SCIENCE CHINA
Mathematics

Progress of Projects Supported by NSFC
• ARTICLES •

November 2020 Vol. 63 No. 11: 2177–2206
https://doi.org/10.1007/s11425-020-1740-1

Spatial propagation in an epidemic model with nonlocal diffusion: The influences of initial data and dispersals

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Received March 20, 2020; accepted July 12, 2020; published online September 8, 2020

Abstract This paper studies an epidemic model with nonlocal dispersals. We focus on the influences of initial data and nonlocal dispersals on its spatial propagation. Here, initial data stand for the spatial concentrations of the infectious agent and the infectious human population when the epidemic breaks out and the nonlocal dispersals mean their diffusion strategies. Two types of initial data decaying to zero exponentially or faster are considered. For the first type, we show that spreading speeds are two constants whose signs change with the number of elements in some set. Moreover, we find an interesting phenomenon: the asymmetry of nonlocal dispersals can influence the propagating directions of the solutions and the stability of steady states. For the second type, we show that the spreading speed is decreasing with respect to the exponentially decaying rate of initial data, and further, its minimum value coincides with the spreading speed for the first type. In addition, we give some results about the nonexistence of traveling wave solutions and the monotone property of the solutions. Finally, some applications are presented to illustrate the theoretical results.

Keywords nonlocal dispersal, epidemic model, spreading speed, initial data, dispersal kernel

MSC(2010) 35C07, 35K57, 92D25

Citation: Xu W-B, Li W-T, Ruan S G. Spatial propagation in an epidemic model with nonlocal diffusion: The influences of initial data and dispersals. Sci China Math, 2020, 63: 2177–2206, https://doi.org/10.1007/s11425-020-1740-1

1 Introduction

To model the spread of cholera in the European Mediterranean regions in 1973, Capasso and Maddalena [8, 9] proposed a system of two parabolic differential equations to describe a positive feedback interaction between the concentration of bacteria and the infectious human population; namely, the high concentration of bacteria leads to the large infection rate of the human population and once infected the human population increases the growth rate of bacteria. Capasso and Kunisch [7] and Capasso and Wilson [10] also applied this mechanism to model other epidemics with fecal-oral transmission (such as

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typhoid fever and hepatitis A). In these studies, the spatial movements of the infectious agent and the infectious human host are described by the Laplacian operators.

In this paper, we use nonlocal convolution operators to represent the spatial movements of the infectious agent and the infectious human host. Then the epidemic model becomes

$$\begin{cases} u_t(t,x) = \mathcal{D}_1 u(t,x) - \alpha u(t,x) + h(v(t,x)), \quad t>0, \quad x\in\mathbb{R}, \\ v_t(t,x) = \mathcal{D}_2 v(t,x) - \beta v(t,x) + g(u(t,x)), \quad t>0, \quad x\in\mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $u(t,x)$ and $v(t,x)$ biologically stand for the spatial concentration of the infectious agent (bacteria or viruses) and the spatial density of the infectious human population at time t and location $x\in\mathbb{R}$, respectively. The constants $\alpha>0$ and $\beta>0$ denote the natural death rates of the infectious agent and the infectious humans, respectively. The function $h(v)$ denotes the growth of the infectious agent caused by the infectious humans. Meanwhile, the function $g(u)$ is the infection rate of the human population under the assumption that the total susceptible human population is a constant during the evolution of the epidemic. The nonlocal dispersals, represented by the following convolution operators:

$$\mathcal{D}_1 u(t,x) \triangleq k_1 * u(t,x) - u(t,x) = \int_{\mathbb{R}} k_1(x-y)u(t,y)dy - u(t,x),$$
$$\mathcal{D}_2 v(t,x) \triangleq k_2 * v(t,x) - v(t,x) = \int_{\mathbb{R}} k_2(x-y)v(t,y)dy - v(t,x)$$

describe the movements of the infectious agent and the infectious humans, respectively, between not only adjacent but also nonadjacent spatial locations. The dispersal kernel k_i with $i\in\{1,2\}$ is nonnegative and stands for the probability of the movement from the spatial location 0 to x, and thus

$$\int_{\mathbb{R}} k_i(x)dx = 1.$$

Here, the movements between nonadjacent spatial locations can be thought as the long-distance movements of the infectious agent and the infectious humans across cities or countries by air-traffic and other long-distance transportation.

1.1 A brief review of related literature

The spatial propagation of the system (1.1) and its variants has been widely studied in the literature. For example, Li et al. [25] and Meng et al. [34] studied traveling wave solutions, spreading speeds and entire solutions of the system (1.1). We refer to Bao and Li [4], Bao et al. [5], Hu et al. [20], Liu and Wang [29], Wang and Castillo-Chavez [39] and Xu et al. [47] for the results on the spreading dynamics of more general nonlocal dispersal systems. Particularly, if the infected humans do not move during the infectious period (for example, they are in sickbeds or quarantined probably), then the system (1.1) reduces to the following partially degenerate system:

$$\begin{cases} u_t(t,x) = k_1 * u(t,x) - u(t,x) - \alpha u(t,x) + h(v(t,x)), \quad t>0, \quad x\in\mathbb{R}, \\ v_t(t,x) = -\beta v(t,x) + g(u(t,x)), \quad t>0, \quad x\in\mathbb{R}. \end{cases} \tag{1.2}$$

This system is a special case of the system (1.1) with $k_2(x)$ being equal to a Dirac function $\delta(x)$ (the movement happens only between every spatial location and itself; namely, there is no movement of the infected humans). Traveling wave solutions and entire solutions of the system (1.2) were studied by Wang et al. [41], Wu and Hsu [45] and Zhang et al. [55]. For other related results on nonlocal dispersal epidemic models, we refer to for example Li and Yang [26] and Yang et al. [51].

In addition, if the movements of the infectious agent and the infectious human population happen only between adjacent spatial locations, the classical Laplace diffusion operators are applied instead of the nonlocal dispersal operators. For the results about classical diffusion epidemic models, we refer to Allen et al. [2], Cui et al. [13], Cui and Lou [14], Hsu and Yang [19], Wang [38], Xu and Zhao [46] and Zhao and Wang [57].

Other fundamental properties involved in this paper such as the existence and uniqueness of the solution in the system (1.1) can be studied following the theories in [3]. The stability of the steady state can be studied following the techniques in [50, 51, 56]. For more classical results about nonlocal dispersal problems, we refer to Andreu-Vaillo et al. [3], Bates [6], Fife [15], Kao et al. [21], Li et al. [24], Murray [35], Shen and Zhang [36], Wang [42] and the references cited therein.

1.2 Preview of the main results

In this paper, we mainly study the influences of two important factors on the spatial propagation in the model (1.1), namely nonlocal dispersals and initial data. Here, initial data stand for the spatial density of the infectious agent and the infectious human population when epidemic breaks out and the nonlocal dispersals mean their diffusion strategies. Our contribution can be summarized in the following three aspects.

First, we consider the dependence of the spatial propagation on the nonlocal dispersals. Usually, we can find the phenomenon of anisotropic dispersal; for example, the avian influenza viruses carried by migratory birds have a higher possibility to move along the flight route. Then we can use the asymmetric dispersal to study this phenomenon. Here, the asymmetric dispersal (kernel) means that for any spatial location $x \in \mathbb{R}$, the probability of organism moving from 0 to x is not equal to that from 0 to $-x$. Since diffusion is the original driving force of the spatial propagation, it is necessary to study the changes of the spatial propagation caused by the asymmetry of dispersals in the system (1.1).

Before it, we recall the known results on spreading speeds of the following scalar equation:

$$u_t = k * u - u + f(u), \tag{1.3}$$

where $f(\cdot)$ is Fisher-KPP (short for Kolmogorov Petrovsky and Piskunov) type and $k(\cdot)$ is asymmetric. Then there are two constants c_l^* and c_r^* such that

$$\lim_{t\to+\infty} u(t, x+ct) = 1 \quad \text{for } c_l^* < c < c_r^*, \qquad \lim_{t\to+\infty} u(t, x+ct) = 0 \quad \text{for } c < c_l^* \quad \text{or} \quad c > c_r^*,$$

where c_l^* and c_r^* are called the *spreading speeds to left* and *right*, respectively (see Lutscher et al. [31], Finkelshtein et al. [16] and Shen and Zhang [36]). Furthermore, Coville et al. [12] showed that asymmetric kernels may cause the nonpositive minimal wave speed for traveling wave solutions (see also Sun et al. [37] and Zhang et al. [53, 54]). As is well known, the minimal wave speed for traveling wave solutions always equals the spreading speed in the Fisher-KPP equations. Therefore, it is worth identifying the signs of spreading speeds when the kernels are asymmetric. Recently, this problem was solved in our paper [49], and furthermore, it was shown that the asymmetry level of the kernel determines the signs of spreading speeds c_l^* and c_r^*, which in turn determine the propagating directions of the solutions and influence the stability of equilibrium states [32].

Motivated by [20, 49], we study the influences of asymmetric kernels on the spatial propagation and identify the signs of spreading speeds. However, such a problem is more difficult than that in the equation (1.3), because the signs of spreading speeds c_l^* and c_r^* in the system (1.1) are actually influenced by two kernels $k_1(\cdot)$ and $k_2(\cdot)$. In order to treat this problem, we define

$$\Lambda = \{\lambda \in \mathbb{R} \mid A(\lambda)B(\lambda) \geqslant g'(0)h'(0),\ A(\lambda) < 0,\ B(\lambda) < 0\},$$

where

$$A(\lambda) = \int_{\mathbb{R}} k_1(x)\mathrm{e}^{\lambda x}dx - 1 - \alpha, \quad B(\lambda) = \int_{\mathbb{R}} k_2(x)\mathrm{e}^{\lambda x}dx - 1 - \beta.$$

Then we show that the signs of c_l^* and c_r^* change with the number of elements in the set Λ (see Theorem 2.2) which is essentially determined by the dispersal kernels $k_1(\cdot)$ and $k_2(\cdot)$. Particularly, when $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric, it follows that $c^* \triangleq c_r^* = -c_l^* > 0$.

We show that in the system (1.1), the asymmetric dispersals can influence the propagating directions of the solutions and the stability of steady states. More precisely, denote the spatial region

$$\Omega(t) \triangleq \{x \in \mathbb{R} \mid (u(t,x), v(t,x)) \geqslant (\nu, \nu)\} \quad \text{for } t \geqslant 0 \text{ with some } \nu \in (0,1), \tag{1.4}$$

and there is an interesting phenomenon: $\Omega(t)$ propagates to both the left and the right of the x-axis for $c_l^* < 0 < c_r^*$, propagates only to the right for $0 < c_l^* < c_r^*$, and propagates only to the left for $c_l^* < c_r^* < 0$. For some appropriate initial data, when $c_l^* < 0 < c_r^*$, the steady state $(u, v) \equiv (1, 1)$ is stable, i.e., $(u(t,x), v(t,x)) \to (1,1)$ as $t \to +\infty$, but when $0 < c_l^* < c_r^*$ or $c_l^* < c_r^* < 0$, we see that $(u(t,x), v(t,x)) \to (0,0)$ as $t \to +\infty$ in any bounded spatial region.

Next, we study the dependence of the spatial propagation on initial data. Consider two types of initial data which decay to zero exponentially or faster as $|x| \to +\infty$, but their decaying rates are different. We establish a relationship between the spreading speed and the exponentially decaying rate λ of initial data. For the first type whose decaying rate is large (this type includes compactly supported functions), we show that spreading speeds are constants c_l^* and c_r^* (see Theorem 3.1). For the second type whose decaying rate is small, when $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric, we show that the spreading speed $c(\lambda)$ is decreasing with respect to λ, and the minimum value of $c(\lambda)$ coincides with c^* (see Theorem 4.2). In addition, we obtain two other results of the system (1.1), namely the nonexistence of traveling wave solutions (see Corollary 3.2) and the monotone property of the solutions (see Theorem 4.1).

These results give us guidance for better control of the spatial propagation of epidemics. We see that even though the spatial concentrations of the infectious agent and the infectious human population are very low at the spatial locations far away from $x = 0$, they have an important influence on the spatial propagation of the system (1.1). Therefore, in order to slow down the spreading speed of epidemics, the prevention in low-density spatial regions is at least as important as the treatment in high-density spatial regions. In addition, there are some applications of the theoretical results to the control of epidemics whose infectious agent is carried by migratory birds. As we shall see in Section 5, it is possible that the epidemic spreads only along the flight route of migratory birds and the spatial propagation against the flight route fails, as long as the infectious humans are kept from moving frequently.

Finally, we show that the spreading speed in this paper is studied by applying the comparison principle (see Lemma 3.4) and constructing new types of upper and lower solutions, instead of the classic theories of spreading speeds which are established by Weinberger [43] and developed by Lewis et al. [22], Li et al. [23], Liang and Zhao [27, 28], Lui [30] and Yi and Zou [52]. Indeed, when we study the dependence of spreading speeds on initial data, the method of upper and lower solutions is more useful because it can deal with more general types of initial data (see, e.g., Hamel and Nadin [17], Hamel and Roques [18] and Xu et al. [49]). We present a new method to construct the lower solution of the system (1.1) which spreads at a speed of c_1 or c_2, where $c_1 \in (c_r^* - \epsilon, c_r^*)$ and $c_2 \in (c_l^*, c_l^* + \epsilon)$. We also apply the new *"forward-backward spreading"* method which was first given in our previous paper [49]. In this method, for any time $T > 0$ and any $\mu \in [0,1]$, we construct a lower solution $U_1(t,x)$ in the first period of time $[0, \mu T]$ which spreads at a speed of c_1, and in the second period of time $[\mu T, T]$ we construct another lower solution $U_2(t,x)$ which spreads at a speed of c_2 and satisfies $U_2(\mu T, x) \leqslant U_1(\mu T, x)$. Then these two lower solutions can be regarded as a lower solution defined in the time period $[0, T]$ whose speed is $\bar{c} = \mu c_1 + (1-\mu) c_2$. Moreover, the arbitrariness of μ guarantees that $\bar{c}$ can be any number in $[c_1, c_2]$.

The methods in this paper could be applicable to the following m-species nonlocal dispersal cooperative system:

$$\begin{cases} \partial_t U(t,x) = K * U(t,x) - U(t,x) + F(U(t,x)), & t > 0, \quad x \in \mathbb{R}, \\ U(0,x) = U_0(x) = (u_{0,1}(x), \ldots, u_{0,m}(x)), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $m \geqslant 2$, $U(t,x) = (u_1(t,x), \ldots, u_m(t,x))$ and $K(x) = (k_1(x), \ldots, k_m(x))$. Here, the function $F(U) = (f_1(U), \ldots, f_m(U))$ is cooperative and $F'(\mathbf{0})$ is an irreducible matrix. Actually, the system (1.1) can be regarded as a special case of the system (1.5) with $m = 2$. The study of the system (1.1) has simpler calculations, but it shows clearer presentations of the new upper and lower solutions and the "forward-backward spreading" method. Moreover, in the system (1.5), if the nonlocal dispersal operators are replaced by Laplacian operators, all the methods still work. However, it is not necessary to apply the "forward-backward spreading" method, since we can use a monotone property similar to Theorem 4.1 instead (see also the proof of Theorem 4.2 for more details).

The rest of this paper is organized as follows. In Section 2, we present the definitions and some mathematical analysis of spreading speeds. Section 3 is devoted to the spatial propagation for the first type of initial data and asymmetric kernels. In Section 4, we study the spatial propagation for the second type of initial data and symmetric kernels. Meanwhile, we also prove some monotone property result for the system (1.1). In Section 5, we give some applications of the theoretical results.

2 The signs of spreading speeds

In this section, we define the notations of spreading speeds and identify their signs. First, we give some assumptions. Let α and β be two positive constants. Throughout this paper, we assume $g(\cdot)$ and $h(\cdot)$ are two functions in $C^1([0,1]) \cap C^{1+\delta_0}([0,p_0])$, where δ_0 and p_0 are two constants in $(0,1)$, and satisfy that

(H1) $g(0)=h(0)=0$, $h(1)/\alpha=g(1)/\beta=1$, $h(g(s)/\beta)-\alpha s>0$ for all $s\in(0,1)$;

(H2) $0<g(u)\leqslant g'(0)u$, $g'(u)\geqslant 0$ for all $u\in(0,1)$; $0<h(v)\leqslant h'(0)v$, $h'(v)\geqslant 0$ for all $v\in(0,1)$.

From (H1) and (H2), the system (1.1) is monostable and $(u(t,x),v(t,x))\equiv(1,1)$ is the unique nontrivial steady state. Moreover, we have $\alpha\beta<h'(0)g'(0)$. Suppose $k_1(\cdot)$ and $k_2(\cdot)$ are two continuous and nonnegative dispersal kernel functions satisfying

(K1) $\int_{\mathbb{R}} k_i(x)dx=1$ and $\int_{\mathbb{R}} k_i(x)\mathrm{e}^{\lambda x}dx<+\infty$ for any $\lambda\in\mathbb{R}$ and $i\in\{1,2\}$;

(K2) there are $x_i^+\in\mathbb{R}^+$ and $x_i^-\in\mathbb{R}^-$ such that $k_i(x_i^\pm)>0$ for each $i\in\{1,2\}$.

We assume the initial data $u_0(\cdot)$ and $v_0(\cdot)$ are two continuous functions which satisfy that $0\leqslant u_0(x)\leqslant 1$, $0\leqslant v_0(x)\leqslant 1$ for all $x\in\mathbb{R}$ and

$$u_0(x)\to 0,\quad v_0(x)\to 0\quad \text{as } |x|\to+\infty.$$

Now define

$$c(\lambda)=\frac{1}{\lambda}D(\lambda)\quad \text{for } \lambda\neq 0, \tag{2.1}$$

where

$$D(\lambda)=\frac{1}{2}[A(\lambda)+B(\lambda)+\sqrt{(A(\lambda)-B(\lambda))^2+4g'(0)h'(0)}]$$

and

$$A(\lambda)=\int_{\mathbb{R}} k_1(x)\mathrm{e}^{\lambda x}dx-1-\alpha,\quad B(\lambda)=\int_{\mathbb{R}} k_2(x)\mathrm{e}^{\lambda x}dx-1-\beta. \tag{2.2}$$

It follows that $D(\lambda)>A(\lambda)$ and $D(\lambda)>B(\lambda)$ for $\lambda\in\mathbb{R}$. Particularly, if $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric, then $c(\lambda)=-c(-\lambda)$ for $\lambda\neq 0$.

Theorem 2.1. *There are two unique constants* $\lambda_r^*\in\mathbb{R}^+$ *and* $\lambda_l^*\in\mathbb{R}^-$ *such that*

$$c_r^*\triangleq c(\lambda_r^*)=\inf_{\lambda\in\mathbb{R}^+}\{c(\lambda)\},\quad c_l^*\triangleq c(\lambda_l^*)=\sup_{\lambda\in\mathbb{R}^-}\{c(\lambda)\}, \tag{2.3}$$

and $c'(\lambda)<0$ *for* $\lambda\in(\lambda_l^*,0)\cup(0,\lambda_r^*)$. *Moreover, we have* $c_l^*<c_r^*$. *Particularly, if* $k_1(\cdot)$ *and* $k_2(\cdot)$ *are symmetric, then* $c^*\triangleq c_r^*=-c_l^*>0$ *and* $\lambda^*\triangleq\lambda_r^*=-\lambda_l^*$.

Proof. This proof is based on some mathematical analysis of the functions $c'(\lambda)$ and $c''(\lambda)$. First, we prove

$$\lim_{\lambda\to 0^+}c'(\lambda)=-\infty\quad\text{and}\quad\lim_{\lambda\to 0^-}c'(\lambda)=-\infty. \tag{2.4}$$

By some simple calculations, we see the functions $A(\lambda)$, $B(\lambda)$, $A'(\lambda)$ and $B'(\lambda)$ are uniformly bounded as $\lambda\to 0$. Then the functions $D(\lambda)$ and $D'(\lambda)$ are also uniformly bounded as $\lambda\to 0$. Therefore, we easily get (2.4) from $c'(\lambda)=\lambda^{-1}D'(\lambda)-\lambda^{-2}D(\lambda)$ and $D(0)>0$.

Now we show that

$$c'(\lambda)>0\quad\text{for } |\lambda| \text{ large enough.} \tag{2.5}$$

From the definitions of the functions $c(\lambda)$ and $D(\lambda)$, we have

$$\begin{aligned}2\lambda^2 c'(\lambda) &= 2(\lambda D'(\lambda) - D(\lambda)) \\ &= (\lambda A' - A) + (\lambda B' - B) + \frac{(A-B)[(\lambda A' - A) - (\lambda B' - B)] - 4g'(0)h'(0)}{[(A-B)^2 + 4g'(0)h'(0)]^{\frac{1}{2}}}.\end{aligned}$$

Then from

$$|A - B| < [(A-B)^2 + 4g'(0)h'(0)]^{\frac{1}{2}},$$

it follows that

$$\lambda^2 c'(\lambda) > \min\{\lambda A'(\lambda) - A(\lambda) - \sqrt{g'(0)h'(0)}, \lambda B'(\lambda) - B(\lambda) - \sqrt{g'(0)h'(0)}\}.$$

By some simple calculations, we have

$$\begin{aligned}\lambda A'(\lambda) - A(\lambda) &= \int_{\mathbb{R}} k_1(x) \mathrm{e}^{\lambda x}(\lambda x - 1) dx + 1 + \alpha \to +\infty \quad \text{as } |\lambda| \to +\infty, \\ \lambda B'(\lambda) - B(\lambda) &= \int_{\mathbb{R}} k_2(x) \mathrm{e}^{\lambda x}(\lambda x - 1) dx + 1 + \beta \to +\infty \quad \text{as } |\lambda| \to +\infty,\end{aligned}$$

which imply that (2.5) holds.

Next, we try to prove that

$$\lambda c''(\lambda) > 0 \quad \text{for } \lambda \neq 0, \quad \text{provided } c'(\lambda) = 0. \tag{2.6}$$

Indeed, since

$$c''(\lambda) = \lambda^{-1}[D''(\lambda) - 2c'(\lambda)],$$

we just need to prove that

$$D''(\lambda) > 0 \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.7}$$

From the definitions of the functions $A(\lambda)$, $B(\lambda)$ and $D(\lambda)$, it follows that for all $\lambda \in \mathbb{R}$, $A''(\lambda) > 0$, $B''(\lambda) > 0$ and

$$2D'' = A'' + B'' + \frac{(A-B)(A''-B'')}{[(A-B)^2 + 4g'(0)h'(0)]^{\frac{1}{2}}} + \frac{4h'(0)g'(0)(A'-B')^2}{[(A-B)^2 + 4g'(0)h'(0)]^{\frac{3}{2}}}.$$

By $|A - B| < [(A-B)^2 + 4g'(0)h'(0)]^{\frac{1}{2}}$, we get

$$D''(\lambda) \geqslant \min\{A''(\lambda), B''(\lambda)\} > 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

Then we get (2.6).

It follows from (2.6) that there is at most one constant λ_r^* in $\mathbb{R}^+$ such that $c'(\lambda_r^*) = 0$. Meanwhile, (2.4) and (2.5) imply the existence of this constant. Similarly, there is a unique constant $\lambda_l^* \in \mathbb{R}^-$ such that $c'(\lambda_l^*) = 0$. Therefore, we have

$$c'(\lambda) \begin{cases} > 0, & \lambda \in (-\infty, \lambda_l^*) \cup (\lambda_r^*, +\infty), \\ = 0, & \lambda = \lambda_l^* \quad \text{or} \quad \lambda = \lambda_r^*, \\ < 0, & \lambda \in (\lambda_l^*, 0) \cup (0, \lambda_r^*). \end{cases} \tag{2.8}$$

Then we obtain (2.3) from (2.8). Moreover, since $c'(\lambda) = \lambda^{-1}[D'(\lambda) - c(\lambda)]$ and $c'(\lambda_l^*) = c'(\lambda_r^*) = 0$, we have

$$c_l^* = c(\lambda_l^*) = D'(\lambda_l^*) \quad \text{and} \quad c_r^* = c(\lambda_r^*) = D'(\lambda_r^*).$$

From (2.7) and $\lambda_l^* < 0 < \lambda_r^*$, it follows that $c_l^* < c_r^*$. Particularly, if $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric, we have $D(\lambda) = D(-\lambda)$ for $\lambda \in \mathbb{R}$. Then $c(\lambda) + c(-\lambda) = 0$ for $\lambda \neq 0$, which implies $\lambda_r^* = -\lambda_l^*$ and $c_r^* = -c_l^* > 0$. $\square$

In order to identify the signs of c_l^* and c_r^*, we define a set

$$\Lambda \triangleq \{\lambda \in \mathbb{R} \mid A(\lambda)B(\lambda) \geqslant g'(0)h'(0),\ A(\lambda) < 0,\ B(\lambda) < 0\}.$$

Now we give a relationship between the set Λ and the signs of c_l^* and c_r^*.

Theorem 2.2. *We have either $\Lambda \subseteq \mathbb{R}^+$ or $\Lambda \subseteq \mathbb{R}^-$. Moreover,*

(i) *if $\Lambda = \emptyset$, then $c_l^* < 0 < c_r^*$;*
(ii) *if $\Lambda \cap \mathbb{R}^+$ is a singleton set, then $c_l^* < c_r^* = 0$;*
(iii) *if $\Lambda \cap \mathbb{R}^-$ is a singleton set, then $0 = c_l^* < c_r^*$;*
(iv) *if $\mathrm{int}(\Lambda) \cap \mathbb{R}^+ \neq \emptyset$, then $c_l^* < c_r^* < 0$;*
(v) *if $\mathrm{int}(\Lambda) \cap \mathbb{R}^- \neq \emptyset$, then $0 < c_l^* < c_r^*$.*

Proof. First, we prove that either $\Lambda \subseteq \mathbb{R}^+$ or $\Lambda \subseteq \mathbb{R}^-$. Since

$$A(0)B(0) = \alpha\beta < h'(0)g'(0),$$

we have $0 \notin \Lambda$. So it is sufficient to prove that the set Λ is a closed interval in $\mathbb{R}$. For this purpose, we denote

$$\Lambda^A = \{\lambda \in \mathbb{R} \mid A(\lambda) < 0\} \quad \text{and} \quad \Lambda^B = \{\lambda \in \mathbb{R} \mid B(\lambda) < 0\}.$$

Then we have $\Lambda \subseteq \Lambda^A \cap \Lambda^B$. Some calculations show that $A''(\lambda) > 0$ and $B''(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, which imply that the sets Λ^A and Λ^B are two open intervals in $\mathbb{R}$. For any $\lambda \in \Lambda^A \cap \Lambda^B$, if

$$(A(\lambda)B(\lambda))' = A'(\lambda)B(\lambda) + A(\lambda)B'(\lambda) = 0,$$

then we have

$$(A(\lambda)B(\lambda))'' = A''(\lambda)B(\lambda) + A(\lambda)B''(\lambda) + 2A'(\lambda)B'(\lambda) < 0. \tag{2.9}$$

Therefore, the set Λ is a closed interval in $\mathbb{R}$, which means that either $\Lambda \subseteq \mathbb{R}^+$ or $\Lambda \subseteq \mathbb{R}^-$.

Now we determine the signs of c_l^* and c_r^*. From the definition of the function $D(\lambda)$, we have

$$D(\lambda) < 0 \Leftrightarrow A(\lambda) + B(\lambda) < 0 \quad \text{and} \quad A(\lambda)B(\lambda) > g'(0)h'(0) \Leftrightarrow \lambda \in \mathrm{int}(\Lambda).$$

Similarly, we can get

$$D(\lambda) = 0 \Leftrightarrow A(\lambda) + B(\lambda) < 0 \quad \text{and} \quad A(\lambda)B(\lambda) = g'(0)h'(0) \Leftrightarrow \lambda \in \partial\Lambda.$$

Then it follows that

$$D(\lambda) > 0 \Leftrightarrow \lambda \notin \Lambda.$$

Therefore, if $\Lambda = \emptyset$, then $D(\lambda) > 0$ for all $x \in \mathbb{R}$, which implies that $c_l^* < 0 < c_r^*$. If there is some constant $\lambda_0 \in \mathbb{R}^+$ such that $\Lambda \cap \mathbb{R}^+ = \{\lambda_0\} = \partial\Lambda$, we have

$$c(\lambda_0) = 0 = \inf_{\lambda \in \mathbb{R}^+}\{c(\lambda)\} = c_r^* > c_l^*.$$

If there is some constant $\lambda_0 \in \mathrm{int}(\Lambda) \cap \mathbb{R}^+$, then it follows that $0 > c(\lambda_0) \geqslant c_r^* > c_l^*$. Similarly, we can get Theorems 2.2(iii) and 2.2(v). □

Remark 2.3. From Theorem 2.2 we can see that the signs of c_l^* and c_r^* change with the number of elements in the set Λ, which is essentially determined by the kernels $k_1(\cdot)$ and $k_2(\cdot)$. Moreover, from Theorem 2.2(i) we have $c_l^* < 0 < c_r^*$ when

$$(1 + \alpha - E(k_1))(1 + \beta - E(k_2)) < g'(0)h'(0), \tag{2.10}$$

where $E(k)$ can describe the asymmetry level of $k(\cdot)$ and is defined by

$$E(k) = \inf\left\{ \int_{\mathbb{R}} k(x)\mathrm{e}^{\lambda x}dx \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

It is easy to check that $E(k) \in [0, 1]$. Particularly, when $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric, we have $E(k_1) = E(k_2) = 1$, which verifies that (2.10) is right by $\alpha\beta < h'(0)g'(0)$.

3 The first type of initial data and the case of asymmetric kernels

In this section, we establish the spatial propagation result of the system (1.1) for the first type of initial data and asymmetric kernels by constructing new types of upper and lower solutions and using the "forward-backward spreading" method. Now we present the main theorem.

Theorem 3.1. *Assume that* (H1), (H2), (K1) *and* (K2) *hold. If* $u_0(\cdot)$ *and* $v_0(\cdot)$ *satisfy that* $u_0(x_0) > 0$, $v_0(x_0) > 0$ *for some constant* $x_0 \in \mathbb{R}$ *and there are two positive constants* x_1 *and* Γ_0 *such that*

$$\max\ \{u_0(x), v_0(x)\}\mathrm{e}^{\lambda_l^* x} \leqslant \Gamma_0 \quad \textit{for } x \leqslant -x_1, \quad \max\{u_0(x), v_0(x)\}\mathrm{e}^{\lambda_r^* x} \leqslant \Gamma_0 \quad \textit{for } x \geqslant x_1,$$

then for any small $\epsilon > 0$ *there is a constant* $\nu \in (0,1)$ *such that the solution of the system* (1.1) *has the following properties*:

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{x-x_0 \leqslant (c_l^*-\epsilon)t} (u(t,x), v(t,x)) = (0,0), \\ \inf\limits_{(c_l^*+\epsilon)t \leqslant x-x_0 \leqslant (c_r^*-\epsilon)t} (u(t,x), v(t,x)) \geqslant (\nu,\nu) \quad \textit{for all } t>0, \\ \lim\limits_{t\to+\infty} \sup\limits_{x-x_0 \geqslant (c_r^*+\epsilon)t} (u(t,x), v(t,x)) = (0,0). \end{cases}$$

Before giving its proof, we show some other results derived from Theorem 3.1. We see that the spreading speeds of the system (1.1) for this type of initial values are c_l^* and c_r^* whose signs are determined by $k_1(\cdot)$ and $k_2(\cdot)$ as stated in Section 2. Therefore, the asymmetric dispersals in the system (1.1) can influence the propagating directions of the solutions and the stability property of steady states. More precisely, the spatial region $\Omega(t)$ defined by (1.4) propagates to both the left and the right of the x-axis for $c_l^* < 0 < c_r^*$, propagates only to the right for $0 < c_l^* < c_r^*$, and propagates only to the left for $c_l^* < c_r^* < 0$. However, if the set $\Omega(t)$ is connected at time $t > 0$, in the case of $0 = c_l^* < c_r^*$, the movement of the left boundary of $\Omega(t)$ is slower than linearity and we cannot identify its propagating direction. Similarly, we cannot identify the propagating direction of the right boundary of $\Omega(t)$ in the case of $c_l^* < c_r^* = 0$ either. Furthermore, for this type of initial data, when $c_l^* < 0 < c_r^*$, the steady state $(u(t,x), v(t,x)) \equiv (1,1)$ is stable, i.e., $(u(t,x), v(t,x)) \to (1,1)$ as $t \to +\infty$, but when $c_l^* < c_r^* < 0$ or $0 < c_l^* < c_r^*$, we see that $(u(t,x), v(t,x)) \to (0,0)$ as $t \to +\infty$ in any bounded spatial region.

From Theorem 3.1 we also obtain the following spatial propagation phenomenon: any small positive perturbation of the steady state $(u(t,x), v(t,x)) \equiv (0,0)$ at some spatial location $x_0 \in \mathbb{R}$ and time $t = 0$ (namely $(u(0,x_0), v(0,x_0)) > (0,0)$ holds) will spread in the spatial region

$$\Omega(t,\epsilon,x_0) \triangleq \{x \in \mathbb{R} \mid (c_l^*+\epsilon)t \leqslant x - x_0 \leqslant (c_r^* - \epsilon)t\} \quad \text{for any } t>0 \text{ and small } \epsilon > 0, \tag{3.1}$$

which means that $(u(t,x), v(t,x)) > (\mu,\mu)$ for $x \in \Omega(t,\epsilon,x_0)$ and some constant $\mu > 0$. From this result, we can get some nonexistence results of traveling wave solutions of the following system:

$$\begin{cases} u_t(t,x) = k_1 * u(t,x) - u(t,x) - \alpha u(t,x) + h(v(t,x)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \\ v_t(t,x) = k_2 * v(t,x) - v(t,x) - \beta v(t,x) + g(u(t,x)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \end{cases} \tag{3.2}$$

Corollary 3.2. *Assume that* (H1), (H2), (K1) *and* (K2) *hold. Suppose that*

$$(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$$

is a traveling wave solution of the system (3.2) *and satisfies* $(\phi,\psi) \not\equiv (0,0)$. *We have*

(i) *if* $(\phi(+\infty), \psi(+\infty)) = (0,0)$, *then* $c \geqslant c_r^*$;

(ii) *if* $(\phi(-\infty), \psi(-\infty)) = (0,0)$, *then* $c \leqslant c_l^*$.

Proof. Let the initial data $(u_0(x), v_0(x))$ in the system (1.1) satisfy

$$(u_0(x), v_0(x)) \leqslant (\phi(x), \psi(x)) \quad \text{for } x \in \mathbb{R}, \quad (u_0(x_0), v_0(x_0)) \gg (0,0) \quad \text{for some } x_0 \in \mathbb{R}.$$

Then Theorem 3.1 and the comparison principle (see Lemma 3.4) show that for any constant $\epsilon > 0$ small enough,

$$(\phi(x-ct), \psi(x-ct)) \geqslant (u(t,x), v(t,x)) \geqslant (\nu, \nu) \quad \text{for } t > 0, \quad x \in \Omega(t, \epsilon, x_0),$$

where $(u(t,x), v(t,x))$ is a solution of the system (1.1) and $\Omega(t, \epsilon, x_0)$ is defined by (3.1).

In Case (i), we suppose $c < c_r^*$. Let ϵ be small enough such that $0 < \epsilon < c_r^* - c$. By taking a constant $c_0 \in \mathbb{R}$ satisfying $\max\{c, c_l^* + \epsilon\} < c_0 < c_r^* - \epsilon$, we get that $x_0 + c_0 t \in \Omega(t, \epsilon, x_0)$ and

$$(\phi(x_0 + c_0 t - ct), \psi(x_0 + c_0 t - ct)) \geqslant (\nu, \nu) \quad \text{for } t > 0.$$

It is a contradiction to $(\phi(+\infty), \psi(+\infty)) = (0, 0)$. Similarly, we can prove Case (ii). □

Remark 3.3. Corollary 3.2 shows that there exists no traveling wave solution $(u(t,x), v(t,x)) = (\phi(x-ct), \psi(x-ct))$ of the system (3.2) satisfying $(\phi(+\infty), \psi(+\infty)) = (0,0)$ and $c \in (-\infty, c_r^*)$. Meanwhile, the system (3.2) has no traveling wave solution satisfying $(\phi(-\infty), \psi(-\infty)) = (0,0)$ and $c \in (c_l^*, +\infty)$ either.

Now we focus on the proof of Theorem 3.1 in the following three subsections.

3.1 Preliminaries

The basic tools in the proof of Theorem 3.1 are the method of upper and lower solutions and the following comparison principle of the system (1.1) whose proof can be found in [25].

Lemma 3.4 (Comparison principle)**.** *Assume that* (H1)*,* (H2) *and* (K1) *hold. For any* $\tau > 0$*, if the continuous functions* $(u_1(t,x), v_1(t,x))$ *and* $(u_2(t,x), v_2(t,x))$ *satisfy*

$$\begin{cases} \partial_t u_1 - k_1 * u_1 + u_1 + \alpha u_1 - h(v_1) \geqslant \partial_t u_2 - k_1 * u_2 + u_2 + \alpha u_2 - h(v_2), \\ \partial_t v_1 - k_2 * v_1 + v_1 + \beta v_1 - g(u_1) \geqslant \partial_t v_2 - k_2 * v_2 + v_2 + \beta v_2 - g(u_2), \\ u_1(0,x) \geqslant u_2(0,x), \quad v_1(0,x) \geqslant v_2(0,x) \end{cases}$$

for $t \in (0, \tau]$*,* $x \in \mathbb{R}$*, then* $(u_1(t,x), v_1(t,x)) \geqslant (u_2(t,x), v_2(t,x))$ *for* $t \in [0, \tau]$ *and* $x \in \mathbb{R}$*.*

Next, we define some notations. For $c \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, denote

$$G(c, \lambda) \triangleq c\lambda - A(\lambda) = c\lambda - \int_{\mathbb{R}} k_1(x) e^{\lambda x} dx + 1 + \alpha, \tag{3.3}$$

$$H(c, \lambda) \triangleq c\lambda - B(\lambda) = c\lambda - \int_{\mathbb{R}} k_2(x) e^{\lambda x} dx + 1 + \beta. \tag{3.4}$$

From (2.1), we get that for $\lambda \neq 0$,

$$G(c(\lambda), \lambda) = D(\lambda) - A(\lambda) > 0, \quad H(c(\lambda), \lambda) = D(\lambda) - B(\lambda) > 0. \tag{3.5}$$

It follows that for $\lambda \neq 0$,

$$G(c(\lambda), \lambda) H(c(\lambda), \lambda) = (D(\lambda) - A(\lambda))(D(\lambda) - B(\lambda)) = g'(0) h'(0). \tag{3.6}$$

Denote the function

$$b(\lambda) \triangleq \frac{1}{2h'(0)} [-A(\lambda) + B(\lambda) + \sqrt{(A(\lambda) - B(\lambda))^2 + 4h'(0) g'(0)}] > 0 \quad \text{for } \lambda \in \mathbb{R}. \tag{3.7}$$

When k_1 and k_2 are symmetric, we have $b(\lambda) = b(-\lambda)$. Then we get from (2.1) that

$$b(\lambda) = \frac{G(c(\lambda), \lambda)}{h'(0)} = \frac{g'(0)}{H(c(\lambda), \lambda)} \quad \text{for } \lambda \neq 0. \tag{3.8}$$

In the construction of new lower solutions, we also need to introduce some new notations. For any $\eta \in (0, \min\{g'(0), h'(0)\})$, we define a function

$$c_\eta(\lambda) = \frac{1}{\lambda} D_\eta(\lambda) \quad \text{for } \lambda \neq 0, \tag{3.9}$$

where

$$D_\eta(\lambda) = \frac{1}{2}[A(\lambda) + B(\lambda) + \sqrt{(A(\lambda) - B(\lambda))^2 + 4(g'(0) - \eta)(h'(0) - \eta)}].$$

Similar to (3.6), we have

$$G(c_\eta(\lambda), \lambda)H(c_\eta(\lambda), \lambda) = (g'(0) - \eta)(h'(0) - \eta) \quad \text{for} \ \ \lambda \neq 0. \tag{3.10}$$

By the same method used in the proof of Theorem 2.1, for any $\eta \in (0, \min\{g'(0), h'(0)\})$, we can define

$$c_r^*(\eta) \triangleq \inf_{\lambda \in \mathbb{R}^+} \{c_\eta(\lambda)\} \quad \text{and} \quad c_l^*(\eta) \triangleq \sup_{\lambda \in \mathbb{R}^-} \{c_\eta(\lambda)\}. \tag{3.11}$$

It follows that $c_l^* < c_l^*(\eta) < c_r^*(\eta) < c_r^*$. Moreover, we have $c_r^*(\eta) \to c_r^*$ and $c_l^*(\eta) \to c_l^*$ as $\eta \to 0$. Then for any $\epsilon > 0$ small enough, there are two small constants $\eta_1, \eta_2 \in (0, \min\{g'(0), h'(0)\})$ such that $c_r^*(\eta_1) = c_r^* - \epsilon, c_l^*(\eta_2) = c_l^* + \epsilon$ and

$$\alpha\beta < (h'(0) - \eta_1)(g'(0) - \eta_1), \quad \alpha\beta < (h'(0) - \eta_2)(g'(0) - \eta_2).$$

For short, we denote

$$g_1 \triangleq g'(0) - \eta_1, \quad h_1 \triangleq h'(0) - \eta_1, \quad g_2 \triangleq g'(0) - \eta_2, \quad h_2 \triangleq h'(0) - \eta_2.$$

The following lemma gives some properties of the functions $G(c, \lambda)$ and $H(c, \lambda)$.

Lemma 3.5. *For any $c_1 \in (c_r^* - \epsilon, c_r^*)$ with $\epsilon > 0$ small enough, there are two unique constants $\zeta_1(c_1) > \gamma_1(c_1) > 0$ (denoted also by ζ_1 and γ_1 for short) such that*

$$G(c_1, \gamma_1)H(c_1, \gamma_1) = G(c_1, \zeta_1)H(c_1, \zeta_1) = g_1 h_1$$

and

$$G(c_1, \rho)H(c_1, \rho) > g_1 h_1, \quad G(c_1, \rho) > 0, \quad H(c_1, \rho) > 0 \quad \text{for all } \rho \in (\gamma_1, \zeta_1).$$

Similarly, for any $c_2 \in (c_l^*, c_l^* + \epsilon)$ with $\epsilon > 0$ small enough, there are two unique constants $\zeta_2(c_2) < \gamma_2(c_2) < 0$ (denoted also by ζ_2 and γ_2 for short) such that

$$G(c_2, \gamma_2)H(c_2, \gamma_2) = G(c_2, \zeta_2)H(c_2, \zeta_2) = g_2 h_2$$

and

$$G(c_2, \rho)H(c_2, \rho) > g_2 h_2, \quad G(c_2, \rho) > 0, \quad H(c_2, \rho) > 0 \quad \text{for all } \rho \in (\zeta_2, \gamma_2).$$

Proof. Similar to the proof of Theorem 2.1, for any constant $\eta \in (0, \min\{g'(0), h'(0)\})$, there are two unique constants $\lambda_r^*(\eta) \in \mathbb{R}^+$ and $\lambda_l^*(\eta) \in \mathbb{R}^-$ such that

$$c_r^*(\eta) = c_\eta(\lambda_r^*(\eta)), \quad c_l^*(\eta) = c_\eta(\lambda_l^*(\eta)),$$

where $c_\eta(\lambda)$, $c_r^*(\eta)$ and $c_l^*(\eta)$ are defined by (3.9) and (3.11). Since $\lambda_r^*(\eta_1) > 0$ and

$$\frac{\partial}{\partial c}G(c, \lambda) = \frac{\partial}{\partial c}H(c, \lambda) = \lambda, \quad c_1 > c_r^* - \epsilon = c_r^*(\eta_1) = c_{\eta_1}(\lambda_r^*(\eta_1)),$$

we get

$$G(c_1, \lambda_r^*(\eta_1)) > G(c_{\eta_1}(\lambda_r^*(\eta_1)), \lambda_r^*(\eta_1)) > 0, \tag{3.12}$$
$$H(c_1, \lambda_r^*(\eta_1)) > H(c_{\eta_1}(\lambda_r^*(\eta_1)), \lambda_r^*(\eta_1)) > 0. \tag{3.13}$$

Then (3.10) implies

$$G(c_1, \lambda_r^*(\eta_1))H(c_1, \lambda_r^*(\eta_1)) > G(c_{\eta_1}(\lambda_r^*(\eta_1)), \lambda_r^*(\eta_1))H(c_{\eta_1}(\lambda_r^*(\eta_1)), \lambda_r^*(\eta_1)) = g_1 h_1.$$

On the other hand, we easily get

$$G(c_1,0)=\alpha>0,\quad H(c_1,0)=\beta>0,\quad G(c_1,0)H(c_1,0)=\alpha\beta<g_1h_1.$$

Since

$$G(c_1,+\infty)<0,\quad H(c_1,+\infty)<0,\quad \frac{\partial^2}{\partial\lambda^2}G(c_1,\lambda)<0,\quad \frac{\partial^2}{\partial\lambda^2}H(c_1,\lambda)<0,$$

from (3.12) and (3.13), there is a unique constant λ_1 in $(\lambda_r^*(\eta_1),+\infty)$ such that $G(c_1,\lambda)>0$, $H(c_1,\lambda)>0$ for $\lambda\in(0,\lambda_1)$ and either $G(c_1,\lambda_1)=0$ or $H(c_1,\lambda_1)=0$. Then it follows that

$$G(c_1,\lambda_1)H(c_1,\lambda_1)=0<g_1h_1.$$

By the arguments above, there are two constants $\gamma_1\in(0,\lambda_r^*(\eta_1))$ and $\zeta_1\in(\lambda_r^*(\eta_1),\lambda_1)$ such that $G(c_1,\gamma_1)H(c_1,\gamma_1)=G(c_1,\zeta_1)H(c_1,\zeta_1)=g_1h_1$. Moreover, if the constant $\lambda_0\in(0,\lambda_1)$ satisfies

$$\frac{\partial}{\partial\lambda}(G(c_1,\lambda)H(c_1,\lambda))\Big|_{\lambda=\lambda_0}=G(c_1,\lambda_0)\frac{\partial}{\partial\lambda}H(c_1,\lambda_0)+H(c_1,\lambda_0)\frac{\partial}{\partial\lambda}G(c_1,\lambda_0)=0,$$

then we can get

$$\begin{aligned}&\frac{\partial^2}{\partial\lambda^2}(G(c_1,\lambda)H(c_1,\lambda))\Big|_{\lambda=\lambda_0}\\&=G(c_1,\lambda_0)\frac{\partial^2}{\partial\lambda^2}H(c_1,\lambda_0)+H(c_1,\lambda_0)\frac{\partial^2}{\partial\lambda^2}G(c_1,\lambda_0)+2\frac{\partial}{\partial\lambda}G(c_1,\lambda_0)\frac{\partial}{\partial\lambda}H(c_1,\lambda_0)\\&<0.\end{aligned}$$

Therefore, we have that γ_1 and ζ_1 are unique and

$$G(c_1,\rho)>0,\quad H(c_1,\rho)>0,\quad G(c_1,\rho)H(c_1,\rho)>g_1h_1\quad\text{for } \rho\in(\gamma_1,\zeta_1).$$

Similarly, we can get the results about ζ_2 and γ_2. □

Now we choose some constants $\rho_1\in(\gamma_1,\zeta_1)$, $\rho_2\in(\zeta_2,\gamma_2)$, $\delta_1>0$ and $\delta_2>0$ such that

$$\gamma_1<\rho_1(1-\delta_1)<\rho_1(1+\delta_1)<\zeta_1,\quad \zeta_2<\rho_2(1+\delta_2)<\rho_2(1-\delta_2)<\gamma_2. \tag{3.14}$$

Then for short, we denote

$$\begin{aligned}&G_1^0\triangleq G(c_1,\rho_1),\quad G_1^+\triangleq G(c_1,\rho_1(1+\delta_1)),\quad G_1^-\triangleq G(c_1,\rho_1(1-\delta_1)),\\&H_1^0\triangleq H(c_1,\rho_1),\quad H_1^+\triangleq H(c_1,\rho_1(1+\delta_1)),\quad H_1^-\triangleq H(c_1,\rho_1(1-\delta_1)),\\&\Delta_1^0=G_1^0H_1^0-g_1h_1>0,\quad \Delta_1^+=G_1^+H_1^+-g_1h_1>0,\quad \Delta_1^-=G_1^-H_1^--g_1h_1>0\end{aligned}$$

and

$$\begin{aligned}&G_2^0\triangleq G(c_2,\rho_2),\quad G_2^+\triangleq G(c_2,\rho_2(1+\delta_2)),\quad G_2^-\triangleq G(c_2,\rho_2(1-\delta_2)),\\&H_2^0\triangleq H(c_2,\rho_2),\quad H_2^+\triangleq H(c_2,\rho_2(1+\delta_2)),\quad H_2^-\triangleq H(c_2,\rho_2(1-\delta_2)),\\&\Delta_2^0=G_2^0H_2^0-g_2h_2>0,\quad \Delta_2^+=G_2^+H_2^+-g_2h_2>0,\quad \Delta_2^-=G_2^-H_2^--g_2h_2>0.\end{aligned}$$

It follows from Lemma 3.5 that $G_i^0H_i^0>g_ih_i$ for each $i\in\{1,2\}$. Therefore, we can choose some constant $\kappa_i>0$ such that

$$\frac{g_i}{H_i^0}<\kappa_i<\frac{G_i^0}{h_i}\quad\text{for each } i\in\{1,2\}.$$

Since

$$G_i^+\to G_i^0,\quad H_i^+\to H_i^0,\quad G_i^-\to G_i^0,\quad H_i^-\to H_i^0\quad\text{as } \delta_i\to0^+,$$

we can retake δ_i small enough such that the constant κ_i also satisfies

$$\frac{g_i}{H_i^+}<\kappa_i<\frac{G_i^+}{h_i}\quad\text{and}\quad\frac{g_i}{H_i^-}<\kappa_i<\frac{G_i^-}{h_i}\quad\text{for each } i\in\{1,2\}. \tag{3.15}$$

Remark 3.6. All the notations defined in this section with subscript "1" will be used to construct the first lower solutions spreading at a speed of $c_1 \in (c_r^* - \epsilon, c_r^*)$; meanwhile, all the notations with subscript "2" will be used to construct the second lower solutions spreading at a speed of $c_2 \in (c_l^*, c_l^* + \epsilon)$.

In addition, we also define an auxiliary function and give its properties in the following lemma.

Lemma 3.7. *Let M, N and L be three positive constants. For any $\delta \in (0,1)$, define*

$$f(y) = My - Ny^{1+\delta} - Ly^{1-\delta} \quad \text{for } y > 0.$$

Then we have the following conclusions:

(i) $F^{\max} > 0$ *when* $M^2 > 4LN$, *and* $F^{\max} = 0$ *when* $M^2 \leqslant 4LN$,

(ii) $F^{\max} \to 0^+$ *and* $S - R \to 0^+$ *as* $M^2 - 4LN \to 0^+$,

where

$$F^{\max} \triangleq \sup_{y>0}\{f(y)\} \quad \text{and} \quad (R,S) \triangleq \{y > 0 \mid f(y) > 0\} \quad \text{when } M^2 > 4LN.$$

Proof. Let y_0 and y_1 denote two constants satisfying

$$y_0 = \left[\frac{M + \sqrt{M^2 - 4LN(1-\delta^2)}}{2(1+\delta)N}\right]^{\frac{1}{\delta}}, \quad y_1 = \left[\frac{M - \sqrt{M^2 - 4LN(1-\delta^2)}}{2(1+\delta)N}\right]^{\frac{1}{\delta}}.$$

Then we have

$$f'(y) \begin{cases} < 0 & \text{for } y \in (0, y_1) \cup (y_0, +\infty), \\ = 0 & \text{for } y = y_0 \quad \text{and} \quad y = y_1, \\ > 0 & \text{for } y \in (y_1, y_0) \end{cases}$$

and

$$F^{\max} \triangleq \sup_{y>0}\{f(y)\} = \max\{0, f(y_0)\}.$$

For the fixed positive constants M and N, we define a function

$$F(L) \triangleq f(y_0) = My_0 - Ny_0^{1+\delta} - Ly_0^{1-\delta} \quad \text{for } L > 0.$$

From some simple calculations, we get

$$F'(L) = f'(y_0)\frac{\partial y_0}{\partial L} - y_0^{1-\delta} = -y_0^{1-\delta} < 0.$$

Notice $F(L) = 0$ when $L = \frac{M^2}{4N}$. Then it follows that

$$F(L) > 0 \quad \text{when } L < \frac{M^2}{4N}, \quad F(L) < 0 \quad \text{when } L > \frac{M^2}{4N}.$$

Therefore, we prove that

$$F^{\max} > 0 \quad \text{when } M^2 > 4LN, \quad \text{and} \quad F^{\max} = 0 \quad \text{when } M^2 \leqslant 4LN$$

and $F^{\max} \to 0^+$ as $M^2 - 4LN \to 0^+$. Since $(R,S) \triangleq \{y > 0 \mid f(y) > 0\}$ when $M^2 > 4LN$, some simple calculations imply that

$$R = \left[\frac{M - \sqrt{M^2 - 4LN}}{2N}\right]^{\frac{1}{\delta}}, \quad S = \left[\frac{M + \sqrt{M^2 - 4LN}}{2N}\right]^{\frac{1}{\delta}}.$$

Then it follows that $S - R \to 0^+$ as $M^2 - 4LN \to 0^+$. This completes the proof. □

3.2 Lower bounds of the spatial propagation

In this subsection, we prove the lower bounds of the spatial propagation in Theorem 3.1. First, we give a new method to construct lower solutions. Let P denote some positive constant satisfying that for each $i \in \{1,2\}$,

$$P > \max\left\{\left(\frac{1}{\kappa_i}\right)^{1+\delta_i}\left[\frac{2(G_i^0 - h_i\kappa_i)^2}{G_i^- - h_i\kappa_i} - (G_i^+ - h_i\kappa_i)\right], \frac{2(H_i^0\kappa_i - g_i)^2}{H_i^-\kappa_i - g_i} - (H_i^+\kappa_i - g_i)\right\}, \tag{3.16}$$

where $g_i = g'(0) - \eta_i$, $h_i = h'(0) - \eta_i$ and κ_i satisfies (3.15). Since g and h are in the function space $C^1[0,1]$, there is some constant $q_0 \in (0,1)$ such that for each $i \in \{1,2\}$,

$$g(u) \geqslant \left(g'(0) - \frac{\eta_i}{2}\right)u \quad \text{for } u \in (0,q_0), \quad h(v) \geqslant \left(h'(0) - \frac{\eta_i}{2}\right)v \quad \text{for } v \in (0,q_0).$$

By taking q_0 smaller such that $q_0 \leqslant \min\{(\frac{\eta_1}{2P})^{-\delta_1}, (\frac{\eta_2}{2P})^{-\delta_2}\}$, we can get

$$g(u) \geqslant g_i u + P u^{1+\delta_i} \quad \text{for } u \in (0,q_0), \quad h(v) \geqslant h_i v + P v^{1+\delta_i} \quad \text{for } v \in (0,q_0). \tag{3.17}$$

Define two sets of lower solutions as follows:

$$\begin{cases} \underline{u}_i(t,x;\xi_i) = \max\{0, f_i(\mathrm{e}^{\rho_i(-x+c_it+\xi_i)})\}, \\ \underline{v}_i(t,x;\xi_i) = \max\{0, \kappa_i f_i(\mathrm{e}^{\rho_i(-x+c_it+\xi_i)})\}, \end{cases} \quad \text{for each } i \in \{1,2\}, \tag{3.18}$$

where $f_i(y) = y - y^{1+\delta_i} - L_i y^{1-\delta_i}$ for $y \in \mathbb{R}^+$, and ρ_i and δ_i are two constants satisfying (3.14). Here, L_i is some constant in $[\frac{1}{8}, \frac{1}{4})$ and $\xi_i \in \mathbb{R}$ is a parameter number, and both will be chosen later. Moreover, we define

$$R_i = \left[\frac{1 - \sqrt{1-4L_i}}{2}\right]^{\frac{1}{\delta_i}}, \quad S_i = \left[\frac{1 + \sqrt{1-4L_i}}{2}\right]^{\frac{1}{\delta_i}}, \quad Y_i = \left[\frac{1 + \sqrt{1-4L_i(1-\delta_i^2)}}{2(1+\delta_i)}\right]^{\frac{1}{\delta_i}}.$$

Then Lemma 3.7 shows that

$$(R_i, S_i) = \{y > 0 \mid f_i(y) > 0\}, \quad Y_i \in (R_i, S_i), \quad F_i^{\max} \triangleq \sup_{y>0}\{f_i(y)\} = f_i(Y_i) > 0.$$

Also from Lemma 3.7, we can take L_i close enough to $\frac{1}{4}$ such that

$$\max\{F_i^{\max}, \kappa_i F_i^{\max}\} \leqslant q_0.$$

Therefore, we obtain from some simple calculations that

$$\begin{cases} \underline{u}_i(t,x;\xi_i) = \underline{v}_i(t,x;\xi_i) = 0 & \text{for } x - c_it \notin \Omega_i, \\ \underline{u}_i(t,x;\xi_i) = \dfrac{1}{\kappa_i}\underline{v}_i(t,x;\xi_i) = f_i(\mathrm{e}^{\rho_i(-x+c_it+\xi_i)}) \in (0, F_i^{\max}] & \text{for } x - c_it \in \Omega_i, \end{cases}$$

where $\Omega_i = (\xi_i - \rho_i^{-1}\ln S_i, \xi_i - \rho_i^{-1}\ln R_i)$.

Next, we prove that the pair of the functions $(\underline{u}_i(t,x;\xi_i), \underline{v}_i(t,x;\xi_i))$ is a lower solution of the system (1.1) for all $\xi_i \in \mathbb{R}$. When $x - c_it \notin \overline{\Omega}_i$, we have $\underline{u}_i(t,x;\xi_i) = \underline{v}_i(t,x;\xi_i) = 0$ and

$$\frac{\partial}{\partial t}\underline{u}_i(t,x;\xi_i) - k_1 * \underline{u}_i(t,x;\xi_i) + \underline{u}_i(t,x;\xi_i) + \alpha\underline{u}_i(t,x;\xi_i) - h(\underline{v}_i(t,x;\xi_i)) \leqslant 0,$$
$$\frac{\partial}{\partial t}\underline{v}_i(t,x;\xi_i) - k_2 * \underline{v}_i(t,x;\xi_i) + \underline{v}_i(t,x;\xi_i) + \beta\underline{v}_i(t,x;\xi_i) - g(\underline{u}_i(t,x;\xi_i)) \leqslant 0.$$

When $x - c_it \in \overline{\Omega}_i$, we have $\underline{u}_i(t,x;\xi_i) = \frac{1}{\kappa_i}\underline{v}_i(t,x;\xi_i) = f_i(\mathrm{e}^{\rho_i(-x+c_it+\xi_i)})$. Then it follows from (3.17) that

$$\begin{aligned} &\frac{\partial}{\partial t}\underline{u}_i(t,x;\xi_i) - k_1 * \underline{u}_i(t,x;\xi_i) + \underline{u}_i(t,x;\xi_i) + \alpha\underline{u}_i(t,x;\xi_i) - h(\underline{v}_i(t,x;\xi_i)) \\ &\quad \leqslant (G_i^0 - h_i\kappa_i)\mathrm{e}^{\rho_i(-x+c_it+\xi_i)} - (G_i^+ - h_i\kappa_i + P\kappa_i^{1+\delta_i})\mathrm{e}^{\rho_i(1+\delta_i)(-x+c_it+\xi_i)} \\ &\qquad - (G_i^- - h_i\kappa_i)L_i\mathrm{e}^{\rho_i(1-\delta_i)(-x+c_it+\xi_i)} \end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\underline{v}_i(t,x;\xi_i)-k_2*\underline{v}_i(t,x;\xi_i)+\underline{v}_i(t,x;\xi_i)+\beta\underline{v}_i(t,x;\xi_i)-g(\underline{u}_i(t,x;\xi_i))\\&\quad\leqslant(H_i^0\kappa_i-g_i)\mathrm{e}^{\rho_i(-x+c_it+\xi_i)}-(H_i^+\kappa_i-g_i+P)\mathrm{e}^{\rho_i(1+\delta_i)(-x+c_it+\xi_i)}\\&\qquad-(H_i^-\kappa_i-g_i)L_i\mathrm{e}^{\rho_i(1-\delta_i)(-x+c_it+\xi_i)}.\end{aligned}$$

From (3.16) and $L_i>\frac{1}{8}$, we have

$$\begin{aligned}&(G_i^0-h_i\kappa_i)^2-4(G_i^+-h_i\kappa_i+P\kappa_i^{1+\delta_i})(G_i^--h_i\kappa_i)L_i<(G_i^0-h_i\kappa_i)^2(1-8L_i)<0,\\&(H_i^0\kappa_i-g_i)^2-4(H_i^+\kappa_i-g_i+P)(H_i^-\kappa_i-g_i)L_i<(H_i^0\kappa_i-g_i)^2(1-8L_i)<0.\end{aligned}$$

Then Lemma 3.7 shows that when $x-c_it\in\overline{\Omega}_i$,

$$\begin{aligned}&\frac{\partial}{\partial t}\underline{u}_i(t,x;\xi_i)-k_1*\underline{u}_i(t,x;\xi_i)+\underline{u}_i(t,x;\xi_i)+\alpha\underline{u}_i(t,x;\xi_i)-h(\underline{v}_i(t,x;\xi_i))\leqslant0,\\&\frac{\partial}{\partial t}\underline{v}_i(t,x;\xi_i)-k_2*\underline{v}_i(t,x;\xi_i)+\underline{v}_i(t,x;\xi_i)+\beta\underline{v}_i(t,x;\xi_i)-g(\underline{u}_i(t,x;\xi_i))\leqslant0.\end{aligned}$$

Therefore, the pair of the functions $(\underline{u}_i(t,x;\xi_i),\underline{v}_i(t,x;\xi_i))$ is a lower solution for any $\xi_i\in\mathbb{R}$.

Finally, we are ready to prove the lower bounds of the spatial propagation in Theorem 3.1. The "forward-backward spreading" method will be applied here.

Proof of Theorem 3.1 (*Lower bounds*). From the assumptions in Theorem 3.1, we have $u_0(x_0)>0$ and $v_0(x_0)>0$ for some constant $x_0\in\mathbb{R}$. By translating the x-axis, we can simply suppose that $x_0=0$. Then there are two constants $q_1>0$ and $d>0$ such that

$$u_0(x)\geqslant q_1,\quad v_0(x)\geqslant q_1\quad\text{for }x\in[-d,d].\tag{3.19}$$

Now we prove that for any small $\epsilon>0$ there is some constant $\nu\in(0,1)$ such that the solution $(u(t,x),v(t,x))$ of the system (1.1) satisfies

$$(u(T,X),v(T,X))\geqslant(\nu,\nu)\quad\text{for all }T>0,\quad X\in[c_2T,c_1T],$$

where $c_1\in(c_r^*-\epsilon,c_r^*)$ and $c_2\in(c_l^*,c_l^*+\epsilon)$. For any given $T>0$ and $X\in[c_2T,c_1T]$, we denote

$$\mu=\frac{X-c_2T}{c_1T-c_2T}\in[0,1].$$

First, we construct a set of lower solutions in the first time period $[0,\mu T]$ as follows:

$$\begin{cases}\underline{u}_1(t,x;\xi_1)=\max\{0,f_1(\mathrm{e}^{\rho_1(-x+c_1t+\xi_1)})\},\\\underline{v}_1(t,x;\xi_1)=\max\{0,\kappa_1f_1(\mathrm{e}^{\rho_1(-x+c_1t+\xi_1)})\},\end{cases}\quad\text{for }t\in[0,\mu T],\quad x\in\mathbb{R},$$

where $\xi_1\in[-d/2+\rho_1^{-1}\ln R_1,d/2+\rho_1^{-1}\ln S_1]$ and L_1 is some constant in $[\frac{1}{8},\frac{1}{4})$, which is close to $\frac{1}{4}$ such that

$$\max\{F_1^{\max},\kappa_1F_1^{\max}\}\leqslant\min\{q_0,q_1\}\quad\text{and}\quad\rho_1^{-1}(\ln S_1-\ln R_1)\leqslant d/2.$$

Then it follows that

$$\begin{cases}\underline{u}_1(t,x;\xi_1)=\underline{v}_1(t,x;\xi_1)=0&\text{for }x-c_1t\notin\Omega_1,\\\underline{u}_1(t,x;\xi_1)=\dfrac{1}{\kappa_1}\underline{v}_1(t,x;\xi_1)=f_1(\mathrm{e}^{\rho_1(-x+c_1t+\xi_1)})>0&\text{for }x-c_1t\in\Omega_1\end{cases}$$

with

$$\Omega_1=(\xi_1-\rho_1^{-1}\ln S_1,\xi_1-\rho_1^{-1}\ln R_1)\subseteq(-d,d).\tag{3.20}$$

From the discussion above, the pair of the functions $(\underline{u}_1(t,x;\xi_1),\underline{v}_1(t,x;\xi_1))$ is a lower solution of the system (1.1). Moreover, we obtain

$$u_1(t,x;\xi_1)\leqslant F_1^{\max}\leqslant q_1,\quad v_1(t,x;\xi_1)\leqslant\kappa_1F_1^{\max}\leqslant q_1\quad\text{for }t\geqslant0,\quad x\in\mathbb{R}.$$

It follows from (3.19) and (3.20) that for every $\xi_1 \in [-d/2 + \rho_1^{-1} \ln R_1, d/2 + \rho_1^{-1} \ln S_1]$,

$$u_0(x) \geqslant \underline{u}_1(0, x; \xi_1), \quad v_0(x) \geqslant \underline{v}_1(0, x; \xi_1), \quad x \in \mathbb{R}.$$

Therefore, by Lemma 3.4 we have

$$u(t, x) \geqslant \underline{u}_1(t, x; \xi_1), \quad v(t, x) \geqslant \underline{v}_1(t, x; \xi_1) \quad \text{for } t \in [0, \mu T], \quad x \in \mathbb{R}.$$

If we denote $z_1(t) = c_1 t + \xi_1 - \rho_1^{-1} \ln Y_1$ for $t \in [0, \mu T]$, then

$$u(t, z_1(t)) \geqslant \underline{u}_1(t, z_1(t); \xi_1) = f_1(Y_1) = F_1^{\max},$$
$$v(t, z_1(t)) \geqslant \underline{v}_1(t, z_1(t); \xi_1) = \kappa_1 f_1(Y_1) = \kappa_1 F_1^{\max}.$$

Furthermore, the arbitrariness of ξ_1 and $R_1 < Y_1 < S_1$ show that

$$u(t, x) \geqslant F_1^{\max}, \quad v(t, x) \geqslant \kappa_1 F_1^{\max} \quad \text{for all } t \in [0, \mu T], \quad x \in [c_1 t - d/2, c_1 t + d/2].$$

Therefore, there is some constant $q_2 = \min\{F_1^{\max}, \kappa_1 F_1^{\max}\}$ such that

$$u(\mu T, x) \geqslant q_2, \quad v(\mu T, x) \geqslant q_2 \quad \text{for } x \in [c_1 \mu T - d/2, c_1 \mu T + d/2]. \tag{3.21}$$

Next, we construct another set of lower solutions in the second time period $[\mu T, T]$ as follows:

$$\begin{cases} \underline{u}_2(t, x; \xi_2) = \max\{0, f_2(\mathrm{e}^{\rho_2(-x + c_2 t + \xi_2)})\}, \\ \underline{v}_2(t, x; \xi_2) = \max\{0, \kappa_2 f_2(\mathrm{e}^{\rho_2(-x + c_2 t + \xi_2)})\}, \end{cases} \quad \text{for } t \in [\mu T, T], \quad x \in \mathbb{R},$$

where $\xi_2 \in [(c_1 - c_2)\mu T + \rho_2^{-1} \ln R_2, (c_1 - c_2)\mu T + \rho_2^{-1} \ln S_2]$ and L_2 is some constant in $[\frac{1}{8}, \frac{1}{4})$, which is close to $\frac{1}{4}$ such that

$$\max\{F_2^{\max}, \kappa_2 F_2^{\max}\} \leqslant q_2 \quad \text{and} \quad \rho_2^{-1}(\ln S_2 - \ln R_2) \leqslant d/2.$$

Then it follows that

$$\begin{cases} \underline{u}_2(t, x; \xi_2) = \underline{v}_2(t, x; \xi_2) = 0 & \text{for } x - c_2 t \notin \Omega_2, \\ \underline{u}_2(t, x; \xi_2) = \dfrac{1}{\kappa_2} \underline{v}_2(t, x; \xi_2) = f_2(\mathrm{e}^{\rho_2(-x + c_2 t + \xi_2)}) > 0 & \text{for } x - c_2 t \in \Omega_2 \end{cases}$$

with $\Omega_2 = (\xi_2 - \rho_2^{-1} \ln S_2, \xi_2 - \rho_2^{-1} \ln R_2)$.

As stated above, the pair of the functions $(\underline{u}_2(t, x; \xi_2), \underline{v}_2(t, x; \xi_2))$ is also a lower solution of the system (1.1). At the time $t = \mu T$, we have

$$\begin{cases} \underline{u}_2(\mu T, x; \xi_2) = \underline{v}_2(\mu T, x; \xi_2) = 0 & \text{for } x \notin c_2 \mu T + \Omega_2, \\ \underline{u}_2(\mu T, x; \xi_2) = \dfrac{1}{\kappa_2} \underline{v}_2(\mu T, x; \xi_2) \in (0, q_2) & \text{for } x \in c_2 \mu T + \Omega_2, \end{cases}$$

where

$$c_2 \mu T + \Omega_2 \triangleq (c_2 \mu T + \xi_2 - \rho_2^{-1} \ln S_2, c_2 \mu T + \xi_2 - \rho_2^{-1} \ln R_2).$$

It follows that $c_2 \mu T + \Omega_2 \subseteq (c_1 \mu T - d/2, c_1 \mu T - d/2)$. Then we get from (3.21) that for every $\xi_2 \in [(c_1 - c_2)\mu T + \rho_2^{-1} \ln R_2, (c_1 - c_2)\mu T + \rho_2^{-1} \ln S_2]$,

$$u(\mu T, x) \geqslant \underline{u}_2(\mu T, x; \xi_2), \quad v(\mu T, x) \geqslant \underline{v}_2(\mu T, x; \xi_2), \quad x \in \mathbb{R}.$$

Therefore, Lemma 3.4 implies that

$$u(t, x) \geqslant \underline{u}_2(t, x; \xi_2), \quad v(t, x) \geqslant \underline{v}_2(t, x; \xi_2) \quad \text{for } t \in [\mu T, T], \quad x \in \mathbb{R}.$$

If we denote $z_2(t) = c_2t + \xi_2 - \rho_2^{-1} \ln Y_2$ for $t \in [\mu T, T]$, then

$$u(t, z_2(t)) \geqslant \underline{u}_1(t, z_2(t); \xi_2) = f_2(Y_2) = F_2^{\max},$$
$$v(t, z_2(t)) \geqslant \underline{v}_1(t, z_2(t); \xi_2) = \kappa_2 f_2(Y_2) = \kappa_2 F_2^{\max}.$$

Furthermore, the arbitrariness of ξ_2 and $R_2 < Y_2 < S_2$ show that

$$u(t, x) \geqslant F_2^{\max}, \quad v(t, x) \geqslant \kappa_2 F_2^{\max} \quad \text{for all } t \in [\mu T, T], \quad x = c_2t + (c_1 - c_2)\mu T.$$

By taking $\nu = \min\{F_2^{\max}, \kappa_2 F_2^{\max}\}$, we get from $X = c_2T + (c_1 - c_2)\mu T$ that

$$u(T, X) \geqslant \nu, \quad v(T, X) \geqslant \nu \quad \text{for } T > 0, \quad X \in [c_2T, c_1T].$$

Therefore, for any small constant $\epsilon > 0$ we have

$$\inf_{(c_l^* + \epsilon)t \leqslant x \leqslant (c_r^* - \epsilon)t} (u(t, x), v(t, x)) \geqslant (\nu, \nu) \quad \text{for } t > 0.$$

This completes the proof. □

3.3 Upper bounds of the spatial propagation

Proof of Theorem 3.1 (*Upper bounds*). In this subsection, we prove that

$$\sup_{x \leqslant (c_l^* - \epsilon)t} (u(t, x), v(t, x)) \to (0, 0) \quad \text{and} \quad \sup_{x \geqslant (c_r^* + \epsilon)t} (u(t, x), v(t, x)) \to (0, 0) \quad \text{as } t \to +\infty. \tag{3.22}$$

First, we define the functions

$$\begin{cases} \bar{u}(t, x) = \min\{1, \Gamma e^{\lambda_l^*(-x + c_l^* t)}, \Gamma e^{\lambda_r^*(-x + c_r^* t)}\}, \\ \bar{v}(t, x) = \min\{1, b(\lambda_l^*)\Gamma e^{\lambda_l^*(-x + c_l^* t)}, b(\lambda_r^*)\Gamma e^{\lambda_r^*(-x + c_r^* t)}\} \end{cases} \tag{3.23}$$

for $t \geqslant 0$ and $x \in \mathbb{R}$, where the function $b(\lambda)$ is defined by (3.7). From the assumptions in Theorem 3.1, we can take Γ large enough such that $\Gamma \geqslant \max\{1, \Gamma_0, \frac{1}{b(\lambda_l^*)}, \frac{1}{b(\lambda_r^*)}\}$ and

$$\bar{u}(0, x) \geqslant u_0(x), \quad \bar{v}(0, x) \geqslant v_0(x) \quad \text{for } x \in \mathbb{R}. \tag{3.24}$$

Next, we prove that the pair of the functions $(\bar{u}(t, x), \bar{v}(t, x))$ is an upper solution of the system (1.1). When $x \leqslant c_l^* t + (\lambda_l^*)^{-1} \ln \Gamma$, we have $\bar{u}(t, x) = \Gamma e^{\lambda_l^*(-x + c_l^* t)}$ and $\bar{v}(t, x) \leqslant b(\lambda_l^*)\Gamma e^{\lambda_l^*(-x + c_l^* t)}$. Then it follows from (H2) and (3.8) that

$$\partial_t \bar{u} - k_1 * \bar{u} + \bar{u} + \alpha \bar{u} - h(\bar{v}) \geqslant [G(c_l^*, \lambda_l^*) - h'(0) b(\lambda_l^*)]\Gamma e^{\lambda_l^*(-x + c_l^* t)} = 0.$$

Similarly, when $x \geqslant c_r^* t + (\lambda_r^*)^{-1} \ln \Gamma$, we get from (H2) and (3.8) that

$$\partial_t \bar{u} - k_1 * \bar{u} + \bar{u} + \alpha \bar{u} - h(\bar{v}) \geqslant [G(c_r^*, \lambda_r^*) - h'(0) b(\lambda_r^*)]\Gamma e^{\lambda_r^*(-x + c_r^* t)} = 0.$$

If $x \in [c_l^* t + (\lambda_l^*)^{-1} \ln \Gamma, c_r^* t + (\lambda_r^*)^{-1} \ln \Gamma]$, then $\bar{u}(t, x) = 1$ and $\bar{v}(t, x) \leqslant 1$, which implies

$$\partial_t \bar{u} - k_1 * \bar{u} + \bar{u} + \alpha \bar{u} - h(\bar{v}) \geqslant \alpha - h(\bar{v}) \geqslant \alpha - h(1) = 0.$$

Therefore, we finally obtain

$$\partial_t \bar{u} - k_1 * \bar{u} + \bar{u} + \alpha \bar{u} - h(\bar{v}) \geqslant 0 \quad \text{for all } t > 0, \quad x \in \mathbb{R}.$$

Similarly, we can obtain

$$\partial_t \bar{v} - k_2 * \bar{v} + \bar{v} + \beta \bar{v} - g(\bar{u}) \geqslant 0 \quad \text{for all } t > 0, \quad x \in \mathbb{R}.$$

From Lemma 3.4 and (3.24), it follows that

$$(u(t,x),v(t,x)) \leqslant (\bar{u}(t,x),\bar{v}(t,x)) \quad \text{for } t \geqslant 0, \quad x \in \mathbb{R}.$$

Then we have

$$\sup_{x\leqslant(c_l^*-\epsilon)t}(u(t,x),v(t,x)) \leqslant \sup_{x\leqslant(c_l^*-\epsilon)t}(\bar{u}(t,x),\bar{v}(t,x)) \leqslant (\Gamma \mathrm{e}^{\lambda_l^*\epsilon t}, b(\lambda_l^*)\Gamma \mathrm{e}^{\lambda_l^*\epsilon t}),$$
$$\sup_{x\geqslant(c_r^*+\epsilon)t}(u(t,x),v(t,x)) \leqslant \sup_{x\geqslant(c_r^*+\epsilon)t}(\bar{u}(t,x),\bar{v}(t,x)) \leqslant (\Gamma \mathrm{e}^{-\lambda_r^*\epsilon t}, b(\lambda_r^*)\Gamma \mathrm{e}^{-\lambda_r^*\epsilon t}).$$

Therefore, using $\lambda_l^* < 0 < \lambda_r^*$, we finish the proof of (3.22). □

Remark 3.8. The irreducibility of the linearized system at zero is a necessary property in this paper. In fact, our idea of the new lower solution (3.18) is from the following system:

$$\begin{cases} u_t = k_1 * u - u - \alpha u + (h'(0)-\eta)v + Pv^{1+\delta}, & t>0, \quad x\in\mathbb{R}, \\ v_t = k_2 * v - v - \beta v + (g'(0)-\eta)u + Pu^{1+\delta}, & t>0, \quad x\in\mathbb{R}, \end{cases}$$

where $\delta > 0$ is an appropriate constant and $\eta > 0$ is a constant small enough (see the condition (3.17)). If the linearized system at zero is reducible (namely, $h'(0)$ or $g'(0)$ is equal to 0), the above system becomes non-cooperative and meanwhile Lemma 3.5 does not hold. Then there are not any ρ_i and δ_i satisfying (3.14). Thus, we cannot construct any lower solution in the form of (3.18). Moreover, in some studies (see, for example, Weinberger et al. [44]) the irreducibility can be replaced by some other assumptions on the matrix in Frobenius form.

Remark 3.9. The linear and nonlinear selection of speed is an important problem in reaction-diffusion systems. In the system (1.1), the condition for linear selection is given by

$$g(u) \leqslant g'(0)u \quad \text{and} \quad h(v) \leqslant h'(0)v. \tag{3.25}$$

However, when (3.25) is not satisfied, the upper solution (3.23) becomes unavailable and thus the upper bound (3.22) of the spatial propagation is no longer right. In order to obtain the upper bound, we can use $g(u) \leqslant \hat{g}u$ and $h(v) \leqslant \hat{h}v$ instead of (3.25), where

$$\hat{g} = \sup_{u\in(0,1]}\{g(u)/u\} \quad \text{and} \quad \hat{h} = \sup_{v\in(0,1]}\{h(v)/v\}.$$

Under the same assumptions except (3.25) as in Theorem 3.1, when k_1 and k_2 are symmetric, we can obtain

$$\begin{cases} \lim\limits_{t\to+\infty}\sup\limits_{|x|\geqslant(c^++\epsilon)t}(u(t,x),v(t,x)) \to (0,0), \\ \inf\limits_{|x|\leqslant(c^--\epsilon)t}(u(t,x),v(t,x)) \geqslant (\nu,\nu) \quad \text{for all } t\geqslant 0, \end{cases}$$

where the constants c^+ and c^- satisfy that $c^+ \geqslant c^-$ and

$$c^+ \leqslant \inf_{\lambda\in\mathbb{R}^+}\left\{\frac{1}{2\lambda}[A(\lambda)+B(\lambda)+\sqrt{(A(\lambda)-B(\lambda))^2+4\hat{h}\hat{g}}]\right\},$$
$$c^- \geqslant \inf_{\lambda\in\mathbb{R}^+}\left\{\frac{1}{2\lambda}[A(\lambda)+B(\lambda)+\sqrt{(A(\lambda)-B(\lambda))^2+4h'(0)g'(0)}]\right\}.$$

However, it is challenging to prove that $c^+ = c^-$. There are more results about the linear and nonlinear selection of speed (see, e.g., Alhasanat and Ou [1], Ma and Ou [33], Ma et al. [32] and Wang et al. [40]).

4 The second type of initial data and symmetric kernels

In this section, under the assumption that k_1 and k_2 are symmetric, we prove the monotone property and the spatial propagation result for the second type of initial data.

4.1 Monotone property

The following theorem gives a monotone property result of the system (1.1).

Theorem 4.1. *If $k_1(\cdot)$, $k_2(\cdot)$, $u_0(\cdot)$ and $v_0(\cdot)$ are symmetric and decreasing on $\mathbb{R}^+$, so are the functions $u(t,\cdot)$ and $v(t,\cdot)$ at any time $t>0$, where $(u(t,x),v(t,x))$ is the solution of (1.1).*

Proof. First, the symmetry properties of $u(t,\cdot)$ and $v(t,\cdot)$ can be obtained easily. Indeed, by considering the system

$$\begin{cases} \dfrac{\partial}{\partial t}w_1(t,x)=k_1*w_1(t,x)-w_1(t,x)-\alpha w_1(t,x)+h(w_2(t,x)), & t>0, \quad x\in\mathbb{R},\\ \dfrac{\partial}{\partial t}w_2(t,x)=k_2*w_2(t,x)-w_2(t,x)-\beta w_2(t,x)+g(w_1(t,x)), & t>0, \quad x\in\mathbb{R},\\ w_1(0,x)=u_0(-x), \quad w_2(0,x)=v_0(-x), \quad x\in\mathbb{R} \end{cases}$$

and by using the uniqueness property of the solution, we have $u(t,x)=w_1(t,x)=u(t,-x)$ and $v(t,x)=w_2(t,x)=v(t,-x)$ for $t\geqslant 0$, $x\in\mathbb{R}$.

Next, we prove the monotone property. For a fixed constant $y>0$, we define

$$m_1(t,x)=u(t,x+2y)-u(t,x), \quad m_2(t,x)=v(t,x+2y)-v(t,x) \quad \text{for } t\geqslant 0, \quad x\in\mathbb{R}.$$

Then the symmetric properties of $u(t,\cdot)$ and $v(t,\cdot)$ imply that

$$m_1(t,-y)=m_2(t,-y)=0 \quad \text{for } t\geqslant 0.$$

At time $t=0$, we easily get that

$$\begin{aligned} &m_1(0,x)\leqslant 0, \quad m_2(0,x)\leqslant 0 \quad \text{for } x>-y,\\ &m_1(0,x)\geqslant 0, \quad m_2(0,x)\geqslant 0 \quad \text{for } x<-y. \end{aligned}$$

In order to show that $u(t,\cdot)$ and $v(t,\cdot)$ are decreasing in $\mathbb{R}^+$, we prove that

$$m_1(t,x)\leqslant 0, \quad m_2(t,x)\leqslant 0 \quad \text{for all } t>0, \quad x>-y. \tag{4.1}$$

Indeed, if (4.1) holds, then $u(t,x+2y)\leqslant u(t,x)$ and $v(t,x+2y)\leqslant v(t,x)$ for all $x>-y$ and $t>0$, which imply that $u(t,\cdot)$ and $v(t,\cdot)$ are decreasing in $\mathbb{R}^+$.

Now we prove (4.1). Since $h(\cdot)\in C^1([0,1])$, there is some constant $M>0$ such that for all $t\geqslant 0$ and $x\in\mathbb{R}$,

$$\begin{aligned} \frac{\partial}{\partial t}m_1(t,x)&=k_1*m_1(t,x)-m_1(t,x)-\alpha m_1(t,x)+h(v(t,x+2y))-h(v(t,x))\\ &\leqslant k_1*m_1(t,x)-m_1(t,x)-\alpha m_1(t,x)+Mm_2(t,x). \end{aligned} \tag{4.2}$$

Now we suppose that (4.1) does not hold, which means that there are two constants $T_0>0$ and $\varepsilon>0$ such that

$$m_1(t,x)<\varepsilon \mathrm{e}^{Kt}, \quad m_2(t,x)<\varepsilon \mathrm{e}^{Kt} \quad \text{for all } t\in(0,T_0), \quad x>-y \tag{4.3}$$

and at least one of the following two results holds:

$$\begin{aligned} &\sup_{x>-y}\{m_1(T_0,x)\}=\varepsilon \mathrm{e}^{KT_0}, \quad m_2(T_0,x)\leqslant \varepsilon \mathrm{e}^{KT_0} \quad \text{for } x>-y; \\ &m_1(T_0,x)\leqslant \varepsilon \mathrm{e}^{KT_0} \quad \text{for } x>-y, \quad \sup_{x>-y}\{m_2(T_0,x)\}=\varepsilon \mathrm{e}^{KT_0}. \end{aligned} \tag{4.4}$$

Here, K is a positive constant satisfying $K>\frac{4}{3}(M+1)-\alpha$. Without loss of generality, we assume (4.4) holds. As stated in the proof of [49, Lemma 2.2], when $m_1(t,x)\geqslant 0$, it holds that

$$k_1*m_1(t,x)-m_1(t,x)\leqslant \varepsilon \mathrm{e}^{Kt} \quad \text{for } t\in(0,T_0], \quad x>-y. \tag{4.5}$$

From (4.4), at least one of the following cases must hold:

Case 1. There is $x_0 \in (-y, +\infty)$ such that $m_1(T_0, x_0) = \sup_{x>-y}\{m_1(T_0, x)\} = \varepsilon \mathrm{e}^{KT_0}$.

Case 2. $\limsup_{x\to+\infty}\{m_1(T_0, x)\} = \varepsilon \mathrm{e}^{KT_0}$.

If Case 1 holds, it follows that

$$\frac{\partial}{\partial t}(m_1(t, x_0) - \varepsilon \mathrm{e}^{Kt})\Big|_{t=T_0} \geqslant 0,$$

which means that

$$\frac{\partial}{\partial t} m_1(T_0, x_0) \geqslant \varepsilon K \mathrm{e}^{KT_0}.$$

Then from (4.4) and (4.5) we get

$$\begin{aligned}&\frac{\partial}{\partial t} m_1(T_0, x_0) - k_1 * m_1(T_0, x_0) + m_1(T_0, x_0) + \alpha m_1(T_0, x_0) - M m_2(T_0, x_0)\\ &\quad \geqslant (K - 1 + \alpha - M)\varepsilon \mathrm{e}^{KT_0} > 0.\end{aligned}$$

It is a contradiction to (4.2), which implies that (4.1) holds.

If Case 2 holds, there is some constant x_1 large enough such that

$$m_1(T_0, x_1) > \frac{3}{4}\varepsilon \mathrm{e}^{KT_0}.$$

For all $\sigma > 0$, we define

$$\rho_\sigma(t, x) = \left[\frac{1}{2} + \sigma q_0(x)\right]\varepsilon \mathrm{e}^{Kt} \quad \text{for } t \in [0, T_0], \quad x \in \mathbb{R},$$

where $q_0(x)$ is a smooth and increasing function satisfying

$$q_0(x) = \begin{cases} 1 & \text{for } x \leqslant x_1,\\ 3 & \text{for } x \geqslant x_1 + 1.\end{cases}$$

Let σ^* be a constant denoted by

$$\sigma^* = \inf\{\sigma > 0 \mid m_1(t, x) - \rho_\sigma(t, x) \leqslant 0 \text{ for } t \in [0, T_0],\ x > -y\}.$$

Moreover, some simple calculations yield that $\frac{1}{4} \leqslant \sigma^* \leqslant \frac{1}{2}$ and

$$\rho_{\sigma^*}(t, x) \geqslant \frac{5}{4}\varepsilon \mathrm{e}^{Kt} > m_1(t, x) \quad \text{for } t \in [0, T_0], \quad x \geqslant x_1 + 1.$$

From the definition of σ^*, there must exist $T_1 \in (0, T_0]$ and $x_2 \in (-y, x_1 + 1)$ such that

$$m_1(T_1, x_2) - \rho_{\sigma^*}(T_1, x_2) = \sup_{t\in[0,T_0], x>-y}\{m_1(t, x) - \rho_{\sigma^*}(t, x)\} = 0.$$

Then we have

$$\begin{aligned}&m_1(T_1, x_2) = \rho_{\sigma^*}(T_1, x_2) \geqslant \rho_{\frac{1}{4}}(T_1, x_2) \geqslant \frac{3}{4}\varepsilon \mathrm{e}^{KT_1},\\ &\frac{\partial}{\partial t} m_1(T_1, x_2) \geqslant \frac{\partial}{\partial t}\rho_{\sigma^*}(T_1, x_2) = K\rho_{\sigma^*}(T_1, x_2) \geqslant K\rho_{\frac{1}{4}}(T_1, x_2) \geqslant \frac{3}{4}K\varepsilon \mathrm{e}^{KT_1}.\end{aligned}$$

From (4.3) and (4.5), it follows that

$$\begin{aligned}&\frac{\partial}{\partial t} m_1(T_1, x_2) - k_1 * m_1(T_1, x_2) + m_1(T_1, x_2) + \alpha m_1(T_1, x_2) - M m_2(T_1, x_2)\\ &\quad \geqslant \left(\frac{3}{4}K - 1 + \frac{3}{4}\alpha - M\right)\varepsilon \mathrm{e}^{KT_1} > 0,\end{aligned}$$

which contradicts (4.2). Therefore, we finish the proof of (4.1). □

4.2 Spatial propagation

In this subsection, we study the spatial propagation of the system (1.1) for the second type of initial data and symmetric kernels. The following theorem is the main result.

Theorem 4.2. *Assume that* (H1) *and* (H2) *hold. Let* k_1 *and* k_2 *satisfy* (K1) *and be symmetric on* $\mathbb{R}$ *and decreasing in* $\mathbb{R}^+$. *If* $u_0(\cdot)$ *and* $v_0(\cdot)$ *are two continuous functions satisfying* $0 < u_0(x) \leqslant 1$, $0 < v_0(x) \leqslant 1$ *for* $x \in \mathbb{R}$ *and*

$$u_0(x) \sim O(\mathrm{e}^{-\lambda|x|}), \quad v_0(x) \sim O(\mathrm{e}^{-\lambda|x|}) \quad \textit{as } |x| \to +\infty \quad \textit{with} \quad \lambda \in (0, \lambda^*),$$

then for any $\epsilon \in (0, c(\lambda))$ *there is some constant* $\nu \in (0,1)$ *such that the solution of the system* (1.1) *has the following properties*:

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x| \geqslant (c(\lambda)+\epsilon)t} (u(t,x), v(t,x)) \to (0,0), \\ \inf\limits_{|x| \leqslant c(\lambda)t} (u(t,x), v(t,x)) \geqslant (\nu, \nu) \quad \textit{for all } t \geqslant 0, \end{cases}$$

where $\lambda^* \triangleq \lambda_r^* = -\lambda_l^*$.

Remark 4.3. From Theorem 4.2 and the definition of $c(\lambda)$ in (2.1), we obtain a relationship between spreading speeds and the exponentially decaying rate of initial data. Moreover, Theorem 2.1 shows that $c'(\lambda) < 0$ for all $\lambda \in (0, \lambda^*)$; namely, the spreading speed $c(\lambda)$ is decreasing with respect to $\lambda \in (0, \lambda^*)$. Meanwhile, we also have $\inf\{c(\lambda) \mid \lambda \in (0, \lambda^*)\} = c^*$, which implies that the minimum value of $c(\lambda)$ coincides with the spreading speed for the first type of initial value and symmetric kernels.

Before proving Theorem 4.2, we give the following lemma.

Lemma 4.4. *For any* $\lambda \in (0, \lambda_r^*)$, *there is a unique constant* $\delta_\lambda > 0$ *such that*

$$c(\lambda) = c(\lambda + \lambda\delta_\lambda) \quad \textit{and} \quad c(\eta) < c(\lambda) \quad \textit{for } \eta \in (\lambda, \lambda + \lambda\delta_\lambda).$$

Similarly, for any $\lambda \in (\lambda_l^*, 0)$, *there is a unique constant* $\delta_\lambda > 0$ *such that*

$$c(\lambda) = c(\lambda + \lambda\delta_\lambda) \quad \textit{and} \quad c(\eta) > c(\lambda) \quad \textit{for } \eta \in (\lambda + \lambda\delta_\lambda, \lambda).$$

Proof. Since $D(\lambda) > A(\lambda)$ for all $\lambda \in \mathbb{R}$ and

$$\lim_{\lambda\to+\infty} \frac{A(\lambda)}{\lambda} = +\infty, \quad \lim_{\lambda\to-\infty} \frac{A(\lambda)}{\lambda} = -\infty$$

from (2.1), we get that $\lim_{\lambda\to+\infty} c(\lambda) = +\infty$ and $\lim_{\lambda\to-\infty} c(\lambda) = -\infty$. On the other hand, from $D(0) \in (0, +\infty)$ it follows that $\lim_{\lambda\to0^+} c(\lambda) = +\infty$ and $\lim_{\lambda\to0^-} c(\lambda) = -\infty$. Therefore, by (2.8), we finish the proof of Lemma 4.4. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. For any $\lambda \in (0, \lambda^*)$, let δ_λ denote the constant in Lemma 4.4. Then $c(\lambda) > c(\lambda(1+\delta))$ for $\delta \in (0, \delta_\lambda)$. We denote $G(c, \lambda)$, $H(c, \lambda)$ and $b(\lambda)$ by (3.3), (3.4) and (3.7), respectively. Since $\frac{\partial}{\partial c} G(c, \lambda) = \frac{\partial}{\partial c} H(c, \lambda) = \lambda \in (0, \lambda^*)$, from (3.5) we get

$$\begin{cases} G(c(\lambda), \lambda(1+\delta)) > G(c(\lambda(1+\delta)), \lambda(1+\delta)) > 0, \\ H(c(\lambda), \lambda(1+\delta)) > H(c(\lambda(1+\delta)), \lambda(1+\delta)) > 0, \end{cases} \text{for } \lambda \in (0, \lambda^*), \quad \delta \in (0, \delta_\lambda).$$

Therefore, it follows from (3.8) that

$$\frac{g'(0)}{H(c(\lambda), \lambda(1+\delta))} < b(\lambda(1+\delta)) < \frac{G(c(\lambda), \lambda(1+\delta))}{h'(0)} \quad \text{for } \lambda \in (0, \lambda^*), \quad \delta \in (0, \delta_\lambda). \tag{4.6}$$

Step 1. Now we prove that

$$\sup_{|x|\geqslant(c(\lambda)+\epsilon)t}(u(t,x),v(t,x))\to(0,0)\quad\text{as } t\to+\infty. \tag{4.7}$$

For any given $\lambda\in(0,\lambda^*)$, define

$$\begin{cases}\bar{u}(t,x)=\min\{1,\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}\},\\ \bar{v}(t,x)=\min\{1,b(\lambda)\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}\},\end{cases}\quad\text{for } t\geqslant 0,\quad x\in\mathbb{R}, \tag{4.8}$$

where the constant Γ is large enough such that $\Gamma\geqslant\max\{1,\frac{1}{b(\lambda)}\}$. By the assumptions about initial data in Theorem 4.2, we can take Γ larger if necessary such that

$$\bar{u}(0,x)\geqslant u_0(x),\quad \bar{v}(0,x)\geqslant v_0(x)\quad\text{for } x\in\mathbb{R}. \tag{4.9}$$

Now we prove that the pair of functions $(\bar{u}(t,x),\bar{v}(t,x))$ is an upper solution of the system (1.1). If $|x|\leqslant c(\lambda)t+\lambda^{-1}\ln\Gamma$, we have $\bar{u}(t,x)=1$ and $\bar{v}(t,x)\leqslant 1$. Then it follows from (H1) and (H2) that

$$\partial_t\bar{u}-k_1*\bar{u}+\bar{u}+\alpha\bar{u}-h(\bar{v})\geqslant\alpha-h(\bar{v})\geqslant\alpha-h(1)=0.$$

If $|x|>c(\lambda)t+\lambda^{-1}\ln\Gamma$, we get $\bar{u}(t,x)=\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}$ and $\bar{v}(t,x)\leqslant b(\lambda)\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}$. By (H2) and (3.8), some simple calculations imply that

$$\partial_t\bar{u}-k_1*\bar{u}+\bar{u}+\alpha\bar{u}-h(\bar{v})\geqslant[G(c(\lambda),\lambda)-h'(0)b(\lambda)]\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}=0.$$

We finally get that

$$\partial_t\bar{u}-k_1*\bar{u}+\bar{u}+\alpha\bar{u}-h(\bar{v})\geqslant 0\quad\text{for all } t>0,\quad x\in\mathbb{R}. \tag{4.10}$$

Meanwhile, if $|x|\leqslant c(\lambda)t+\lambda^{-1}\ln(b(\lambda)\Gamma)$, we have $\bar{v}(t,x)=1$ and $\bar{u}(t,x)\leqslant 1$. Then it follows from (H1) and (H2) that

$$\partial_t\bar{v}-k_2*\bar{v}+\bar{v}+\beta\bar{v}-g(\bar{u})\geqslant\beta-g(\bar{u})\geqslant\beta-g(1)=0.$$

If $|x|>c(\lambda)t+\lambda^{-1}\ln(b(\lambda)\Gamma)$, we get $\bar{v}(t,x)=b(\lambda)\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}$ and $\bar{u}(t,x)\leqslant\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}$. By (H2) and (3.8), some simple calculations show

$$\partial_t\bar{v}-k_2*\bar{v}+\bar{v}+\beta\bar{v}-g(\bar{u})\geqslant[H(c(\lambda),\lambda)b(\lambda)-g'(0)]\Gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}=0.$$

We finally get that

$$\partial_t\bar{v}-k_2*\bar{v}+\bar{v}+\beta\bar{v}-g(\bar{u})\geqslant 0\quad\text{for all } t>0,\quad x\in\mathbb{R}. \tag{4.11}$$

Therefore, $(\bar{u}(t,x),\bar{v}(t,x))$ is an upper solution of the system (1.1).

By (4.9)–(4.11), Lemma 3.4 shows that

$$(u(t,x),v(t,x))\leqslant(\bar{u}(t,x),\bar{v}(t,x))\quad\text{for } t\geqslant 0,\quad x\in\mathbb{R}.$$

Then we have

$$\sup_{|x|\geqslant(c(\lambda)+\epsilon)t}(u(t,x),v(t,x))\leqslant\sup_{|x|\geqslant(c(\lambda)+\epsilon)t}(\bar{u}(t,x),\bar{v}(t,x))\leqslant(\Gamma\mathrm{e}^{-\lambda\epsilon t},b(\lambda)\Gamma\mathrm{e}^{-\lambda\epsilon t}),$$

which implies that (4.7) holds.

Step 2. Next, we prove that

$$(u(t,x),v(t,x))\geqslant(\nu,\nu)\quad\text{for all } t\geqslant 0,\quad |x|\leqslant c(\lambda)t.$$

From the assumptions in Theorem 4.2, there exists a continuous symmetric function $w_0(x)$, which is decreasing in $\mathbb{R}^+$ and satisfies that

$$u_0(x)\geqslant w_0(x),\quad v_0(x)\geqslant w_0(x)\quad\text{for } x\in\mathbb{R},\quad w_0(x)=\begin{cases}\gamma_0\mathrm{e}^{-\lambda|x|}, & |x|\geqslant y_0,\\ p_1\triangleq\gamma_0\mathrm{e}^{-\lambda y_0}, & |x|\leqslant y_0,\end{cases}$$

where γ_0 and y_0 are two positive constants. Let p and δ denote two constants satisfying $p=\min\{p_0,p_1\}$ and $0<\delta<\min\{\delta_0,\delta_\lambda\}$. Then by $g(\cdot)$, $h(\cdot)\in C^{1+\delta_0}([0,p_0])$, we can find some constant $M>0$ such that

$$g(u)\geqslant g'(0)u-Mu^{1+\delta}\quad \text{for } u\in[0,p],\quad h(v)\geqslant h'(0)v-Mv^{1+\delta}\quad \text{for } v\in[0,p]. \tag{4.12}$$

Let $(w_1(t,x),w_2(t,x))$ denote the solution of the following system:

$$\begin{cases}\partial_t w_1(t,x)=k_1*w_1(t,x)-w_1(t,x)-\alpha w_1(t,x)+h(w_2(t,x)), & t>0,\quad x\in\mathbb{R},\\ \partial_t w_2(t,x)=k_2*w_2(t,x)-w_2(t,x)-\beta w_2(t,x)+g(w_1(t,x)), & t>0,\quad x\in\mathbb{R},\\ w_1(0,x)=w_0(x),\quad w_2(0,x)=w_0(x),\quad x\in\mathbb{R}.\end{cases}$$

Then Lemma 3.4 implies that

$$(u(t,x),v(t,x))\geqslant(w_1(t,x),w_2(t,x))\quad\text{for all } t\geqslant0,\quad x\in\mathbb{R}. \tag{4.13}$$

Since $k_1(\cdot)$ and $k_2(\cdot)$ are symmetric and decreasing on $\mathbb{R}^+$, it follows from Theorem 4.1 that $w_1(t,\cdot)$ and $w_2(t,\cdot)$ are also symmetric and decreasing on $\mathbb{R}^+$ at any time $t\geqslant0$.

For any given $\lambda\in(0,\lambda^*)$, we define

$$\begin{cases}\underline{u}(t,x)=\max\{0,\gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}-\gamma L\mathrm{e}^{\lambda(1+\delta)(-|x|+c(\lambda)t)}\},\\ \underline{v}(t,x)=\max\{0,\gamma b(\lambda)\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}-\gamma Lb(\lambda(1+\delta))\mathrm{e}^{\lambda(1+\delta)(-|x|+c(\lambda)t)}\}\end{cases}$$

for all $t\geqslant0$ and $x\in\mathbb{R}$, where $b(\lambda)$ is defined by (3.7), γ is some positive constant satisfying

$$0<\gamma\leqslant\min\left\{\gamma_0,\frac{\gamma_0}{b(\lambda)}\right\},$$

and $L\in\mathbb{R}^+$ is large enough such that

$$\begin{aligned}L\geqslant\max\bigg\{&1,\frac{b(\lambda)}{b(\lambda(1+\delta))},\gamma^\delta p^{-\delta},\gamma^\delta p^{-\delta}\frac{[b(\lambda)]^{1+\delta}}{b(\lambda(1+\delta))},\\ &\frac{M\gamma^\delta[b(\lambda)]^{1+\delta}}{G(c(\lambda),\lambda(1+\delta))-h'(0)b(\lambda(1+\delta))},\frac{M\gamma^\delta}{b(\lambda(1+\delta))H(c(\lambda),\lambda(1+\delta))-g'(0)}\bigg\}.\end{aligned} \tag{4.14}$$

We easily get that

$$\underline{u}(0,x)\leqslant\gamma_0\mathrm{e}^{-\lambda|x|},\quad \underline{v}(0,x)\leqslant\gamma_0\mathrm{e}^{-\lambda|x|}\quad\text{for all } x\in\mathbb{R}.$$

If we consider the function $f(y)=Ay-By^{1+\delta}$ for $y\in\mathbb{R}^+$ with $A,B\in\mathbb{R}^+$, whose maximum value equals $f^{\max}\triangleq A^{\frac{1+\delta}{\delta}}B^{-\frac{1}{\delta}}\delta(1+\delta)^{-\frac{1+\delta}{\delta}}$, then we have

$$\begin{aligned}&\underline{u}(t,x)\leqslant f_1^{\max}\triangleq\gamma L^{-\frac{1}{\delta}}\delta(1+\delta)^{-\frac{1+\delta}{\delta}}\leqslant p\leqslant p_1,\\ &\underline{v}(t,x)\leqslant f_2^{\max}\triangleq\gamma L^{-\frac{1}{\delta}}[b(\lambda)]^{\frac{1+\delta}{\delta}}[b(\lambda(1+\delta))]^{-\frac{1}{\delta}}\delta(1+\delta)^{-\frac{1+\delta}{\delta}}\leqslant p\leqslant p_1\end{aligned}$$

for all $t\geqslant0$ and $x\in\mathbb{R}$. Therefore, the definition of $w_0(\cdot)$ shows that

$$w_0(x)\geqslant\underline{u}(0,x),\quad w_0(x)\geqslant\underline{v}(0,x)\quad\text{for all } x\in\mathbb{R}. \tag{4.15}$$

We now verify that $(\underline{u}(t,x),\underline{v}(t,x))$ is a lower solution of the system (1.1). When $|x|\leqslant c(\lambda)t+(\lambda\delta)^{-1}\ln L$, we easily get $\underline{u}(t,x)=0$. Then from (H1) and (H2), it follows that

$$\partial_t\underline{u}-k_1*\underline{u}+\underline{u}+\alpha\underline{u}-h(\underline{v})\leqslant-h(\underline{v})\leqslant0.$$

When $|x|>c(\lambda)t+(\lambda\delta)^{-1}\ln L$, we have

$$\begin{aligned}&\underline{u}(t,x)=\gamma\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}-\gamma L\mathrm{e}^{\lambda(1+\delta)(-|x|+c(\lambda)t)},\\ &\underline{v}(t,x)\geqslant\gamma b(\lambda)\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}-\gamma Lb(\lambda(1+\delta))\mathrm{e}^{\lambda(1+\delta)(-|x|+c(\lambda)t)}.\end{aligned}$$

Then by (4.12), some simple calculations imply that

$$\begin{aligned}&\partial_t\underline{u}-k_1*\underline{u}+\underline{u}+\alpha\underline{u}-h(\underline{v})\\&\quad\leqslant\gamma[G(c(\lambda),\lambda)-h'(0)b(\lambda)]\mathrm{e}^{\lambda(-|x|+c(\lambda)t)}\\&\qquad\times\{\gamma L[G(c(\lambda),\lambda(1+\delta))-h'(0)b(\lambda(1+\delta))]-M[\gamma b(\lambda)]^{1+\delta}\}\mathrm{e}^{\lambda(1+\delta)(-|x|+c(\lambda)t)}.\end{aligned}$$

From (3.8), (4.6) and (4.14), it follows that

$$\partial_t\underline{u}-k_1*\underline{u}+\underline{u}+\alpha\underline{u}-h(\underline{v})\leqslant0\quad\text{for}\quad|x|>c(\lambda)t+(\lambda\delta)^{-1}\ln L.$$

Therefore, we finally prove that

$$\partial_t\underline{u}-k_1*\underline{u}+\underline{u}+\alpha\underline{u}-h(\underline{v})\leqslant0\quad\text{for all }t>0,\quad x\in\mathbb{R}.\tag{4.16}$$

Similarly, we can also prove

$$\partial_t\underline{v}-k_2*\underline{v}+\underline{v}+\beta\underline{v}-g(\underline{u})\leqslant0\quad\text{for all }t>0,\quad x\in\mathbb{R}.\tag{4.17}$$

From (4.15)–(4.17), Lemma 3.4 shows that

$$(w_1(t,x),w_2(t,x))\geqslant(\underline{u}(t,x),\underline{v}(t,x))\quad\text{for }t\geqslant0,\quad x\in\mathbb{R}.$$

Then some simple calculations imply that

$$\begin{aligned}&w_1(t,x)\geqslant\underline{u}(t,x)=f_1^{\max},\quad\text{when }|x|=c(\lambda)t+(\lambda\delta)^{-1}\ln[(1+\delta)L],\\&w_2(t,x)\geqslant\underline{v}(t,x)=f_2^{\max},\quad\text{when }|x|=c(\lambda)t+(\lambda\delta)^{-1}\ln\left[(1+\delta)L\frac{b(\lambda(1+\delta))}{b(\lambda)}\right].\end{aligned}$$

Since $w_1(t,\cdot)$ and $w_2(t,\cdot)$ are symmetric and decreasing in $\mathbb{R}^+$ at any time $t\geqslant0$, by taking $\nu=\min\{f_1^{\max},f_2^{\max}\}>0$, we can get from $L\geqslant\max\{1,\frac{b(\lambda)}{b(\lambda(1+\delta))}\}$ that

$$w_1(t,x)\geqslant\nu,\quad w_2(t,x)\geqslant\nu\quad\text{for }t\geqslant0,\quad|x|\leqslant c(\lambda)t.$$

Therefore, by (4.13) we prove that $(u(t,x),v(t,x))\geqslant(\nu,\nu)$ for all $t\geqslant0$, $|x|\leqslant c(\lambda)t$. □

Remark 4.5. In Theorem 4.2, we assume that the initial data u_0 and v_0 have the same exponentially decaying behavior. When they have different decaying behavior, the spatial propagation problem is more difficult and there are some interesting phenomena. For example, our paper [48] showed that the component with exponentially unbounded initial data (for example, decaying algebraically) can accelerate the component with exponentially decaying initial data. However, to the best of our knowledge, when all the components decay exponentially but their decaying rates are different, there is no study about the interaction among the components. We think that the component with a smaller decaying rate could accelerate that with a bigger decaying rate. The fundamental reason of this acceleration phenomenon is that the growth sources of one component could come from other components. For more results about the acceleration among the components, see, e.g., Coulon and Yangari [11] and Xu et al. [47].

5 Applications

In this section, we give some applications of the theoretical results to the control of epidemic whose infectious agent is carried by migratory birds. We consider the question whether it is possible that the epidemic spreads only along the flight route of migratory birds and the spatial propagation against the flight route fails. Throughout this section, we suppose that the positive parameters α, β, $g'(0)$ and $h'(0)$ in the system (1.1) have already been determined. Now we assume some specific forms of the kernel functions k_1 and k_2.

5.1 Normal distribution

Suppose that the migratory birds fly at a constant speed $a \in \mathbb{R}$ and the infectious agent has its own moving ability. In the system (1.1), we assume that k_1 and k_2 satisfy

$$k_1(x) = \frac{1}{\sqrt{2\pi\sigma_1}} \exp\bigg(-\frac{(x-a)^2}{2\sigma_1}\bigg) \quad \text{and} \quad k_2(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\bigg(-\frac{x^2}{2\sigma}\bigg).$$

Here, the expectation a of k_1 represents the movements of the infectious agent caused by migratory flight and the variance $\sigma_1 \in \mathbb{R}^+$ describes the strength of its own moving ability. The expectation of k_2 is 0 because humans usually return after leaving their own residences. The variance $\sigma \in \mathbb{R}^+$ describes the intensity of the movements of the infectious humans.

By observing the migration flight of birds and the moving ability of the infectious agent, we suppose that the parameters a and σ_1 can be determined. We also suppose that $a \geqslant 0$; otherwise just consider the new spatial variable $y = -x$. Finally, our question becomes how to restrict the movements of the infectious humans such that the epidemic spreads only along the flight route and the spatial propagation against the flight route fails; namely we need to find a proper parameter σ such that $0 < c_l^* < c_r^*$.

Define a constant r which can describe the asymmetry level of k_1 as follows:

$$r \triangleq a/\sqrt{2\sigma_1}.$$

Remark 5.1. Intuitively, the asymmetry level of a probability density function k could be measured by the ratio of $M_1(k) = \int_{\mathbb{R}^+} k(x)xdx$ to $M_2(k) = \int_{\mathbb{R}^-} k(x)|x|dx$. By some calculations, we have

$$M_1(k_1)/M_2(k_1) = \varphi(r) \triangleq 2\bigg(\frac{\exp(-r^2)}{r\sqrt{\pi}} + \operatorname{erf}(r) - 1\bigg)^{-1} + 1,$$

where $\operatorname{erf}(\cdot)$ is the error function defined by $\operatorname{erf}(r) = \frac{2}{\sqrt{\pi}} \int_0^r \exp(-t^2)dt$. It is easy to check that $\varphi(\cdot)$ is strictly increasing. Therefore, we can use r to describe the asymmetry level of k_1.

We define another important constant of the system (1.1) by

$$\mathcal{K} \triangleq \beta(\alpha + 1 - \exp(-r^2))/(g'(0)h'(0)) \in \mathbb{R}^+.$$

Note that $\mathcal{K}$ is strictly increasing with respect to r. Next, we show that $\mathcal{K}$ can describe the change of the spatial propagation of the system (1.1) caused by the asymmetry of k_1.

Corollary 5.2. *If $\mathcal{K} > 1$, then there is a constant $\sigma^* \in \mathbb{R}^+$ such that*

(i) *when $0 < \sigma < \sigma^*$, the spatial propagation against the flight route fails; namely $0 < c_l^* < c_r^*$,*

(ii) *when $\sigma > \sigma^*$, the spatial propagation happens along two directions (along and against the flight route); namely $c_l^* < 0 < c_r^*$,*

(iii) *when $\sigma = \sigma^*$, it is the critical state; namely $0 = c_l^* < c_r^*$.*

Moreover, if $\mathcal{K} \leqslant 1$, then $c_l^* < 0 < c_r^*$ holds for any $\sigma \in (0, +\infty)$.

Proof. From (2.2), some calculations show that

$$A(\lambda) = \int_{\mathbb{R}} k_1(x)e^{\lambda x}dx - 1 - \alpha = \exp\bigg(a\lambda + \frac{\sigma_1}{2}\lambda^2\bigg) - 1 - \alpha,$$
$$B(\lambda) = \int_{\mathbb{R}} k_2(x)e^{\lambda x}dx - 1 - \beta = \exp\bigg(\frac{\sigma}{2}\lambda^2\bigg) - 1 - \beta.$$

Recall the following sets defined in the proof of Theorem 2.2:

$$\Lambda^A = \{\lambda \in \mathbb{R} \mid A(\lambda) < 0\}, \quad \Lambda^B = \{\lambda \in \mathbb{R} \mid B(\lambda) < 0\},$$
$$\Lambda = \{\lambda \in \mathbb{R} \mid A(\lambda)B(\lambda) \geqslant g'(0)h'(0),\, A(\lambda) < 0,\, B(\lambda) < 0\}.$$

We know that Λ^A and Λ^B are two open intervals and Λ is a closed interval in $\mathbb{R}$. Moreover, it is easy to check that $\Lambda \subseteq \Lambda^A \cap \Lambda^B$. Since

$$\frac{\partial}{\partial\lambda}(B(\lambda)A(\lambda))\bigg|_{\lambda=0} = -a\beta \leqslant 0,$$

we get that $\Lambda \subseteq \mathbb{R}^-$ when $a > 0$ and $\Lambda = \emptyset$ by $A(0)B(0) < h'(0)g'(0)$ when $a = 0$.

Next, in order to study the relation between Λ and σ, we consider a function $\Lambda(\cdot) : \sigma \mapsto \Lambda$ which is from $\mathbb{R}^+$ to the set that consists of all closed intervals in $\mathbb{R}$. From

$$\frac{\partial B}{\partial \sigma} = \frac{1}{2}\lambda^2 \exp\left(\frac{\sigma}{2}\lambda^2\right) > 0 \quad \text{for } \lambda \in \mathbb{R},$$
$$\frac{\partial |AB|}{\partial \sigma} = A\frac{\partial B}{\partial \sigma} < 0 \quad \text{for } \lambda \in \Lambda^A \cap \Lambda^B,$$

it follows that

$$\Lambda(\sigma') \subseteq \Lambda(\sigma) \quad \text{for any } \sigma' > \sigma \tag{5.1}$$

and this inclusion is strict when $\Lambda(\sigma) \neq \emptyset$. By the continuity of B with respect to σ, we know that $\Lambda(\cdot)$ is also continuous, which means that both its lower bound and upper bound are continuous with respect to σ when $\Lambda \neq \emptyset$.

When $\mathcal{K} > 1$, first, we consider $\sigma \to 0^+$ and $\lambda = -a/\sigma_1$. Then

$$\lim_{\sigma \to 0^+} A(-a/\sigma_1)B(-a/\sigma_1) = \beta\left(1 + \alpha - \exp\left(-\frac{a^2}{2\sigma_1}\right)\right) > g'(0)h'(0).$$

Therefore, there is a positive constant σ_0 small enough such that $\mathrm{int}\Lambda(\sigma_0) \cap \mathbb{R}^- \neq \emptyset$. Next, we consider $\sigma \to +\infty$. Then $\lambda_B^+ \to 0^+$ and $\lambda_B^- \to 0^-$, where

$$\lambda_B^{\pm} = \pm\sqrt{\frac{2}{\sigma}\ln(1 + \beta)} \quad \text{and} \quad \Lambda^B = (\lambda_B^-, \lambda_B^+).$$

It follows that

$$\lim_{\sigma \to +\infty} A(\lambda)B(\lambda) \leqslant \alpha\beta < g'(0)h'(0) \quad \text{for any } \lambda \in \Lambda^A \cap \Lambda^B. \tag{5.2}$$

Therefore, there is a positive constant σ_∞ large enough such that $\Lambda(\sigma_\infty) \cap \mathbb{R} = \emptyset$. Finally, by Theorem 2.2 and (5.1), we finish the proof of Corollaries 5.2(i)–5.2(iii).

When $\mathcal{K} \leqslant 1$, we have

$$A(\lambda)B(\lambda) \leqslant \beta\left(1 + \alpha - \exp\left(-\frac{a^2}{2\sigma_1}\right)\right) \leqslant g'(0)h'(0) \quad \text{for } \lambda \in \Lambda^A \cap \Lambda^B.$$

In the above inequalities, the first equality holds only if $a = 0$, which implies that the second equality does not hold. Then

$$A(\lambda)B(\lambda) < g'(0)h'(0) \quad \text{for } \lambda \in \Lambda^A \cap \Lambda^B,$$

which means that $\Lambda \neq \emptyset$. From Theorem 2.2, it follows that $c_l^* < 0 < c_r^*$. □

Now we give more details of the change of the spatial propagation caused by the asymmetry of k_1. When k_1 is symmetric (namely $r = 0$), it follows that $\mathcal{K} = \alpha\beta/(h'(0)g'(0)) < 1$ and the propagation always happens along two directions. When the asymmetry of k_1 becomes stronger (namely, r becomes larger), $\mathcal{K}$ becomes larger. If $\mathcal{K} > 1$, the asymmetry of k_1 is strong enough to change the spreading dynamics of the system (1.1). It is possible that the epidemic spreads only along the flight route of migratory birds and the spatial propagation against the flight route fails, as long as the infectious humans are kept from moving frequently such that $\sigma < \sigma^*$. Moreover, we point out that if $(1 + \alpha)\beta \leqslant g'(0)h'(0)$, then $\mathcal{K} < 1$ always holds for any k_1, which means that the reaction terms play a more important role and the asymmetry of dispersal cannot change the spreading dynamics of the system (1.1).

Finally, the critical number σ^* can be calculated by some numerical methods. For example, suppose that $\alpha = 0.2$, $\beta = 0.1$, $h'(0)g'(0) = 0.22$, $a = 0.5$ and $\sigma_1 = 1$. Then we have $\mathcal{K} = 1.4432$ and $\sigma^* = 2.2098$.

5.2 Uniform distribution

Suppose that k_1 and k_2 are given by

$$k_1(x) = \begin{cases} \dfrac{1}{a-b} & \text{for } x \in [b,a], \\ 0 & \text{for } x \notin [b,a], \end{cases} \quad \text{and} \quad k_2(x) = \begin{cases} \dfrac{1}{2\sigma} & \text{for } x \in [-\sigma,\sigma], \\ 0 & \text{for } x \notin [-\sigma,\sigma], \end{cases}$$

where the constants $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^-$ stand for the farthest distances of movements of the infectious agent during a unit time period along and against the flight route, respectively. The average moving speed is

$$\int k_1(x)x dx = \frac{(a+b)}{2}.$$

The constant $\sigma \in \mathbb{R}^+$ stands for the farthest distance of movements of the infectious human during a unit time period. Similar to the normal distribution case, it holds that

$$\int k_2(x)x dx = 0.$$

Here, the uniform distribution means that every distance in the moving range has the same probability to happen.

Similar to the normal distribution case, we suppose that the parameters a and b have already been determined by some experimental data and $a+b \geqslant 0$; otherwise, just consider the new spatial variable $y=-x$. Now we show how to choose the parameter σ such that $0 < c_l^* < c_r^*$.

From (2.2), some calculations show that

$$A(\lambda) = \begin{cases} \dfrac{\mathrm{e}^{a\lambda} - \mathrm{e}^{b\lambda}}{(a-b)\lambda} - 1 - \alpha, & \lambda \neq 0, \\ -\alpha, & \lambda = 0, \end{cases}$$

$$B(\lambda) = \begin{cases} \dfrac{\mathrm{e}^{\sigma\lambda} - \mathrm{e}^{-\sigma\lambda}}{2\sigma\lambda} - 1 - \beta, & \lambda \neq 0, \\ -\beta, & \lambda = 0. \end{cases}$$

When $a+b>0$, denote

$$r = -a/b \in (1,+\infty),$$

which describes the asymmetry level of k_1. Indeed, we have that $M_1(k_1)/M_2(k_1) = r^2$ and it is strictly increasing with respect to r, where $M_1(k_1)$ and $M_2(k_1)$ are defined in Remark 5.1.

Before giving the result in this case, we need to prove the following lemma.

Lemma 5.3. *Let $\omega(z) = (z-1)\mathrm{e}^z$ with $z \in \mathbb{R}$. Then for any $r \in (1,+\infty)$, there is a unique constant $z_r \in \mathbb{R}$ such that $\omega(z_r) = \omega(-rz_r)$ and $z_r \neq 0$. Moreover, we have $z_r \in (1-1/r, 1)$. In addition, when $r=1$, $\omega(z) > \omega(-z)$ for $z \in \mathbb{R}^+$.*

Proof. For $r \in (1,+\infty)$, define a function

$$\bar{\omega}(z) = \omega(z) - \omega(-rz) = (z-1)\mathrm{e}^z + (rz+1)\mathrm{e}^{-rz} \quad \text{for } z \in \mathbb{R}.$$

It follows that

$$\bar{\omega}'(z) = z\mathrm{e}^z - r^2 z\mathrm{e}^{-rz} \quad \text{for } z \in \mathbb{R}.$$

Denote $z_1 = 0$ and $z_2 = 2(1+r)^{-1}\ln r \in (0,1)$. Then some calculations imply that $\bar{\omega}'(z_1) = \bar{\omega}'(z_2) = 0$ and

$$\bar{\omega}'(z) < 0, \quad z \in (z_1,z_2) \quad \text{and} \quad \bar{\omega}'(z) > 0, \quad z \in \mathbb{R}\backslash[z_1,z_2].$$

It is easy to check that

$$\bar{\omega}(1) = (r+1)\mathrm{e}^{-r} > 0$$

and it follows from $r - 1/r > 2\ln r$ for $r > 1$ that

$$\bar{\omega}(1 - 1/r) = \frac{e^{1-r}}{r}(r^2 - e^{r-1/r}) < 0 \quad \text{for } r > 1.$$

Then we can find a unique constant $z_r \in (1 - 1/r, 1)$ such that $\bar{\omega}(z_r) = 0$; namely $\omega(z_r) = \omega(-rz_r)$. Moreover, when $r = 1$, we have $z_1 = z_2$ and $\bar{\omega}$ is strictly increasing in $\mathbb{R}$. Then $\omega(z) > \omega(-z)$ for $z \in \mathbb{R}^+$ by $\bar{\omega}(0) = 0$. □

Now define $\omega(z) = (z - 1)e^z$ with $z \in \mathbb{R}$. From Lemma 5.3, let z_r denote the constant satisfying $\omega(z_r) = \omega(-rz_r)$. In view of

$$A'(\lambda) = \frac{1}{(a-b)\lambda^2}(\omega(a\lambda) - \omega(b\lambda))$$

from $\omega(z_r) = \omega(-rz_r)$, it follows that $A'(z_r/b) = 0$ and

$$A(z_r/b) = \min\{A(z); z \in \mathbb{R}\} = \frac{e^{z_r}}{1 + rz_r} - 1 - \alpha \leqslant A(0) < 0.$$

Now we can define the constant $\mathcal{K}$ which describes the change of the spatial propagation of the system (1.1) caused by the asymmetry of k_1 as follows:

$$\mathcal{K} \triangleq \frac{-\beta \min\{A(z); z \in \mathbb{R}\}}{g'(0)h'(0)} = \frac{-\beta A\left(z_r/b\right)}{g'(0)h'(0)} > 0.$$

When $a + b = 0$, by $\min\{A(z); z \in \mathbb{R}\} = -\alpha$, we can simply denote $\mathcal{K} = \alpha\beta/(g'(0)h'(0))$. From the following result, we see that $\mathcal{K}$ is strictly increasing with respect to r.

Proposition 5.4. $\frac{\partial}{\partial r}\mathcal{K} > 0$ for $r > 1$.

Proof. It suffices to prove that $\frac{\partial}{\partial r}A(z_r/b) < 0$ for $r > 1$. By differentiating the equation $\omega(z_r) = \omega(-rz_r)$ with respect to r, we have

$$\frac{dz_r}{dr} = \frac{rz_r}{e^{(1+r)z_r} - r^2}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{e^{z_r}}{1 + rz_r}\right) &= \frac{e^{z_r}(1 - r + rz_r)}{(1 + rz_r)^2} \cdot \frac{dz_r}{dr} - \frac{e^{z_r} z_r}{(1 + rz_r)^2} \\ &= \frac{e^{z_r} z_r}{(1 + rz_r)^2(e^{(1+r)z_r} - r^2)}(r + r^2 z_r - e^{(1+r)z_r}).\end{aligned}$$

Also from $\omega(z_r) = \omega(-rz_r)$, it holds that $e^{(1+r)z_r} = (1 + rz_r)/(1 - z_r)$. Then by $z_r \in (1 - 1/r, 1)$, we have

$$r + r^2 z_r - e^{(1+r)z_r} = \frac{1 + rz_r}{1 - z_r}(r - rz_r - 1) < 0.$$

From the proof of Lemma 5.3, it holds that

$$z_r > z_2 = 2(1 + r)^{-1}\ln r;$$

namely $e^{(1+r)z_r} - r^2 > 0$. Therefore, $\frac{\partial}{\partial r}A\left(z_r/b\right) < 0$, which completes the proof. □

Now we give the result on the change of the spatial propagation caused by the asymmetry of k_1.

Corollary 5.5. *All the results in Corollary* 5.2 *hold for the uniform distribution case.*

Proof. Although this proof is similar to the proof of Corollary 5.2, we need to check some details. Let the sets Λ, Λ_A and Λ_B and the function $\Lambda(\cdot) : \sigma \mapsto \Lambda$ be the same notations as in the proof of Corollary 5.2. By some calculations and Lemma 5.3, we have

$$\frac{\partial B}{\partial \sigma} = \frac{\omega(\lambda\sigma) - \omega(-\lambda\sigma)}{2\lambda\sigma^2} > 0 \quad \text{for } \lambda \in \mathbb{R}.$$

Then it follows that (5.1) holds and this inclusion is strict when $\Lambda(\sigma) \neq \emptyset$.

When $\mathcal{K} > 1$, consider $\sigma \to 0^+$ and $\lambda = z_r/b$. Then

$$\lim_{\sigma\to 0^+} A(z_r/b)B(z_r/b) = -\beta A(z_r/b) > g'(0)h'(0).$$

Considering $\sigma \to +\infty$, we have $B(\lambda) \to +\infty$ for any $\lambda \in \mathbb{R}^+ \cup \mathbb{R}^-$. Then $\lambda_B^+ \to 0^+$ and $\lambda_B^- \to 0^-$ where $\Lambda^B = (\lambda_B^-, \lambda_B^+)$, which means that (5.2) holds. The rest of this proof can be obtained similarly. □

From Corollary 5.5, we have some similar discussions to those from Corollary 5.2 in the normal distribution case. In addition, here the critical number σ^* can also be calculated by a numerical method. For example, when $\alpha = \beta = 0.2$, $g'(0)h'(0) = 0.06$, $a = 2$ and $b = -1$, we have $\mathcal{K} = 1.1952$ and $\sigma^* = 0.8423$.

Remark 5.6. For the more general form of k_1, when k_2 is symmetric, we think that Corollary 5.2 remains true, as long as we define

$$\mathcal{K} \triangleq \beta(\alpha + 1 - E(k_1))/(h'(0)g'(0))$$

and $\sigma \triangleq \mathrm{Var}(k_2)$, where

$$E(k_1) = \inf\left\{\int_{\mathbb{R}} k_1(x)\mathrm{e}^{\lambda x}dx; \lambda \in \mathbb{R}\right\}$$

and $\mathrm{Var}(k_2)$ is the variance of k_2.

We have presented some applications of the theoretical results to the control of epidemics whose infectious agents (bacteria or viruses) are carried by migratory birds. These applications demonstrate that the frequent movements of the infectious humans accelerate the spreading of the epidemics. Moreover, it is possible that the epidemic spreads only along the flight route of migratory birds and the spatial propagation against the flight route fails as long as the infectious humans are kept from moving frequently.

Acknowledgements The first author was supported by China Postdoctoral Science Foundation (Grant No. 2019M660047). The second author was supported by National Natural Science Foundation of China (Grant Nos. 11731005 and 11671180). The third author was supported by National Science Foundation of USA (Grant No. DMS-1853622). The authors thank the reviewers for their helpful comments and Dr. Ru Hou (Peking University) for her helpful discussion.

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