum solution of the quadratic programming relaxation of QAP may not be a permutation matrix. Thus we round the optimal solution to $\mathcal{S}_n$ by projection: $\min_{\Pi \in \mathcal{S}_n} \|\Pi - \widehat{D}\|_F^2$, which is a linear assignment problem and efficiently solvable via max-weighted bipartite matching.

For the SP method, we compute the eigenvectors u of A and v of B corresponding to the largest eigenvalue. Then we align u and v , by finding the permutation π that minimizes the Euclidean distance $\sum_{i \in [n]} |u_i - v_{\pi(i)}|^2$. This is equivalent to $\min_{\Pi \in \mathcal{S}_n} \|\Pi - uv^\top\|_F^2$, which again can be efficiently solved via max-weighted bipartite matching.

For each method, we can potentially boost its accuracy using the iterative clean-up procedure described in Algorithm 5.

Algorithm 5 Iterative clean-up procedure

- 1: **Input:** Graphs A and B on n vertices; a permutation π on $[n]$; and the maximum number of iterations T ;
- 2: **Output:** A permutation $\widehat{\pi}$ on $[n]$.
- 3: (Initialization) Initialize Π_0 to be the permutation matrix corresponding to π
- 4: **for** $t = 1, \dots, T$ **do**
- 5: Solve the linear assignment problem

$$\Pi_{t+1} \in \arg \max_{\Pi \in \mathcal{S}_n} \langle \Pi, A \Pi_t B \rangle \quad (164)$$

- 6: **end for**
 - 7: Output $\widehat{\pi}$ to be the permutation corresponding to Π_{T+1} .
-

Note that $(A\Pi_t B)_{ik}$ in (164) can be viewed as the number of “common” neighbors j between i and k under the permutation π_t in the sense that j is i ’s neighbor in A and $\pi_t(j)$ is k ’s neighbor in B . Hence, (164) finds the matching which maximizes the total sum of “common” neighbors under π_t . This resembles the second stage of Algorithm 3 for seeded graph matching. Alternatively, by rewriting the objective in (164) as $\text{vec}(\Pi)^\top (B \otimes A) \text{vec}(\Pi)$, where $B \otimes A$ denotes the Kronecker product and $\text{vec}(\Pi) \in \mathbb{R}^{n^2}$ denotes the vectorized version of the matrix Π , we can reduce (164) to the projected power iteration discussed in [44].

For ease of notation, we denote by DP+ the degree profile matching algorithm followed by the iterative clean-up procedure. Similarly, we define QP+ and SP+. We run the iterative clean-up procedure up to $T = 100$ iterations. Also, for the sake of computational efficiency, instead of using the max-weighted bipartite matching algorithm to solve (164) exactly, we use the following standard greedy matching algorithm to approximately solve (164) with input weight matrix being $A\Pi_t B$.

Algorithm 6 Greedy Matching

- 1: **Input:** A bipartite graph with $n \times n$ symmetric edge weight matrix W ;
 - 2: **Output:** A $n \times n$ permutation matrix Π .
 - 3: (Initialization) Initialize $\mathcal{M} = \emptyset$
 - 4: **for** all (i, j) in decreasing order of W_{ij} **do**
 - 5: Add (i, j) to $\mathcal{M}$ if $\mathcal{M}$ forms a matching
 - 6: **end for**
 - 7: Output Π , where $\Pi_{ij} = 1$ if $(i, j) \in \mathcal{M}$ and $\Pi_{ij} = 0$ otherwise.
-

5.1 Wigner matrices

We evaluate the performance of all three algorithms as well as their cleaned-up version on the correlated Wigner model given in Sect. 2. The results are shown in Fig. 2 as a function of the noise magnitude σ with $n = 1000$ fixed. Clearly, QP dominates DP, which, in turn, significantly outperforms SP in term of the matching accuracy. Furthermore, the iterative clean-up procedure significantly boosts the accuracy rates for all three methods. Computationally, QP needs to solve a quadratic program, where the Hessian matrix in the objective function involves Kronecker product $B \otimes A$ and thus is of dimension $n^2 \times n^2$. Hence, QP is much more computationally expensive and memory costly than either DP and SP. In our simulation of QP, we developed a fast solver for QP based on the alternating direction method of multipliers (ADMM) algorithm [10, Section 5.2] and that avoids computing $B \otimes A$; nevertheless, even with this fast solver, to generate the simulation results in Fig. 2, QP takes around 85 minutes, while DP takes about 7 minutes, and SP takes about 23 seconds.

Next we simulate the performance of DP and DP+ for different matrix sizes ranging from 100 up to 1600. The results are depicted in Fig. 3. Since our theory predicts that DP succeeds in exact recovery when $\sigma \log n \leq c$ for a small constant c , we rescale the x -axis as $\sigma \log n$. As we can see, the curves for different n align well with each other. Moreover, the accuracy rate of DP gradually drops off from 1 to 0 when $\sigma \log n$

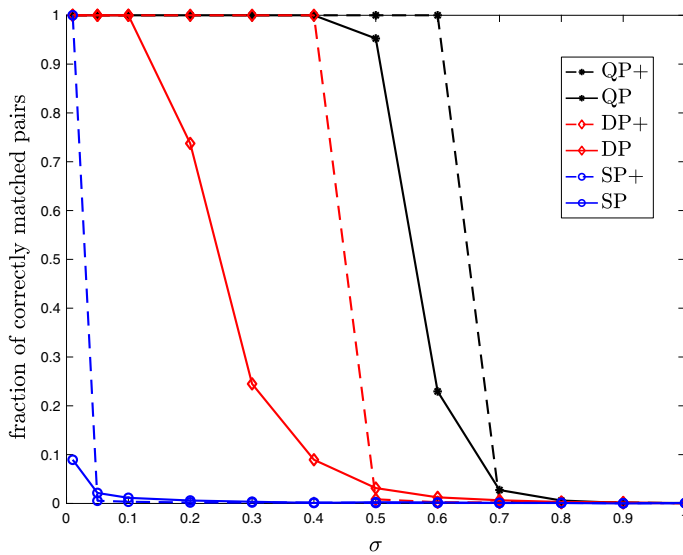


Fig. 2 Simulated correlated Wigner model with $n = 1000$ and varying σ . For each value of σ , the accurate rate shown is the median of 10 independent runs

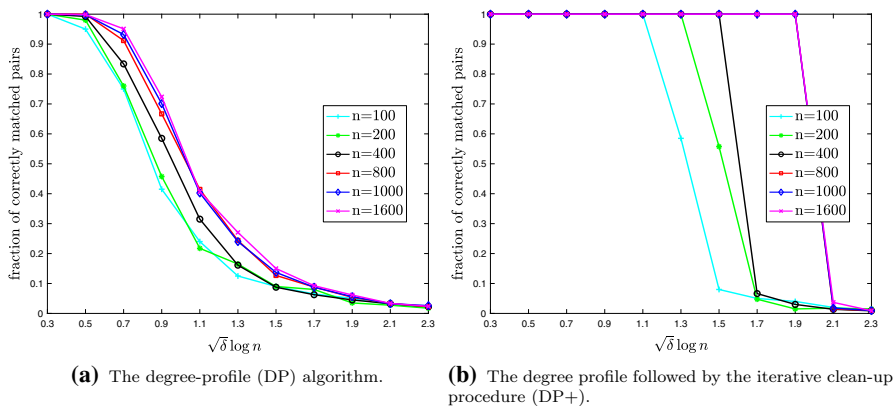


Fig. 3 Simulated correlated Wigner model with varying n and σ . For each value of σ , the accurate rate shown is the median of 10 independent runs

is above 0.7, while that of DP+ sharply drops off from 1 to 0 when $\sigma \log n$ is above 3.3.

5.2 Erdős-Rényi graphs

We evaluate the performance of all three algorithms as well as their cleaned-up version on the correlated Erdős-Rényi graph model $\mathcal{G}(n, q; s)$. We focus on sparse graphs where the edge probability of the parent graph is fixed to be $p \triangleq q/s = \log^2(n)/n$. The simulation results for dense graphs (such as $p = 1/2$) are similar and thus omitted.

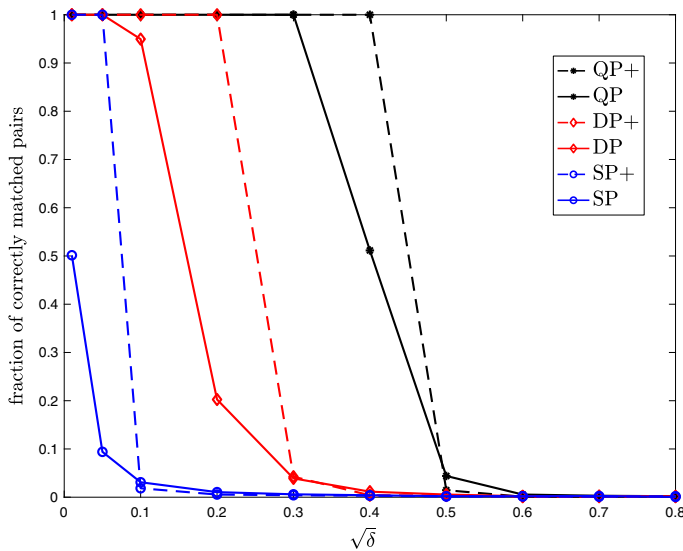


Fig. 4 Simulated correlated Erdős-Rényi graph model $\mathcal{G}(n, q; s)$ with $n = 1000$, $p \triangleq q/s = \log^2(n)/n$, and varying $\sqrt{\delta} = \sqrt{1 - s}$. For each value of $\sqrt{\delta}$, the accurate rate shown is the median of 10 independent runs

The results are shown in Fig. 4 as a function of the edge deletion probability $\delta \triangleq 1 - s$ with $n = 1000$ fixed. Analogous to the Wigner case, QP dominates DP, which, in turn, significantly outperforms SP in term of the matching accuracy, and the iterative clean-up procedure significantly boosts the accuracy rates for all three methods. Computationally, to generate the simulation results in Fig. 4, QP takes around 51 minutes, DP takes about 2 minutes, and SP takes about 12 seconds. Note that each of these methods is run on the same architecture under the same conditions.

Next we simulate the performance of DP and DP+ for different graph sizes ranging from 100 up to 1600. The results are depicted in Fig. 5. Since our theory predicts that DP succeeds in exact recovery when $\sqrt{\delta} \log n \leq c$ for a small constant c , we rescale the x -axis as $\sqrt{\delta} \log n$. As we can see, the curves for different n align well with each other. Analogous to the Wigner case, the accuracy rate of DP gradually drops off from 1 to 0 when $\sqrt{\delta} \log n$ exceeds 0.5, while that of DP+ sharply drops off from 1 to 0 when $\sqrt{\delta} \log n$ exceeds 2.

5.3 Subsampled real networks

In this section, we generate two graphs A and B by independently subsampling a real parent graph G .

Inspired by previous work [28], we consider the Slashdot network. The Slashdot network contains links between the users of Slashdot (a technology-related news website). The network was obtained in February 2009 and is available on Stanford Large Network Dataset Collection (SNAP) [48]. To generate the parent graph G , we first focus on the subnetwork induced by the users whose ID is at most 750, and then

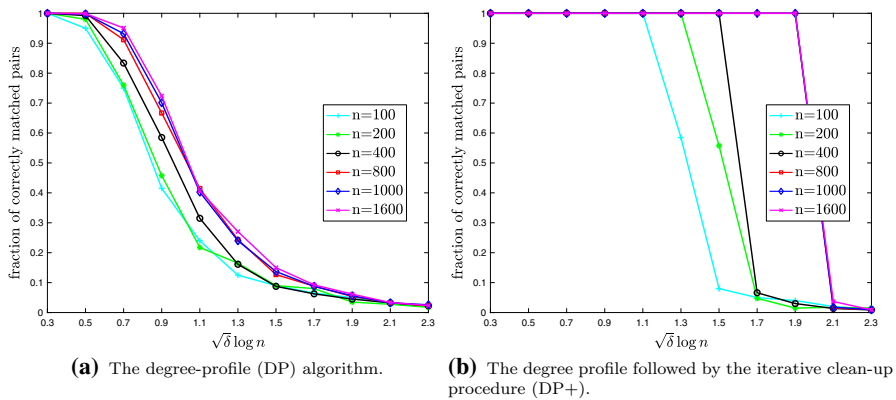


Fig. 5 Simulated correlated Erdős-Rényi graph model with varying n and δ with edge probability in the parent graph fixed to be $\log^2(n)/n$. For each value of δ , the accurate rate shown is the median of 10 independent runs

connect user i and user j if either i has a directed link to j or vice versa. This gives rise to a graph G with 750 vertices and 3338 edges. The graph G is connected and has a heavy-tailed degree distribution. In particular, there are 216 degree-1 vertices, 102 degree-2 vertices, and the average degree is around 9, while the maximum degree is 524 and there are 9 vertices whose degree is at least 100.

To obtain two correlated graphs A and B , we first independently subsample the edges of G twice with probability s , and then relabel the vertices in B according to a random permutation π^* .

We simulate the performance of the three algorithms (DP, QP, and SP) as well as their cleaned-up version, with inputs A and B . The edge subsampling probability s varies from 0.6 to 1, or equivalently δ varies from 0 to 0.4, and the results are shown in Fig. 6.

Note that in the noiseless case of $\delta = 0$, the accuracy rates of all three algorithms as well as their cleaned-up version are about the same and around 0.62. However, in the noisy case, QP dominates DP, which, in turn, significantly outperforms SP in term of the matching accuracy; this is consistent with the observations in the previous two subsections. In particular, as soon as δ becomes positive, the accuracy of SP drops off sharply as expected because the leading eigenvectors of A and B are highly sensitive to the perturbation. In contrast, the accuracy rates of DP and QP drop off gradually as δ increases.

Analogous to our synthetic experiments, the iterative clean-up procedure significantly improves the accuracy of all three methods. In fact, the accuracy rates of all three methods after clean-up (QP+, DP+, and SP+) are about the same for all $\delta \leq 0.225$. At $\delta = 0.25$, the accuracy rate of SP+ drops off sharply, while the accuracy rates of QP+ and DP+ continue to decrease gradually and match each other until $\delta \leq 0.3$. At $\delta = 0.325$, the accuracy rate of DP+ drops off sharply, while the accuracy rate of QP+ continues to decrease gradually.

Computationally, to generate the simulation results in Fig. 6, QP takes about 290 minutes, DP takes about 2 minutes, and SP takes about 18 seconds.

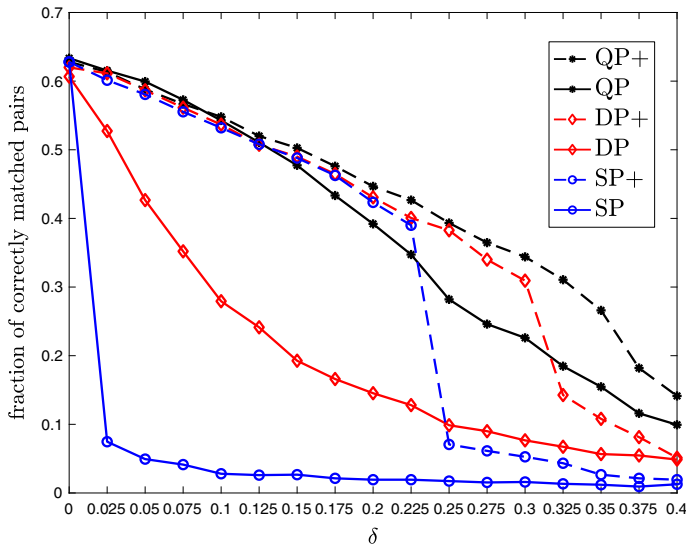


Fig. 6 Slashdot network with $n = 750$ and varying $\delta = 1 - s$. For each value of δ , the accurate rate shown is the median of 10 independent runs

Appendix A Auxiliary results

Recall the following tail bound for binomial random variable $X \sim \text{Binom}(n, p)$ [37, Theorems 4.4, 4.5]

$$\begin{aligned}\mathbb{P}\{X \geq (1+t)np\} &\leq e^{-\frac{t^2}{3}np}, \quad 0 \leq t \leq 1 \\ \mathbb{P}\{X \leq (1-t)np\} &\leq e^{-\frac{t^2}{2}np}, \quad 0 \leq t \leq 1\end{aligned}\quad (165)$$

and

$$\mathbb{P}\{X \geq R\} \leq 2^{-R}, \quad R \geq 6np. \quad (166)$$

Theorem 5 ([43]) *Let $X \sim \text{Bin}(n, p)$. It holds that*

$$\mathbb{P}\{X \leq nt\} \leq \exp\left(-n\left(\sqrt{p} - \sqrt{t}\right)^2\right), \quad \forall 0 \leq t \leq p \quad (167)$$

$$\mathbb{P}\{X \geq nt\} \leq \exp\left(-2n\left(\sqrt{t} - \sqrt{p}\right)^2\right), \quad \forall p \leq t \leq 1. \quad (168)$$

Appendix B Analysis for seeded graph matching

In this section we analyze Algorithm 3 for seeded graph matching. Note that when Algorithm 3 is used as a subroutine in Algorithm 2, the seed set S is obtained from

Algorithm 1 based on matching degree profiles, which can potentially depend on the edges between the non-seeded vertices. To deal with this dependency, the following lemma gives a sufficient condition for the seeded graph matching subroutine (Algorithm 3) to succeed, even if the seed set is chosen adversarially:

Lemma 18 (Seeded graph matching) *Assume $n \geq 4$, $s \geq 30q$, and*

$$n(qs)^2 \geq 2^{11} \times 3 \log^2 n. \quad (169)$$

If the number of seeds satisfies $m \geq \frac{96 \log n}{qs}$, then with probability $1 - 5n^{-1}$, the following holds: for any $\pi_0 : S \rightarrow T$ that coincides with true permutation π^ on the seed set S , (i.e. $\pi_0 = \pi^*|_S$) with $|S| = m$, Algorithm 3 with π_0 as the seed set and threshold $\kappa = \frac{1}{2}mqs$ outputs $\hat{\pi} = \pi$.*

We start by analyzing the first stage of Algorithm 3, which upgrades a partial (but correct) permutation $\pi_0 : S \rightarrow T$ to a full permutation $\pi_1 : [n] \rightarrow [n]$ with at most $O(\log n/q)$ errors, even if the seed set S is adversarially chosen.

Lemma 19 *Assume $n \geq 2$, $mqs \geq 96 \log n$, and $s \geq 12q$. Recall the threshold $\kappa = \frac{1}{2}mqs$ in Algorithm 3. Then with probability at least $1 - 2n^{-m}$, the following holds in Algorithm 3: for any partial permutation $\pi_0 : S \rightarrow T$ such that $\pi_0 = \pi^*|_S$ and $|S| = m$, π_1 is guaranteed to have at most $\frac{192 \log n}{qs}$ errors with respect to π^* , i.e., $|\{i \in [n] : \pi_1(i) \neq \pi^*(i)\}| \leq \frac{192 \log n}{qs}$.*

Proof (Proof of Lemma 19) Without loss of generality, we assume π^* is the identity permutation.

Fix a seed set S of cardinality m . Since $\pi_0 = \pi^*|_S$, it follows that

$$n_{ik} = \sum_{j \in S} A_{ij} B_{k\pi_0(j)} = \sum_{j \in S} A_{ij} B_{k\pi^*(j)}.$$

Recall that according to the definition of the weights in (35), we have

$$w(\pi^*) = \sum_{i \in S^c} \mathbf{1}_{\{n_{ii} \geq \kappa\}}.$$

First, we show that

$$\mathbb{P} \left\{ w(\pi^*) \leq n - m - \frac{32 \log n}{qs} \right\} \leq \exp(-2m \log n), \quad (170)$$

Indeed, for $i \in S^c$ we have $n_{ii} \stackrel{\text{i.i.d.}}{\sim} \text{Binom}(m, qs)$. It follows from the Chernoff bound (165) that

$$\mathbb{P} \{n_{ii} \leq \kappa\} = \mathbb{P} \left\{ n_{ii} \leq \frac{1}{2}mqs \right\} \leq \exp \left(-\frac{1}{8}mqs \right).$$

Therefore,

$$(n - m) - w(\pi^*) = \sum_{i \in S^c} \mathbf{1}_{\{n_{ii} < \kappa\}} \stackrel{s.t.}{\leq} \text{Binom} \left(n - m, \exp \left(-\frac{1}{8} mqs \right) \right).$$

Using the following fact (which follows from a simple union bound)

$$\mathbb{P} \{ \text{Binom}(n, p) \geq t \} \leq \binom{n}{t} p^t, \quad (171)$$

we get that

$$\begin{aligned} \mathbb{P} \{ (n - m) - w(\pi^*) \geq t \} &\leq \binom{n - m}{t} \exp \left(-\frac{t}{8} mqs \right) \leq n^t \exp \left(-\frac{t}{8} mqs \right) \\ &\leq \exp \left(-\frac{t}{16} mqs \right), \end{aligned}$$

where the last inequality holds due to the assumption that $mqs \geq 16 \log n$. Setting $t = \frac{32 \log n}{qs}$, we arrive at the desired (170).

Next, fix any permutation π such that $\pi|_S = \pi_0$ and it has ℓ non-fixed points. Since by assumption $\pi_0 = \pi^*|_S$ and π^* is the identity permutation, it follows that $\pi(i) = i$ for all $i \in S$. Let $F = \{i \in S^c : \pi(i) = i\}$ denote the set of fixed points in S^c . Then $|F| = n - m - \ell$ and $|S^c \setminus F| = \ell$. Thus

$$w(\pi) = \sum_{i \in F} \mathbf{1}_{\{n_{ii} \geq \kappa\}} + \sum_{i \in S^c \setminus F} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}} \leq n - m - \ell + \sum_{i \in S^c \setminus F} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}}.$$

Note that for each $i \in S^c \setminus F$, $n_{i\pi(i)} \sim \text{Binom}(m, q^2)$. Since by assumption $s \geq 12q$, it follows that $\kappa = mqs/2 \geq 6mq^2$. Hence, the Chernoff bound (166) yields that for each $i \in S^c \setminus F$,

$$\mathbb{P} \{ n_{i\pi(i)} \geq \kappa \} \leq 2^{-mqs/2} \leq \exp \left(-\frac{1}{4} mqs \right).$$

Note that $\{n_{i\pi(i)} : i \in S^c \setminus F\}$ are not mutually independent. For instance, $n_{i\pi(i)}$ and $n_{\pi(i), \pi(\pi(i))}$ are dependent. To deal with this dependency issue, we construct a subset $\mathcal{I} \subset S^c \setminus F$ with $|\mathcal{I}| \geq \ell/3$ such that $\{n_{i\pi(i)} : i \in \mathcal{I}\}$ are mutually independent. In particular, consider the canonical cycle decomposition of permutation $\pi|_{S^c \setminus F}$. Let $\mathcal{C}_1, \dots, \mathcal{C}_a$ denote the cycles. Since π has no fixed point in $S^c \setminus F$, each cycle $\mathcal{C}_i$ has length $\ell_i \geq 2$. Let Γ denote the graph formed by the union of these cycles. Each cycle $\mathcal{C}_i$ has an independent set $\mathcal{I}_i$ of size $\lfloor \ell_i/2 \rfloor \geq \ell_i/3$. Let $\mathcal{I} = \cup_{i=1}^a \mathcal{I}_i$. Then $\mathcal{I}$ is an independent set in Γ and $|\mathcal{I}| \geq \sum_{i=1}^a \ell_i/3 = \ell/3$. Since $\mathcal{I}$ is an independent set, it follows that $\{i, \pi(i)\} \cap \{j, \pi(j)\} = \emptyset$ for all $i \neq j \in \mathcal{I}$. Therefore, $\{n_{i\pi(i)} : i \in \mathcal{I}\}$

are mutually independent. Therefore,

$$\sum_{i \in \mathcal{I}} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}} \stackrel{s.t.}{\leq} \text{Binom} \left(|\mathcal{I}|, \exp \left(-\frac{1}{4} mqs \right) \right).$$

Note that

$$w(\pi) \leq n - m - \ell + \sum_{i \in S^c \setminus F} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}} \leq n - m - |\mathcal{I}| + \sum_{i \in \mathcal{I}} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}}$$

Using (171) again, we have

$$\begin{aligned} & \mathbb{P} \left\{ w(\pi) \geq n - m - \frac{32 \log n}{qs} \right\} \\ & \leq \mathbb{P} \left\{ \sum_{i \in \mathcal{I}} \mathbf{1}_{\{n_{i\pi(i)} \geq \kappa\}} \geq |\mathcal{I}| - \frac{32 \log n}{qs} \right\} \\ & \leq \binom{|\mathcal{I}|}{|\mathcal{I}| - \frac{32 \log n}{qs}} \exp \left(-\frac{1}{4} mqs \left(|\mathcal{I}| - \frac{32 \log n}{qs} \right) \right) \\ & \leq 2^\ell \exp \left(-\frac{1}{4} mqs \left(\frac{\ell}{3} - \frac{32 \log n}{qs} \right) \right) \leq 2^\ell \exp \left(-\frac{1}{24} mqs \ell \right), \end{aligned}$$

where the last inequality holds provided $\ell qs \geq 192 \log n$. Let Π_ℓ denote the set of permutations π which has ℓ non-fixed points and satisfies $\pi|_S = \pi_0$. Then $|\Pi_\ell| \leq \binom{n-m}{\ell} \ell! \leq n^\ell$. By the union bound, we have that for any $\ell \geq \frac{192 \log n}{qs}$,

$$\mathbb{P} \left\{ \max_{\pi \in \Pi_\ell} w(\pi) \geq n - m - \frac{32 \log n}{qs} \right\} \leq (2n)^\ell \exp \left(-\frac{1}{24} mqs \ell \right) \leq \exp \left(-\frac{1}{48} mqs \ell \right),$$

where the last inequality holds due to the assumption that $mqs \geq 96 \log n$ and $n \geq 2$. Applying the union bound again over ℓ , we get that

$$\begin{aligned} \mathbb{P} \left\{ \max_{\ell \geq \frac{192 \log n}{qs}} \max_{\pi \in \Pi_\ell} w(\pi) \geq n - m - \frac{32 \log n}{qs} \right\} & \leq \sum_{\ell \geq \frac{192 \log n}{qs}} \exp \left(-\frac{1}{48} mqs \ell \right) \\ & \leq \frac{\exp(-4m \log n)}{1 - \exp(-4m \log n)} \\ & \leq \exp(-2m \log n), \end{aligned}$$

where the last inequality holds due to $m \log n \geq \log 2$.

Combining the last displayed equation with (170) we get that with probability at least $1 - 2n^{-2m}$, π_1 has at most $192 \log n / (qs)$ errors with respect to π^* .

Finally, applying a simple union bound over all the $\binom{n}{m} \leq n^m$ possible choices of seed set S with $|S| = m$, we complete the proof. $\square$

The second stage of Algorithm 3 upgrades an almost exact full permutation $\pi_1 : [n] \rightarrow [n]$ to an exact full permutation $\widehat{\pi} : [n] \rightarrow [n]$. The following lemma provides a worst-case guarantee even if π_1 is adversarially chosen.

Lemma 20 *Let $0 \leq \ell \leq n$. Assume that $(\ell - 1)qs \geq 12nq^2 + 2$ and $(\ell - 1)qs \geq 16 \max\{1, n - \ell\} \log n$. Then with probability at least $1 - 3n^{-1}$, the following holds for Algorithm 3: for any π_1 with at most $n - \ell$ errors with respect to the true permutation π^* , we have $\widehat{\pi} = \pi^*$.*

Proof Without loss of generality, we assume π^* is the identity permutation.

We first fix a permutation π_1 which has at least ℓ fixed points. Let $F \subset [n]$ denote the set of fixed points of π_1 . Then $|F| \geq \ell$. Recall that

$$w_{ik} = \sum_{j \in [n]} A_{ij} B_{k\pi_1(j)}.$$

Then for $i = k$,

$$w_{ii} \geq \sum_{j \in F \setminus \{i\}} A_{ij} B_{ij} \stackrel{s.t.}{\geq} \text{Binom}(|F| - 1, qs).$$

Similarly, for $i \neq k$, note that $A_{ij} B_{k\pi_1(j)} = 0$ if $j = i$ or $j = \pi_1^{-1}(k)$. Thus, $w_{ik} = \sum_{j \in [n] \setminus \{i, \pi_1^{-1}(k)\}} A_{ij} B_{k\pi_1(j)}$. Moreover, $A_{ij} B_{k\pi_1(j)} \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(q^2)$ for all $j \in [n] \setminus \{i, \pi_1^{-1}(k), k\}$. Therefore,

$$w_{ik} \leq \sum_{j \in [n] \setminus \{i, \pi_1^{-1}(k), k\}} A_{ij} B_{k\pi_1(j)} + 1 \stackrel{s.t.}{\leq} \text{Binom}(n - 2, q^2) + 1.$$

It follows from the Chernoff bound (165) that

$$\begin{aligned} \mathbb{P} \left\{ w_{ii} \leq \frac{1}{2}(\ell - 1)qs \right\} &\leq \mathbb{P} \left\{ \text{Binom}(|F| - 1, qs) \leq \frac{1}{2}(\ell - 1)qs \right\} \\ &\leq \exp \left(-\frac{1}{8}(\ell - 1)qs \right). \end{aligned}$$

Thus, by the union bound,

$$\mathbb{P} \left\{ \min_{i \in [n]} w_{ii} \leq \frac{1}{2}(\ell - 1)qs \right\} \leq n \exp \left(-\frac{1}{8}(\ell - 1)qs \right) \leq \exp \left(-\frac{1}{16}(\ell - 1)qs \right),$$

where the last inequality holds due to the assumption that $(\ell - 1)qs \geq 16 \log n$. Moreover, since by assumption $(\ell - 1)qs/2 - 1 \geq 6nq^2$, it follows that the Chernoff bound (166) that for any $i \neq k$,

$$\begin{aligned}\mathbb{P}\left\{w_{ik} \geq \frac{1}{2}(\ell-1)qs\right\} &\leq \mathbb{P}\left\{\text{Binom}(n-2, q^2) \geq \frac{1}{2}(\ell-1)qs-1\right\} \\ &\leq 2^{-(\ell-1)qs/2+1} \leq 2 \exp\left(-\frac{1}{4}(\ell-1)qs\right).\end{aligned}$$

Thus, by the union bound again,

$$\mathbb{P}\left\{\max_{i \neq k} w_{ik} \geq \frac{1}{2}(\ell-1)qs\right\} \leq 2n^2 \exp\left(-\frac{1}{4}(\ell-1)qs\right) \leq 2 \exp\left(-\frac{1}{8}(\ell-1)qs\right).$$

In conclusion, for a fixed permutation π_1 with at least ℓ fixed points, with probability at least $1 - 3 \exp\left(-\frac{1}{8}(\ell-1)qs\right)$,

$$\min_{i \in [n]} w_{ii} > \max_{i \neq k} w_{ik},$$

and hence $\hat{\pi} = \pi^*$.

Finally, applying a simple union bound over all the $\binom{n}{n-\ell}(n-\ell)! \leq n^{n-\ell}$ possible choices of permutation π_1 with at least ℓ fixed points, we get that even if π_1 is adversarially chosen, $\hat{\pi} = \pi^*$ with probability at least

$$1 - 3n^{n-\ell} \exp\left(-\frac{1}{8}(\ell-1)qs\right) \geq 1 - 3 \exp\left(-\frac{1}{16}(\ell-1)qs\right) \geq 1 - 3n^{-1},$$

where the first inequality holds due to $(\ell-1)qs \geq 16(n-\ell) \log n$ and the last inequality holds due to $(\ell-1)qs \geq 16 \log n$. $\square$

We now prove Lemma 18:

Proof (Proof of Lemma 18) In view of Lemma 19, we get that with probability at least $1 - 2n^{-m}$, π_1 is guaranteed to have at most $192 \log n/(qs)$ errors with respect to π^* , even if π_0 , or equivalently the seed set S , is adversarially chosen.

We next apply Lemma 20 with $\ell = n - 192 \log n/(qs)$. In view of the assumption $n(qs)^2 \geq 2^{11} \times 3 \log^2 n$ and $n \geq 4$, we have $(\ell-1) \geq n/2$. Thus $(\ell-1)qs \geq nqs/2 \geq 16 \log n$, and $(\ell-1)qs \geq nqs/2 \geq 12nq^2 + 2$ in view of $s \geq 30q$ and $nqs \geq 20$. Moreover, $(\ell-1)qs \geq nqs/2 \geq 2^{10} \times 3 \log^2 n/(qs) = 16(n-\ell) \log n$. Therefore, all assumptions of Lemma 20 are satisfied. It follows from Lemma 20 that with probability at least $1 - 3n^{-1}$, $\hat{\pi} = \pi^*$, even if π_1 is adversarially chosen.

In conclusion, we get that with probability at least $1 - 5n^{-1}$, Algorithm 3 with π_0 as the seed set outputs $\hat{\pi} = \pi$. $\square$

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