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Note

A note on weak delta systems

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ABSTRACT

Let $\mathcal{F}$ be a family of n-element sets. In 1995, Axenovich, Fon-Der-Flaass and Kostochka established an upper bound on the size of $\mathcal{F}$ that does not contain a Δ-system with $q = 3$ sets. Using the ideas of their proof we extend the results to an arbitrary q.

1. Introduction

A family of sets $\mathcal{D}$ is called an (n, q)-Δ-system if $|\mathcal{D}| = q$, each member of $\mathcal{D}$ has cardinality n (i.e., $\mathcal{D}$ is *n-uniform*), and the intersection of any two sets in $\mathcal{D}$ is the same. Similarly, a family of sets $\mathcal{D}$ is called a *weak* (n, q)-Δ-system if $|\mathcal{D}| = q$, each member of $\mathcal{D}$ has cardinality n, and the intersection of any two sets in $\mathcal{D}$ has the same cardinality. Denote by $f(n, q)$ the maximum cardinality of an n-uniform family $\mathcal{F}$ which contains no (n, q)-Δ-system, and denote by $g(n, q)$ the maximum cardinality of an n-uniform family $\mathcal{F}$ which contains no weak (n, q)-Δ-system. Upper and lower bounds for $f(n, q)$ and $g(n, q)$ have been studied extensively. A survey by Kostochka can be found in [7].

Axenovich, Fon-Der-Flaass and Kostochka [4] proved that for any $\epsilon > 0$, there is a constant $C(\epsilon)$, such that

$$g(n, 3) \leq C(n!)^{\frac{1}{2}+\epsilon}.$$

Based on ideas of [4], we were able to obtain an upper bound for general q, namely Theorem 2.1 which says that for any integer q and $\epsilon \in (0, 1/(q-1)]$ there is a constant C, such that

$$g(n, q) \leq C(n!)^{1-\frac{1}{q-1}+\epsilon}.$$

In Section 3 we prove that

$$g(n, q) \geq \begin{cases} (q(q-1))^{n/2} & \text{if } n \text{ is even} \\ (q-1)\left((q-1)^2 + \frac{q-2}{2}\right)^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

As mentioned before, the idea of the proof of Theorem 2.1 is quite similar to that of [4] and is done by induction on n. We finish this section with the outline of that idea. The base case is validated by choosing C large enough. Constants K, L, M, α, that depend on ϵ and q but not on n, are chosen sequentially. Based on $\mathcal{F}$ that does not contain (n, q)—Δ-system we construct a "relatively large" subfamily $\mathcal{G} \subset \mathcal{F}$ i.e. such that

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(i) $|\mathcal{G}|/|\mathcal{F}|$ is bounded from below by $1/(Ln^{\alpha})^{(q-2)n^{\alpha}}$,

and which additionally satisfies

(ii) the ratio $|\{Y : |Y \cap X| \geq n^{\alpha}\}|/|\mathcal{G}|$ is at least $1 - 1/L$ for all $X \in \mathcal{G}$.

Note that the way we construct $\mathcal{G}$ is the main difference between [4] and our paper. Due to (i), an upper bound on $|\mathcal{G}|$ yields also an upper bound on $|\mathcal{F}|$, so the rest of the proof is focused on bounding $|\mathcal{G}|$. Inside $\mathcal{G}$ there exists a family $\mathcal{A} = \{A_1, \ldots, A_K\}$ with $|A_i \cap A_j| \leq Mn^{\alpha}$. Family $\mathcal{A}$ is used to decompose $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ as follows:

$$\mathcal{G}_0 = \{X \in \mathcal{G} : \exists j \in [K] \text{ such that} |X \cap A_j| < n^{\alpha}\},$$
$$\mathcal{G}_1 = \{X \in \mathcal{G} : \forall j \in [K]\; n^{\alpha} \leq |X \cap A_j| < Mn^{\alpha}\},$$
$$\mathcal{G}_2 = \{X \in \mathcal{G} : \exists j \in [K] \text{ such that} |X \cap A_j| \geq Mn^{\alpha}\}.$$

Bounding of $|\mathcal{G}_0|$ will be based on use of inequality (ii) and bound on $\mathcal{G}_2$ will follow from Lemma 2.4 (in this lemma $k = n^{\alpha}$, $\delta = \frac{1}{q-1} - \epsilon$ and ϵ_1 is a small constant).

To estimate $|\mathcal{G}_1|$ set $B = \bigcup_{i,j} A_i \cap A_j$ and $N(b, a_1, \ldots, a_K)$ to be the number of $X \in \mathcal{G}_1$ such that $|X \cap B| = b$ and $|X \cap (A_i \setminus B)| = a_i$. Hence

$$|\mathcal{G}_1| \leq \sum_{(b,a_1,\ldots,a_K)\in D} N(b, a_1, \ldots, a_K),$$

where $\mathcal{D}$ is a set of all possible choices of $(b, a_1, \ldots, a_K)$. To estimate each of $N(b, a_1, \ldots, a_K)$ consider $\tilde{B}, \tilde{A}_1, \ldots \tilde{A}_K$, subsets of $B, A_1/B, \ldots, A_k/B$ of size $b, a_1, \ldots, a_K$ respectively, and $X = \cup_{i=1}^{K} \tilde{A}_i \cup \tilde{B}$. An important observation that allows to incorporate induction hypothesis is that for $t = (n-b-a_1-\cdots-a_K)$ a family $\mathcal{G}_1(X) = \{F \backslash X : F \in \mathcal{G}_1, X \subseteq F\}$ is t-uniform and does not contain a weak (t, q)-Δ-system. So $N(b, a_1, \ldots, a_K)$ is bounded by estimating the number of possible choices of $B', A'_1, \ldots, A'_K$ and using induction hypothesis for each such choice. Finally, $|\mathcal{G}_1|$ is bounded by estimating a size of $\mathcal{D}$ and using derived bounds for $N(b, a_1, \ldots, a_K)$.

2. Main theorem

Theorem 2.1. *Let $q \geq 3$ be an integer and $\epsilon \in (0, 1/(q-1)]$. There exists $C = C(\epsilon, q)$ such that if $\mathcal{F}$ is an n-uniform family of sets that does not contain weak Δ system of size q, then*

$$|\mathcal{F}| \leq C\,(n!)^{1-\frac{1}{q-1}+\epsilon}.$$

Proof. For simplicity of the argument set $\delta = \frac{1}{q-1} - \epsilon$ and note that $\delta \in [0, 1/(q-1))$. We work with δ instead of ϵ for the rest of the proof. Given q and δ, we will now fix some other constants that are useful to prove Theorem 2.1. We choose an integer K such that

$$K > \frac{(q-2)(1-\delta)}{1-(q-1)\delta}. \tag{1}$$

Since $\delta \geq 0$ we observe that $K \geq q-1$. In order to select the other constants, we now make the following Claim:

Claim 2.2. *Inequality (1) implies that for all $M > (q-2)/K$,*

$$\delta + \frac{1-\delta}{K} > \frac{M-1+\delta(KM-M+1)}{KM-q+2} \tag{2}$$

We postpone the proof of Claim 2.2 until we have finished selecting constants. Next we choose M large enough ($M > \max\{q-2, e^2\}$) so that inequality (3) holds:

$$\delta + \frac{1-\delta}{K} > \frac{M\delta}{M-q+2}. \tag{3}$$

Consequently, we choose α which is smaller than the left hand side of inequalities (2) and (3) and which is simultaneously larger than the right hand sides of both. Such α satisfies

$$\alpha > \frac{M\delta}{M-q+2},$$
$$\alpha < \delta + \frac{1-\delta}{K},$$
$$\alpha > \frac{M-1+\delta(KM-M+1)}{KM-q+2}.$$

Note that $\alpha > \delta$, and that the three inequalities above imply that

$$\epsilon_1 = \alpha(M - q + 2) - M\delta > 0, \tag{4}$$

$$\epsilon_2 = -\alpha K + \delta K + 1 - \delta > 0, \tag{5}$$

$$\epsilon_3 = \alpha(KM - q + 2) - M + 1 - \delta(KM - M + 1) > 0. \tag{6}$$

Finally, we choose $L > 2K$ (say $L = 3K$).

We now include the proof of Claim 2.2:

Proof of Claim 2.2.

$$\begin{aligned}
\delta + \frac{1-\delta}{K} &> \frac{M - 1 + \delta(KM - M + 1)}{KM - q + 2} && \Leftrightarrow \\
\delta + \frac{1-\delta}{K} &> \delta + \frac{M - 1 + \delta(q - 2 - M + 1)}{KM - q + 2} && \Leftrightarrow \\
\frac{1-\delta}{K} &> \frac{M - 1 + \delta(q - 1 - M)}{KM - q + 2} && \Leftrightarrow \\
(1-\delta)(KM - q + 2) &> KM - K + \delta(q - 1 - M)K && \Leftrightarrow \\
-q + 2 + (-\delta)(KM - q + 2) &> -K + \delta(q - 1 - M)K && \Leftrightarrow \\
-q + 2 - \delta(-q + 2) &> -K + \delta(q - 1)K && \Leftrightarrow \\
(-q + 2)(1 - \delta) &> -((q - 1)\delta - 1)K && \Leftrightarrow \\
K &> \frac{(q-2)(1-\delta)}{(q-1)\delta - 1}. \quad \square
\end{aligned}$$

The proof of Theorem 2.1 is by induction on n. Let n_0 be such that for all $n \geq n_0$

$$n^{1-\alpha} > 3M, \quad n^{\epsilon_1} > 2L^{2(q-2)}, \quad n^{\epsilon_2} > (2M)^K, \quad \frac{n^\alpha}{\ln n} \geq \frac{K+1}{MK^2}, \quad n^{\epsilon_3} > 4e^{7MK^2}. \tag{7}$$

Take

$$C = \max_{n \leq n_0}\{\frac{|\mathcal{F}|}{(n!)^{1-\delta}} \;:\; \mathcal{F} \text{ is } n\text{-uniform}, \; \mathcal{F} \text{ has no weak } (n, q)\text{-}\Delta\text{-system}\}.$$

Observe that such a choice of C establishes the base case of induction.

Now assume that for all integers smaller than n we have proved Theorem 2.1. Let $\mathcal{F}$ be an n-uniform system that does not contain a weak $(n, q) - \Delta$-system. The following claim will be used extensively throughout the proof.

Claim 2.3. *Let X be a set of size x and $\mathcal{F}' = \{F \in \mathcal{F} : X \subset F\}$, then*

$$|\mathcal{F}'| \leq C\,((n - x)!)^{1-\delta}.$$

Proof of Claim. Note that $\tilde{\mathcal{F}} = \{F \backslash X : F \in \mathcal{F}'\}$ is $(n - x)$-uniform and does not contain a weak $(n - x, q)$-Δ-system. The claim then follows from the induction hypothesis. $\square$

Set

$$k = n^\alpha \quad \text{and} \quad \ell = Lk.$$

The following lemma is also needed.

Lemma 2.4. *For every $A \in \mathcal{F}$*

$$|\{X \in \mathcal{F} : \; |A \cap X| \geq Mk\}| \leq n^{-\epsilon_1 n^\alpha} \frac{C\,(n!)^{1-\delta}}{k^{(q-2)k}}.$$

Proof of Lemma. Note that $\epsilon_1 > 0$ by (4). Observe that in view of Claim 2.3 and induction hypothesis, we have

$$|\{X \in \mathcal{F} : \; |X \cap A| \geq Mk\}| \leq \sum_{i \geq Mk} \binom{n}{i} C\,((n - i)!)^{1-\delta}.$$

In order to further estimate this upper bound we set

$$\phi(i) = \binom{n}{i}\,((n - i)!)^{1-\delta}.$$

Then, for all $i \geq Mk$

$$\frac{\phi(i+1)}{\phi(i)} = \frac{\binom{n}{i+1}\left((n-i-1)!\right)^{1-\delta}}{\binom{n}{i}\left((n-i)!\right)^{1-\delta}} = \frac{n-i}{i+1}\frac{1}{(n-i)^{1-\delta}}$$
$$= \frac{(n-i)^{\delta}}{i+1} \leq \frac{n^{\delta}}{Mn^{\alpha}+1} < \frac{1}{M},$$

with last inequality following from $\alpha > \delta$. Consequently,

$$\sum_{i \geq Mk} \phi(i) \leq \frac{M}{M-1}\phi(Mk) \leq 2\phi(Mk).$$

Hence,

$$|\{X \in \mathcal{F} : \ |X \cap A| \geq Mk\}| \leq 2C\phi(Mn^{\alpha}).$$

It is now sufficient to show that

$$2C\phi(Mn^{\alpha}) \leq n^{-\epsilon_1 n^{\alpha}} \frac{C\,(n!)^{1-\delta}}{k^{(q-2)k}},$$

or equivalently that

$$P := 2n^{\epsilon_1 n^{\alpha}} k^{(q-2)k} \phi(Mn^{\alpha})/(n!)^{1-\delta} < 1.$$

Indeed,

$$P = 2\, n^{\epsilon_1 n^{\alpha}} n^{\alpha(q-2)n^{\alpha}} \binom{n}{Mn^{\alpha}} (n-Mn^{\alpha})!^{1-\delta}/(n!)^{1-\delta}$$
$$\leq 2\, n^{\epsilon_1 n^{\alpha}} n^{\alpha(q-2)n^{\alpha}} \frac{(n^{Mn^{\alpha}})^{\delta}}{Mn^{\alpha}!}$$
$$\leq 2\, n^{\epsilon_1 n^{\alpha}} n^{\alpha(q-2)n^{\alpha}} n^{Mn^{\alpha}\delta} \frac{e^{Mn^{\alpha}}}{(Mn^{\alpha})^{Mn^{\alpha}}}$$
$$\leq 2\, \left(\frac{e}{M}\right)^{Mn^{\alpha}} n^{(\epsilon_1 + \alpha(q-2) + \delta M - \alpha M)n^{\alpha}}.$$

By (4) the exponent of n above is equal to zero. Also by the choice of M we have that $M > e^2$ so we infer that

$$P \leq 2\left(\frac{e}{M}\right)^{Mn^{\alpha}} < 1. \quad \square$$

Our next goal is to obtain $\mathcal{G}$ with following two properties:

$$\text{for all } A \in \mathcal{G},\ |\{X \in \mathcal{G} : |A \cap X| < k\}| < \frac{|\mathcal{G}|}{L}, \tag{8}$$

$$|\mathcal{G}| \geq \frac{|\mathcal{F}|}{\ell^{(q-2)k}}. \tag{9}$$

In order to obtain a family $\mathcal{G}$ we construct families

$$\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_m = \mathcal{G}$$

with $m \leq (q-2)k$ and auxiliary multisets

$$\emptyset = I_0 \subset I_1 \subset \cdots I_m \subseteq \{0, 1, \ldots, k-1\}^{q-2}.$$

Take $\mathcal{F}_0 = \mathcal{F}$ and $I_0 = \emptyset$. For $j > 0$, if there is $F_j \in \mathcal{F}_{j-1}$ and $x_j \in \{0, 1, \ldots, k-1\}$ such that

$$|\{X \in \mathcal{F}_{j-1} : |X \cap F_j| = x_j\}| \geq \frac{\mathcal{F}_{j-1}}{\ell}, \tag{10}$$

then set $I_j = I_{j-1} \cup x_j$ and

$$\mathcal{F}_j = \{X \in \mathcal{F}_{j-1} : |X \cap F_j| = x_j\}. \tag{11}$$

Otherwise stop the process, and set $m = j - 1$.

This process ends after at most $(q-2)k$ steps as a consequence of the following claim.

Claim 2.5. *No value of $x \in \{0, \ldots, k-1\}$ can appear in some I_j more than $(q-2)$ times.*

Proof of Claim. The proof is by contradiction. Assume x appears in an I_j at least $(q-1)$ times. Let $x = x_{i_1} = \cdots = x_{i_{q-1}}$ for some $1 \leq i_1 < \cdots < i_{q-1} \leq j$, and let $F_{i_1} \in \mathcal{F}_{i_1-1}, \ldots, F_{i_{q-1}} \in \mathcal{F}_{i_{q-1}-1}$ be the sets used in the process of creating $\mathcal{F}_{i_1}, \ldots, \mathcal{F}_{i_{q-1}}$. It follows from (11) that sets $F_{i_1}, \ldots, F_{i_{q-1}}$ are different and from (10) that all constructed sets $\mathcal{F}_i$ are nonempty. Hence for an $X \in \mathcal{F}_{i_{q-1}}$ we infer that $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{q-1}}, X\}$ is a weak Δ-system of size q with a common intersections of size x. This contradicts the assumption of Theorem 2.1 □

Hence, the process described above stops after $m \leq (q-2)k$ steps.

Recall that $\mathcal{G} = \mathcal{F}_m$. We now want to show, that $\mathcal{G}$ satisfies (8) and (9). Note that (9) is straightforward, since in each step of the construction we take $1/l$ portion of the previous family, i.e. $|\mathcal{F}_j| \geq \frac{|\mathcal{F}_{j-1}|}{l}$ holds. We prove that (8) holds by contradiction. Assuming that (8) fails, it means that there exist $F_{m+1} \in \mathcal{G} = \mathcal{F}_m$ such that

$$|\{X \in \mathcal{G} : |X \cap F_{m+1}| < k\}| > \frac{|\mathcal{F}_m|}{L}.$$

Hence there exists $x_{m+1} < k$, such that

$$|\{X \in \mathcal{G} : |X \cap F_{m+1}| = x_{m+1}\}| > \frac{|\mathcal{F}_m|}{\ell}.$$

This, however, means that we should have continued in the construction, contradicting our assumption that $\mathcal{G} = \mathcal{F}_m$.

Construction of $\mathcal{A}$. Next we will construct a set $\mathcal{A} = \{A_1, \ldots, A_K\} \subseteq \mathcal{G}$ with the property that

$$|A_i \cap A_j| \leq Mk \text{ for all } 1 \leq i < j \leq K. \tag{12}$$

(This set will be subsequently used to bound $|\mathcal{F}|$.) We choose $A_1 \in \mathcal{G}$ arbitrarily and then pick $A_2, \ldots, A_K$ consequently. Assume that for some A_j with $j < K$ we cannot find $A_{j+1} \in \mathcal{G}$ that will satisfy (12), then any $X \in \mathcal{G}$ intersects some A_i, $i = 1, \ldots, j$ in at least Mk elements. Hence, by (9), Lemma 2.4 and (7)

$$\begin{aligned}|\mathcal{F}| &\leq \ell^{(q-2)k}|\mathcal{G}| \\ &\leq L^{(q-2)k}k^{(q-2)k}K \max_{i\in[j]} |\{X \in \mathcal{G} : |X \cap A_i| \geq Mk\}| \\ &\leq L^{2(q-2)n^{\alpha}} n^{-\epsilon_1 n^{\alpha}} C(n!)^{1-\delta} < C(n!)^{1-\delta}.\end{aligned}$$

In other words, if we could not find K sets $A_1, \ldots, A_K$ satisfying (12), then

$$|\mathcal{F}| < C(n!)^{1-\delta},$$

establishing the inductive step.

Consequently, we may assume that the process of selecting members of the family $\{A_1, A_2, \ldots, A_K\}$ does not stop before A_K is chosen. Having constructed $\mathcal{A} = \{A_1, A_2, \ldots, A_K\}$ we will set

$$\begin{aligned}\mathcal{G}_0 &= \{X \in \mathcal{G} : \exists j \in [K] \text{ such that} |X \cap A_j| < k\}, \\ \mathcal{G}_1 &= \{X \in \mathcal{G} : \forall j \in [K]\ k \leq |X \cap A_j| < Mk\}, \\ \mathcal{G}_2 &= \{X \in \mathcal{G} : \exists j \in [K] \text{ such that} |X \cap A_j| \geq Mk\}.\end{aligned}$$

Note that $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. By (8) and choice of L, we infer

$$|\mathcal{G}_0| \leq K\frac{|\mathcal{G}|}{L} < \frac{|\mathcal{G}|}{2}.$$

Consequently, $|\mathcal{G}| \leq |\mathcal{G}_0| + |\mathcal{G}_1| + |\mathcal{G}_2|$ implies $|\mathcal{G}| < 2|\mathcal{G}_1| + 2|\mathcal{G}_2|$, so by (9)

$$|\mathcal{F}| \leq \ell^{(q-2)k}|\mathcal{G}| < 2\ell^{(q-2)k}(|\mathcal{G}_1| + |\mathcal{G}_2|). \tag{13}$$

We will first bound $2\ell^{(q-2)k}|\mathcal{G}_2|$. By Lemma 2.4 and recalling that $L > 2K$,

$$\begin{aligned}2\ell^{(q-2)k}|\mathcal{G}_2| &\leq 2L^{(q-2)k}k^{(q-2)k}K \max_{j\in[K]} |\{X \in \mathcal{G} : |X \cap A_j| \geq Mk\}| \\ &\leq L^{2(q-2)n^{\alpha}} n^{-\epsilon_1 n^{\alpha}} C(n!)^{1-\delta} \\ &\leq \left(L^{2(q-2)} n^{-\epsilon_1}\right)^{n^{\alpha}} C(n!)^{1-\delta}.\end{aligned}$$

So, by (4) we get

$$2\ell^{(q-2)k}|\mathcal{G}_2| < \frac{1}{2}C(n!)^{1-\delta}. \tag{14}$$

Next, we bound $2\ell^{(q-2)k}|\mathcal{G}_1|$. Set $B = \bigcup_{1\leq i<j\leq K}(A_i \cap A_j)$. Recalling (12), we have $|B| < K^2Mk$. Set $A_i' = A_i/B$ for $i \in [K]$. Note that

$$\mathcal{G}_1 = \bigcup_{\substack{b,a_1,\ldots,a_K\in\mathbb{Z}_{\geq 0} \\ b\leq|B|,\ k\leq b+a_i,\ a_i<Mk}} \{X \in \mathcal{G} : \forall j\ |X \cap A_j'| = a_j, |X \cap B| = b\}. \tag{15}$$

Next, we estimate the number of $X \in \mathcal{G}$ that have a prescribed intersection size with each of $B, A_1, \ldots, A_K$. To this end set

$$N(b, a_1, \ldots, a_K) = |\{X \in \mathcal{G} : \forall j\ |X \cap A'_j| = a_j,\ |X \cap B| = b\}|.$$

Let $D = \{(b, a_1, \ldots, a_K) : b < MK^2k,\ k \le b + a_i,\ a_i < Mk\}$. Note that (15) implies

$$|\mathcal{G}_1| \le \sum_{(b,a_1,\ldots,a_K)\in D} N(b, a_1, \ldots, a_K). \tag{16}$$

In order to further estimate $|\mathcal{G}_1|$ we provide a bound for $N(b, a_1, \ldots, a_K)$ over the set D. Consider $\tilde{B}, \tilde{A}_1, \ldots \tilde{A}_K$, subsets of $B, A'_1, \ldots, A'_k$ of size $b, a_1, \ldots, a_K$ respectively, and set $X = \cup_{i=1}^K \tilde{A}_i \cup \tilde{B}$. Define a family $\mathcal{G}_1(X) = \{F \setminus X : F \in \mathcal{G}_1, X \subseteq F\}$, which, for $t = n - b - \sum_{i=1}^K a_i$, is t-uniform and does not contain a weak (t, q)-Δ-system (due to Claim 2.3). Applying induction hypothesis to $\mathcal{G}_1(X)$, we have

$$N(b, a_1, \ldots, a_K) \le \binom{|B|}{b} \prod_{i=1}^{K} \binom{|A_i|}{a_i} C\,[(n - b - a_1 - \cdots - a_K)!]^{1-\delta},$$

which implies

$$N(b, a_1, \ldots, a_K) \le 2^{MK^2k} \prod_{i=1}^{K} \binom{n}{a_i} C\,[(n - b - a_1 - \cdots - a_K)!]^{1-\delta}. \tag{17}$$

In order to further bound $N(b, a_1, \ldots, a_K)$, recall that $(b, a_1, \ldots, a_K) \in D$. We set $a = \lceil \sum_1^K a_i/K \rceil$. Since $a_i < Mk$ for every $j = 1, \ldots, K$ we have that $a \le Mk$. We note that $\sum a_i \ge Ka - K$. In view of definition of $\mathcal{G}_1$ we have $b + a_i \ge k$ and hence $b + a \ge k$, or equivalently $-b \le a - k$. Hence, we infer that

$$-b - a_1 - \cdots - a_K \le -(K-1)a + K - k.$$

Using the log-concavity of binomial coefficients we infer that

$$\begin{aligned} N(b, a_1, \ldots, a_K) &\le 2^{MK^2k} \prod_{i=1}^{K} \binom{n}{a_i} C\,[(n - (K-1)a + K - k)!]^{1-\delta} \\ &\le 2^{MK^2k} \binom{n}{a}^K C\,[(n - (K-1)a + K - k)!]^{1-\delta}. \end{aligned}$$

In order to further bound $N(b, a_1, \ldots, a_K)$ we set

$$\psi(a) = \binom{n}{a}^K C\,[(n - (K-1)a + K - k)!]^{1-\delta} \tag{18}$$

and verify the following

Claim 2.6.

$$\max_{a \le Mk} \psi(a) = \psi(Mk).$$

Proof of Claim. Indeed, observe that for $a \le Mk$

$$\begin{aligned} \frac{\psi(a)}{\psi(a-1)} &= \frac{\binom{n}{a}^K C\,[(n - (K-1)a + K - k)!]^{1-\delta}}{\binom{n}{a-1}^K C\,[(n - (K-1)(a-1) + K - k)!]^{1-\delta}} \\ &\ge \left(\frac{n - a + 1}{a}\right)^K \left(\frac{1}{n^{K-1}}\right)^{1-\delta}. \end{aligned}$$

Since $a \le Mk$ and because of (4) we have

$$\frac{n - a + 1}{a} \ge \frac{n}{a} - 1 \ge \frac{n}{Mk} - 1 \ge \frac{n}{2Mk},$$

so we get

$$\begin{aligned} \frac{\psi(a)}{\psi(a-1)} &\ge \left(\frac{n}{2Mk}\right)^K \left(\frac{1}{n^{K-1}}\right)^{1-\delta} \\ &= \frac{1}{(2M)^K} n^{(1-\alpha)K - (K-1)(1-\delta)} \end{aligned}$$

$$= \frac{1}{(2M)^K} n^{-\alpha K + K\delta + (1-\delta)}$$
$$\text{(by (5))} = \frac{1}{(2M)^K} n^{\epsilon_2}.$$

By (7) the last expression is greater than 1. Consequently $\phi(a+1) \geq \phi(a)$, which establishes Claim 2.6. $\square$

Claim 2.6 implies that for $(b, a_1, \ldots, a_K) \in D$

$$N(b, a_1, \ldots, a_K) \leq 2^{MK^2k} \psi(Mk),$$

and by (18)

$$N(b, a_1, \ldots, a_K) \leq 2^{MK^2k} \binom{n}{Mk}^K C\left[(n-(K-1)Mk+K-k)!\right]^{1-\delta}.$$

This, together with (16) implies the following bound on $|\mathcal{G}_1|$.

$$\begin{aligned} |\mathcal{G}_1| &\leq MK^2k\,(Mk)^K\, 2^{MK^2k} \binom{n}{Mk}^K C\left[(n-(K-1)Mk+K-k)!\right]^{1-\delta} \\ &\leq (MK^2k)(Mk)^K\, e^{MK^2k} \binom{n}{Mk}^K C\left[(n-(K-1)Mk+K-k)!\right]^{1-\delta} \\ &\leq e^{2MK^2k} \binom{n}{Mk}^K C\left[(n-(K-1)Mk+K-k)!\right]^{1-\delta}, \end{aligned} \tag{19}$$

with the last inequality due to the fact that we have $xK^2 \cdot x^K \leq e^{K^2x}$ for all $x \geq 1$ and $K \geq 1$. Finally, we establish the following

Claim 2.7.

$$2\ell^{(q-2)k}|\mathcal{G}_1| < \frac{1}{2}C(n!)^{1-\delta}.$$

Proof of Claim. Indeed, by (19)

$$2\ell^{(q-2)k}|\mathcal{G}_1| \leq 2\ell^{(q-2)k} e^{2MK^2k} \binom{n}{Mk}^K C\left[(n-(K-1)Mk+K-k)!\right]^{1-\delta}.$$

To establish Claim 2.7 it is sufficient to show

$$4\ell^{(q-2)k} e^{2MK^2k} \binom{n}{Mk}^K < \left(\frac{n!}{(n-(K-1)Mk+K-k)!}\right)^{1-\delta}. \tag{20}$$

To estimate the left side of (20) we recall that $L = 3K$, $K > (q-2)$, $M > e^2$ and note that $L^{q-2} = (3K)^{q-2} < e^{3K(q-2)} \leq e^{3K^2M}$.

$$\begin{aligned} 4\ell^{(q-2)k} e^{2MK^2k} \binom{n}{Mk}^K &\leq 4L^{(q-2)k} k^{(q-2)k} e^{2MK^2k} \left(\frac{ne}{Mk}\right)^{MkK} \\ &\leq 4e^{5K^2Mk} k^{(q-2)k} n^{(1-\alpha)KMk} \left(\frac{e}{M}\right)^{MKk} \\ &\leq 4e^{5K^2Mk} n^{(\alpha(q-2)+(1-\alpha)KM)k}. \end{aligned} \tag{21}$$

On the other hand, the right side of (20) can be bounded using $n\left(\frac{n}{e}\right)^n > n! > \left(\frac{n}{e}\right)^n$:

$$\begin{aligned} \left(\frac{n!}{(n-(K-1)Mk+K-k)!}\right)^{1-\delta} &\geq \left(\left(\frac{n}{e}\right)^n \frac{1}{n}\left(\frac{e}{n}\right)^{n-(K-1)Mk+k-k}\right)^{1-\delta} \\ &\geq \frac{1}{n}\left(\left(\frac{n}{e}\right)^{(K-1)Mk-K+k}\right)^{1-\delta} \\ &\geq \frac{1}{n^{K+1}} \frac{1}{e^{MK^2k}} n^{((K-1)Mk+k)(1-\delta)}. \end{aligned}$$

Moreover, $n^{K+1} \leq e^{(K+1)\ln n} \leq e^{MK^2k}$ by (7), so

$$\left(\frac{n!}{(n-(K-1)Mk+K-k)!}\right)^{1-\delta} \geq \frac{1}{e^{2MK^2k}} n^{((K-1)Mk+k)(1-\delta)}. \tag{22}$$

Recall, that our goal is to prove (20), and according to (21) and (22) it is sufficient to show that

$$4e^{5K^2Mk}n^{(\alpha(q-2)+(1-\alpha)KM)k} \le \frac{1}{e^{2MK^2k}}n^{((K-1)Mk+k)(1-\delta)},$$

or by rearranging that

$$4e^{7K^2Mk} \le n^{((KM-M+1)(1-\delta)-(\alpha(q-2-KM)+KM))k}. \tag{23}$$

Note that the coefficient of the exponent of the right side after rearranging terms is equal to

$$KM - M + 1 + (KM - M + 1)(-\delta) - \alpha(q - 2 + KM) - KM =$$
$$\alpha(KM - q + 2) - M + 1 - \delta(KM - M + 1) = \epsilon_3.$$

Hence, (23) is equivalent to $4e^{7K^2Mk} < n^{\epsilon_3 k}$ which is true by (7). □

Finally, recall that by (13)

$$|\mathcal{F}| < 2\ell^{(q-2)k}|\mathcal{G}_1| + 2\ell^{(q-2)k}|\mathcal{G}_2|.$$

By (14) and Claim 2.7 both terms on the right side are smaller than $\frac{1}{2}C(n!)^{1-\delta}$, which implies

$$|\mathcal{F}| < C(n!)^{1-\delta}.$$

Therefore, the induction step holds, and this finishes the proof of Theorem 2.1. □

3. Lower bound

In this section, we will show that estimates on $f(n, q)$ imply lower bounds on $g(n, q)$.
Trivially, $f(1, q) = q - 1$ and in [3], Abbott, Hanson, and Sauer settled exactly the values of $f(2, q)$:

Theorem 3.1. *For all $q \ge 0$, we have:*

$$f(2, q) = \begin{cases} q(q-1) & \text{if } n \text{ is even} \\ (q-1)^2 + \frac{q-2}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, in [2] Abbott and Hanson showed the following:

Theorem 3.2 (*Abbott, Hanson, 1977*)**.** *For all $r, s, q \ge 0$, we have:*

$$g(r + s, q) \ge g(r, q)g(s, q).$$

In [5], Deza showed that the only *large* weak-Δ-systems are also Δ-systems. More precisely:

Theorem 3.3 (*Deza, 1974*)**.** *For all $n > 0$ and $q > n^2 - n + 1$, if $\mathcal{F}$ is a weak (n, q)-Δ-system, then $\mathcal{F}$ is an (n, q)-Δ-system. In particular, we have:*

$$f(n, q) = g(n, q).$$

In [6], Erdős, Milner, and Rado showed that $g(n, q) \ge (q - 1)^n$ for all $n, q > 0$. Here we remark that the result of Theorem 3.3 shows that for $q > n^2 - n + 1$, the lower bounds for $f(n, q)$ are also lower bounds on $g(n, q)$. We observe now that the inequality in Theorem 3.2 along with the results of Theorem 3.3 can be used to prove lower bounds on $g(n, q)$ whenever $q \le n^2 - n + 1$.

Theorem 3.4. *Fix $n > 0$ and let $3 < q \le n^2 - n + 1$. Then we have:*

$$g(n, q) \ge \begin{cases} (q(q-1))^{n/2} & \text{if } n \text{ is even} \\ (q-1)\left((q-1)^2 + \frac{q-2}{2}\right)^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Claim. For ease of notation, suppose n is even, and set $n = 2t$. By iterated application of the inequality in Theorem 3.2 and observing that $q > 2^2 - 2 + 1$, we have:

$$\begin{aligned} g(n, q) &= g(\underbrace{2 + 2 + \cdots + 2}_{t}, q) \\ &\ge \underbrace{g(2, q)g(2, q)\ldots g(2, q)}_{t} \\ &= \underbrace{f(2, q)f(2, q)\ldots f(2, q)}_{t} \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{(q(q-1))\,(q(q-1))\ldots(q(q-1))}_{t} \\
&= (q(q-1))^{t} \\
&= (q(q-1))^{n/2}.
\end{aligned}
$$

as desired. The case when n is odd resolves similarly by writing $n = 2t + 1$ and applying $f(1, q) = (q - 1)$. □

Abbott and Exoo [1] obtained bounds better than in Theorem 3.4 for $q = 4$, $q = 5$ and odd n. Namely they showed that for odd n

$$g(n, 4) \geq 31(10)^{(n-3)/2} \quad \text{and} \quad g(n, 5) \geq 79(20)^{(n-3)/2}.$$

Their bounds are based on $g(3, 4) \geq 31$, $g(3, 5) \geq 79$ and $g(2, 5) \geq 20$.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] H.L. Abbott, G. Exoo, On set systems not containing delta systems, Graphs Combin. 8 (1992) 1–9.
[2] H.L. Abbott, D. Hanson, On finite Δ-systems. II, Discrete Math. 17 (2) (1977) 121–126.
[3] H.L. Abbott, D. Hanson, N. Sauer, Intersection theorems for systems of sets, J. Combin. Theory Ser. A 12 (1972) 381–389.
[4] M. Axenovich, D. Fon-Der-Flaass, A. Kostochka, On set systems without weak 3-Δ-subsystems, in: 14th British Combinatorial Conference (Keele, 1993), Discrete Math. 138 (1–3) (1995) 57–62.
[5] M. Deza, Solution d'un problème de Erdős-Lovász, J. Combin. Theory Ser. B 16 (1974) 166–167.
[6] P. Erdős, E.C. Milner, R. Rado, Intersection theorems for systems of sets. III, in: Collection of Articles Dedicated to the Memory of Hanna Neumann, IX, J. Austral. Math. Soc. 18 (1974) 22–40.
[7] A.V. Kostochka, Extremal problems on Δ-systems, in: Numbers, Information and Complexity, Bielefeld, 1998,, Kluwer Acad. Publ., Boston, MA, 2000, pp. 143–150.