Teachers' Knowledge of Fraction Magnitude

Abstract

This article explores three attributes of teachers' understanding of fraction magnitude: the accuracy and reasonableness of teachers' estimations in response to fraction arithmetic tasks as well as the alignment of the estimation strategies they used with the concept of fraction magnitude. The data were collected from a national sample of mathematics teachers in Grades 3–7 in which fraction concepts were taught ($N = 603$). The results indicated the teachers' estimations were only partially accurate and reasonable, particularly when fraction division was involved. Furthermore, teachers' credentials and the grade level at which they taught mathematics were significantly related to teachers' understanding of fraction magnitude.

Key words: fractions, teachers, integrated theory of numerical development, fraction magnitude, estimations

Introduction

Fractions are one of the most, if not the most, challenging topics to teach and learn (Lamon 2007; Ma, 1999; Mazzocco & Devlin, 2008; Newton, 2008; Rinne, Ye, & Jordan, 2019). Indeed, many students have difficulty learning even the basic properties of fractions. Yet studies have documented that students' understanding of fraction magnitude, as captured by the accuracy with which they place a given fraction on a number line, is foundational for developing competence with fractions (e.g., Siegler, Thompson, & Schneider, 2011; Torbeyns, Schneider, Xin, & Siegler, 2015).

Given that teachers' own understanding of the subject matter is associated with the learning opportunities they create for their students (e.g., Borko et al., 1992; Fisher, 1988), exploring teachers' understanding of fraction magnitude may provide insights into how students develop their sense of fraction magnitude. Yet, empirical research *explicitly* aiming to explore teachers' knowledge of fraction magnitude is limited. In fact, in a review of research on preservice teachers' knowledge of fractions, Olanoff, Lo, and Tobias (2014) noted that only three studies prior to 2011 had focused on preservice teachers' understanding of fraction magnitude. Although researchers have recognized the importance of teachers' knowledge of fraction magnitude and have included items capturing it in studies focusing on teachers' knowledge of fractions (e.g., Bartell, Webel, Bowen, & Dyson, , 2013; Lehrer & Franke, 1992; Toluk-Ucar, 2009; Zhou, Peverly, & Xin, 2006), teachers' understanding of fraction magnitude has not been the focus of these studies. Even though research interest in teachers' understanding of fraction magnitude has increased in recent years, such studies are still quite scarce, especially with in-service teachers (Lemonidis, Tsakiridou, & Meliopoulou, 2018; Siegler & Lortie-Forgues, 2015; Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009).

This study aimed to contribute to the literature in three major ways. First, some of the prior work on teachers' knowledge of fraction magnitude has attempted to capture teachers' understanding of fraction magnitude by using fractions teachers may be familiar with (e.g., Lemonidis, Tsakiridou, & Meliopoulou, 2018; Tsao, 2005; Whitacre & Nickerson, 2016). Unlike students, teachers deal with common fractions throughout their careers; thus, using certain fraction pairs might not capture the extent to which teachers understand fraction magnitude. Furthermore, some of the fraction pairs used in these studies have had either the same numerator or the same denominator, which tends to lead to the use of strategies that do not require attending to fraction magnitude (Siegler & Lortie-Forgues, 2015). For instance, in one of the few studies conducted with in-service teachers, Lemonidis and colleagues (2018) used fraction comparison items that were used in a prior study conducted with children. Seventy fifth- and sixth-grade Greek mathematics teachers were asked to mentally compare a set of fraction pairs with denominators of 2, 4, 5, 6, 7, 8, and 9, some of which had the same numerator or denominator. They found that the participating teachers performed well on comparison problems, especially with problems that had the same numerators and denominators (above 95%). The success rate for comparing fractions with different numerators and denominators was lower, and the lowest was for comparing 3/4 and 7/9 (59%). Therefore, I hypothesized that using fractions with different numerators and denominators could provide greater insights into teachers' understanding of fraction magnitude.

Second, teachers' understanding of fraction magnitude has commonly been captured by asking teachers to order and compare fractions (e.g., Lemonidis et al., 2018; Whitacre & Nickerson, 2016; for a review, see Olanoff et al., 2014). However, another indicator of fraction magnitude is locating a fraction on a number line, a task that has been supported by evidence from fraction magnitude studies (e.g., Siegler et al., 2011) and acknowledged by mathematics educators (e.g., Behr, Wachsmuth, & Post, 1985). Similarly, estimating the outcome of an operation with fractions is considered an indicator of fraction magnitude (e.g., Behr et al., 1985). Furthermore, prior work focusing on the strategies that teachers use to make estimations has revealed that the strategies teachers use could provide insights into their understanding of fraction magnitude that would not be captured by their final answers (Thanheiser et al., 2016; Yang et al., 2009). For instance, Yang and colleagues (2009) examined the strategies used by 280 Taiwanese preservice teachers from one university to estimate the answer to a word problem that included a comparison of fractions (30/31 and 36/37). They found that although 95% of the participants answered the question correctly, only 36% of them applied a strategy using the number sense in which both fractions were only one-unit fraction (1/31 and 1/37) away from 1. Thus, I hypothesized that analyzing teachers' strategies could reveal insights into teachers' understanding of fraction magnitude.

Third, the majority of prior studies have been conducted with preservice teachers (e.g., Lee & Lee, 2020; Thanheiser et al., 2016; Whitacre & Nickerson, 2016; for a review, see Olanoff et al., 2014). This gap in the literature creates the potential for in-service teachers' understanding of fractions to be radically different from that of preservice teachers because they may have developed a deeper understanding of these concepts over time from teaching the concepts, interacting with students and the curriculum materials, and participating in professional development. Therefore, exploring *in-service* teachers' understanding of fraction magnitude could help us ascertain why students might be struggling with fraction magnitude and could help us identify the best practices to enhance teachers' understanding of fractions.

In sum, this study aimed to contribute to the literature by investigating how a national sample of U.S. in-service teachers (*N* = 603) who taught mathematics in Grades 3–7 dealt with fraction magnitudes by adapting an approach that was found useful for capturing such understanding in prior work (e.g., Siegler et al., 2011). The following research questions guided this study:

1) What is the nature of teachers' understanding of fraction magnitude?

2) What is the association between teachers' educational backgrounds and their understanding of fraction magnitude?

Conceptual Framework

Fractions, in their basic definition, are nonnegative rational numbers. While I acknowledge that alternate interpretations or subconstructs of rational numbers exist (Lamon, 2007), here I focus on the meaning of fractions as numbers that can be defined as "how much there is of a quantity relative to a specified unit of that quantity" (Behr, Lesh, Post & Silver, 1983, p. 99). Thus, understanding fractions as numbers involves understanding their relative amounts (Lamon, 2007). Whether the same fraction is written as 1/2, 4/8, 5/10, or 13/26 is not as important as that they all convey the same amount. Similarly, when fractions are compared or ordered, the same relative magnitude plays a key role in understanding this issue. Thus, linking a visual representation to a symbolic representation, such as placing a fraction on a number line, is an indicator of understanding the fraction magnitude.

Such a conceptualization of fractions is also at the core of Siegler's (2016) integrated theory of numerical development. According to this theory, "numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines" (Siegler et al., 2011, p. 274). Unlike prior theories, which have emphasized the differences between whole numbers and fractions and have assumed that the properties of whole numbers create difficulties for children as they learn fraction concepts (e.g., DeWolf & Vosniadou, 2015; Van Hoof, Verschaffel, & Van Dooren, 2015), Siegler's integrated theory focuses on both the similarities and differences between whole numbers and fractions. Specifically, it underscores one feature—the magnitude of the number—that connects whole numbers and fractions together with the characteristics that differentiate whole numbers from fractions, such as that different procedures are used for addition and division or that infinite numbers can exist between two fractions. Thus, rather than seeing children's knowledge of whole numbers as a barrier to developing their knowledge of fractions, the integrated theory recognizes that the magnitude of numbers can have a positive impact on the learning of fractions.

According to this theory, students' understanding of the size of a fraction has the potential to improve their conceptual understanding of fractions. Students' knowledge of the fraction magnitude can help them evaluate the reasonableness of their solutions to fraction arithmetic problems, such as rejecting an implausible answer of 4/6, which could be obtained by adding numerators and denominators across (1/2 + 3/4 = 4/6), because the sum of two positive amounts cannot lead to a smaller amount than either of the addends. Given that many errors in children's thinking are associated with considering the numerator and denominator as separate numbers (e.g., Post, Cramer, Behr, Lesh, & Harel, 1993) rather than as a single number, coordinating the numerator and denominator of a given fraction to represent a single number with a magnitude can help students overcome difficulties with fraction arithmetic.

Linking the symbolic value of a fraction to its magnitude with accuracy is at the heart of the integrated theory. Thus, this theory considers that being able to accurately represent the magnitude of a fraction on a number line is a crucial indicator of understanding fractions because "all real numbers have magnitudes and can be represented on number lines" (Siegler, 2016, p. 343). The number line is also considered an important visual representation that links conceptual and procedural knowledge (National Mathematics Advisory Panel, 2008).

Thus, given that prior empirical evidence has suggested that individuals' understanding of fraction magnitude can be captured by assessing how accurately they can locate a fraction on a given number line, I decided to use number line tasks, which are favored in the integrated theory. Scholars in mathematics education also agree that locating a fraction on a number line is one indicator of understanding fraction magnitude (e.g., Behr et al., 1985), as are other indicators, such as estimating the outcome of an operation with fractions, ordering fractions and recognizing equivalent fractions (e.g., Behr et al., 1985).

To capture teachers' understanding of fraction magnitude thoroughly, I modified tasks used in prior studies to capture several attributes of teachers' understanding of fraction magnitude. Specifically, locating a fraction precisely on a number line (e.g., Siegler et al., 2011), estimating the outcome of an operation with fractions, and ordering fractions are important indicators that a person understands fraction magnitude. Teachers were presented with two unfriendly fractions on a number line (the same ones used by Siegler & Lortie-Forgues, 2015) and were asked to estimate the answers to arithmetic problems involving these fractions, locate their estimated answers on a number line, and report how they arrived at their answers. Through this process, I was able to capture three indicators of fraction magnitude understanding.

The first indicator was the *accuracy* of the placement of their estimations on a number line similar to the one used by Sigler and others. The second indicator was the *reasonableness* of their estimations of the outcome. For instance, locating the sum on the left side of both positive fractions would suggest that the teacher lacked an understanding of the magnitude of the given fractions because the sum would be greater than either of the two positive fractions. The third indicator of teachers' understanding of fraction magnitude was *the alignment of strategies* they used to estimate the outcome of the operation with fraction magnitude. The estimation strategies they used could reveal their understanding of the fraction size because "the numerator, the denominator, and the relationship between them must all three be coordinated in order to estimate correctly or understand the size of a given rational number" (Behr et al., 1986, p. 103). For instance, a teacher who rounds the numerator and denominator separately may have only a partial understanding of fraction magnitude in that fraction size is based on the quotient of the numerator and denominator. In fact, interviews conducted with 20 in-service teachers using the two tasks in this study suggested that those who were using rounding strategies, for instance, were not paying attention to the fraction sizes.

As mentioned, using a fraction pair that did not have an equal numerator or denominator or that was not easily computed by creating a common denominator and numerator could reveal teachers' understanding of fraction magnitude. For easier and more frequently used fractions, teachers might already associate the size of the fraction with its numerical value, which would make it less likely for them to reveal their sense of fraction magnitude (possibly because of repeated exposure to smaller and friendlier fractions). That is why, I used unfamiliar fractions (19/35 and 41/66) so that the cognitive process could be studied more accurately. Because unfamiliar fractions were given, teachers needed to assign a magnitude to the given symbols by utilizing some estimation strategies.

In sum, the accuracy of teachers' placement of their estimations on a number line, the reasonableness of their answers, and their use of strategies focusing on number magnitude, such as noticing that the location of a fraction on a number line indicates its size or using friendlier fractions with a similar size (e.g., ½ for 19/35 2/3 for 41/66), would be indicators of teachers' understanding of fraction magnitude. To test this premise, I chose two fraction operations: addition and division.

Method

Sample

The data used in this study were collected from 603 U.S. elementary and middle school teachers who were teaching fractions in Grades 3–7 at the time of data collection. Teachers were recruited from across the United States via email[1]. The study sample is similar to a nationally representative sample of U.S. elementary school teachers in public schools terms of gender and racial or ethnic background (Snyder, de Brey, & Dillow, 2019). Half of the sample majored in education (50.6%). About 67.2% of the sample held multiple subject teaching credentials, whereas 17.4% of teachers held a credential in teaching mathematics; 76.2% of the sample held a full teaching credential, whereas 12.4% of teachers held a preliminary teaching certification. Teachers in the study had an average of 6.6 years of experience in teaching mathematics (*SD* = 7.2).

Estimation Tasks Used to Capture Teachers' Understanding of Fraction Magnitude

Teachers were presented with two fractions on a number line, 19/35 and 41/66, and were asked to estimate the sum of 19/35 and 41/66 and the division of 41/66 by 19/35 (see Figure 1).[2] The number line ranged from −1 to 3, with whole numbers and midpoints between each whole

[1] Participants were first completed a screening survey to ensure the data were collected from the targeted sample. The screening survey began with general questions regarding the participant's career and was followed by specific questions, such as at what grade level the teacher taught mathematics. Those who were eligible to participate in the study were allowed to continue to take the survey, and they completed an additional set of questions regarding their educational background. Teachers who participated in the study were compensated with an online gift card.

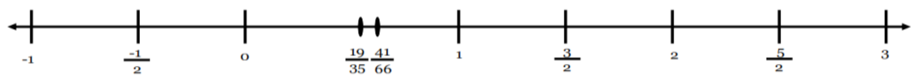
[2] Teachers saw these questions in a randomized order to reduce the error in their responses.

number positioned on the line. Teachers located their responses by moving the cursor to the desired location on the number line and then clicking the mouse. They were asked to explain their answers immediately so that they could accurately report how they generated their answers.

Estimating the sum of fractions

1. The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line.

   Without computing, please estimate the sum of $\frac{19}{35}+\frac{41}{66}$ by placing a dot on the number line where you think the sum would be found. *Explain your answer.*



Estimating the quotient of fractions

2. The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line.

   Without computing, please estimate the quotient of $\frac{41}{66}\div\frac{19}{35}$ by placing a dot on the number line where you think the quotient would be found. *Explain your answer.*

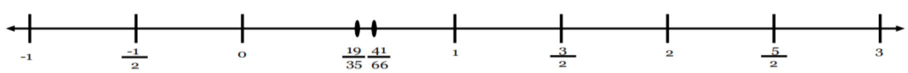


Figure 1. Estimation tasks used in the study.

After a scholar with expertise in fractions and I developed these items, we conducted interviews with 20 mathematics teachers to ensure that the strategies teachers reported using were the ones they actually used, that the locations of their answers matched their reported estimations, and that their estimations were aligned with where they placed the dots. Furthermore, their responses guided the coding in terms of which strategies were in alignment with the concept of fraction magnitude.

Three Indicators of Teachers’ Understanding of Fraction Magnitude

Teachers’ responses to these tasks were analyzed according to the three indicators fraction magnitude understanding: (1) the accuracy of their estimations, (2) the reasonableness of their estimations, and (3) the alignment of their estimation strategies with the concept of fraction magnitude.

The ***accuracy*** of the estimation was computed by using a method developed by Siegler and colleagues (2011) for similar tasks. Specifically, teachers' accuracy of estimation was calculated by the absolute difference between the location of the teacher's answer on the given number line and the location of the exact answer, divided by the length of 0 to 1 on the number line. Note that a score of 0 meant that the estimate equaled the correct answer, whereas a higher score on this scale indicated the extent to which the teachers' estimation deviated from the actual answer. Thus, a higher score on this scale indicated less accuracy.

The ***reasonableness*** of their estimations was measured by the extent to which their estimations fell within a reasonable range for the given fraction pairs and operations. Specifically, for the addition problem, the sum would be greater than 1 because both fractions were larger than ½ and that it would not be higher than 4/3 because both fractions were less than ⅔. Similarly, for the division problem, that the quotient would be greater than 1 because the divisor was smaller than the dividend and that it would be less than 4/3 because ⅔ divided by ½ would result in 4/3, given that the dividend was less than ⅔, whereas the divisor was greater than ½. Thus, the reasonableness score was 0 if a teacher's estimation was not in a reasonable range and 1 if it was in a reasonable range.

The *alignment of the estimation strategies with the fraction magnitude concept* (i.e., the third indicator) was measured by identifying the strategies teachers used and then grouping these strategies by the extent to which they were aligned with the concept of fraction magnitude.

Identifying teachers' estimation strategies. A set of estimation strategies was developed based on prior work (e.g., Siegler et al., 2011) and existing resources (e.g., Beckmann, 2018) to identify possible strategies teachers might use. A subsample of participating teachers' responses was coded to test that these categories were comprehensive enough to capture different strategies. The categories were finalized through an iterative process of redefining and coding teachers' responses.

All responses were coded by two raters, one of whom was the author. The initial training included a discussion of descriptions of these categories and how to code teachers' responses accurately by first coding them together and then separately, followed by discussion. After reaching 90% agreement on coding, we coded the remaining responses separately, and our overall agreement reached 97%. All responses were binary coded against the aforementioned categories, and each response was assigned to one estimation strategy.

The teachers in the study used eight estimation strategies (see the Supplementary Materials). The *numerical rational number* category included strategies in which teachers found a more convenient fraction with a similar size to work with, such as selecting ½ for 19/35 and ⅔ for 41/66. Teachers using this strategy seemed to relate these unfriendly fractions to easier fractions with similar magnitudes. For instance, some of the teachers who used rational number strategies utilized the given number line to more accurately find easier fractions to work with, such as by dividing the space between 0 and 1 into 4, 5, or 10 pieces. Some came up with an easier fraction by focusing on the relationship between the numerator and denominator in the given fractions, such as by noticing that 41/66 is less than ⅔ because 44/66 equals ⅔. Responses in the *nonnumerical rational number* strategies category did not include specific numerical estimates for the given fractions; rather, teachers focused on the relationship between the given fractions or they used their knowledge of the location of a fraction on a number line to represent the magnitude of the fraction. For example, teachers who applied this strategy measured the distance between 0 and 19/35 to see how many units of this segment could fit into the space between 0 and 41/66 to estimate the division problem. These two strategies are indicators of teachers’ understanding of fraction magnitude in that the teachers either selected fractions with a similar magnitude or used the location of the fractions to capture the fraction magnitude.

Unlike teachers who used fractions with similar magnitudes, teachers who used *benchmark fraction* strategies selected 0, ½, and 1 as benchmark numbers to estimate both fractions. I created a separate category for benchmark fractions because teachers in this category were using the same benchmark fractions to estimate *both* fractions, which indicated they were paying relatively less attention to the magnitude of each fraction. As an example, most of the teachers applying the benchmark fraction strategy reported that the sum would greater than one because both fractions were greater than ½. Teachers who used the *common denominator* strategy changed the denominators of the two given fractions to similar ones, relatively less attention seemed to be given to the numerators, therefore, the overall magnitude determined by the quotient of the numerator and denominator. Teachers who used this strategy usually changed the fraction 19/35 to 19/33 to find a common denominator with 41/66, or they changed 41/66 to 41/70 to find a common denominator with 19/35.

Finally, teachers used other strategies that did not seem cognizant of the fraction magnitudes, such as the *rounding strategy*, in which participants rounded the numerators and denominators to the nearest 10 or 5. Teachers who used rounding strategies rounded the numerators and denominators separately, rather than focusing on the fraction magnitude. In fact, during the interviews with 20 teachers, I found that those who used the rounding strategy did not pay attention to the numerators and denominators simultaneously. The *none*/*guess* category indicated that teachers did not report how they found the answer, or they mentioned that they were simply guessing. The *flawed strategy* category included responses indicating the teachers held a misconception or used flawed reasoning, such as that division would make things smaller or that they could add the numerator and denominator across fractions. Finally, I used an *other* category when the teacher's strategy did not fall into any of the categories mentioned.

Grouping estimation strategies based on their alignment with the fraction magnitude concept. The extent to which the strategies were aligned with the fraction magnitude was captured based on the natural alignment of strategies with fraction magnitude (e.g., nonnumerical strategies) and explanations the teachers provided during the interviews and in their written answers. Three hierarchical levels were created. Level 3 included nonnumerical and rational number strategies that indicated the teachers were using either the size of a given fraction (nonnumerical strategies) or a fraction of a similar magnitude (rational number strategies) in their estimations. Level 2 categories included the common denominator and benchmark fraction strategies because teachers who were using these categories were not fully attending to the numerator or denominator of a given fraction simultaneously in their estimation or were using rough estimates for fraction magnitudes. Finally, rounding, incorrect, and other strategies were all coded as Level 1 because these strategies indicated teachers were attending to the numerator and denominator as separate quantities, indicating they were not attending to the fraction magnitude.

While these three indicators revealed different elements of teachers' understanding of fraction magnitude, a composite scale is created using these indicators to capture the overall understanding. Scores for each indicator and each task were converted to *Z*-scores[3], and these *Z*-

[3] Teachers' accuracy score was reverse-coded because a higher score indicated lower accuracy.

scores were averaged to create a fraction magnitude score. Cronbach's alpha (i.e., internal consistency) of this composite scale was .77.

Analytic Strategy

To investigate teachers' understanding of fraction magnitude, I first focused separately on each of its three indicators. To investigate the accuracy and reasonableness of teachers' estimations (i.e., two of the indicators of teachers' understanding of fraction magnitude), I first reported descriptive statistics for both indicators by operation. To investigate the third indicator (i.e., the alignment of the strategies with the fraction size), I reported the frequency with which teachers used the strategies based on their alignment with the fraction magnitude concept for each operation.

To examine the relationships between teacher's professional background indicators and their overall understanding of fraction magnitude, I used an ordinary linear regression, with the composite scale score being the outcome variable. Specifically, as shown in Equation 1, I added a dummy-coded indicator of whether teachers majored in education and were fully certified, the credential level (*other areas*, *generalist*, and *mathematics*, with *other areas* being the reference category), the grade level of mathematics being taught, and years of mathematics teaching experience as predictors of teachers' scores on the fraction magnitude scale:

$$\begin{aligned}\text{fractionmagnitude}_t &= \beta_0 + \beta_1\text{educationmajor}_t + \beta_2\text{fullycertified}_t + \beta_3\text{generalist}_t + \beta_4\text{mathcredential}_t + \\ &\quad \beta_5\text{gradelevel}_t + \beta_6\text{mathteachingexp}_t + \varepsilon_t. \end{aligned} \tag{1}$$

Results

Teachers' Understanding of Fraction Magnitude

The ***accuracy*** of teachers' estimations for the addition of fractions was, on average, .15 with a standard deviation of .22 (*N* = 586). In contrast, the accuracy was lower for the division estimation, with a mean of .29 and a standard deviation of .35 ($N = 577$). This means that, on average, the difference in teachers' estimations and the actual answer for addition was 15% of the distance between 0 and 1, whereas the difference in their estimations and the actual answer for division was 29% of the distance between 0 and 1. The correlation between the accuracy of teachers' estimations for both operations was low ($r = .24$), suggesting that providing a precise estimation for one operation may not indicate providing an estimation of similar accuracy for the other, even when the same fraction pairs were given for both operations. This result suggests that teachers' understanding of fraction magnitude based on the first indicator is partial, particularly when interpreting the quotient of two fractions.[4]

Analysis of teachers' responses according to the second indicator of fraction magnitude suggested that 75.6% of teachers' estimations for addition were in a *reasonable* range, whereas this rate was only 54.1% for their estimations of division for the same fraction pairs.[5] As shown in Figure 2, 45% of the sample provided a reasonable estimation for both operations, whereas 14.5% did not provide a reasonable estimation for either operation.

[4] Indeed, the accuracy of teachers' estimations for the division of fractions was statistically lower than the accuracy of their estimations for the addition problem, $t(559) = -9.01$, $p < .001$.

[5] A significantly higher percentage of teachers provided a *reasonable* estimation for the addition of fractions (76.6%) compared with the division of fractions (54.0%), McNemar's Chi2(559) = 71.05, $p < .0001$.

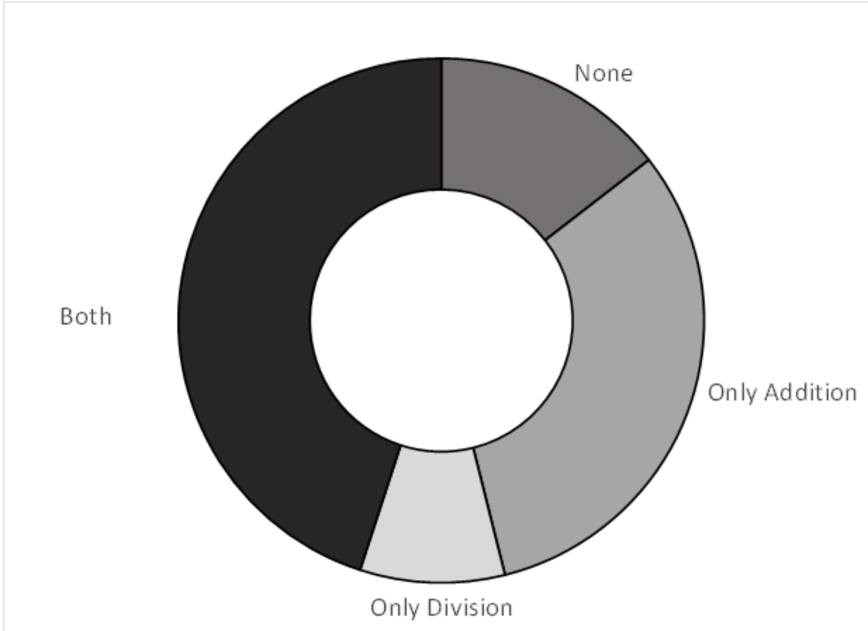


Figure 2. Percentage of teachers who provided reasonable estimations for fraction addition and division.

As shown in Figure 3, the alignment of teachers' estimation strategies with the fraction magnitude (i.e., the third indicator of teachers' understanding of fraction magnitude) revealed that the teachers used Level 3 and 2 strategies more frequently when estimating the sum of two fractions, whereas they used Level 3 and 1 strategies for estimating the quotient of two fractions. Moreover, the two most common strategies chosen for the addition problem were using a rational number of a similar size (Level 3) and using benchmark numbers (Level 2; 30.4% and 28.2%, respectively). Teachers who worked with easier rational numbers generally used ½ or .5 for 19/35 and ⅔ or .6 for 41/66. The third most common strategy for estimating addition was another level 2 strategy: creating a common denominator (e.g., changing 19/35 to 20/33 and estimating the sum of 40/66 and 41/66).

For estimating the division of fractions, finding more convenient rational numbers (a Level 3 strategy) was the most common approach (27.8%) and rounding (a Level 1 strategy) became the second most common approach (16.6%). Furthermore, guessing (a Level 1 strategy) and nonnumerical estimations (a Level 3 strategy) were strategies more frequently applied for division estimations (14% and 13.3% for guessing and using nonnumerical strategies, respectively).

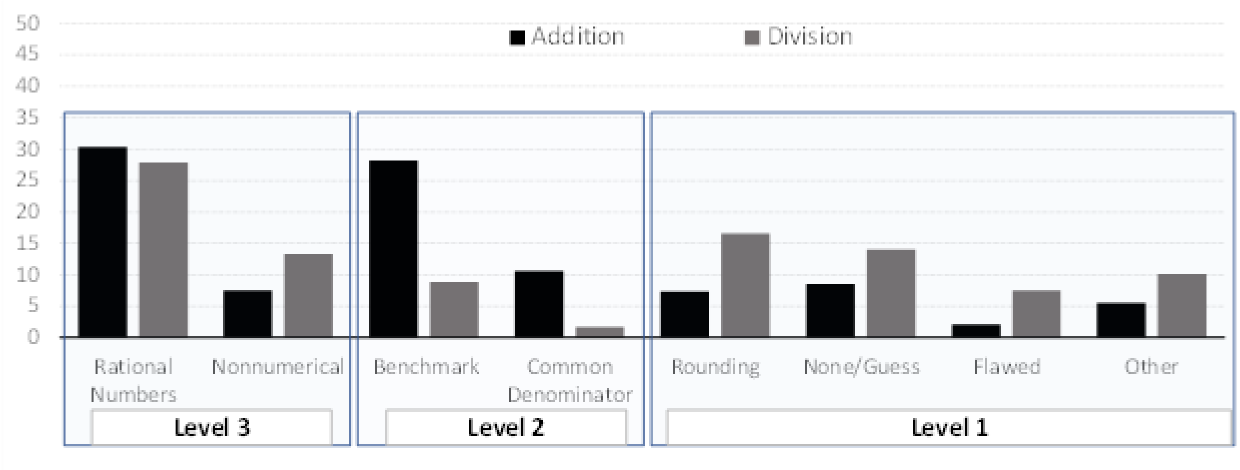


Figure 3. Frequency with which teachers used various strategies for estimations.

An analysis of the strategies same individuals used for both operations indicated that 83.6% of the teachers who used a Level 1 estimation strategy for addition also used a Level 1 estimation strategy for division, and 71.1% of the teachers who used a Level 3 strategy for addition also used a Level 3 strategy for division. However, only 19.2% of those who used Level 2 strategies for addition used Level 2 strategies for division. More than half the teachers (51.9%) who used a Level 2 strategy for addition switched to a Level 1 strategy for division, such as the one coded as *other* in which teachers first applied the division algorithm ([41 × 35]/[66 × 19]) and then estimated the division of number pairs (e.g., 41 ÷ 19 and 35 ÷ 66, which is 2/2).

Teachers' estimation strategies also provided insights into why their understanding of fraction magnitude is particularly limited when the division of fractions was involved. Indeed, one reason seemed to be that they overgeneralized the patterns for dividing whole numbers by applying them to fractions. Of teachers whose responses were categorized as flawed or who used incorrect reasoning ($N = 43$), several reported that dividing fractions would make a number smaller ($N = 12$). For instance, one teacher wrote, "The answer could not become smaller than 0 because they are both positive integers. However, the answer cannot become larger than 1 because you are dividing one fraction by another and it will give you a smaller fraction." In contrast, another group of teachers commented that the division of fractions would lead to larger numbers ($N = 11$). For instance, one teacher noted, "When you divide fractions, the opposite rule happens and your number always increases." Finally, some teachers used multiplication rather than division ($N = 13$). These teachers usually found ½ of the dividend. As one teacher said, "19/35 is greater than ½; the answer would be ½ of 41/66."

Linking Teachers' Background Characteristics to Their Understanding of Fraction Magnitude

Of the teachers' background indicators, the grade level teachers taught and credential type were significantly related to their knowledge of fraction magnitude (see Table 2). Specifically, the higher the grade level at which teachers were teaching mathematics was associated with a higher score on the fraction magnitude scale ($b = .15$). Teachers who held a mathematics teaching credential or teachers who held multiple subject teaching credentials had significantly higher scores than those holding credentials in other subjects, such as special education ($b = .26$ and $b = .20$, respectively). However, the scores of teachers who held mathematics teaching credentials were not statistically different from the scores of teachers holding multiple subject teaching credentials ($p = .46$). Fully certified teachers had marginally higher scores than did those who were not ($p = .08$, $b = .13$), whereas teachers who majored in education in their undergraduate years seemed to have marginally lower scores than did those who had not (p = .07, a b = -.11). The number of years of mathematics teaching experience was also not related to teachers' scores on the fraction magnitude scale.

Table 2. *Linear Regression Results for Teachers' Understanding of Fraction Magnitude as Predicted by Their Educational Background*

Background indicator	Fraction magnitude scale score (*SE*)	
Major in education	$-.11^{\sim}$	(.06)
Fully certified	$.13^{\sim}$	(.08)
Credential (multiple subjects)	**.20***	**(.09)**
Credential (mathematics)	**.26****	**(.11)**
Grade level taught	**.15*****	**(.04)**
Mathematics teaching experience	.01	(.004)

Note. Numbers in boldface indicate statistically significant results. *SE* = standard error.
$^{\sim}p < 0.10$. $^{*}p < 0.05$. $^{**}p < 0.01$. $^{***}p < 0.001$.

Discussion

In this study, three attributes of teachers' understanding of fraction magnitude were investigated: the accuracy of teachers' estimations, reasonableness of their estimations, and alignment of the strategies they used with the fraction magnitude concept. Before discussing the study findings, I would like to call attention to the study limitations. First, the study was based on teachers' responses to two tasks that included the same fraction pairs; therefore, more research is needed to test whether similar results would be obtained if teachers' understanding were captured through multiple items that included various fractions. Similarly, both tasks used in the study relied on only number line representations. Thus, teachers' understanding of fraction magnitude through the presentation of other methods is also needed, particularly given that presentational flexibility is an important competency in fraction understanding (Deliyianni, Gagatsis, Elia, & Panaoura, 2016).

The findings of this study suggest that teachers' understanding of fraction magnitudes may have been overestimated in prior work, given that this construct was generally measured by fractions that teachers were familiar with (e.g., Lemonidis, Tsakiridou, & Meliopoulou, 2017; Tsao, 2005; Whitacre & Nickerson, 2016). Analysis of teachers' understanding of fraction magnitude when using unfamiliar fractions showed that teachers might not connect the size of a fraction precisely to its symbolic representation (i.e., the first indicator), even when they seemed to find a reasonable size for that particular fraction (i.e., the second indicator). More than three-fourths of the sample provided reasonable estimations for fraction addition, and their estimations were, on average, 15% off the size of the distance between 0 and 1. However, for division, their estimations were, on average, off by almost one-third the size of the one unit (i.e., the distance between 0 and 1), and half the sample did not provide a reasonable answer for the outcome of fraction division. Furthermore, the third indicator, understanding the concept of fraction magnitude, provided evidence that this issue was partly related to teachers' incorrect generalization of the division of numbers. Some teachers did not consider that the quotient could be larger than the dividend or divisor.

One important implication of this study was related to how a teacher's understanding of fraction magnitude could be conceptualized and measured. Precisely locating an unfamiliar fraction on a number line, as conceptualized and measured in the integrated number theory, seemed to be an important indicator of the teacher's understanding of fraction magnitude (e.g., Siegler & Lortie-Forgues, 2015). Yet the findings also suggest that the extent to which the teachers were able to provide reasonable estimates as well as the extent to which their estimation strategies were in line with the fraction size revealed somewhat distinct insights into their understanding of fraction magnitude. In particular, when teachers' understanding was examined in terms of the reasonableness of their estimates, the results revealed that half the sample provided reasonable estimates for both operations, whereas only 15 % were unable to provide a reasonable estimate for either. Similarly, although both the accuracy and reasonableness indicators suggested that teachers had difficulty with fraction magnitude, especially when division was involved (similar to what was found in prior work; Siegler & Lortie-Forgues, 2015), analyzing teachers' strategies offered insights. Specifically, more than half the teachers who used estimation strategies that were partially aligned with the fraction magnitude of a relatively easier concept (i.e., fraction addition) tended to switch strategies that treat the numerator and denominator as separate quantities when dealing with a more difficult concept (i.e., fraction division). Taken altogether, the study results suggest that focusing on the three indicators of teachers' understanding of fraction magnitude could reveal greater insights into teachers' understanding of this concept. It is also important to point out that the task used in this study could be used in future work to gather rich information regarding different attributes of teachers' understanding of fraction magnitude.

The study findings also underscore the importance of learning and teaching fractions by focusing on the association between the symbolic representations of fractions and the relative amounts they refer to, and they echo prior work on the importance of connecting symbolic notations to other representations (Deliyianni et al., 2016). One implication of this study is that teacher education and professional development programs could provide more opportunities for teachers to develop a sense of fraction magnitude. For instance, teachers could connect symbolic notations for fractions to relative amounts by learning estimation strategies for identifying friendlier fractions of relatively the same size when presented with unfriendly fractions. Similarly, drawn images and number lines could be used strategically to connect relative amounts to the fraction notations to enhance teachers' understanding of fraction magnitudes.

Further studies are needed to capture the effectiveness of different learning opportunities in developing teachers' fraction magnitude sense.

Some of the study findings are not in line with research conducted with preservice teachers or students (cf. Siegler & Lortie-Forgues , 2015; Yang et al., 2009). For instance, a significant portion of the teachers in this study used strategies that were aligned with fraction magnitude, whereas prior work with preservice teachers indicated that only a small percentage of preservice teachers used strategies based on number magnitude (Yang et al., 2009). This finding underscores the need to conduct research on content-related issues with in-service teachers because teachers' understanding of the content is likely different from that of students and prospective teachers. It also cautions researchers and teacher educators to consider the potential differences in thinking and understanding between prospective teachers and in-service teachers.

The findings also indicated that the grade teachers were presently teaching was associated with their scores on the fraction magnitude scale, suggesting that the higher the grade they taught, the higher the score they received on the scale. These findings also concur with the results of studies showing that teachers' mathematical knowledge was associated with the grade the teachers taught (Hill, 2010) and their credentials (Copur-Gencturk, 2021; Hill, 2007; Izsák, Jacobson, & Bradshaw, 2019). Teachers holding credentials in other areas, such as special education, had significantly lower scores than did those holding multiple subject and mathematics teaching credentials. These results imply that special education teachers may need more opportunities to develop fraction magnitude sense.

In sum, this study aimed to contribute to the literature by offering insights into how teachers' understanding of fraction magnitude could be conceptualized and measured. It provides evidence focusing on the precision and reasonableness of teachers' estimations of fraction operations that involve unfamiliar fractions and the alignment of estimation strategies with the fraction magnitude concept could reveal insights into teachers' understanding of fraction magnitude. These findings imply that teachers need more professional learning opportunities to connect fraction notations to the relative fraction amounts and to estimate fractions with greater precision.

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