Improved $\bar{\mu}$-scheme effective dynamics of full loop quantum gravity

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We propose a new derivation from full loop quantum gravity (LQG) to the loop quantum cosmology improved $\bar{\mu}$-scheme effective dynamics, based on the reduced phase-space formulation of LQG and a proposal of an effective Hamiltonian/action in full LQG. A key step of our program is an improved regularization of full LQG Hamiltonian on a cubic lattice. The improved Hamiltonian uses a set of "dressed holonomies" $h_\Delta(\mathfrak{s})$ which depend on both the connection A and the length of the curve $\mathfrak{s}$. With the improved Hamiltonian, we propose a quantum effective action and derive a new set of effective equations of motion (EOMs) for full LQG. Then, we show that these new EOMs imply the $\bar{\mu}$-scheme effective dynamics for both the homogeneous-isotropic and Bianchi-I cosmologies, and predict a bounce and Planckian critical density. As a byproduct, although the model is defined on a cubic lattice, we find that the improved effective Hamiltonian of cosmology is invariant under lattice refinement. The cosmological effective dynamics, predictions for the bounce, and critical density are results in the continuum limit.

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I. INTRODUCTION

Loop quantum gravity (LQG) is a promising candidate for a nonperturbative and background-independent theory of quantum gravity (see, e.g., Refs. [1,2] for reviews). Among many important achievements of LQG, one of the most profound physical predictions is the resolution of singularities [3–14]. It is well known that classical Einstein gravity breaks down at singularities, while the purpose of quantum gravity is to extend the gravitational theory to describe the physics of singularities.

A well-developed method of singularity resolution is loop quantum cosmology (LQC), where the big bang singularity is replaced by a quantum bounce (see, e.g., Refs. [15–17] for reviews). LQC applies the LQG method to the homogeneous and isotropic sector of gravity. The homogeneous and isotropic sector is given by a classical symmetry reduction from the infinitely many degrees of freedom of gravity to a single degree of freedom (the scale factor). The quantum dynamics of LQC have been studied extensively, and it turns out that it can be efficiently described by an effective equation, which reduces to the classical Friedmann equation at low energy density and modifying the Friedmann equation at high energy density [18]. The solution of the effective equation demonstrates that the big bang singularity is resolved and replaced by a big bounce, where the curvature is finite and Planckian. The time evolution of cosmology governed by the LQC effective equation is often called the effective dynamics.

Due to the theme of symmetry reduction before quantization, LQC has suffered from the long-standing issue regarding the relation with the full theory of LQG. Symmetry-reduced models of loop quantum black holes share the same issues. There has been interesting recent progress concerning this relation [14,19–28]. Although there is a top-down derivation from the full LQG to LQC μ_0-scheme effective dynamics [14], the LQG derivation to the improved $\bar{\mu}$-scheme effective dynamics of LQC is still largely open [29]. The $\bar{\mu}$ scheme is physically preferred because it predicts a constant Planckian energy density at the bounce, while the density at the bounce in the μ_0 scheme can vary and possibly be non-Planckian.

Our present work takes one step further toward resolving the above issue, and proposes a new derivation of the LQC improved $\bar{\mu}$-scheme effective dynamics from full LQG. Our derivation is based on the reduced phase-space formulation of LQG [30,31]. A key step of our program is an improved regularization of the full LQG Hamiltonian on a cubic lattice. The improved Hamiltonian uses a set of "dressed holonomies" $h_\Delta(\mathfrak{s})$ which depend on both the connection A and the length of the curve $\mathfrak{s}$. With the improved Hamiltonian, we propose a quantum effective action and

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derive a new set of effective equations of motion (EOMs) for full LQG. Then, we show that these new EOMs imply the $\bar{\mu}$-scheme effective dynamics for both the homogeneous-isotropic and Bianchi-I cosmologies, and predict a bounce and Planckian critical density. Moreover, although the model is defined on a cubic lattice, we find that the improved effective Hamiltonian of cosmology is invariant under lattice refinement, and has a trivial continuum limit. The predictions for the bounce and critical density are also independent of the lattice refinement.

The idea of constructing the improved Hamiltonian $\mathbf{H}_\Delta$ is to regularize the curvature $F(A)$ of the Ashtekar-Barbero connection A in a nonconventional manner. In the standard regularization [32], $F(A)$ is replaced by a loop holonomy, which is a functional of A only. In our approach, we denote by $S(\Delta)$ the surface enclosed by the loop holonomy $h(\Delta)$, and require $h(\Delta)$ to not only be a functional of A but also depend on the geometry of the surface $S(\Delta)$. Inspired by LQC, we introduce an area scale Δ such that the area of $S(\Delta)$ is fixed to be Δ. Here Δ is a free parameter of dimension $(\text{length})^2$ and may be chosen to be the minimal area gap in LQG. $h(\Delta)$ depends on Δ, and thus so does the Hamiltonian $\mathbf{H}_\Delta$ constructed from $h(\Delta)$. Our regularization of $\mathbf{H}_\Delta$ is described in Sec. III.

Interestingly, $\mathbf{H}_\Delta$ is a *non-graph-changing Hamiltonian which changes the graph*. $\mathbf{H}_\Delta$ is non-graph-changing because it is a function on the phase space $\mathcal{P}_\gamma$ of holonomies and fluxes on a fixed cubic lattice (a graph whose vertices are all 6-valent) γ. The cubic graph is preferred by the semiclassical analysis of the canonical LQG [14,33], but it changes the graph γ because $h(\Delta)$ contains holonomies along curves that do not belong to γ. The simple way to make these two aspects of $\mathbf{H}_\Delta$ consistent is to define $h(\Delta)$ as a phase-space function on $\mathcal{P}_\gamma$. Indeed, a desired $h(\Delta)$ can be defined by comparing the continuum approximations of $h(\Delta)$ and the holonomy around a plaquette in γ. The procedure involves a certain gauge fixing. The strategy of constructing $h(\Delta)$ is discussed in Secs. IV and V.

In Sec. VI, we propose a canonical quantum effective action S_{eff} with the improved Hamiltonian $\mathbf{H}_\Delta$. $\mathbf{H}_\Delta$ and S_{eff} explicitly depend on the scale Δ which is similar to UV cutoffs in quantum effective actions of quantum field theories. This suggests that S_{eff} should be viewed as a quantum effective action which takes quantum effects into account. We derive a new set of EOMs of full LQG from the variational principle of S_{eff}. These EOMs are *improved effective equations* because they come from the improved Hamiltonian and relate to the $\bar{\mu}$-scheme effective dynamics in LQC.

In Sec. VII, we look for spatially homogeneous solutions of our improved effective equations of LQG. When inserting an ansatz that respects the spatially homogeneous symmetry, we show that the effective equation reduces to the $\bar{\mu}$-scheme effective equations of Bianchi-I LQC. If we further restrict the solution to be isotropic, the effective equation reduces to the $\bar{\mu}$-scheme effective equations of standard LQC. We demonstrate the singularity resolution and bounce in Sec. VIII, and reproduce a constant critical density ρ_c at the bounce, $\rho_c = \frac{16-\beta^2\Delta\Lambda}{\beta^2\Delta\kappa}$ (where Λ is the positive cosmological constant). ρ_c is Planckian and corresponds to the Planckian curvature when $\Delta \sim \ell_P^2$. Moreover, although the $\mathbf{H}_\Delta$ of the full theory involves gauge fixing, the cosmological effective dynamics is gauge invariant and independent of the gauge fixing.

Additionally, in Sec. IX we observe that although $\mathbf{H}_\Delta$ is defined on the lattice γ, it is invariant under lattice refinement and therefore has a trivial continuum limit, at least when evaluating at homogeneous solutions. The effective equations, predictions for the bounce, and critical density are also invariant under lattice refinement, and so can be understood as results at the continuum limit. The key point here is that $h(\Delta)$ is defined around a surface with fixed area and is invariant under lattice refinement. This lattice independence of $\mathbf{H}_\Delta$ suggests that the theory at the homogeneous solution is possibly a fix point of the Hamiltonian renormalization in Ref. [34]. Moreover, this invariance indicates scaling invariance from the viewpoint of lattice field theory, and relates to conformal invariance in three dimensions.

II. REDUCED PHASE-SPACE FORMULATION

Our work is based on the reduced phase-space formulation of LQG. The reduced phase-space formulation couples gravity to matter fields (clock fields), followed by the deparametrization procedure that parametrizes the gravity variables by clock fields. In this paper, we mainly focus on two scenarios: coupling gravity to Brown-Kuchař and Gaussian dust fields [31,35–37]. The following briefly reviews the procedure in the reduced phase-space formulation.

We denote the action of the Brown-Kuchař dust model as S_{BKD}:

$$S_{\text{BKD}}[\rho, g_{\mu\nu}, T, S^j, W_j] = -\frac{1}{2}\int \mathrm{d}^4x \sqrt{|\det(g)|}\rho[g^{\mu\nu}U_\mu U_\nu + 1], \quad (2.1)$$

$$U_\mu = -\partial_\mu T + W_j \partial_\mu S^j, \quad (2.2)$$

where the scalars $T, S^{j=1,2,3}$ form the dust coordinates of time and space to parametrize physical fields. ρ, W_j are Lagrangian multipliers. ρ is interpreted as the dust energy density. When we couple S_{BKD} to gravity (or gravity coupled to some other matter fields) and carry out a Hamiltonian analysis [37], we obtain the following constraints:

$$C^{\text{tot}} = C + \frac{1}{2}\left[\frac{P^2/\rho}{\sqrt{\det(q)}} + \sqrt{\det(q)}\rho(q^{\alpha\beta}U_\alpha U_\beta + 1)\right] = 0, \quad (2.3)$$

$$C^{\rm tot}_\alpha = C_\alpha + PT_{,\alpha} - P_j S^j_{,\alpha} = 0, \tag{2.4}$$

$$\rho^2 = \frac{P^2}{\det(q)}(1+q^{\alpha\beta}U_\alpha U_\beta)^{-1}, \tag{2.5}$$

$$W_j = P_j/P, \tag{2.6}$$

where α, β are spatial coordinate indices, $q_{\alpha\beta}$ is the 3-metric, P, P_j are momenta conjugate to T, S^j, and C, C_α are Hamiltonian and diffeomorphism constraints of gravity (or gravity coupled to some other matter fields). First, Eq. (2.5) can be solved as

$$\rho = \varepsilon\frac{P}{\sqrt{\det(q)}}(1+q^{\alpha\beta}U_\alpha U_\beta)^{-1/2}, \quad \varepsilon = \pm 1. \tag{2.7}$$

ε can be fixed to $\varepsilon = 1$ by the physical requirement that U is timelike and future pointing [30], so $\mathrm{sgn}(P) = \mathrm{sgn}(\rho)$. Inserting this solution into Eq. (2.3) and using Eq. (2.6) leads to

$$C = -P\sqrt{1+q^{\alpha\beta}C_\alpha C_\beta/P^2}. \tag{2.8}$$

Thus, $-\mathrm{sgn}(C) = \mathrm{sgn}(P) = \mathrm{sgn}(\rho)$. When we consider dust coupled to pure gravity, we must have $C < 0$ and the physical dust $\rho, P > 0$ to fulfill the energy condition, as in Ref. [35]. However, we may couple some additional matter fields (e.g., scalars, fermions, gauge fields, etc.) to make $C > 0$; then, $\rho, P < 0$ corresponds to the phantom dust, as in Refs. [30,31]. The case of phantom dust may not violate the usual energy condition due to the presence of additional matter fields. We can solve P, P_j from Eqs. (2.3) and (2.4),

$$P = \begin{cases} h & \text{physical dust,} \\ -h & \text{phantom dust,} \end{cases} \qquad h = \sqrt{C^2 - q^{\alpha\beta}C_\alpha C_\beta}, \tag{2.9}$$

$$P_j = -S^\alpha_j (C_\alpha - hT_{,\alpha}), \tag{2.10}$$

which are strongly Poisson-commutative constraints. S^α_j is the inverse matrix of $\partial_\alpha S^j$ ($\alpha = 1, 2, 3$). In deriving the above constraints, we find at an intermediate step that $P^2 = C^2 - q^{\alpha\beta}C_\alpha C_\beta > 0$ constrains the argument of the square root to be positive. Moreover, the physical dust requires $C < 0$, while the phantom dust requires $C > 0$.

We use $A^a_\alpha(x), E^\alpha_a(x)$ as the canonical variables of gravity, where $A^a_\alpha(x)$ is the Ashtekar-Barbero connection and $E^\alpha_a(x) = \sqrt{\det q}\, e^\alpha_a(x)$ is the densitized triad. $a = 1$, 2, 3 is the Lie algebra index of su(2). Gauge-invariant Dirac observables are constructed relationally by parametrizing (A, E) with values of the dust fields $T(x) \equiv \tau$, $S^j(x) \equiv \sigma^j$, i.e., $A^a_j(\sigma,\tau) = A^a_j(x)|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}$ and $E^j_a(\sigma,\tau) = E^j_a(x)|_{T(x)\equiv\tau, S^j(x)\equiv\sigma^j}$, where σ, τ are physical space and time coordinates in the dust reference frame. Here $j = 1$, 2, 3 is the dust coordinate index (e.g., $A_j = A_\alpha S^\alpha_j$).

Both $A^a_j(\sigma,\tau)$ and $E^j_a(\sigma,\tau)$ are invariant under gauge transformations generated by diffeomorphism and Hamiltonian constraints on the constraint surface [38,39]. They satisfy the standard Poisson bracket in the dust frame:

$$\{E^i_a(\sigma,\tau), A^b_j(\sigma',\tau)\} = \frac{1}{2}\kappa\beta\delta^i_j\delta^b_a\delta^3(\sigma,\sigma'), \tag{2.11}$$

where β is the Barbero-Immirzi parameter [40] and $\kappa = 16\pi G$. The phase space $\mathcal{P}$ of $A^a_j(\sigma,\tau), E^j_a(\sigma,\tau)$ is free of Hamiltonian and diffeomorphism constraints. All SU(2) gauge-invariant phase-space functions are Dirac observables.

The evolution in physical time τ is generated by the classical physical Hamiltonian $\mathbf{H}_0$ given by integrating h on the constant $T = \tau$ slice $\mathcal{S}$. The constant τ slice $\mathcal{S}$ is coordinated by the value of the dust scalars $S^j = \sigma^j$ and thus is referred to as the dust space [31,37]. From Eq. (2.9), we find that $\mathbf{H}_0$ is negative for physical dust and positive for phantom dust. We flip the direction of the time flow $\tau \to -\tau$ and thus $\mathbf{H}_0 \to -\mathbf{H}_0$ for physical dust, so we have a positive Hamiltonian in every case:

$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{C(\sigma,\tau)^2 - \frac{1}{4}\sum_{a=1}^{3} C_a(\sigma,\tau)^2}. \tag{2.12}$$

Here C and $C_a = 2e^\alpha_a C_\alpha$ (e^α_a is the triad) are parametrized in the dust frame. In terms of $A^a_j(\sigma,\tau)$ and $E^j_a(\sigma,\tau)$,

$$C = \frac{1}{\kappa}[F^a_{jk} - (\beta^2+1)\varepsilon_{ade}K^d_j K^e_k]\varepsilon_{abc}\frac{E^j_b E^k_c}{\sqrt{\det(q)}} + \frac{2\Lambda}{\kappa}\sqrt{\det(q)}, \tag{2.13}$$

$$C_a = \frac{4}{\kappa\beta}F^b_{jk}\frac{E^j_a E^k_b}{\sqrt{\det(q)}}, \tag{2.14}$$

where K^a_j is the extrinsic curvature, F^a_{jk} is the curvature of A^a_j, and Λ is the cosmological constant.

Coupling gravity to the Gaussian dust model can be analyzed similarly, so we do not present the details here (but details can be found in Ref. [37]). For Gaussian dust the physical Hamiltonian has the simpler expression

$$\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma C(\sigma,\tau). \tag{2.15}$$

In order to discuss both the Brown-Kuchař and Gaussian dusts in a unified manner, we express the physical Hamiltonian as

$$
\begin{aligned}
\mathbf{H}_0 &= \int_{\mathcal{S}} \mathrm{d}^3\sigma h(\sigma,\tau), \\
h(\sigma,\tau) &= \sqrt{C(\sigma,\tau)^2 - \frac{\alpha}{4}\sum_{a=1}^{3} C_a(\sigma,\tau)^2}, \\
&\begin{cases} \alpha = 1, & \text{Brown-Kuchař dust}, \\ \alpha = 0, & \text{Gaussian dust}. \end{cases}
\end{aligned} \tag{2.16}
$$

The physical Hamiltonian $\mathbf{H}_0$ is manifestly positive in Eq. (2.16). When $C < 0$, Eq. (2.16) is different from Eq. (2.15) by an overall minus sign and thus reverses the time flow $\tau \to -\tau$ for the Gaussian dust.

In both scenarios, the physical Hamiltonian $\mathbf{H}_0$ generates the τ-time evolution,

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \{f, \mathbf{H}_0\}, \tag{2.17}$$

for all phase-space functions f of $A_j^a(\sigma,\tau)$ and $E_a^j(\sigma,\tau)$. In particular, the Hamilton equations are

$$\frac{\mathrm{d}A_j^a(\sigma,\tau)}{\mathrm{d}\tau} = -\frac{\kappa\beta}{2}\frac{\delta\mathbf{H}_0}{\delta E_a^j(\sigma,\tau)}, \qquad \frac{\mathrm{d}E_a^j(\sigma,\tau)}{\mathrm{d}\tau} = \frac{\kappa\beta}{2}\frac{\delta\mathbf{H}_0}{\delta A_j^a(\sigma,\tau)}. \tag{2.18}$$

The functional derivatives on the right-hand sides of Eq. (2.18) can be computed as

$$\delta\mathbf{H}_0 = \int_{\mathcal{S}} \mathrm{d}^3\sigma\left(\frac{C}{h}\delta C - \frac{\alpha}{4}\frac{C_a}{h}\delta C_a\right), \tag{2.19}$$

where C/h is negative (positive) for physical (phantom) dust. Comparing $\delta\mathbf{H}$ to the variation of the Hamiltonian H_{GR} of pure gravity in the absence of dust motivates us to view the following as the physical lapse function and shift vector:

$$N = \frac{C}{h}, \qquad N_a = -\frac{\alpha}{4}\frac{C_a}{h}. \tag{2.20}$$

Therefore, N is negative (positive) for physical (phantom) dust. The negative N for the physical dust is related to the flip $\tau \to -\tau$ that comes from making the Hamiltonian positive.

Our framework in this work focuses on pure gravity coupled to dusts, and thus we only work with physical dusts in order to not violate the energy condition. The time generated by $\mathbf{H}_0$ flows backward from the future to the past.

III. IMPROVED HAMILTONIAN

We propose to define the effective theory of LQG as a discretization of the above reduced phase-space framework. In the effective theory, dynamical variables are discretized on a lattice, and their time evolutions are generated by $\mathbf{H}$, the discretization of the physical Hamiltonian $\mathbf{H}_0$. In this section and Secs. IV and V we describe a new discretization of $\mathbf{H}_0$ that is different from the existing procedure in, e.g., Refs. [31,32,41]. The new discretization turns out to be crucial for the relation with $\bar{\mu}$-scheme LQC.

Our model is defined on a cubic lattice γ partitioning $\mathcal{S}$ which may be finite or infinite.[1] $E(\gamma)$ and $V(\gamma)$ denote the sets of edges and vertices in γ. For the purpose of relating LQC at $k = 0$, we consider γ to be a partition of a 3-torus. The dynamical variables on γ are holonomies $h(e)$ and gauge-covariant fluxes $p^a(e)$ [42], which discretize the fields $A_j^a(\tau,\sigma)$, $E_a^j(\tau,\sigma)$:

$$h(e) \coloneqq \mathcal{P}\exp\int_e A^a\frac{\tau^a}{2}, \tag{3.1}$$

$$p^a(e) \coloneqq -\frac{1}{2\beta a^2}\mathrm{tr}\left[\tau^a\int_{S_e}\varepsilon_{ijk}d\sigma^i \wedge \mathrm{d}\sigma^j h(\rho_e(\sigma))E_b^k(\sigma)\tau^b h(\rho_e(\sigma))^{-1}\right], \tag{3.2}$$

where S_e is a 2-face in the dual lattice γ^*, and $\tau^a = -i(\text{Pauli matrix})^a$. $\rho_e(\sigma)$ is a path starting at the beginning point of e and traveling along e until $e \cap S_e$, then running in S_e until σ. a is a length unit for making $p^a(e)$ dimensionless. $h(e)$ and $p^a(e)$ satisfy the holonomy-flux algebra

$$
\begin{aligned}
\{h(e), h(e')\} &= 0, \\
\{p^a(e), h(e')\} &= \frac{\kappa}{a^2}\delta_{e,e'}\frac{\tau^a}{2}h(e'), \\
\{p^a(e), p^b(e')\} &= -\frac{\kappa}{a^2}\delta_{e,e'}\varepsilon_{abc}p^c(e'),
\end{aligned} \tag{3.3}
$$

where $\kappa = 16\pi G_N$. Our work is developed from the reduced phase-space framework, so $h(e)$ and $p^a(e)$ are Dirac observables.

We focus on the deparametrized model of gravity coupled to dust, and thus the discrete physical Hamiltonian on γ can be written as (see, e.g., Ref. [14])

$$\mathbf{H} = \sum_{v\in V(\gamma)}\mathbf{H}_v, \quad \mathbf{H}_v = \sqrt{C_v^2 - \frac{\alpha}{4}\sum_{a=1}^{3}C_{a,v}^2}, \tag{3.4}$$

$$C_v = -\frac{1}{\beta^2}C_{0,v} - \frac{1+\beta^2}{\beta^2\kappa}{}^3\mathcal{R}_v + \frac{\Lambda}{\kappa}V_v, \tag{3.5}$$

[1] One may consider generalizing this work to noncubic graphs. The generalization seems to be straightforward except that analyzing the relation to $\bar{\mu}$-scheme LQC can become quite involved.

where C_v and $C_{a,v}$ are the discretization of $C(\sigma)$ and $C_a(\sigma)$ in Eqs. (2.13) and (2.14). $\alpha = 1$, 0 corresponds to Brown-Kuchař or Gaussian dust. There are nonholonomic constraints $C_v < 0$ and $C_v^2 - \frac{\alpha}{4} C_{a,v}^2 \geq 0$. We focus on the physical dust with positive energy density, and the physical time flow is backward to make $\mathbf{H}$ positive, as shown in Sec. II. We have included the cosmological constant term ΛV_v in the Hamiltonian. Here we employ the regularization of C_v from Refs. [43,44]. ${}^3\mathcal{R}_v$ is the discrete 3-curvature:

$$
{}^3\mathcal{R}_v = \sum_{I \neq J} \sum_{s_1, s_2 = \pm 1} L_v(I, s_1; J, s_2) \left(\frac{2\pi}{\alpha} - \pi + \arccos\left[\frac{\vec{p}(e_{v;Is_1}) \cdot \vec{p}(e_{v;Js_2})}{p(e_{v;Is_1}) p(e_{v;Js_2})} \right] \right), \tag{3.6}
$$

$$
L_v(I, s_1; J, s_2) = \frac{1}{V_v} \sqrt{\varepsilon^{abc} p_b(e_{v;Is_1}) p_c(e_{v;Js_2}) \varepsilon^{ab'c'} p_{b'}(e_{v;Is_1}) p_{c'}(e_{v;Js_2})}. \tag{3.7}
$$

In our notation $e_{v;I,s}$, with $I = 1, 2, 3$, $s = \pm$, and vertex $v \in V(\gamma)$ denotes an edge starting at v oriented toward the (I, s) direction (see Fig. 1). ${}^3\mathcal{R}_v$ is motivated by the Regge calculus in three dimensions, and contains L as the length of the edge, while the other part is related to the deficit angle hinged at the edge [43]. V_v here is the 3-volume at v:

$$
V_v = (|Q_v|)^{1/2}, \tag{3.8}
$$

$$
Q_v = \beta^3 a^6 \varepsilon^{abc} \frac{p_a(e_{v;1+}) - p_a(e_{v;1-})}{2} \frac{p_b(e_{v;2+}) - p_b(e_{v;2-})}{2} \frac{p_c(e_{v;3+}) - p_c(e_{v;3-})}{2}. \tag{3.9}
$$

Standard discretizations of $C_{0,v}$, $C_{a,v}$ (used in, e.g., Refs. [14,31,41]) are given by

$$
\begin{aligned}
C_{\mu,v} = \frac{-4}{3\beta\kappa^2} \sum_{s_1,s_2,s_3=\pm 1} & s_1 s_2 s_3 \varepsilon^{I_1 I_2 I_3} \mathrm{Tr}[\tau_\mu (h(\square_{v;I_1 s_1, I_2 s_2}) - h(\square_{v;I_1 s_1, I_2 s_2})^{-1}) \\
& \times h(e_{v;I_3 s_3}) \{ h(e_{v;I_3 s_3})^{-1}, V_v \}],
\end{aligned} \tag{3.10}
$$

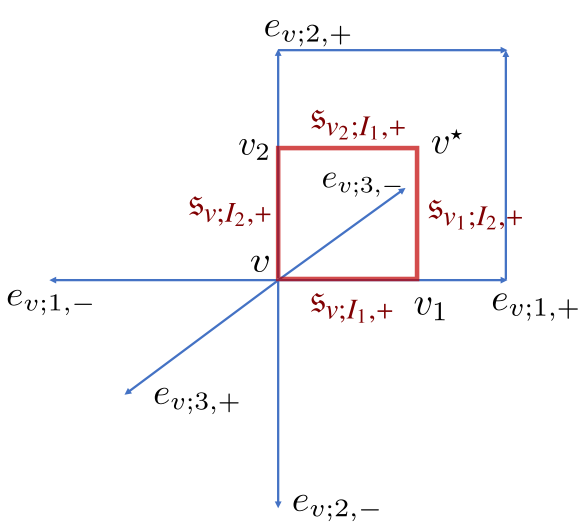

FIG. 1. A neighborhood of a vertex v in the cubic lattice γ. The square bounded by red edges is $S(\Delta)$ of Planckian area Δ, while the square bounded by blue edges is the minimal plaquette $\square$. The loop holonomy $h(\Delta)$ is along $\partial S(\Delta)$.

where $\mu = 0, 1, 2, 3$, $\tau_\mu = (\mathbf{1}, \tau_a)$ and $\tau_a = -i(\text{Pauli matrix})_a$. $h(\square_{v;I_1 s_1, I_2 s_2})$ is the loop holonomy around a minimal plaquette $\square_{v;I_1 s_1, I_2 s_2}$ (bounded by $e_{v;I_1,s_1}, e_{v;I_2,s_2}$) in γ.

$\mathbf{H}$ depends on two types of variables—the flux $p^a(e)$ and loop holonomies $h(\square)$—while $h(e)\{h(e)^{-1}, V_v\}$ leads to an expression involving only fluxes:

$$
\begin{aligned}
& h(e_{v;J,s}) \{ h(e_{v;J,s})^{-1}, V_v \} \\
& \quad = \mathrm{sgn}(Q_v) \frac{s\beta^3 a^4 \kappa}{2|Q_v|^{1/2}} \varepsilon^{JMN} \varepsilon_{abc} \left[\frac{\tau^a/2}{2} \right] \frac{X_M^b(v)}{2} \frac{X_N^c(v)}{2},
\end{aligned} \tag{3.11}
$$

$$
X_M^b(v) = p^b(e_{v;M,+}) - p^b(e_{v;M,-}). \tag{3.12}
$$

Note that, because $p^a(e)$ is a covariant flux, we have

$$
p^a(e_{v;I,-}) = \frac{1}{2} \mathrm{Tr}[\tau^a h(e_{v-\hat{I};I,+})^{-1} p^b(e_{v-\hat{I};I,+}) \tau^b h(e_{v-\hat{I};I,+})], \tag{3.13}
$$

where $\hat{I}$ is the lattice vector of γ along the I direction.

In the following, we are going to construct a new discrete Hamiltonian $\mathbf{H}_\Delta$ as a discretization of $\mathbf{H}_0$. $\mathbf{H}_\Delta$ is different from H in that it has different discretizations of $C_{0,v}$, $C_{a,v}$. The modification comes from a different regularization of the curvature $F_{ij}(A)$ in $C_{\mu,v}$. The new regularization replaces the loop holonomy $h(\square)$ by a different expression.

Given the smooth fields A_j^a and E_a^j, the loop holonomy $h(\square)$ of A is related to F_{ij} when the scale of the plaquette $\square$ is small (measured by the metric q_{ij} determined by E_a^j),

$$\frac{h(\square)-h(\square)^{-1}}{2} \simeq \frac{1}{2}\int_\square F_{ij}\mathrm{d}\sigma^i\wedge\mathrm{d}\sigma^j = \int_\square F_{12}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2, \tag{3.14}$$

where we use $\square = \square_{v;1,2}$ as an example. We multiply and divide the integrand $\sqrt{\det(\mathfrak{h})}$, where $\mathfrak{h}$ is the metric on $\square$ induced by the 3-metric q_{ij} (determined by E_a^j), and assume that $\square$ is small enough so that F_{12} and $\det(\mathfrak{h})$ are approximately constant,

$$\begin{aligned}\frac{h(\square)-h(\square)^{-1}}{2} &\simeq \int_\square \frac{F_{12}}{\sqrt{\det(\mathfrak{h})}}\sqrt{\det(\mathfrak{h})}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2 \simeq \frac{F_{12}}{\sqrt{\det(\mathfrak{h})}}(v)\int_\square\sqrt{\det(\mathfrak{h})}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2\\ &= \frac{F_{12}}{\sqrt{\det(\mathfrak{h})}}(v)\mathrm{Ar}(\square) = \frac{F_{12}(v)\Delta}{\sqrt{\det(\mathfrak{h}(v))}}\frac{\mathrm{Ar}(\square)}{\Delta}.\end{aligned} \tag{3.15}$$

In the last step, we multiply and divide a quantum area scale $\Delta \simeq \ell_P^2 = \hbar\kappa$ which may be chosen as the LQG minimal area gap: $\Delta = \frac{\sqrt{3}}{8}\beta\ell_P^2$. We may write $\Delta = \int_{S(\Delta)}\sqrt{\det(\mathfrak{h})}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2$, where $S(\Delta)$ is a small two-dimensional surface of Planckian size (the square bounded by red edges in Fig. 1). In order to derive the discretization, we assume that the size of $\square$ is fine enough that F_{12} and $\det(\mathfrak{h})$ are approximately constant on $\square$ and thus constant on the smaller surface $S(\Delta)$. We can repeat the trick in Eq. (3.15),

$$\begin{aligned}\frac{h(\square)-h(\square)^{-1}}{2} &\simeq \frac{\mathrm{Ar}(\square)}{\Delta}\frac{F_{12}(v)}{\sqrt{\det(\mathfrak{h}(v))}}\int_{S(\Delta)}\sqrt{\det(\mathfrak{h})}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2\\ &\simeq \frac{\mathrm{Ar}(\square)}{\Delta}\int_{S(\Delta)}F_{12}\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2 \simeq \frac{\mathrm{Ar}(\square)}{\Delta}\frac{h(\Delta)-h(\Delta)^{-1}}{2},\end{aligned} \tag{3.16}$$

where $h(\Delta)$ is the holonomy along the boundary of $S(\Delta)$. Here we set

$$\mathrm{Ar}(\square_{v;I_1s_1,I_2s_2}) = \frac{1}{2}\beta a^2\sum_{s_3=\pm}\sqrt{p^a(e_{v;I_3,s_3})p^a(e_{v;I_3,s_3})} \quad (I_3\neq I_1,I_2). \tag{3.17}$$

The error from the approximations (3.14)–(3.16) is bounded by $O(\ell_P^3)$,[2] while $h(\Delta)-h(\Delta)^{-1}\sim O(\ell_P^2)$. As an interesting perspective, the dependence on the lattice γ is only reflected by the plaquette area $\mathrm{Ar}(\square)$, while $h(\Delta)$ is independent of the sizes of plaquettes in γ and thus is invariant under lattice refinement. The price is that the free parameter Δ has to be introduced. However, it seems to us that Δ labels the UV energy scale where this theory is defined.

We denote by $h(\Delta_{v;I_1s_1,I_2s_2})$ the holonomy around the Planckian surface $S(\Delta_{v;I_1s_1,I_2s_2})$ adapted in $\square_{v;I_1s_1,I_2s_2}$. $S(\Delta_{v;I_1s_1,I_2s_2})$ and $\square_{v;I_1s_1,I_2s_2}$ share the same vertex v. Figure 1 shows a choice for $S(\Delta_{v;I_1s_1,I_2s_2})$ in which two boundary edges of $S(\Delta_{v;I_1s_1,I_2s_2})$ are contained in lattice edges $e_{v;I_1,s_1}$ and $e_{v;I_2,s_2}$. Using the new holonomies $h(\Delta)$, we define the new Hamiltonian $\mathbf{H}_\Delta$ as

$$\mathbf{H}_\Delta = \sum_{v\in V(\gamma)}\mathbf{H}_{\Delta,v}, \qquad \mathbf{H}_{\Delta,v} = \sqrt{(C_v^\Delta)^2 - \frac{\alpha}{4}\sum_{a=1}^3(C_{a,v}^\Delta)^2}, \tag{3.18}$$

$$C_v^\Delta = -\frac{1}{\beta^2}C_{0,v}^\Delta - \frac{1+\beta^2}{\beta^2\kappa}{}^3\mathcal{R}_v + \frac{\Lambda}{\kappa}V_v, \tag{3.19}$$

$$\begin{aligned}C_{\mu,v}^\Delta = \frac{-4}{3\beta\kappa^2}&\sum_{s_1,s_2,s_3=\pm1}s_1s_2s_3\varepsilon^{I_1I_2I_3}\frac{\mathrm{Ar}(\square_{v;I_1s_1,I_2s_2})}{\Delta}\\ &\times\mathrm{Tr}[\tau_\mu(h(\Delta_{v;I_1s_1,I_2s_2})-h(\Delta_{v;I_1s_1,I_2s_2})^{-1})\\ &\times h(e_{v;I_3s_3})\{h(e_{v;I_3s_3})^{-1},V_v\}].\end{aligned} \tag{3.20}$$

[2]Given any two-dimensional integral of a bounded function $f(x)$ whose upper and lower bounds are $f(a)=f_{\max}, f(b)=f_{\min}$, we have $|\int_S\mathrm{d}^2xf(x)-\int_S\mathrm{d}^2xf(v)|\leq\int_S\mathrm{d}^2x|f(x)-f(v)|\leq\mathrm{Ar}(S)|f_{\max}-f_{\min}|\leq C\mathrm{Ar}(S)||a-b||$ for some constant C, where $\mathrm{Ar}(S)\sim\ell_P^2$ and $||a-b||\sim\ell_P$ for $S=S(\Delta)$ or $\square$.

The derivation in Eq. (3.16) assumes that the size of $\Box$ is comparable to $S(\Delta)$. This indicates that $\mathbf{H}_\Delta$ is defined at the UV or deep quantum level where $\mathrm{Ar}(\Box) \sim \Delta \sim \ell_P^2$. In the semiclassical and continuum limit $\Delta \sim \ell_P^2 \to 0$ and for shrinking $\Box$, the loop $\partial S(\Delta)$ shrinks and $h(\Delta) - h(\Delta)^{-1} \to 2\int_{S(\Delta)} F \simeq \frac{2\Delta}{\sqrt{\det(\mathfrak{h}(v))}} F_{12}(v)$, as well as $\mathrm{Ar}(\Box) \simeq \sqrt{\det(\mathfrak{h}(v))} \int_\Box \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2$. Then, we recover the classical continuum expression of the physical Hamiltonian.

It is clear that $\mathbf{H}_\Delta$ in Eq. (3.18) is a different discretization compared to $\mathbf{H}$ in Eq. (3.4). $\mathbf{H}_\Delta$ and $\mathbf{H}$ coincide only when taking the continuum limit (ignoring the size of $\Box$ and Δ). In the continuum limit, both $\mathbf{H}_\Delta$ and $\mathbf{H}$ reduce to the continuum expression $\mathbf{H}_0$. Our new Hamiltonian $\mathbf{H}_\Delta$ is more likely to be related to a *quantum effective action* because $\mathbf{H}_\Delta$ explicitly depends on ℓ_P (in Δ). Indeed, in Sec. VI we propose a candidate quantum effective action for full LQG by using $\mathbf{H}_\Delta$, and show in the following section that $\mathbf{H}_\Delta$ can lead to the $\bar{\mu}$-scheme LQC effective dynamics, while the old Hamiltonian $\mathbf{H}$ can only give the μ_0-scheme LQC effective dynamics (as shown in Ref. [14]).

IV. HOLONOMY ALONG THE PLANCKIAN SEGMENT

As preparation for defining $h(\Delta)$ in $\mathbf{H}_\Delta$, in this section we construct the length-dependent holonomy $h_\Delta(\mathfrak{s})$ along the segment $\mathfrak{s}$ whose length is fixed as $\sqrt{\Delta}$. We denote by $h_\Delta(\mathfrak{s}_{v;I,s})$ with $s = +/-$ the holonomy along $\mathfrak{s}$ in the positive/negative Ith direction at v. $\mathfrak{s}_{v;I,s} \subset e_{v;I,s}$ and shares the same source v with $e_{v;I,s}$ ($\mathfrak{s}_{v;I,s}$ is the red segment along $e_{v;I,s}$ in Fig. 1). In contrast to $h(e_{v;I,s})$ (where the length of $e_{v;I,s}$ is dynamical), $h_\Delta(\mathfrak{s}_{v;I,s})$ has $\mathfrak{s}_{v;I,s}$ with fixed length $\sqrt{\Delta}$:

$$h_\Delta(\mathfrak{s}_{v;I,s}) = \mathcal{P}\exp\left[\int_0^1 du \sum_{J=1}^3 \frac{d\sigma^J}{du} A_J^a(\vec{\sigma}(u)) \frac{\tau^a}{2}\right],$$
$$\sqrt{\Delta} = \int_0^1 du \sqrt{q_{IJ} \frac{d\sigma^I}{du}\frac{d\sigma^J}{du}}, \tag{4.1}$$

where $\sigma^J(u)$ ($J = 1, 2, 3$) is a parametrization of the segment $\mathfrak{s}_{v;I,s}$ such that the source and target of the segment correspond to $u = 0, 1$. For simplicity, we adapt the coordinates σ^J to the lattice γ, such that the σ^J coordinate axis is along $e_{v;I,s}$ or $\mathfrak{s}_{v;I,s}$. The tangent vector of $\mathfrak{s}_{v;I,s}$ has the only nonzero component $d\sigma^I/du$ ($d\sigma^J/du = 0$ for $J \neq I$). We assume $d\sigma^I/du > 0$ without loss of generality. $q_{IJ} = e_I^a e_J^a$ are components of the 3-metric on $\mathcal{S}$ in the coordinate σ^J. The e_I^a are related to E_a^I by $e_I^a = \mathrm{sgn}(\det(e)) \frac{1/2}{\sqrt{|\det(E)|}} \epsilon_{IJK}\epsilon^{abc} E_b^J E_c^K$. Equation (4.1) simplifies to

$$h_\Delta(\mathfrak{s}_{v;I,s}) = \mathcal{P}\exp\left[\int_0^1 du \frac{d\sigma^I}{du} A_I^a(\sigma^I(u), 0, 0) \frac{\tau^a}{2}\right] \quad \text{(no sum in } I\text{)}, \tag{4.2}$$

$$\sqrt{\Delta} = \int_0^1 du \frac{d\sigma^I}{du} \sqrt{q_{II}}. \tag{4.3}$$

Equation (4.3) can be solved as

$$\frac{d\sigma^I}{du} = \sqrt{\frac{\Delta}{q_{II}}} = \sqrt{\frac{\Delta \det(q)}{\frac{1}{2}\epsilon_{Imn}E_b^m E_c^n \epsilon_{Ipq} E_b^p E_c^q}} \quad \text{(no sum in } I\text{)}, \tag{4.4}$$

where we have expressed the metric component $q_{II} = e_I^a e_I^a$ in terms of densitized triads E_a^j, using $e_i^a = \mathrm{sgn}(\det(e)) \frac{1/2}{\sqrt{\det(q)}} \epsilon_{ijk}\epsilon^{abc} E_b^j E_c^k$. It is clear that the solution to Eq. (4.3) is not unique. We make the choice (4.4) and insert it into Eq. (4.2) as a definition of $h_\Delta(\mathfrak{s})$. Different choices lead to different definitions of $h(\mathfrak{s})$, and thus different definitions of $\mathbf{H}_\Delta$. We prefer the present choice since $\mathbf{H}_\Delta$ is related to $\bar{\mu}$-scheme LQC (see Sec. VII) and leave the study of other choices to future work. Moreover, we regularize $d\sigma^I/du$ to express it in terms of fluxes. For instance,

$$\begin{aligned} \frac{d\sigma^1}{du} &= \sqrt{\frac{\Delta \det(q)}{(E^2 \cdot E^2)(E^3 \cdot E^3) - (E^2 \cdot E^3)^2}} \\ &= \delta\sigma^1 \sqrt{\frac{\Delta \det(q)}{(E^2 \cdot E^2)(E^3 \cdot E^3) - (E^2 \cdot E^3)^2} \frac{\delta\sigma^1\delta\sigma^2\delta\sigma^3}{\delta\sigma^1\delta\sigma^3\delta\sigma^1\delta\sigma^2}} \\ &\simeq \lambda \sqrt{\frac{|Q_v|/(\beta^3 a^6)}{[p(e_{v;2,s}) \cdot p(e_{v;2,s})][p(e_{v;3,s}) \cdot p(e_{v;3,s})] - [p(e_{v;2,s}) \cdot p(e_{v;3,s})]^2}} \equiv \bar{\mu}_{v;1,s}, \end{aligned} \tag{4.5}$$

$$\lambda = \frac{\delta\sigma^1}{\beta^{1/2} a} \sqrt{\Delta}, \tag{4.6}$$

where $\delta\sigma^I$ is the coordinate scale of the 2-surface $S_{e_{v;I,s}}$ defining $p(e_{v;I,s})$. We set $\delta\sigma^I = 1$ since the coordinate length of $e_{v;I,s}$ is set to be 1. Thus, λ is a constant. It is clear that $\bar{\mu}_{v;1,s}$ reduces to $d\sigma^1/du$ when taking the continuum limit. Similarly, we have

$$\frac{d\sigma^2}{du} \simeq \bar{\mu}_{v;2,s} = \lambda\sqrt{\frac{|Q_v|/(\beta^3 a^6)}{[p(e_{v;1,s})\cdot p(e_{v;1,s})][p(e_{v;3,s})\cdot p(e_{v;3,s})] - [p(e_{v;1,s})\cdot p(e_{v;3,s})]^2}}, \tag{4.7}$$

$$\frac{d\sigma^3}{du} \simeq \bar{\mu}_{v;3,s} = \lambda\sqrt{\frac{|Q_v|/(\beta^3 a^6)}{[p(e_{v;1,s})\cdot p(e_{v;1,s})][p(e_{v;2,s})\cdot p(e_{v;2,s})] - [p(e_{v;1,s})\cdot p(e_{v;2,s})]^2}}. \tag{4.8}$$

We obtain the following result when inserting the above $\bar{\mu}_{v;I,s}$ in $h_\Delta(\mathfrak{s}_{v;I,s})$.

Lemma 4.1: A gauge transformation can lead to the following expression for $h_\Delta(\mathfrak{s}_{v;I,s})$ while leaving $h(e_{v;I,s})$ invariant:

$$h_\Delta(\mathfrak{s}_{v;I,s}) = e^{\bar{\mu}_{v;I,s}\theta^a(e_{v;I,s})\tau^a/2}, \qquad \mathfrak{s}_{v;I,s} \subset e_{v;I,s}, \tag{4.9}$$

where $\theta^a(e_{v;I,s})$ is given by the lattice edge holonomy $h(e_{v;I,s})$,

$$h(e_{v;I,s}) = e^{\theta^a(e_{v;I,s})\tau^a/2}. \tag{4.10}$$

Proof: We locally impose the following axial gauge to the connection A along every edge $e_{v;I,s}$, i.e., at a given $e_{v;I,s}$, the restriction of A along the edge, A_I^a, is fixed to be a constant:

$$A_I^a(\vec{\sigma}) = \theta^a(e_{v;I,s}), \qquad \vec{\sigma} \in e_{v;I,s}. \tag{4.11}$$

Indeed, Eq. (4.11) can be obtained by a gauge transformation from generic A:

$$g(\sigma^I)A_I(\sigma^I)g(\sigma^I)^{-1} - \partial_I g(\sigma^I)g(\sigma^I)^{-1} = \theta(e_{v;I,s}), \tag{4.12}$$

where $A_I(\sigma^I) = A_I^a(\sigma^I)\frac{\tau^a}{2}$ and $\theta(e_{v;I,s}) = \theta^a(e_{v;I,s})\frac{\tau^a}{2}$. Equivalently, the above gauge transformation is a first-order ordinary differential equation of $g(\sigma^I)$ along $e_{v;I,s}$:

$$\partial_I g(\sigma^I) = g(\sigma^I)A_I(\sigma^I) - \theta(e_{v;I,s})g(\sigma^I). \tag{4.13}$$

The solution to the above equation is given by[3]

$$g(\sigma^I) = e^{-\sigma^I\theta(e_{v;I,s})}\mathcal{P}\exp\left[\int_0^{\sigma^I} d\sigma'^I A_I(\sigma'^I)\right]. \tag{4.14}$$

If the constant $\theta(e_{v;I,s})$ is chosen such that

$$e^{\theta(e_{v;I,s})} = \mathcal{P}\exp\left[\int_0^1 d\sigma'^I A_I(\sigma'^I)\right] = h(e_{v;I,s}), \tag{4.15}$$

we obtain that gauge transformations are identities at the source and target of $e_{v;I,s}$,

$$g(0) = g(1) = 1. \tag{4.16}$$

Therefore, the gauge transformation $g(\vec{\sigma})$ leaves all lattice edge holonomies $h(e_{v;I,s})$ invariant, while locally transforming the connection A. Inserting Eq. (4.11) and the definition of $\bar{\mu}_{v;I,s}$ into $h_\Delta(\mathfrak{s}_{v;I,s})$ gives Eq. (4.9). ■

V. LOOP HOLONOMY AROUND THE PLANCKIAN PLAQUETTE

The curvature $F(A)$ smeared on the Planckian size $S(\Delta)$ is related to the loop holonomy $h(\Delta)$ along $\partial S(\Delta)$. Here we assume the shape of $S(\Delta)$ to be a square, i.e., $\partial S(\Delta)$ is made by four edges (see Fig. 1), and $h(\Delta)$ is given by

$$\begin{aligned} h(\Delta_{v;I_1,s_1,I_2,s_2}) = {}& h_\Delta(\mathfrak{s}_{v;I_1,s_1})h_\Delta(\mathfrak{s}_{v_1;I_2,s_2})h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})^{-1} \\ & \times h_\Delta(\mathfrak{s}_{v;I_2,s_2})^{-1}, \end{aligned} \tag{5.1}$$

where v_1, v_2 are vertices of $\partial S(\Delta)$ (see Fig. 1). $h_\Delta(\mathfrak{s}_{v;I_1,s_1})$ and $h_\Delta(\mathfrak{s}_{v;I_2,s_2})$ with $\mathfrak{s}_{v;I_1,s_1} \subset e_{v;I_1,s_1}$ and $\mathfrak{s}_{v;I_2,s_2} \subset e_{v;I_2,s_2}$ are given by Eq. (4.9). However, $h_\Delta(\mathfrak{s}_{v_1;I_2,s_2})$ and $h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})$ are based on segments $\mathfrak{s}_{v_1;I_2,s_2}, \mathfrak{s}_{v_2;I_1,s_1}$ which do not belong to any edge in γ. To construct these two holonomies, we write

$$h_\Delta(\mathfrak{s}_{v_2;I_1,s_1}) = e^{X_1}, \qquad h_\Delta(\mathfrak{s}_{v_1;I_2,s_2}) = e^{X_2}, \tag{5.2}$$

and expand $h(\Delta)$ since the size of $S(\Delta)$ is Planckian:

$$\begin{aligned} h(\Delta_{v;I_1,s_1,I_2,s_2}) = {}& 1 + (\bar{\mu}_{v;I_1,s_1}\theta(e_{v;I_1,s_1}) - X_1) \\ & - (\bar{\mu}_{v,I_2,s_2}\theta(e_{v;I_2,s_2}) - X_2) \\ & + \bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v,I_2,s_2}[\theta(e_{v;I_1,s_1}), \theta(e_{v;I_2,s_2})] \\ & + O(\ell_P^3), \end{aligned} \tag{5.3}$$

[3]The holonomy $h(\sigma^I) \equiv \mathcal{P}\exp[\int_0^{\sigma^I} d\sigma'^I A_I(\sigma'^I)]$ satisfies $\partial_I h(\sigma^I) = h(\sigma^I)A_I(\sigma^I)$.

where we have ignored higher orders in size of $S(\Delta)$. $h(\Delta_{v;I_1,s_1,I_2,s_2})-1$ approximates the curvature integrated on $S(\Delta)$,

$$(\bar{\mu}_{v;I_1,s_1}\theta(e_{v;I_1,s_1})-X_1)-(\bar{\mu}_{v,I_2,s_2}\theta(e_{v;I_1,s_1})-X_2)+\bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v,I_2,s_2}[\theta(e_{v;I_1,s_1}),\theta(e_{v;I_2,s_2})]$$
$$\simeq\frac{1}{2}\int_{S(\Delta_{v;I_1,s_1,I_2,s_2})}\mathrm{d}\sigma^I\wedge\mathrm{d}\sigma^J F_{IJ}(\vec{\sigma}). \tag{5.4}$$

On the other hand, the loop holonomy along the lattice edges can be expanded similarly,

$$\begin{aligned}h(\square_{v;I_1,s_1,I_2,s_2})&=h(e_{v;I_1,s_1})h(e_{v+s_1\hat{I}_1;I_2,s_2})h(e_{v+s_2\hat{I}_2;I_1,s_1})^{-1}h(e_{v;I_2,s_2})^{-1}\\&\simeq 1+(\theta(e_{v;I_1,s_1})-\theta(e_{v+s_2\hat{I}_2;I_1,s_1}))-(\theta(e_{v;I_2,s_2})-\theta(e_{v+s_1\hat{I}_1;I_2,s_2}))\\&\quad+[\theta(e_{v;I_1,s_1}),\theta(e_{v;I_2,s_2})]\end{aligned} \tag{5.5}$$

up to higher orders in lattice size. The curvature integrated on the lattice plaquette $\square$ is approximated by $h(\square_{v;I_1,s_1,I_2,s_2})-1$:

$$\begin{aligned}\frac{1}{2}\int_{\square_{v;I_1,s_1,I_2,s_2}}\mathrm{d}\sigma^I\wedge\mathrm{d}\sigma^J F_{IJ}(\vec{\sigma})&\simeq(\theta(e_{v;I_1,s_1})-\theta(e_{v+s_2\hat{I}_2;I_1,s_1}))-(\theta(e_{v;I_2,s_2})-\theta(e_{v+s_1\hat{I}_1;I_2,s_2}))\\&\quad+[\theta(e_{v;I_1,s_1}),\theta(e_{v;I_2,s_2})].\end{aligned} \tag{5.6}$$

Consistent with our regularization (3.15) and (3.16), $F_{IJ}(\vec{\sigma})$ are approximately constant on both $S(\Delta)$ and $\square$. Therefore,

$$\int_{S(\Delta_{v;I_1,s_1,I_2,s_2})}\mathrm{d}\sigma^I\wedge\mathrm{d}\sigma^J F_{IJ}(\vec{\sigma})\simeq\bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v,I_2,s_2}\int_{\square_{v;I_1,s_1,I_2,s_2}}\mathrm{d}\sigma^I\wedge\mathrm{d}\sigma^J F_{IJ}(\vec{\sigma}). \tag{5.7}$$

This relation and comparing Eqs. (5.4) and (5.6) suggest the following definitions of X_1 and X_2:

$$X_1\coloneqq\bar{\mu}_{v;I_1,s_1}\theta(e_{v;I_1,s_1})-\bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v;I_2,s_2}(\theta(e_{v;I_1,s_1})-\theta(e_{v+s_2\hat{I}_2;I_1,s_1})), \tag{5.8}$$

$$X_2\coloneqq\bar{\mu}_{v;I_2,s_2}\theta(e_{v;I_2,s_2})-\bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v;I_2,s_2}(\theta(e_{v;I_2,s_2})-\theta(e_{v+s_1\hat{I}_1;I_2,s_2})). \tag{5.9}$$

With the above X_1, X_2, the holonomies $h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})=e^{X_1}$, $h_\Delta(\mathfrak{s}_{v_1;I_2,s_2})=e^{X_2}$ reduce to $h_\Delta(\mathfrak{s}_{v;I_1,s_1})$, $h_\Delta(\mathfrak{s}_{v;I_2,s_2})$ when the connection A is constant on $\square$, and reduce to $h(e_{v+s_1\hat{I}_1;I_2,s_2})$, $h(e_{v+s_2\hat{I}_2;I_1,s_1})$ when $\bar{\mu}_{v;I_1,s_1}=\bar{\mu}_{v;I_2,s_2}=1$, i.e., when $\square$ coincides with $S(\Delta)$.

To interpret $h_\Delta(\mathfrak{s}_{v_1;I_2,s_2})$ and $h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})$ as holonomies, we define the following connection field in the plaquette $\square_{v,I,J}$ bounded by $e_{v;I_1,s_1}$ $e_{v;I_2,s_2}$ $(I,J=1,\,2,\,3)$:

$$\begin{aligned}A(\vec{\sigma})&=[(1-\sigma^{I_2})\theta(e_{v;I_1,s_1})+\sigma^{I_2}\theta(e_{v+s_2\hat{I}_2;I_1,s_1})]\mathrm{d}\sigma^{I_1}\\&\quad+[(1-\sigma^{I_1})\theta(e_{v;I_2,s_2})+\sigma^{I_1}\theta(e_{v+s_1\hat{I}_1;I_2,s_2})]\mathrm{d}\sigma^{I_2}.\end{aligned} \tag{5.10}$$

A_{I_1} reduces to $\theta(e_{v;I_1,s_1})$ (or $\theta(e_{v+s_2\hat{I}_2;I_1,s_1})$) along $e_{v;I_1,s_1}$ at $\sigma^{I_2}=0$ (or $e_{v+s_2\hat{I}_2;I_1,s_1}$ at $\sigma^{I_2}=1$), while A_{I_2} reduces to $\theta(e_{v;I_2,s_2})$ (or $\theta(e_{v+s_1\hat{I}_1;I_2,s_2})$) along $e_{v;I_2,s_2}$ at $\sigma^{I_1}=0$ (or $e_{v+s_1\hat{I}_1;I_2,s_2}$ at $\sigma^{I_1}=1$). Therefore, A is an extension of the connection along lattice edges to the plaquette $\square$. It may be generalized to a connection in the three-dimensional cube by

$$\begin{aligned}A(\vec{\sigma})&=[(1-\sigma^2-\sigma^3)\theta(e_{v;1,+})+\sigma^2\theta(e_{v+\hat{2};1,+})+\sigma^3\theta(e_{v+\hat{3};1,+})]\mathrm{d}\sigma^1\\&\quad+[(1-\sigma^1-\sigma^3)\theta(e_{v;2,+})+\sigma^1\theta(e_{v+\hat{1};2,+})+\sigma^3\theta(e_{v+\hat{3};2,+})]\mathrm{d}\sigma^2\\&\quad+[(1-\sigma^1-\sigma^2)\theta(e_{v;3,+})+\sigma^1\theta(e_{v+\hat{1};3,+})+\sigma^2\theta(e_{v+\hat{2};3,+})]\mathrm{d}\sigma^3.\end{aligned} \tag{5.11}$$

The above proposed relation between A and $\theta(e)$ is not gauge invariant. One may notice that A in Eq. (5.11) satisfies $\sum_{I=1}^3\partial_I A_I=0$ which cannot always be satisfied by all gauge transformations of A. Therefore, the expression of A in Eq. (5.10) or Eq. (5.11) in terms of $\theta(e)$ depends on a certain gauge fixing. Since we are going to apply $\mathbf{H}_\Delta$ to cosmology, we impose the following gauge-fixing condition to $\theta(e)$:

$$\sum_{a=1}^{3} \delta_a^{I+1} \theta^a(e_{v;I,+}) = 0. \tag{5.12}$$

In Sec. VII, when we perturb $\theta^a(e_{v;I,+})$ from the homogeneous variables $\theta^a(e_{v;I,+}) = [\theta_I + \delta\theta_{\|}(e_{v;I,+})]\delta_I^a + \delta\theta_{\perp}^a(e_{v;I,+})$, where $\delta\theta_{\perp}^a(e_{v;I,+})$ is perpendicular to δ_I^a, the gauge condition fixes one component of $\delta\theta_{\perp}^a(e_{v;I,+})$ to zero.

In terms of the $\vec{\sigma}$ coordinate, $v = (0,0)$, $v_1 = (\bar{\mu}_{v,1,s_1}, 0)$, $v_2 = (0, \bar{\mu}_{v,2,s_2})$, and $v^\star = (\bar{\mu}_{v,1,s_1}, \bar{\mu}_{v,2,s_2})$, where $v^\star$ is the intersection between $\mathfrak{s}_{v_1;I_2,s_2}$ and $\mathfrak{s}_{v_2;I_1,s_1}$. Along $\mathfrak{s}_{v_1;I_2,s_2}$ (or $\mathfrak{s}_{v_2;I_1,s_1}$), A_{I_2} (or A_{I_2}) is constant:

$$A_{I_2}(\bar{\mu}_{v;I_1,s_1}, \sigma^{I_2}) = [(1-\bar{\mu}_{v;I_1,s_1})\theta(e_{v;I_2,s_2}) + \bar{\mu}_{v;I_1,s_1}\theta(e_{v+s_1\hat{I}_1;I_2,s_2})] \text{along} \mathfrak{s}_{v_1;I_2,s_2}, \tag{5.13}$$

$$A_{I_1}(\sigma^{I_1}, \bar{\mu}_{v;I_2,s_2}) = [(1-\bar{\mu}_{v;I_2,s_2})\theta(e_{v;I_1,s_1}) + \bar{\mu}_{v;I_2,s_2}\theta(e_{v+s_2\hat{I}_2;I_1,s_1})] \text{along} \mathfrak{s}_{v_2;I_1,s_1}. \tag{5.14}$$

Therefore, the holonomies along $\mathfrak{s}_{v_1;I_2,s_2}$ and $\mathfrak{s}_{v_2;I_1,s_1}$ reproduce

$$h_\Delta(\mathfrak{s}_{v_1;I_2,s_2}) = \mathcal{P}\exp\left[\int_0^1 du \frac{d\sigma^{I_2}}{du} A_{I_2}(\bar{\mu}_{v;I_1,s_1}, \sigma^{I_2})\right] = e^{X_2}, \tag{5.15}$$

$$h_\Delta(\mathfrak{s}_{v_2;I_1,s_1}) = \mathcal{P}\exp\left[\int_0^1 du \frac{d\sigma^{I_1}}{du} A_{I_1}(\sigma^{I_1}, \bar{\mu}_{v;I_2,s_2})\right] = e^{X_1}, \tag{5.16}$$

where $\frac{d\sigma^I}{du}$ is the same as $\bar{\mu}_{v;I,s}$ at v. Similarly, the connection also reproduces the holonomies $h_\Delta(\mathfrak{s}_{v;I_2,s_2})$ and $h_\Delta(\mathfrak{s}_{v;I_1,s_1})$ in Eq. (4.9).

In summary, we have obtained the following definition of the loop holonomy $h(\Delta)$ around a Planckian-size plaquette $S(\Delta)$:

$$h(\Delta_{v;I_1,s_1,I_2,s_2}) = h_\Delta(\mathfrak{s}_{v;I_1,s_1}) h_\Delta(\mathfrak{s}_{v_1;I_2,s_2}) h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})^{-1} h_\Delta \times (\mathfrak{s}_{v;I_2,s_2})^{-1}, \tag{5.17}$$

where holonomies along segments are given by

$$h_\Delta(\mathfrak{s}_{v;I_1,s_1}) = e^{\bar{\mu}_{v;I_1,s_1}\theta(e_{v;I_1,s_1})}, \tag{5.18}$$

$$h_\Delta(\mathfrak{s}_{v_1;I_2,s_2}) = e^{\bar{\mu}_{v;I_2,s_2}\theta(e_{v;I_2,s_2}) - \bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v;I_2,s_2}(\theta(e_{v;I_2,s_2}) - \theta(e_{v+s_1\hat{I}_1;I_2,s_2}))}, \tag{5.19}$$

$$h_\Delta(\mathfrak{s}_{v_2;I_1,s_1})^{-1} = e^{-\bar{\mu}_{v;I_1,s_1}\theta(e_{v;I_1,s_1}) + \bar{\mu}_{v;I_1,s_1}\bar{\mu}_{v;I_2,s_2}(\theta(e_{v;I_1,s_1}) - \theta(e_{v+s_2\hat{I}_2;I_1,s_1}))}, \tag{5.20}$$

$$h_\Delta(\mathfrak{s}_{v;I_2,s_2})^{-1} = e^{-\bar{\mu}_{v;I_2,s_2}\theta(e_{v;I_2,s_2})}. \tag{5.21}$$

$h(\Delta)$ is the loop holonomy of the connection A in Eq. (5.10). Although $h(\Delta)$ contains holonomies that do not along edges of the lattice γ, we are able to expresses $h(\Delta)$ in terms of lattice variables $\theta(e) = \theta^a(e)\tau^a/2$ and $p^a(e)$ by the above construction. Inserting the above loop holonomy into Eqs. (3.18) and (3.20) defines the improved Hamiltonian $\mathbf{H}_\Delta$.

Simple expressions of $h_\Delta(\mathfrak{s})$ in terms of $\theta(e)$ [in Eqs. (5.18)–(5.21)] rely on the expression of A [in Eq. (5.10)], which depends on the gauge fixing. Although a generic gauge transformation leaves $\mathbf{H}_\Delta$ invariant, it may change the expressions of $h_\Delta(\mathfrak{s})$ by adding terms with higher orders in $\bar{\mu}$ to their exponents, as suggested by some numerical tests.

Although $\mathbf{H}_\Delta$ with $h_\Delta(\mathfrak{s})$ in Eqs. (5.18)–(5.21) depends on the gauge-fixing condition, the effective dynamics of cosmology derived from $\mathbf{H}_\Delta$ turns out to be gauge invariant [the Gaussian constraint is preserved by the dynamics; see the discussions below Eq. (7.12)] and independent of gauge fixing.

VI. EFFECTIVE EQUATIONS OF FULL LQG

We propose a discrete (canonical) effective action that governs the effective dynamics of full LQG,

$$S_{\text{eff}}[g,h] = \sum_{i=0}^{N+1} K(g_{i+1}, g_i) - \frac{i\kappa}{a^2}\sum_{i=1}^{N} \delta\tau \mathbf{H}_\Delta[g_i], \tag{6.1}$$

where $g_i = \{g_i(e)\}_{e\in E(\gamma)}$, $g_i(e) = e^{-ip^a(e)\tau^a/2} e^{\theta^a(e)\tau^a/2}$, and $i = 1, \ldots, N$ labels steps of discrete time evolution $\delta\tau$. a is a length unit. The kinetic term in the action is implied by the coherent-state path integral in Ref. [14]:

$$K(g_{i+1}, g_i) = \sum_{e\in E(\gamma)} \left[z_{i+1,i}(e)^2 - \frac{1}{2}p_{i+1}(e)^2 - \frac{1}{2}p_i(e)^2\right], \tag{6.2}$$

$$z_{i+1,i}(e) = \operatorname{arccos} h(x_{i+1,i}(e)),$$
$$x_{i+1,i}(e) = \frac{1}{2}\operatorname{tr}[g_{i+1}(e)^\dagger g_i(e)]. \tag{6.3}$$

The effective action S_{eff} is designed in analogy with the "classical action" in the path-integral formula derived in Ref. [14], while here we employ the improved Hamiltonian $\mathbf{H}_\Delta$. Due to the dependence of the scale $\Delta \sim \ell_P^2$, S_{eff} is viewed as an "quantum effective action" defined at the high energy scale corresponding to Δ. $S_{\rm eff}$ contains quantum effects and depends on the scale Δ as an analog of the UV cutoff in quantum field theory. The variables $p^a(e)$ and $\theta^a(e)$ are vacuum expectation values (VEVs) satisfying $\delta S_{\rm eff}=0$. We expect that $S_{\rm eff}$ might be derived from the path integral in Ref. [14] by the standard procedure of a quantum effective action in quantum field theory.

The variational principle $\delta S_{\rm eff}=0$ gives the following equation of motion to determine the VEVs $p^a(e),\theta^a(e)$ [14]:

(1) For $i=1,\ldots,N$, at every edge $e\in E(\gamma)$,

$$\frac{1}{\delta\tau}\left[\frac{z_{i+1,i}(e)\mathrm{tr}[\tau^a g_{i+1}(e)^\dagger g_i(e)]}{\sqrt{x_{i+1,i}(e)-1}\sqrt{x_{i+1,i}(e)+1}} - \frac{p_i(e)\mathrm{tr}[\tau^a g_i(e)^\dagger g_i(e)]}{\sinh(p_i(e))}\right] = \frac{i\kappa}{a^2}\frac{\partial \mathbf{H}_\Delta[g_i^\varepsilon]}{\partial\varepsilon^a(e)}\bigg|_{\vec{\varepsilon}=0}. \tag{6.4}$$

(2) For $i=2,\ldots,N+1$, at every edge $e\in E(\gamma)$,

$$\frac{1}{\delta\tau}\left[\frac{z_{i,i-1}(e)\mathrm{tr}[\tau^a g_i(e)^\dagger g_{i-1}(e)]}{\sqrt{x_{i,i-1}(e)-1}\sqrt{x_{i,i-1}(e)+1}} - \frac{p_i(e)\mathrm{tr}[\tau^a g_i(e)^\dagger g_i(e)]}{\sinh(p_i(e))}\right] = -\frac{i\kappa}{a^2}\frac{\partial \mathbf{H}_\Delta[g_i^\varepsilon]}{\partial\bar{\varepsilon}^a(e)}\bigg|_{\vec{\varepsilon}=0}. \tag{6.5}$$

On the right-hand sides of Eqs. (6.4) and (6.5),

$$g_i^\varepsilon(e) = g_i(e)e^{\varepsilon_i^a(e)\tau^a}. \tag{6.6}$$

The above equations of motion are understood as quantum effective equations since they are derived from the quantum effective action $S_{\rm eff}$. These effective equations determine the improved effective dynamics of full LQG. These equations are complemented by the gauge condition for defining $\mathbf{H}_\Delta$.

VII. HOMOGENEOUS EFFECTIVE DYNAMICS

In order to make contact with LQC, we study solutions of effective equations that are homogeneous (nonisotropic) at every time slice $\mathcal{S}$. We impose the following ansatz at every $\mathcal{S}$:

$$p^a(e_{v;I,s}) = s p_I(\tau)\delta_I^a,\qquad \theta^a(e_{v;I,s}) = s\theta_I(\tau)\delta_I^a, \tag{7.1}$$

where $p_I(\tau),\theta_I(\tau)$ ($I=1,2,3$) are six constants on $\mathcal{S}$ that evolve in time. It is easy to see that four $h_\Delta(\mathfrak{s})$'s in $h(\Delta)$ reduce to the holonomies in $\bar{\mu}$-scheme LQC:

$$h_\Delta(\mathfrak{s}_{v;I,s}) = \exp\left(s\lambda\sqrt{\left|\frac{p_I}{p_{I+1}p_{I+2}}\right|}\theta_I\tau_I/2\right), \tag{7.2}$$

where $I, I+1, I+2$ are defined mod 3.

We insert the homogeneous ansatz (7.1) into the effective equations (6.4) and (6.5) and take the continuous limit $\delta\tau\to 0$. The effective equations reduce to differential equations of homogeneous variables $p_I(\tau),\theta_I(\tau)$. A part of this computation is a simple generalization of the computation for homogeneous and isotropic cosmology in Ref. [14], and is sketched as follows.

First of all, inserting Eq. (7.1) into Eqs. (6.4) and (6.5) gives

$$\begin{aligned}\delta_I^a\left[\frac{d\theta_I}{d\tau}+i\frac{dp_I}{d\tau}\right] &= \frac{i\kappa}{a^2}\frac{\partial\mathbf{H}_\Delta[g^\varepsilon]}{\partial\varepsilon^a(e_I(v))}\bigg|_{\vec{\varepsilon}=0},\\ \delta_I^a\left[\frac{d\theta_I}{d\tau}-i\frac{dp_I}{d\tau}\right] &= -\frac{i\kappa}{a^2}\frac{\partial\mathbf{H}_\Delta[g^\varepsilon]}{\partial\bar{\varepsilon}^a(e_I(v))}\bigg|_{\vec{\varepsilon}=0},\end{aligned} \tag{7.3}$$

where $e_I(v)\equiv e_{v;I,+}$.

Since $\mathbf{H}_\Delta$ is conveniently expressed as a function of $p^a(e),\theta^a(e)$, we would like to write the right-hand sides of Eq. (7.3) in terms of derivatives of $p^a(e),\theta^a(e)$. Equation (6.6) can be rewritten in the polar-decomposition form where we extract perturbations of p^a, θ^a:

$$g^\varepsilon(e_I(v)) = e^{(\theta-ip)\frac{\tau^I}{2}}e^{\varepsilon^a(e_I(v))\tau^a} = e^{-ip^a(e_I(v))\tau^a/2}e^{\theta^a(e_I(v))\tau^a/2}, \tag{7.4}$$

where p^a, θ^a contains longitudinal perturbations $\delta p_\parallel,\delta\theta_\parallel$ and transverse perturbations $\delta p_\perp^a,\delta\theta_\perp^a$ with $a=I+1, I+2$ mod 3,

$$p^a(e_I(v)) = [p_I+\delta p_\parallel(e_I(v))]\delta_I^a + \delta p_\perp^a(e_I(v)), \tag{7.5}$$

$$\theta^a(e_I(v)) = [\theta_I+\delta\theta_\parallel(e_I(v))]\delta_I^a + \delta\theta_\perp^a(e_I(v)). \tag{7.6}$$

$\delta p_\parallel\delta_I^a,\delta\theta_\parallel\delta_I^a$, and $\delta p_\perp^a,\delta\theta_\perp^a$ are perpendicular and related to ε^a up to $O(\varepsilon^2)$ by

$$\begin{aligned}\delta p_\parallel(e_I(v)) &= i[\varepsilon^I(e_I(v))-\bar{\varepsilon}^I(e_I(v))],\\ \delta\theta_\parallel(e_I(v)) &= \varepsilon^I(e_I(v))+\bar{\varepsilon}^I(e_I(v)),\end{aligned} \tag{7.7}$$

$$\begin{pmatrix}\delta p_\perp^{I+1}(e_I(v))\\ \delta p_\perp^{I+2}(e_I(v))\end{pmatrix} = \frac{ip}{\sinh(p)}\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\varepsilon^{I+1}(e_I(v))-\bar{\varepsilon}^{I+1}(e_I(v))\\ \varepsilon^{I+2}(e_I(v))-\bar{\varepsilon}^{I+2}(e_I(v))\end{pmatrix}, \tag{7.8}$$

$$\begin{pmatrix} \delta\theta_\perp^{I+1}(e_I(v)) \\ \delta\theta_\perp^{I+2}(e_I(v)) \end{pmatrix} = \frac{\theta/2}{\sin(\theta/2)}\left[\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\begin{pmatrix} \varepsilon^{I+1}(e_I(v)) + \bar{\varepsilon}^{I+1}(e_I(v)) \\ \varepsilon^{I+2}(e_I(v)) + \bar{\varepsilon}^{I+2}(e_I(v)) \end{pmatrix}\right.$$
$$\left. + i\tanh(p/2)\begin{pmatrix} \sin(\theta/2) & \cos(\theta/2) \\ -\cos(\theta/2) & \sin(\theta/2) \end{pmatrix}\begin{pmatrix} \varepsilon^{I+1}(e_I(v)) - \bar{\varepsilon}^{I+1}(e_I(v)) \\ \varepsilon^{I+2}(e_I(v)) - \bar{\varepsilon}^{I+2}(e_I(v)) \end{pmatrix}\right]. \tag{7.9}$$

The above linear transformation between ε^a and $\delta p, \delta\theta$ is nondegenerate. By changing variables, Eq. (7.3) reduces to the following:

(1) The diagonals $a = I$ give time evolution equations,

$$\frac{\mathrm{d}\theta_I}{\mathrm{d}\tau} = -\frac{\kappa}{a^2}\frac{\partial \mathbf{H}_\Delta[g^\varepsilon]}{\partial\delta p_\parallel(e_I(v))}\bigg|_{\delta\theta=\delta p=0}, \qquad \frac{\mathrm{d}p_I}{\mathrm{d}\tau} = \frac{\kappa}{a^2}\frac{\partial \mathbf{H}_\Delta[g^\varepsilon]}{\partial\delta\theta_\parallel(e_I(v))}\bigg|_{\delta\theta=\delta p=0}. \tag{7.10}$$

(2) The off-diagonals $a \neq I$ give constraint equations,

$$\frac{\partial \mathbf{H}_\Delta[g^\varepsilon]}{\partial\delta p_\perp^a(e_I(v))}\bigg|_{\delta\theta=\delta p=0} = 0, \quad \frac{\partial \mathbf{H}_\Delta[g^\varepsilon]}{\partial\delta\theta_\perp^a(e_I(v))}\bigg|_{\delta\theta=\delta p=0} = 0, \quad a \neq I. \tag{7.11}$$

The gauge condition (5.12) sets some components of $\delta\theta_\perp$ to zero. So Eq. (7.11) only needs to take into account derivatives of nonzero $\delta\theta_\perp$. However, we can show (below) that Eq. (7.11) is satisfied for all $\delta\theta_\perp$ even without imposing the gauge condition. Moreover, the (discrete) Gauss constraint (namely, the closure condition)

$$-\sum_{e,s(e)=v} p_1^a(e) + \sum_{e,t(e)=v} \Lambda_b^a(\vec{\theta}_1(e))p_1^b(e) = 0, \quad \text{where } e^{\theta^a\tau^a/2}\tau^a e^{-\theta^a\tau^a/2} = \Lambda_b^a(\vec{\theta})\tau^b \tag{7.12}$$

is satisfied by the ansatz at all τ. Therefore, when we find a solution (as shown below) by inserting the ansatz into Eqs. (6.4) and (6.5), the effective dynamics implied by the solution is gauge invariant since the Gauss constraint is preserved by the dynamics.

Computing derivatives of $\mathbf{H}_\Delta$ leads to the following result.

Theorem 7.1. The time evolution equations (7.10) are equivalent to the following Hamiltonian equations:

$$\frac{\mathrm{d}\theta_I}{\mathrm{d}\tau} = -\frac{\kappa}{a^2}\frac{\partial}{\partial p_I}H_\Delta(\vec{\theta},\vec{p}), \qquad \frac{\mathrm{d}p_I}{\mathrm{d}\tau} = \frac{\kappa}{a^2}\frac{\partial}{\partial\theta_I}H_\Delta(\vec{\theta},\vec{p}), \tag{7.13}$$

where $H_\Delta(\vec{\theta},\vec{p}) = |C_v^\Delta||_{\delta\theta=\delta p=0}$,

$$H_\Delta(\vec{\theta},\vec{p}) = \frac{16a^3}{3\kappa\beta^{1/2}\Delta}\sqrt{|p_1p_2p_3|}\left|\sin\left(\frac{\theta_2\lambda p_2}{\sqrt{|p_1p_2p_3|}}\right)\sin\left(\frac{\theta_3\lambda p_3}{\sqrt{|p_1p_2p_3|}}\right) + \sin\left(\frac{\theta_1\lambda p_1}{\sqrt{|p_1p_2p_3|}}\right)\sin\left(\frac{\theta_2\lambda p_2}{\sqrt{|p_1p_2p_3|}}\right)\right.$$
$$\left. + \sin\left(\frac{\theta_1\lambda p_1}{\sqrt{|p_1p_2p_3|}}\right)\sin\left(\frac{\theta_3\lambda p_3}{\sqrt{|p_1p_2p_3|}}\right) - \frac{3}{16}\beta^2\Delta\Lambda\mathrm{sgn}(p_1p_2p_3)\right|, \tag{7.14}$$

while Eq. (7.11) is satisfied automatically.

Proof: For a shorthand notation, $\xi^A = (\delta p_\parallel^a(e), \delta p_\perp^a(e), \delta\theta_\parallel(e), \delta\theta_\perp(e))$ denotes a vector of all perturbations. We have the following expansion:

$$\mathbf{H} = \sum_v \sqrt{\left|\left[C_v^\Delta|_0 + \frac{\partial C_v^\Delta}{\partial\xi^A}\bigg|_0\xi^A + O(\xi^2)\right]^2 - \frac{\alpha}{4}\left[C_{j,v}^\Delta|_0 + \frac{\partial C_{a,v}^\Delta}{\partial\xi^A}\bigg|_0\xi^A + O(\xi^2)\right]^2\right|}, \tag{7.15}$$

where $|_0$ means evaluation at $\xi^A = 0$. The contributions to $\partial C_v^\Delta/\partial\xi^A|_0$ only come from the Euclidean Hamiltonian and cosmological constant terms in C_v^Δ. ${}^3\mathcal{R}_v$ does not contribute because ${}^3\mathcal{R}_v = O(\xi^2)$. Moreover, $C_{j,v}^\Delta|_0 = 0$ so the diffeomorphism constraint has no contribution to the linear order in ξ^A, and the result is independent of α. We expand $\mathbf{H}$ and ignore $O(\xi^2)$,

$$
\begin{aligned}
\mathbf{H} &= \sum_v \sqrt{(C_v^\Delta)^2|_0 + 2C_v^\Delta|_0 \frac{\partial C_v^\Delta}{\partial \xi^A}\bigg|_0 \xi^A + O(\xi^2)} \\
&= \sum_v \sqrt{(C_v^\Delta)^2|_0} + \sum_v \mathrm{sgn}(C_v^\Delta|_0) \frac{\partial C_v^\Delta}{\partial \xi^A}\bigg|_0 \xi^A + O(\xi^2) \\
&= \sum_v |C_v^\Delta|_0| + \sum_v \frac{\partial |C_v^\Delta|}{\partial \xi^A}\bigg|_0 \xi^A + O(\xi^2). \qquad (7.16)
\end{aligned}
$$

$\sum_v \partial C_v^\Delta / \partial \xi^A|_0$ can be obtained by a straightforward *Mathematica* computation (the *Mathematica* code can be downloaded at Ref. [45]):

$$
\sum_v \frac{\partial |C_v^\Delta|}{\partial \delta p_\perp^a(e_I(v))}\bigg|_0 = \sum_v \frac{\partial |C_v^\Delta|}{\partial \delta \theta_\perp^a(e_I(v))}\bigg|_0 = 0, \qquad (7.17)
$$

$$
\begin{aligned}
\sum_v \frac{\partial |C_v^\Delta|}{\partial \delta p_\parallel^a(e_I(v))}\bigg|_0 &= \frac{\partial H_\Delta(\vec{\theta}, \vec{p})}{\partial p_I}, \\
\sum_v \frac{\partial |C_v^\Delta|}{\partial \delta \theta_\parallel^a(e_I(v))}\bigg|_0 &= \frac{\partial H_\Delta(\vec{\theta}, \vec{p})}{\partial \theta_I}, \qquad (7.18)
\end{aligned}
$$

where $H_\Delta(\vec{\theta}, \vec{p})$ is given by Eq. (7.14). Equation (7.17) implies that the constraint equations (7.11) are automatically satisfied, while Eq. (7.18) implies Eq. (7.13). ■

The absolute values in the square roots in Eq. (7.14) may be related to Refs. [46,47]. $\mathrm{sgn}(p_1 p_2 p_3)$ in the cosmological constant term is anticipated because, in the absence of a cosmological constant ($\Lambda = 0$) the Hamiltonian constraint $C_v|_{\Lambda=0}$ discretizes $\mathrm{sgn}(e)\mathcal{C}|_{\Lambda=0}$ instead of $\mathcal{C}|_{\Lambda=0}$, where $\mathcal{C}|_{\Lambda=0}$ denotes the classical smooth Hamiltonian constraint without Λ, and $\mathrm{sgn}(e) = \pm 1$ is the sign of the determinant of the triad e_a^j [1]. The presence of the parity-odd factor $\mathrm{sgn}(e)$ is due to Thiemann's trick of regularizing $\mathcal{C}|_{\Lambda=0}$ [$\mathrm{sgn}(e)$ shows up when introducing the Poisson bracket of A_j^a and volume in $\mathcal{C}|_{\Lambda=0}$]. However, the discretization of the cosmological constant term $\Lambda\sqrt{\det q}$ gives ΛV_v, where V_v is the volume but does not involve $\mathrm{sgn}(e)$. When the cosmological constant is included, $C_v = C_v|_{\Lambda=0} + \Lambda V_v$ is a discretization of $\mathrm{sgn}(e)\mathcal{C}|_{\Lambda=0} + \Lambda\sqrt{\det q}$. Due to C_v^2 in $\hat{\mathbf{H}}$, flipping $\mathrm{sgn}(e)$ changes the sign of the cosmological constant. It suggests that here Λ is related to the cosmological constant up to a sign. Even if one fixes $\Lambda > 0$, the effective dynamics still contain both positive and negative cosmological constants, which are related by parity transformation.

To make contact with the effective dynamics in the Bianchi-I model of LQC [48], we define

$$
\bar{\mu}_1 = \lambda \sqrt{\left|\frac{p_1}{p_2 p_3}\right|}, \qquad \bar{\mu}_2 = \lambda \sqrt{\left|\frac{p_2}{p_1 p_3}\right|},
$$

$$
\bar{\mu}_3 = \lambda \sqrt{\left|\frac{p_3}{p_1 p_2}\right|}, \qquad \lambda = \frac{\sqrt{\Delta}}{\beta^{1/2} a}. \qquad (7.19)
$$

In terms of the conventional variables C_I, P_I used in LQC,

$$
\theta_I = C_I, \qquad p_I = \frac{P_I}{\beta a^2}, \qquad (7.20)
$$

$$
\bar{\mu}_1 = \sqrt{\Delta} \sqrt{\left|\frac{P_1}{P_2 P_3}\right|}, \qquad \bar{\mu}_2 = \sqrt{\Delta} \sqrt{\left|\frac{P_2}{P_1 P_3}\right|},
$$

$$
\bar{\mu}_3 = \sqrt{\Delta} \sqrt{\left|\frac{P_3}{P_1 P_2}\right|}. \qquad (7.21)
$$

Then, $H_\Delta(\vec{\theta}, \vec{p})$ is reduced to the effective Hamiltonian in LQC up to an overall sign:

$$
\begin{aligned}
H_\Delta(\vec{\theta}, \vec{p}) = \bigg| &\frac{16 a}{3\beta^{3/2}\kappa} \frac{1}{\sqrt{|p_1 p_2 p_3|}} \\
&\times \left(\frac{\sin(\bar{\mu}_1 \theta_1)}{\bar{\mu}_1} \frac{\sin(\bar{\mu}_2 \theta_2)}{\bar{\mu}_2} p_1 p_2 + \text{cyclic terms}\right) \\
&- \mathrm{sgn}(p_1 p_2 p_3) \frac{a^3 \beta^{3/2}}{\kappa} \Lambda \sqrt{|p_1 p_2 p_3|} \bigg| \qquad (7.22)
\end{aligned}
$$

$$
\begin{aligned}
= \bigg| &\frac{16}{3\beta^2 \kappa} \frac{1}{\sqrt{|P_1 P_2 P_3|}} \\
&\times \left(\frac{\sin(\bar{\mu}_1 C_1)}{\bar{\mu}_1} \frac{\sin(\bar{\mu}_2 C_2)}{\bar{\mu}_2} P_1 P_2 + \text{cyclic terms}\right) \\
&- \mathrm{sgn}(P_1 P_2 P_3) \frac{\Lambda}{\kappa} \sqrt{|P_1 P_2 P_3|} \bigg|. \qquad (7.23)
\end{aligned}
$$

The absolute value here comes from the fact that the physical Hamiltonian is always positive definite, which is due to the choice of physical dust and τ flowing backward, corresponding to a negative lapse (see Sec. II for details).

VIII. COSMIC BOUNCE IN THE $\bar{\mu}$ SCHEME

We set $\Lambda > 0$ and simplify to the homogenous and isotropic cosmology by identifying $p_I = p$ and $\theta_I = \theta$ for $I = 1, 2, 3$. Equations (7.13) and (7.14) reduce to

$$
\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{\kappa}{a^2} \frac{\partial}{\partial p} H_\Delta(\theta, p), \frac{\mathrm{d}p}{\mathrm{d}\tau} = \frac{\kappa}{a^2} \frac{\partial}{\partial \theta} H_\Delta(\theta, p), \qquad (8.1)
$$

$$
\begin{aligned}
H_\Delta(\theta, p) = \frac{1}{3} H_\Delta(\vec{\theta}, \vec{p})\bigg|_{p_I = p, \theta_I = \theta} &= \frac{16 a^3}{3\beta^{1/2}\kappa\Delta} \sqrt{|p|^3} \\
&\times \left[\sin^2\left(\frac{\lambda}{\sqrt{|p|}}\theta\right) - \mathrm{sgn}(p) \frac{\beta^2 \Delta \Lambda}{16}\right]. \qquad (8.2)
\end{aligned}
$$

It is convenient to make the following change of variables:

$$
V = (|p| a^2 \beta)^{3/2}, \qquad b = \frac{\lambda}{\sqrt{|p|}} \theta. \qquad (8.3)
$$

H_Δ reduces to the LQC Hamiltonian up to a possible overall sign:

$$H_\Delta = \frac{16}{3\beta^2\kappa\Delta} V \sin^2(b) - \mathrm{sgn}(p)\frac{\Lambda}{3\kappa} V. \tag{8.4}$$

In terms of (V, b), the evolution equations become

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = \frac{8V\sin(2b)}{\beta\sqrt{\Delta}}, \qquad \frac{\mathrm{d}b}{\mathrm{d}\tau} = \frac{\beta^2\Delta\Lambda + 8\cos(2b) - 8}{2\beta\sqrt{\Delta}}. \tag{8.5}$$

H_Δ is conserved in the time evolution:

$$\frac{16}{3\beta^2\kappa\Delta} V \sin^2(b) - \frac{\Lambda}{3\kappa} V = \frac{\mathcal{E}}{3} = \frac{\rho V}{3}, \tag{8.6}$$

where ρ is the energy of the physical dust. This conservation law can be used to solve for $\sin(b)$, whose solution can be inserted into the first evolution equation. The resulting equation in terms of the scale factor $\mathfrak{a} = V^{1/3}$ gives the modified Friedmann equation

$$\left(\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}\right)^2 = \frac{16}{9}\left[\Lambda\left(1 - \frac{\beta^2\Delta\Lambda}{16}\right) + \kappa\rho\left(1 - \frac{\beta^2\Delta\Lambda}{8}\right) - \frac{\beta^2\Delta\kappa^2}{16}\rho^2\right], \tag{8.7}$$

where $\dot{\mathfrak{a}} = \mathrm{d}\mathfrak{a}/\mathrm{d}\tau$ or $\mathrm{d}\mathfrak{a}/\mathrm{d}(-\tau)$. The modified Friedmann equation reduces to the standard Friedmann equation (up to a rescaling τ) at low density $\rho \ll 1$, with the renormalized gravitational constant $\bar{\kappa}$ and cosmological constant $\bar{\Lambda}$:

$$\bar{\kappa} = \kappa\left(1 - \frac{\beta^2\Delta\Lambda}{8}\right), \qquad \bar{\Lambda} = \Lambda\left(1 - \frac{\beta^2\Delta\Lambda}{16}\right). \tag{8.8}$$

The backward time evolution stops at $\dot{\mathfrak{a}} = 0$ and gives the Planckian critical density

$$\rho_c = \frac{16 - \beta^2\Delta\Lambda}{\beta^2\Delta\kappa}, \tag{8.9}$$

which is a nonzero constant independent of the conserved quantity $\mathcal{E}$, in contrast to the μ_0-scheme effective dynamics obtained in Ref. [14]. A nonzero ρ_c at $\dot{\mathfrak{a}} = 0$ indicates that the big bang singularity is resolved and replaced by a bounce.

IX. LATTICE INDEPENDENCE

In this section we focus on the homogeneous variables (7.1) and study the behavior of $\mathbf{H}_\Delta$ by refining the cubic lattice γ. We focus on the cubic graph because the cubic graph is preferred by the semiclassical analysis of the canonical LQG, [33], and because of the close relation between this work and Ref. [14].

We define a sequence of cubic lattices γ_r, $r = 0, 1, 2, \ldots$, with $\gamma_0 = \gamma$. If edges in γ have unit coordinate lengths, every edge in γ_r has a coordinate length r^{-1} in the same coordinate system. The continuum limit is given by $r \to \infty$.

A key observation in the improved Hamiltonian $\mathbf{H}_\Delta$ is that the loop holonomy $h(\Delta)$ is invariant under lattice refinement. By definition, $h(\Delta)$ is a holonomy around a surface with fixed area and thus is independent of the lattice size. More specifically, for homogeneous variables, we have on γ_r

$$\theta_I^{(r)} = r^{-1}\theta_I, \qquad p_I^{(r)} = r^{-2}p_I, \qquad \bar{\mu}_I^{(r)} = r\bar{\mu}_I, \tag{9.1}$$

where the superscript (r) labels quantities on γ_r. $h_\Delta(\mathfrak{s})$ in Eqs. (5.18)–(5.21) are indeed invariant under lattice refinement. Consequently, $\sin(\frac{\theta_I\lambda p_I}{\sqrt{p_I p_{I+1} p_{I+2}}})$ in Eq. (7.14) does not scale under refinement. Therefore, C_v^Δ scales the same as a volume,

$$C_v^{\Delta\,(r)} = r^{-3} C_v^\Delta. \tag{9.2}$$

$C_v^{\Delta\,(r)}$ is constant at all v by homogeneity, while the number of vertices scales as $|V(\gamma_r)| = r^3|V(\gamma)|$. We obtain the invariance of the Hamiltonian

$$\mathbf{H}_\Delta^{(r)} = \mathbf{H}_\Delta. \tag{9.3}$$

Therefore, the continuum limit of $\mathbf{H}_\Delta$ is trivial for homogeneous variables. Evaluating $\mathbf{H}_\Delta$ on any cubic lattice is equivalent to the evaluation on the continuum.

Because H_Δ in Eqs. (7.13) and (8.1) is equal to $-C_v^\Delta$ evaluated for homogeneous variables, the scalings (9.1) and (9.2) imply that Eqs. (7.13) and (8.1) are invariant under lattice refinement. Therefore, the cosmological effective dynamics, the predictions for the bounce, and the critical density are independent of the lattice refinement, and thus can be understood as results at the continuum limit.

If there exists an operator $\hat{\mathbf{H}}_\Delta{}^{(r)}$ such that $\mathbf{H}_\Delta^{(r)} = \langle\psi^{(r)}|\hat{\mathbf{H}}_\Delta{}^{(r)}|\psi^{(r)}\rangle$ with a sequence of states $\psi^{(r)}$ representing the homogeneous spatial geometry in the LQG Hilbert space on γ_r, Eq. (9.3) becomes

$$\langle\psi^{(r)}|\hat{\mathbf{H}}_\Delta{}^{(r)}|\psi^{(r)}\rangle = \langle\psi^{(0)}|\hat{\mathbf{H}}_\Delta{}^{(0)}|\psi^{(0)}\rangle, \tag{9.4}$$

which suggests that $\mathbf{H}_\Delta$ for homogeneous variables is a fixed point in the Hamiltonian renormalization proposed in Ref. [34]. However, constructions of the operator and states such as $\hat{\mathbf{H}}_\Delta$ and $\psi^{(r)}$ are beyond the scope of this paper.

Interestingly, from the viewpoint of lattice field theory, the triviality of the lattice refinement (9.3) suggests that the theory is scaling invariant. Then it is conformal invariant at the state of homogeneous and isotropic spatial geometry.[4] This three-dimensional conformal invariance might be related to the AdS/CFT correspondence.

X. OUTLOOK

In this section, we discuss a few interesting perspectives which have not yet been addressed in this paper, but will be studied and reported in the future.

First, as mentioned a few times in the above discussion, it is useful to develop an operator formalism for $\mathbf{H}_\Delta$ and find a series of semiclassical states $\psi^{(r)}$ to realize Eq. (9.4), in particular from the perspective of the Hamiltonian renormalization.

Moreover, we conjecture that $S_{\rm eff}$, being a quantum effective action, should be related to the path-integral formulation derived in Ref. [14]. This path-integral formula contains a "classical action" S, while $S_{\rm eff}$ can be seen as S plus quantum corrections of $O(\ell_P^2)$. A quantum derivation of $S_{\rm eff}$ from this path-integral formula is currently underway.

Second, another topic of active research is to generalize the effective dynamics from homogeneous cosmology to other spacetimes, since the effective equations we obtained in Sec. VI are for the full theory. We have applied these equations to study, e.g., cosmological perturbations and spherically symmetric black holes (in both Kantowski-Sachs foliation and global Kruskal foliation). This direction has two strategies:

(1) Similar to the present strategy, we may implement an ansatz that respects the symmetry of the expected solution, and then simplify and solve the effective equations [49,50]. It is also similar to the strategy of analytically solving the Einstein equation to obtain, e.g., black holes and cosmology.

(2) A different strategy is similar to numerical relativity. We are developing a numerical code that implements the effective equations (6.4)–(6.5) of the full theory. Equations (6.4) and (6.5) can be cast into a formulation similar to the evolution equations used in numerical relativity, and thus standard numerical methods such as fourth-order Runge-Kutta methods can be applied to our effective equations. Numerical solutions can be generated by specifying suitable initial conditions of $p^a(e)$ and $\theta^a(e)$. As an initial application of the numerical code, we find that the cosmological solution is unique for homogeneous and isotropic initial data [51] (see also [52]). The path integral in Ref. [14] has a unique critical point when initial and final cosmological coherent states can be related by effective equations (and thus has an oscillatory behavior as $t \to 0$), or has no critical points if they are not related by effective equations (and thus is exponentially suppressed as $t \to 0$).

Last, the analysis of this paper focused on improving the Hamiltonian whose Lorentzian part is the scalar curvature, as in Refs. [43,44]. The next step may be the generalization to Thiemann's Lorentzian Hamiltonian which involves K as the commutator between the Euclidean Hamiltonian and volume.

[4]Scaling, translational, and rotational invariance imply conformal invariance.

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[1] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2007).
[2] M. Han, W. Huang, and Y. Ma, Fundamental structure of loop quantum gravity, Int. J. Mod. Phys. D **16**, 1397 (2007).
[3] M. Bojowald, Absence of Singularity in Loop Quantum Cosmology, Phys. Rev. Lett. **86**, 5227 (2001).
[4] A. Ashtekar, T. Pawlowski, and P. Singh, Quantum nature of the big bang: Improved dynamics, Phys. Rev. D **74**, 084003 (2006).
[5] P. Singh, Are loop quantum cosmos never singular?, Classical Quantum Gravity **26**, 125005 (2009).
[6] M. Assanioussi, A. Dapor, K. Liegener, and T. Pawlowski, Emergent de Sitter epoch of the loop quantum cosmos: A detailed analysis, Phys. Rev. D **100**, 084003 (2019).
[7] A. Ashtekar, J. Olmedo, and P. Singh, Quantum extension of the Kruskal spacetime, Phys. Rev. D **98**, 126003 (2018).
[8] A. Ashtekar, J. Olmedo, and P. Singh, Quantum Transfiguration of Kruskal Black Holes, Phys. Rev. Lett. **121**, 241301 (2018).
[9] M. Assanioussi, A. Dapor, and K. Liegener, Perspectives on the dynamics in loop effective black hole interior, Phys. Rev. D **101**, 026002 (2020).
[10] R. Gambini, J. Olmedo, and J. Pullin, Quantum black holes in loop quantum gravity, Classical Quantum Gravity **31**, 095009 (2014).
[11] J. B. Achour, F. Lamy, H. Liu, and K. Noui, Polymer Schwarzschild black hole: An effective metric, Europhys. Lett. **123**, 20006 (2018).
[12] C. Rovelli and F. Vidotto, Planck stars, Int. J. Mod. Phys. D **23**, 1442026 (2014).
[13] M. Han and M. Zhang, Spinfoams near a classical curvature singularity, Phys. Rev. D **94**, 104075 (2016).

[14] M. Han and H. Liu, Effective dynamics from coherent state path integral of full loop quantum gravity, Phys. Rev. D **101**, 046003 (2020).
[15] M. Bojowald, Loop quantum cosmology, Living Rev. Relativity **8**, 11 (2005).
[16] A. Ashtekar, Loop quantum cosmology: An overview, Gen. Relativ. Gravit. **41**, 707 (2009).
[17] I. Agullo and P. Singh, Loop quantum cosmology, in *Loop Quantum Gravity: The First 30 Years*, edited by A. Ashtekar and J. Pullin (WSP, Singapore, 2017), pp. 183–240.
[18] V. Taveras, Corrections to the Friedmann equations from LQG for a Universe with a free scalar field, Phys. Rev. D **78**, 064072 (2008).
[19] E. Alesci and F. Cianfrani, Quantum-reduced loop gravity: Cosmology, Phys. Rev. D **87**, 083521 (2013).
[20] N. Bodendorfer, Quantum reduction to Bianchi I models in loop quantum gravity, Phys. Rev. D **91**, 081502 (2015).
[21] N. Bodendorfer, An embedding of loop quantum cosmology in (b, v) variables into a full theory context, Classical Quantum Gravity **33**, 125014 (2016).
[22] E. Alesci and F. Cianfrani, Improved regularization from quantum reduced loop gravity, arXiv:1604.02375.
[23] A. Dapor and K. Liegener, Cosmological effective Hamiltonian from full loop quantum gravity dynamics, Phys. Lett. B **785**, 506 (2018).
[24] J. Engle, Relating loop quantum cosmology to loop quantum gravity: Symmetric sectors and embeddings, Classical Quantum Gravity **24**, 5777 (2007).
[25] M. Hanusch, Invariant connections and symmetry reduction in loop quantum gravity, arXiv:1601.05531.
[26] C. Fleischhack, Loop quantization and symmetry: Configuration spaces, Commun. Math. Phys. **360**, 481 (2018).
[27] C. Rovelli and F. Vidotto, Stepping out of homogeneity in loop quantum cosmology, Classical Quantum Gravity **25**, 225024 (2008).
[28] G. Calcagni, Loop quantum cosmology from group field theory, Phys. Rev. D **90**, 064047 (2014).
[29] A. Dapor, K. Liegener, and T. Pawlowski, Challenges in recovering a consistent cosmology from the effective dynamics of loop quantum gravity, Phys. Rev. D **100**, 106016 (2019).
[30] K. Giesel, S. Hofmann, T. Thiemann, and O. Winkler, Manifestly gauge-invariant general relativistic perturbation theory. I. Foundations, Classical Quantum Gravity **27**, 055005 (2010).
[31] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG). IV. Reduced phase space quantisation of loop quantum gravity, Classical Quantum Gravity **27**, 175009 (2010).
[32] T. Thiemann, Quantum spin dynamics (QSD), Classical Quantum Gravity **15**, 839 (1998).
[33] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG). III. Semiclassical perturbation theory, Classical Quantum Gravity **24**, 2565 (2007).
[34] T. Lang, K. Liegener, and T. Thiemann, Hamiltonian renormalisation. I: Derivation from Osterwalder-Schrader reconstruction, Classical Quantum Gravity **35**, 245011 (2018).
[35] J. D. Brown and K. V. Kuchar, Dust as a standard of space and time in canonical quantum gravity, Phys. Rev. D **51**, 5600 (1995).
[36] K. V. Kuchar and C. G. Torre, Gaussian reference fluid and interpretation of quantum geometrodynamics, Phys. Rev. D **43**, 419 (1991).
[37] K. Giesel and T. Thiemann, Scalar material reference systems and loop quantum gravity, Classical Quantum Gravity **32**, 135015 (2015).
[38] B. Dittrich, Partial and complete observables for Hamiltonian constrained systems, Gen. Relativ. Gravit. **39**, 1891 (2007).
[39] T. Thiemann, Reduced phase space quantization and Dirac observables, Classical Quantum Gravity **23**, 1163 (2006).
[40] J. Barbero, Real Ashtekar variables for Lorentzian signature space times, Phys. Rev. D **51**, 5507 (1995).
[41] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG). I. Conceptual setup, Classical Quantum Gravity **24**, 2465 (2007).
[42] T. Thiemann, Quantum spin dynamics (QSD): 7. Symplectic structures and continuum lattice formulations of gauge field theories, Classical Quantum Gravity **18**, 3293 (2001).
[43] E. Alesci, M. Assanioussi, and J. Lewandowski, Curvature operator for loop quantum gravity, Phys. Rev. D **89**, 124017 (2014).
[44] M. Assanioussi, J. Lewandowski, and I. Makinen, New scalar constraint operator for loop quantum gravity, Phys. Rev. D **92**, 044042 (2015).
[45] M. Han and H. Liu, https://github.com/LQG-Florida-Atlantic-University/Hamiltonian, 2019.
[46] L. Garay, M. Martin-Benito, and G. M. Marugan, Inhomogeneous loop quantum cosmology: Hybrid quantization of the Gowdy model, Phys. Rev. D **82**, 044048 (2010).
[47] M. Martin-Benito, G. A. Marugan, and J. Olmedo, Further improvements in the understanding of isotropic loop quantum cosmology, Phys. Rev. D **80**, 104015 (2009).
[48] A. Ashtekar and E. Wilson-Ewing, Loop quantum cosmology of Bianchi I models, Phys. Rev. D **79**, 083535 (2009).
[49] M. Han, H. Li, and H. Liu, Manifestly gauge-invariant cosmological perturbation theory from full loop quantum gravity, arXiv:2005.00883.
[50] A. Dapor, M. Han, and H. Liu, Effective dynamics of black holes in full loop quantum gravity (to be published).
[51] M. Han and H. Liu, The uniqueness of cosmological solution in full loop quantum gravity (to be published).
[52] H. Liu, Semiclassical limit of new path integral formulation from reduced phase space loop quantum gravity, Phys. Rev. D **102**, 024083 (2020).