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The complex-time Segal–Bargmann transform [☆,☆☆]

Bruce K. Driver [a], Brian C. Hall [b], Todd Kemp [a,*]

[a] Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, United States of America
[b] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, United States of America

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ABSTRACT

We introduce a new form of the Segal–Bargmann transform for a connected Lie group K of compact type. We show that the heat kernel $(\rho_t(x))_{t>0,x\in K}$ has a space-time analytic continuation to a holomorphic function

$$(\rho_{\mathbb{C}}(\tau,z))_{\mathrm{Re}\,\tau>0,z\in K_{\mathbb{C}}},$$

where $K_{\mathbb{C}}$ is the complexification of K. The new transform is defined by the integral

$$(B_\tau f)(z)=\int_K \rho_{\mathbb{C}}(\tau,zk^{-1})f(k)\,dk,\quad z\in K_{\mathbb{C}}.$$

If $s>0$ and $\tau\in\mathbb{D}(s,s)$ (the disk of radius s centered at s), this integral defines a holomorphic function on $K_{\mathbb{C}}$ for each $f\in L^2(K,\rho_s)$. We construct a heat kernel density $\mu_{s,\tau}$ on $K_{\mathbb{C}}$ such that, for all s,τ as above, $B_{s,\tau}:=B_\tau|_{L^2(K,\rho_s)}$ is an isometric isomorphism from $L^2(K,\rho_s)$ onto the space of holomorphic functions in $L^2(K_{\mathbb{C}},\mu_{s,\tau})$. When $\tau=t=s$, the transform $B_{t,t}$ coincides with the one introduced by the second author for compact groups and extended by the first author to groups of compact type. When $\tau=t\in(0,2s)$, the

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* Corresponding author.
E-mail addresses: bdriver@ucsd.edu (B.K. Driver), bhall@nd.edu (B.C. Hall), tkemp@ucsd.edu (T. Kemp).

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transform $B_{s,t}$ coincides with the one introduced by the first two authors.

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1. Introduction

1.1. The classical Segal–Bargmann transform

This paper concerns a generalization of the Segal–Bargmann transform over compact-type Lie groups, to allow the time parameter of the transform to be complex. We begin by briefly discussing the history of the transform. For $t > 0$ and $d \in \mathbb{N}$, let ρ_t denote the variance-t Gaussian density on $\mathbb{R}^d$:

$$\rho_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right).$$

This is the *heat kernel* on $\mathbb{R}^d$: the solution u of the heat equation $\partial_t u = \frac{1}{2}\Delta u$ with (sufficiently integrable) initial condition f is given in terms of ρ_t by

$$u(t,x) = (\rho_t * f)(x) = \int_{\mathbb{R}^d} \rho_t(x-y) f(y)\, dy. \tag{1.1}$$

The function ρ_t admits an explicit entire analytic continuation to $\mathbb{C}^d$, which we call $(\rho_t)_{\mathbb{C}}$: it is simply the function

$$(\rho_t)_{\mathbb{C}}(z) = (2\pi t)^{-d/2} \exp\left(-\frac{z \cdot z}{2t}\right),$$

where $z \cdot z = \sum_{j=1}^{d} z_j^2$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and of sufficiently slow growth, then the integral

$$(B_t f)(z) := \int_{\mathbb{R}^d} (\rho_t)_{\mathbb{C}}(z - y) f(y)\, dy \tag{1.2}$$

converges and defines an entire holomorphic function on $\mathbb{C}^d$.

The map $f \mapsto B_t f$ is equivalent to the **Segal–Bargmann transform**, invented and explored by the eponymous authors of [1,2,47–49]. Note that neither Segal nor Bargmann explicitly connected the transform to the heat kernel, nor did they write the transform precisely as in (1.2). Nevertheless, their transforms can easily be rewritten in the form (1.2) by simple changes of variable; cf. [24].

We consider also the heat kernel on $\mathbb{C}^d \cong \mathbb{R}^{2d}$ (with time-parameter rescaled by a factor of 2), which we refer to as μ_t:

$$\mu_t(z) = (\pi t)^{-d} \exp(-|z|^2/t).$$

(Note that the real, positive function μ_t on $\mathbb{C}^d$ is not the same as the holomorphic function $(\rho_t)_{\mathbb{C}}$.) The main theorem about this transform is that B_t is an isometric isomorphism from $L^2(\mathbb{R}^d, \rho_t)$ onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ — the space of holomorphic functions in $L^2(\mathbb{C}^d, \mu_t)$. (For precisely this form of the theorem, see Theorem 6.3 in [24].) For more information about the classical Segal–Bargmann transform, see, for example, [24,29].

1.2. The Segal–Bargmann transform for Lie groups of compact type

In [22], the second author introduced an analog of the Segal–Bargmann transform on an arbitrary compact Lie group. Then, in [10], the first author extended the results of [22] to a Lie group K of compact type (Section 2), a class that includes both compact groups and $\mathbb{R}^d$. The idea of [22] and [10] is the same as in the $\mathbb{R}^d$ case: the heat kernel ρ_t on K has an entire analytic continuation $(\rho_t)_{\mathbb{C}}$ to the *complexification* $K_{\mathbb{C}}$ of K. The transform B_t is defined by the group convolution formula generalizing (1.2):

$$(B_t f)(z) = \int_{K} (\rho_t)_{\mathbb{C}}(zk^{-1}) f(k)\, dk. \tag{1.3}$$

The theorem is that B_t is an isometric isomorphism from $L^2(K, \rho_t)$ onto the holomorphic space $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$, where μ_t is the (time-rescaled) heat kernel on $K_{\mathbb{C}}$. If $K = \mathbb{R}^d$, then B_t is precisely the classical Segal–Bargmann transform of Section 1.1.

Later, in [14,23], the authors made a further generalization related to the time parameter t. One can use a different time $s \neq t$ to measure the functions f in the domain, while still using the analytically continued heat kernel at time t to define the transform, as in (1.3). The resulting map,

$$B_{s,t}\colon L^2(K,\rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,t})$$

is still an isometric isomorphism for an appropriate two-parameter heat kernel density $\mu_{s,t}$, provided $0 < t < 2s$. Note that the formula for the transform $B_{s,t}$ does not depend on s; this parameter only indicates the inner product to be used on the domain and range spaces. In the special case that $K = \mathbb{R}^d$, the two-parameter heat kernel density $\mu_{s,t}$ in the range is a Gaussian measure with different variances in the real and imaginary directions. (Take $u = 0$ in (1.15) below.)

Remark 1.1. For a complex manifold M, let $\mathcal{H}(M)$ denote the space of holomorphic functions on M. If μ is a measure on M having a strictly positive, continuous density with respect to the Lebesgue measure in each holomorphic local coordinate system, it is not hard to show that $\mathcal{H}L^2(M,\mu) := \mathcal{H}(M) \cap L^2(M,\mu)$ is a closed subspace of $L^2(M,\mu)$ and is therefore a Hilbert space. Furthermore, the *pointwise evaluation map* $F \mapsto F(z)$ is continuous for each $z \in M$, and the norm of this functional is locally bounded as a function of z. (See, for example, Theorem 3.2 and Corollary 3.3 in [11] or Theorem 2.2 in [24].)

1.3. The complex-time Segal–Bargmann transform

The topic of the present paper is a new generalization that modifies the transform $B_{s,t}$ as well; in particular, we show that the time parameter t can also be extended into the complex plane, and there is still an isomorphism between real and holomorphic L^2 spaces of associated heat kernel measures. This generalization is natural and, in a certain sense, a completion of Segal–Bargmann transform theory, as explained below. (See Theorem 3.2. See also Section 1.5 for further motivation for this generalization.)

Let K be a connected compact-type Lie group with Lie algebra $\mathfrak{k}$, and fix an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ (Section 2). This induces a bi-invariant Riemannian metric on K, and an associated Laplace operator Δ_K, which is bi-invariant, elliptic, and essentially self-adjoint in $L^2(K)$. There is an associated **heat kernel**, $\rho_t \in C^\infty(K,(0,\infty))$, satisfying

$$\left(e^{\Delta_K/2} f\right)(x) = \int\limits_K \rho_t(xy^{-1}) f(y)\,dk \qquad \text{for all } f \in L^2(K) \text{ and } t > 0. \tag{1.4}$$

Our first theorem is that the heat kernel can be complexified in both space and time.

Theorem 1.2. *Let K be a connected Lie group of compact type, with a given* $\mathrm{Ad}(K)$*-invariant inner product on its Lie algebra $\mathfrak{k}$, and let $(\rho_t)_{t>0}$ be the associated heat kernel. Let*

$\mathbb{C}_+$ denote the right half-plane $\{\tau = t + iu\colon t > 0, u \in \mathbb{R}\}$. There is a unique holomorphic function

$$\rho_{\mathbb{C}}\colon \mathbb{C}_+ \times K_{\mathbb{C}} \to \mathbb{C}$$

such that $\rho_{\mathbb{C}}(t, x) = \rho_t(x)$ for all $t > 0$ and $x \in K \subset K_{\mathbb{C}}$.

Theorem 1.2 is proved in Section 5, as part of Theorem 5.13.

Following the pattern described above for the $\mathbb{R}^d$ case, we make the following definition.

Notation 1.3 *(Complex-time Segal–Bargmann transform).* For $\tau \in \mathbb{C}_+$ and $z \in K_{\mathbb{C}}$, define

$$(B_\tau f)(z) := \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk \qquad \text{for } z \in K_{\mathbb{C}} \tag{1.5}$$

for all measurable functions $f\colon K \to \mathbb{C}$ satisfying

$$\int_K \left|\rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\right| \, dk < \infty. \tag{1.6}$$

Further let $\mathcal{D}(B_\tau)$ denote the vector space of measurable functions $f\colon K \to \mathbb{C}$ such that (1.6) holds for all $z \in K_{\mathbb{C}}$ and such that $B_\tau f \in \mathcal{H}(K_{\mathbb{C}})$.

As defined, $\mathcal{D}(B_\tau)$ is a linear subspace of the measurable $\mathbb{C}$-valued functions on K, and $B_\tau\colon \mathcal{D}(B_\tau) \to \mathcal{H}(K_{\mathbb{C}})$ is a linear map. The main theorem of this paper (Theorem 1.6) identifies L^2-Hilbert subspaces of $\mathcal{D}(B_\tau)$ and $\mathcal{H}(K_{\mathbb{C}})$ which are unitarily equivalent to one another under the action of B_τ. To describe the relevant subspaces of $\mathcal{H}(K_{\mathbb{C}})$ we need a little more notation.

As on K, we fix once and for all a right Haar measure λ on $K_{\mathbb{C}}$, and typically write dz for $\lambda(dz)$ and $L^2(K_{\mathbb{C}})$ for $L^2(K_{\mathbb{C}}, \lambda)$. When $s > 0$, let $\mathbb{D}(s, s) \subset \mathbb{C}_+$ denote the open disk of radius s centered at s.

Definition 1.4. Let $s > 0$ and $\tau = t + iu \in \mathbb{C}$. The (s, τ)**-Laplacian** $\Delta_{s,\tau}$ on $K_{\mathbb{C}}$ is the left-invariant differential operator

$$\Delta_{s,\tau} = \sum_{j=1}^{\dim \mathfrak{k}} \left[\left(s - \frac{t}{2}\right) \tilde{X}_j^2 + \frac{t}{2} \tilde{Y}_j^2 - u\, \tilde{X}_j \tilde{Y}_j \right] \tag{1.7}$$

where $\{X_j\}_{j=1}^{\dim \mathfrak{k}}$ is any orthonormal basis of $\mathfrak{k}$, and $Y_j = JX_j$ where J is the operation of multiplication by i on $\mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}(K_{\mathbb{C}})$. Here, for any $Z \in \mathfrak{k}_{\mathbb{C}}$, we let $\tilde{Z}$ denote the left-invariant vector field on $K_{\mathbb{C}}$ whose value at the identity is Z.

Remark 1.5. Given $s > 0$ and $\tau = t + iu \in \mathbb{C}_+$, from (1.7), it is not difficult to show that the operator $\Delta_{s,\tau}$ is elliptic if and only if

$$\alpha(s,\tau) := \det \begin{bmatrix} s - t/2 & -u/2 \\ -u/2 & t/2 \end{bmatrix} = \frac{1}{4}(2st - t^2 - u^2) > 0. \tag{1.8}$$

This can be written equivalently as

$$2s > t + u^2/t \tag{1.9}$$

or, more succinctly, as $\tau \in \mathbb{D}(s,s)$ (the disk of radius s, centered at s). Further notice that $\mathbb{D}(s,s) \uparrow \mathbb{C}_+$ as $s \uparrow \infty$.

If the conditions in Remark 1.5 hold, then there exists a heat kernel density $\mu_{s,\tau} \in C^\infty(K_{\mathbb{C}}, (0,\infty))$ such that

$$\left(e^{\Delta_{s,\tau}/2} f\right)(w) = \int_{K_{\mathbb{C}}} \mu_{s,\tau}(w^{-1}z)\, f(z)\, dz \quad \text{for all } \; f \in L^2(K_{\mathbb{C}}).$$

We are now prepared to state the main theorem of this paper.

Theorem 1.6 *(Complex-time Segal–Bargmann transform). Let K be a connected, compact-type Lie group. For $s > 0$ and $\tau \in \mathbb{D}(s,s)$, $L^2(K,\rho_s) \subset \mathcal{D}(B_\tau)$; i.e., $B_\tau f$ is holomorphic on $K_{\mathbb{C}}$ for each $f \in L^2(K,\rho_s)$. The image of B_τ on this domain is $B_\tau\left(L^2(K,\rho_s)\right) = \mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$. Moreover,*

$$B_{s,\tau} := B_\tau|_{L^2(K,\rho_s)}$$

is a unitary isomorphism from $L^2(K,\rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$.

Theorem 1.6 is proved in Section 5. The $\tau = t \in \mathbb{R}$ case of Theorem 1.9 was established in [14, Theorem 5.3]. (See also [23, Theorem 2.1].)

Remark 1.7. The condition in [14,23] for the two-parameter Segal–Bargmann transform $B_{s,t}$ to be a well-defined unitary map was $t > 0$ and $s > t/2$, or equivalently $t \in (0,2s)$. It is therefore natural that, in complexifying t to τ, the optimal condition is that $\tau \in \mathbb{D}(s,s)$, the most symmetric region whose intersection with $\mathbb{R}$ is the interval $(0,2s)$.

In the case that the group K is compact, there is a limiting $s \to \infty$ variant (Theorem 1.9) of Theorem 1.6. To state this variant, as in [22], we first introduce a one parameter family of "K-averaged heat kernels."

Definition 1.8. Let K be a compact Lie group. For $t > 0$, define the **K-averaged heat kernel** ν_t on $K_{\mathbb{C}}$ by

$$\nu_t(z) = \int_K \mu_{t,t}(zk)\,dk \quad \text{for all } \; z \in K_{\mathbb{C}}$$

where dk denotes the Haar probability measure on K.

In fact, one can replace $\mu_{t,t}$ by $\mu_{s,\tau}$ for any $\tau \in \mathbb{D}(s,s)$ in the above integral, and the resulting K-averaged density ν_t is the same: it only depends on $t = \operatorname{Re}\tau$; see Proposition 5.15.

Theorem 1.9 *(Large-s limit). Let K be a compact connected Lie group. For all $s > 0$ and $\tau = t + iu \in \mathbb{D}(s,s)$, we have $L^2(K) = L^2(K, \rho_s)$ and $L^2(K_{\mathbb{C}}, \mu_{s,\tau}) = L^2(K_{\mathbb{C}}, \nu_t)$ (equalities as sets). Furthermore, for all $f \in L^2(K)$ and all $F \in L^2(K_{\mathbb{C}}, \nu_t)$, we have*

$$\lim_{s\to\infty} \|f\|_{L^2(K,\rho_s)} = \|f\|_{L^2(K)}$$
$$\lim_{s\to\infty} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} = \|F\|_{L^2(K_{\mathbb{C}},\nu_t)}\,.$$

It follows that $B_{\infty,\tau} := B_\tau|_{L^2(K)}$ is a unitary isomorphism from $L^2(K)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

This theorem is proved in Section 5.3 below.

Remark 1.10. The unitarity of the map $B_{\infty,\tau}$ was previously established in [19, Prop. 2.3]. Indeed, this unitarity result follows easily from the unitarity of the "C-version" Segal–Bargmann transform in [22] and the unitarity of the operator $e^{iu\Delta/2} : L^2(K) \to L^2(K)$. The significance of Theorem 1.9 is that the unitary map $B_{\infty,\tau}$ is, in a strong sense, the $s \to \infty$ limit of the unitary map $B_{s,\tau}$.

1.4. An outline of the proof

We now give a heuristic proof of the isometricity portion of Theorem 1.6, in the Euclidean case $K = \mathbb{R}^d$, for motivation. The argument is a generalization of the method used in the appendix of [25]. By (1.5), if we restrict to real time $\tau = t > 0$ and look at the transform $(B_{s,t}f)(x)$ at a point $x \in \mathbb{R}^d$, we simply have $(B_{s,t}f)(x) = \int_{\mathbb{R}^d} \rho_t(x-y)f(y)\,dy$; in other words, restricted to real time and K, $B_{s,t}f$ is just the heat operator applied to f, $B_{s,t}f = e^{\frac{t}{2}\Delta}f$ where Δ is the standard Laplacian on $\mathbb{R}^d$. Therefore, in general the transform can be described as "apply the heat operator, then analytically continue in space and time". But if the function f itself already possesses a holomorphic extension $f_{\mathbb{C}}$ to all of $\mathbb{C}^d$ (e.g., if f is a polynomial), then at least informally we should have

$$B_{s,\tau} f = e^{\frac{\tau}{2}\Delta} f_{\mathbb{C}},$$

where now Δ (the sum of squares of the $\mathbb{R}^d$-derivatives) is acting on functions on $\mathbb{C}^d$.

Let $F = B_{s,\tau} f$; we need to compute $|F|^2 = F\bar{F}$. Since $f_{\mathbb{C}}$ is holomorphic, we have $\frac{\partial}{\partial x_j} f_{\mathbb{C}} = \frac{\partial}{\partial z_j} f_{\mathbb{C}}$, and so $\Delta f_{\mathbb{C}} = \sum_{j=1}^d \frac{\partial^2}{\partial z_j^2} f_{\mathbb{C}} =: \partial^2 f_{\mathbb{C}}$; similarly $\Delta \bar{f}_{\mathbb{C}} = \sum_{j=1}^d \frac{\partial^2}{\partial \bar{z}_j^2} \bar{f}_{\mathbb{C}} =: \bar{\partial}^2 \bar{f}_{\mathbb{C}}$. Again, since $f_{\mathbb{C}}$ is holomorphic and $\bar{f}_{\mathbb{C}}$ is antiholomorphic, $\partial^2 \bar{f}_{\mathbb{C}} = 0 = \bar{\partial}^2 f_{\mathbb{C}}$; so we have

$$(F\bar{F}) = (e^{\frac{\tau}{2}\partial^2} f_{\mathbb{C}})(e^{\frac{\bar{\tau}}{2}\bar{\partial}^2} \bar{f}_{\mathbb{C}}) = e^{(\frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2)} f_{\mathbb{C}} \bar{f}_{\mathbb{C}}. \tag{1.10}$$

Now, we measure f in $L^2(\mathbb{R}^d, \rho_s)$; setting $x = 0$ in the (additive form of) (1.4) defining the heat operator, we can compute

$$\|f\|^2_{L^2(\mathbb{R}^d,\rho_s)} = \int_{\mathbb{R}^d} \rho_s(y)|f(y)|^2\, dy = \left(e^{\frac{s}{2}\Delta}|f|^2\right)(0) = \left(e^{\frac{s}{2}\Delta}|f_{\mathbb{C}}|^2\right)(0). \tag{1.11}$$

Similarly, we measure F in $L^2(\mathbb{C}^d, \mu_{s,\tau})$, meaning

$$\|F\|^2_{L^2(\mathbb{C}^d,\mu_{s,\tau})} = \left(e^{\frac{1}{2}\Delta_{s,\tau}}|F|^2\right)(0). \tag{1.12}$$

Combining (1.10) and (1.12), and commuting partial derivatives to combine the exponentials, we therefore have

$$\|B_{s,\tau} f\|^2_{L^2(\mathbb{C}^d,\mu_{s,\tau})} = \left(e^{\frac{1}{2}\Delta_{s,\tau} + \frac{\tau}{2}\partial^2 + \frac{\bar{\tau}}{2}\bar{\partial}^2}|f_{\mathbb{C}}|^2\right)(0). \tag{1.13}$$

Comparing (1.11) with (1.13), we see that to prove the isometry in Theorem 1.6, it suffices to have

$$s\Delta = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2.$$

Expressing the operators ∂^2 and $\bar{\partial}^2$ in terms of real partial derivatives, we can then solve for $\Delta_{s,\tau}$; this is how (1.7) arises. In the present Euclidean setting, we have

$$\Delta_{s,\tau} = \sum_{j=1}^d \left[\left(s - \frac{t}{2}\right)\frac{\partial^2}{\partial x_j^2} + \frac{t}{2}\frac{\partial^2}{\partial y_j^2} - u\frac{\partial^2}{\partial x_j \partial y_j}\right]. \tag{1.14}$$

As in Remark 1.5, it is easily verified that $\Delta_{s,\tau}$ is elliptic precisely when $\tau \in \mathbb{D}(s,s)$. Moreover, by a standard Fourier transform argument, one shows that $e^{\frac{1}{2}\bar{\Delta}_{s,\tau}} f = f * \mu_{s,\tau}$ where

$$\mu_{s,\tau}(z) = (2\pi\sqrt{\alpha})^{-d}\left(-\frac{t/2}{2\alpha}|x|^2 - \frac{s-t/2}{2\alpha}|y|^2 - \frac{u}{2\alpha}x\cdot y\right), \tag{1.15}$$

where $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d = \mathbb{C}^d$, and $\alpha := \alpha(s, \tau)$ as in Eq. (1.8).

When $u = 0$, the density $\mu_{s,\tau}$ becomes a product of a Gaussian in the x variable and a Gaussian in the y variable, but with typically unequal variances. If $u = 0$ and $s = t$, the formula for $\mu_{s,\tau}$ reduces to

$$\mu_{t,t}(z) = (\pi t)^{-d} e^{-|z|^2/t},$$

which is the density for the standard Segal–Bargmann space over $\mathbb{C}^d$.

For a general Lie group K of compact type, we replace the partial derivatives in the preceding argument with left-invariant vector fields. The heuristic argument then goes through unchanged, *except* that we must remember that left-invariant vector fields do not, in general, commute. Thus, we must also verify that the particular operators involved in the calculation do, in fact, commute, allowing us to combine the exponents as above. For this, we need to use an inner product on the Lie algebra of K that is Ad-invariant; this is the reason for the assumption that K be of compact type.

Most of this paper is devoted to making the above argument rigorous. The key is to introduce a dense subspace (consisting of matrix entries; see Section 4.2) of the domain Hilbert space on which integration against the heat kernel can be computed rigorously by a power series in the relevant Laplacian. This argument can be found in Section 5.

The operator $\Delta_{s,\tau}$ was the starting point for the current investigation. It is the Laplacian for a left-invariant Riemannian metric on $K_{\mathbb{C}}$ for which the corresponding inner product on the Lie algebra is invariant under the adjoint action of K. While the Lie algebra of the complexified Lie group $K_{\mathbb{C}}$ does not possess a fully Ad-invariant inner product (unless K is commutative), it does possess many inner products that are invariant under the adjoint action of K. These are the most natural from the perspective of diffusion processes, particularly in high dimension (cf. [36]). In fact, there is a natural three (real) parameter family of $\mathrm{Ad}(K)$-invariant inner products on $\mathrm{Lie}(K_{\mathbb{C}})$ (see (3.8) for the relation to the Segal–Bargmann transform parameters s and $\tau = t + iu$). In the case that K is simple, this is a complete characterization of all such invariant inner products; this is the statement of Theorem 3.2 below. It was this fact that led the authors backward to discover the complex-time Segal–Bargmann transform, which is therefore a natural completion of the versions of the transform previously introduced by Segal, Bargmann, and the first two authors of the present paper.

1.5. Motivation

In the case $K = \mathrm{U}(n)$ and $K_{\mathbb{C}} = \mathrm{GL}(n; \mathbb{C})$, we may give one motivation for the complex-time Segal–Bargmann transform as follows: choosing matrices at random from $\mathrm{GL}(n; \mathbb{C})$ with distribution $\mu_{s,\tau}$ is an interesting random matrix model and the transform is a tool for studying that model. We now elaborate on this statement, starting by thinking of the heat kernel measure on $\mathrm{GL}(n; \mathbb{C})$ as giving a random matrix model. The heat kernel measure $\mu_{s,\tau}(g)\, dg$ on $\mathrm{GL}(n; \mathbb{C})$ is just the group analog of a Gaussian measure on its Lie algebra, the space of all $n \times n$ matrices. In the two-parameter case (i.e., with $\tau = t \in \mathbb{R}$), the Gaussian measure is a scaled version of the Ginibre ensemble. In the large-n limit, the eigenvalues of a random matrix chosen according to this Gaussian measure are uniformly distributed on an ellipse with axes lying along the real and imaginary axes. One can certainly add a third parameter to the Gaussian measure, but one does not really get anything new by doing so: The resulting random matrix is just the two-parameter case multiplied by a fixed complex number. Thus, the limiting eigenvalue distribution is uniform over an ellipse in $\mathbb{C}$—but an ellipse that has been rotated so its axes no longer lie along the real and imaginary axes.

For the heat kernel measure on $\mathrm{GL}(n;\mathbb{C})$, the problem is much richer. In the two-parameter case (i.e., with $\tau = t \in \mathbb{R}$), the second and third authors have used [30] the large-n Segal–Bargmann transform developed in [3,15,32] to identify the domain $\Sigma_{s,t}$ in $\mathbb{C}$ on which the "Brown measure" of the limiting object is supported. We expect that this is the domain into which the eigenvalues of random matrices chosen from $\mathrm{GL}(n;\mathbb{C})$ and distributed as $\mu_{s,t}$ cluster in the $n \to \infty$ limit. In the case $s = t$, the authors then computed the Brown measure—not just its support—in [9].

Already in the two-parameter case, the domains $\Sigma_{s,t}$ display an interesting structure, changing from simply connected to doubly connected at $s = 4$. If we then allow τ to be complex, the associated random matrix model is no longer just a complex number times the two-parameter case. Thus the domain into which the eigenvalues cluster will not be simply a rotation of $\Sigma_{s,t}$. Rather, simulations indicated that the domain gets twisted around in a much more complicated (and therefore interesting) way. The large-n limit of the complex-time Segal–Bargmann transform has already been developed in [6]. We expect that this limiting transform will be an important tool in studying the large-n eigenvalue distribution of $\mu_{s,\tau}$, in the same way that the large-n limit of the two-parameter transform was used in [30].

In the rest of this subsection, we provide motivation for considering the complex-time transform for a fixed, finite-dimensional Lie group of compact type. The Segal–Bargmann transform $(B_\tau f)(z)$ is computed by integration of f against the function

$$\chi_\tau^z(x) := \rho_{\mathbb{C}}(\tau, x^{-1}z). \tag{1.16}$$

These functions may be thought of as "coherent states" on K. In the case $K = \mathbb{R}^1$, coherent states are often defined as minimum uncertainty states, namely those giving equality in the classic Heisenberg uncertainty principle. There is, however, a stronger form of the uncertainty principle, due to Schrödinger [46], which says that

$$(\Delta_\chi X)^2 (\Delta_\chi P)^2 \geq \frac{\hbar^2}{4} + |\mathrm{Cov}_\chi(X,P)|^2, \tag{1.17}$$

where $\Delta_\chi X$ is the uncertainty of the observable X in state χ, and

$$\mathrm{Cov}_\chi(X,P) := \langle (XP+PX)/2 \rangle_\chi - \langle X \rangle_\chi \langle P \rangle_\chi$$

is the quantum covariance. (The classic Heisenberg principle omits the covariance term on the right-hand side of (1.17).)

States that give equality in (1.17) are Gaussian wave packets, but where the quadratic term in the exponent can be complex, as follows:

$$\chi(x) = C\exp\{iax^2 - b(x-c)^2 + idx\} \tag{1.18}$$

with $a, b, c, d \in \mathbb{R}$ and $b > 0$. This class of states is actually more natural than the usual ones with $a = 0$, because the collection of states of the form (1.18) is invariant under the metaplectic representation; that is, the natural (projective) unitary action of the group of symplectic linear transformations of $\mathbb{R}^2$.

If we specialize the states in (1.16) to the $\mathbb{R}^d$ case, we find that they are Gaussian wave packets, and that if $\operatorname{Im}\tau \neq 0$ then the quadratic part of the exponent is complex. We see, then, that allowing the time-parameter in the Segal–Bargmann transform to be complex amounts to considering a larger and more natural family of coherent states. In the $\mathbb{R}^d$ case, unitary Segal–Bargmann-type transforms using general Gaussian wave packets were constructed by J. Sjöstrand [50] and L. Hörmander [34], with applications to semiclassical analysis. In these works, it is essential to allow the quadratic part of the exponent to be complex, in order to achieve invariance of the theory under symplectic linear transformations.

In the $s \to \infty$ transform $B_{\infty,t+iu}$ of Theorem 1.9, the domain Hilbert space is $L^2(K)$. Since $e^{iu\Delta/2}$ is a unitary map of $L^2(K)$ to itself, in this case it is possible to derive the complex-time transform from the real one $B_{\infty,t}$ (denoted as the C-version of the transform C_t in [22]) by the decomposition $e^{\frac{1}{2}(t+iu)\Delta} = e^{t\Delta/2}e^{iu\Delta/2}$. This possibility has been exploited, for example, in the papers [18,19] of C. Florentino, J. Mourão, and J. Nunes on the quantization of nonabelian theta functions on $\mathrm{SL}(n,\mathbb{C}) = \mathrm{SU}(n)_{\mathbb{C}}$. The authors show that these functions arise as the image of certain distributions on $\mathrm{SU}(n)$ under the heat operator, evaluated at a complex time, and use the Segal–Bargmann transform in the complexification process. These papers, then, show the utility of introducing a complex time-parameter into the (C-version) Segal–Bargmann transform. The present paper extends this complex time-parameter to the two-parameter transform.

Meanwhile, the Segal–Bargmann transform for K is related to the study of complex structures on the cotangent bundle $T^*(K)$. There is a natural one-parameter family of "adapted complex structures" on $T^*(K)$ arising from a general construction of Guillemen–Stenzel [20,21] and Lempert–Szőke [40,51]. Motivated by ideas of Thiemann [52], the second author and W. Kirwin in [31] showed that these structures arise from the "imaginary-time geodesic flow" on $T^*(K)$. The Segal–Bargmann transform can then be understood [16,17,26] as a quantum counterpart of the construction in [31].

As observed in [41], the adapted complex structures on $T^*(K)$ extend to a two-parameter family, by including both a real and an imaginary part to the time-parameter in the geodesic flow in [31]. The corresponding quantum construction has been done in [42] and can be thought of as adding a complex parameter to the C-version of the Segal–

Bargmann transform for K. (Compare work of Kirwin and Wu [38] in the $\mathbb{R}^d$ case.) The present paper then extends the complex-time transform to its most natural range, in which the domain Hilbert space is taken to be L^2 of K with respect to a heat kernel measure.

Finally, we mention the paper [28], which shows that certain operators on $L^2(K_\mathbb{C}, \nu_t)$ of the form $C_t A C_t^{-1}$, where A is an operator on $L^2(K)$, can be represented as Toeplitz operators. Here C_t, for $t \in \mathbb{R}$, is the C-version Segal–Bargmann transform, which coincides with the limiting transform $B_{\infty,t}$ in Theorem 1.9. Using the results of the present paper, a similar analysis can be performed for operators of the form $C_{t+iu} A C_{t+iu}^{-1}$, where C_{t+iu} is the limiting transform $B_{\infty,t+iu}$ in Theorem 1.9.

2. Compact-type Lie groups and their complexifications

We now introduce the class of Lie groups in which we are interested: those of compact type and their complexifications.

Definition 2.1. A connected Lie group K with Lie algebra $\mathfrak{k}$ is said to be of **compact type** if there exists an Ad-K-invariant inner product on $\mathfrak{k}$; that is, an inner product such that

$$\langle \mathrm{Ad}_x X, \mathrm{Ad}_x Y \rangle = \langle X, Y \rangle, \quad \forall x \in K,\ X, Y \in \mathfrak{k}.$$

Clearly a commutative group is of compact type. Furthermore, every compact group is of compact type, since any inner product on its Lie algebra can be made Ad-invariant by averaging over the adjoint action. A key result says that products of these two examples account for all Lie groups of compact type.

Proposition 2.2 *([43], Lemma 7.5). If K is a compact-type Lie group with a specified* Ad*-invariant inner product, then K is isometrically isomorphic to a direct product group: $K \cong K_0 \times \mathbb{R}^d$ for some compact Lie group K_0 and some non-negative integer d.*

If G is a connected real Lie group, a **complexification** of G is a pair $(G_\mathbb{C}, \iota)$ consisting of a complex Lie group $G_\mathbb{C}$ and a smooth homomorphism $\iota : G \to G_\mathbb{C}$ such that the following universal property holds: for any complex Lie group H and any smooth homomorphism $\Phi : G \to H$, there is a unique holomorphic homomorphism $\Phi_\mathbb{C} : G_\mathbb{C} \to H$ such that

$$\Phi_\mathbb{C} \circ \iota = \Phi.$$

Suppose $K = K_0 \times \mathbb{R}^d$ is a connected Lie group of compact type. It is known ([33, XVII Theorem 5.1] or [4, Theorem 4.1, Propositions 8.4 and 8.6]) that the Lie algebra of the complexification of K_0 is the complexification of its Lie algebra $\mathfrak{k}_0$—that is, $\mathrm{Lie}((K_0)_\mathbb{C}) = \mathfrak{k} + i\mathfrak{k}$—and that ι maps K_0 injectively into its complexification. Meanwhile, the complexification of $\mathbb{R}^d$ is $\mathbb{C}^d$, with ι being the obvious inclusion map. Thus, the Lie algebra of $K_{\mathbb{C}}$ is the complexification of its Lie algebra, and $\iota : K \to K_{\mathbb{C}}$ is injective. From now on, we always identify K with the subgroup $\iota(K)$ of $K_{\mathbb{C}}$.

Example 2.3. The compact Lie groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, and $\mathrm{U}(n)$ have the following complexifications:

$$\mathrm{SO}(n)_{\mathbb{C}} = \mathrm{SO}(n;\mathbb{C}), \quad \mathrm{SU}(n)_{\mathbb{C}} = \mathrm{SL}(n;\mathbb{C}), \quad \mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}(n;\mathbb{C}).$$

We recall that a Lie group is called **unimodular** if every left Haar measure is also right invariant.

Proposition 2.4. *If K is a connected Lie group of compact type, both K and $K_{\mathbb{C}}$ are unimodular.*

Proof. The existence of an invariant inner product guarantees that the Lie algebra $\mathfrak{k}$ of K decomposes as the Lie algebra direct sum of a commutative algebra and a semisimple algebra [27, Proposition 7.6]. It then follows from Corollary 8.31 in [39] that K is unimodular. Meanwhile, the complexification of each simple summand in $\mathfrak{k}$ is also simple as a real Lie algebra [27, Theorem 7.32 and Exercise 12]. Thus, $\mathfrak{k}_{\mathbb{C}}$, when viewed as a real Lie algebra, is also the direct sum of a commutative algebra and a semisimple algebra and is therefore unimodular. $\square$

Let K be a connected Lie group of compact type and let $K_{\mathbb{C}}$ its complexification. It is convenient, for reasons that will be apparent shortly, to write the "multiplication by i" map on $\mathfrak{k}_{\mathbb{C}}$ as $J : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$. (Thus, $J^2 = -I$.) Since $\mathfrak{k}_{\mathbb{C}}$ is a complex Lie algebra, the bracket on $\mathfrak{k}_{\mathbb{C}}$ is bilinear over $\mathbb{C}$, and in particular

$$[JX, Y] = J[X, Y] \tag{2.1}$$

for all $X, Y \in \mathfrak{k}_{\mathbb{C}}$.

For any $X \in \mathfrak{k}_{\mathbb{C}}$, the left-invariant vector field $\tilde{X}$ is given by

$$(\tilde{X} f)(g) = \left.\frac{d}{dt} f(ge^{tX})\right|_{t=0} \tag{2.2}$$

for any smooth real- or complex-valued function f on $K_{\mathbb{C}}$. We may now appreciate the utility of the notion J for the "multiplication by i" map on $\mathfrak{k}_{\mathbb{C}}$: in general, $\widetilde{JX} f$ is not the same as $i\tilde{X} f$ (for example, if f is real valued). On the other hand, a complex-valued function f on $K_{\mathbb{C}}$ is holomorphic if and only if the differential of f at each point $g \in K_{\mathbb{C}}$ is a complex-linear map from $T_g(K_{\mathbb{C}})$ to $\mathbb{C}$. Thus, if f is holomorphic, then for all $X \in \mathfrak{g}$ and $g \in K_{\mathbb{C}}$, we have

$$\widetilde{JX} f(g) = i\tilde{X} f(g) \quad (f \text{ holomorphic}). \tag{2.3}$$

3. Invariant metrics on K and $K_{\mathbb{C}}$

3.1. Invariant metrics

If G is a Lie group with Lie algebra $\mathfrak{g}$ and $K \subseteq G$ is a compact Lie subgroup, one can produce an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$ by averaging any inner product over the adjoint representation of K, as above. This raises the question: how many $\mathrm{Ad}(K)$-invariant inner products does $\mathfrak{g}$ possess? We now answer this question in the case that K is simple (and compact type), and $G = K_{\mathbb{C}}$ is the complexification of K.

Fix a compact-type Lie group K, and an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on its Lie algebra $\mathfrak{k}$. Let $K_{\mathbb{C}}$ denote the complexification of K (cf. Section 2); in particular $\mathfrak{k}_{\mathbb{C}} \equiv \mathrm{Lie}(K_{\mathbb{C}}) = \mathfrak{k} \oplus J\mathfrak{k}$. Consider the following three-parameter family of inner products on $K_{\mathbb{C}}$:

$$\langle X_1 + JY_1, X_2 + JY_2\rangle_{a,b,c} := a\langle X_1, X_2\rangle_{\mathfrak{k}} + b\langle Y_1, Y_2\rangle_{\mathfrak{k}} + c(\langle X_1, Y_2\rangle_{\mathfrak{k}} + \langle X_2, Y_1\rangle_{\mathfrak{k}}) \tag{3.1}$$

for $X_1, X_2, Y_1, Y_2 \in \mathfrak{k}$, where $a, b > 0$ and $c^2 < ab$. It is straightforward to verify that the symmetric bilinear forms in (3.1) are real inner products on $\mathfrak{k}_{\mathbb{C}}$ (precisely under the conditions on a, b, c stated below the equation), and are all $\mathrm{Ad}(K)$-invariant. The main theorem of this section is that, in the case that K is simple, this is a complete characterization of all $\mathrm{Ad}(K)$-invariant inner products on $K_{\mathbb{C}}$.

Definition 3.1. A Lie group K is called *simple* if $\dim K \geq 2$, and the Lie algebra $\mathfrak{k}$ of K has no nontrivial ideals.

Theorem 3.2. *Let K be a simple (or 1-dimensional) Lie group of compact type. Then $\mathfrak{k}$ has a unique (up to scale)* Ad*-invariant real inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$. Furthermore,* all *$\mathrm{Ad}(K)$-invariant real inner products on $\mathfrak{k}_{\mathbb{C}}$ have the form* (3.1)*.*

Remark 3.3. For example, $K = \mathrm{SU}(n)$ is simple, with complexification $K_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. Hence (3.1) characterizes all $\mathrm{Ad}(\mathrm{SU}(n))$-invariant inner products on $\mathrm{SL}(n, \mathbb{C})$, where $\langle X, Y\rangle_{\mathfrak{su}(n)} = \mathrm{Tr}(XY^*) = -\mathrm{Tr}(XY)$ is the unique (up to scale) Ad-invariant inner product on $\mathfrak{su}(n)$. In that case, the family can be written explicitly in terms of the trace as

$$\langle A, B\rangle_{a,b,c} = \frac{1}{2}(b + a)\mathrm{Re}\,\mathrm{Tr}(AB^*) + \frac{1}{2}\mathrm{Re}\left[(b - a + 2ic)\mathrm{Tr}(AB)\right]. \tag{3.2}$$

Extending to $\mathrm{U}(n)$ and its complexification $\mathrm{GL}(n, \mathbb{C})$, it is easy to compute that all $\mathrm{Ad}(\mathrm{U}(n))$-invariant inner products on $\mathfrak{gl}(n, \mathbb{C})$ are of the form (3.1) plus one more term, involving the 1-dimensional subspace spanned by the identity matrix; extending (3.2), there is one more term involving $\mathrm{Tr}(A)\mathrm{Tr}(B)$. In [5,37,36], the third author studied the large-n limits of the diffusion processes on $\mathrm{GL}(n, \mathbb{C})$ invariant with respect to the inner products $\langle\ ,\ \rangle_{a,b,0}$. Part of the motivation for the present work was the question of whether those were the largest class of appropriately invariant diffusions; the answer provided by Theorem 3.2 is no.

Remark 3.4. The first statement of Theorem 3.2, that the Ad-invariant inner product on K is unique up to scale when K is simple, is well known; it was proved, for example, in [43, Lemma 7.6].

We will use Schur's lemma as a tool in the proof of Theorem 3.2, but this is complicated by the fact that the inner products in question are *real*. We must therefore be careful about how and when we complexify.

Lemma 3.5. *If K is a simple (real) Lie group with Lie algebra $\mathfrak{k}$, then the (real) adjoint representation of K on $\mathfrak{k}$ is irreducible. Moreover, if K is compact type, then the (complex) adjoint representation of K on $\mathfrak{k}_{\mathbb{C}}$ is also irreducible.*

Proof. If $\mathcal{I} \subseteq \mathfrak{k}$ is an invariant real subspace for $\mathrm{Ad}(K)$, then $\mathrm{Ad}_{e^{tX}}(Y) \in \mathcal{I}$ for all $t \in \mathbb{R}$, $X \in \mathfrak{k}$, and $Y \in \mathcal{I}$. Taking the derivative at $t = 0$ shows that $\mathrm{ad}_X(Y) = [X, Y] \in \mathcal{I}$ for all $X \in \mathfrak{k}$ and $Y \in \mathcal{I}$, which means $\mathcal{I} \subseteq \mathfrak{k}$ is an ideal in $\mathfrak{k}$. Thus $\mathcal{I} \in \{0, \mathfrak{k}\}$, yielding the first statement of the lemma.

Now, [27, Theorem 7.32] states that the simplicity of $\mathfrak{k}$ implies that $\mathfrak{k}_{\mathbb{C}}$ is also simple as a complex Lie algebra. (The statement given there assumes K is compact, but the proof only uses the fact that it is compact type.) So, let $\mathcal{J} \subseteq \mathfrak{k}_{\mathbb{C}}$ be an invariant complex subspace for $\mathrm{Ad}(K)$. The same argument above shows that $[X, W] \in \mathcal{J}$ for all $X \in \mathfrak{k}$ and $W \in \mathcal{J}$. Any $Z \in \mathfrak{k}_{\mathbb{C}}$ has the form $Z = X + JY$ for $X, Y \in \mathfrak{k}$, and by (2.1), we therefore have

$$[Z, W] = [X + JY, W] = [X, W] + J[Y, W] \in \mathcal{J} + J\mathcal{J} = \mathcal{J}, \qquad \forall\, Z \in \mathfrak{k}_{\mathbb{C}}, W \in \mathcal{J}$$

where the final equality follows from the fact that $\mathcal{J}$ is a *complex* subspace. Hence $\mathcal{J}$ is a complex ideal in $\mathfrak{k}_{\mathbb{C}}$, and therefore $\mathcal{J} \in \{0, \mathfrak{k}_{\mathbb{C}}\}$. This concludes the proof of the second statement. $\square$

We now prove the algebraic result that constitutes most of the proof of Theorem 3.2.

Proposition 3.6. *Let K be a simple (or 1-dimensional) real compact-type Lie group, and fix an Ad-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on its Lie algebra $\mathfrak{k}$. If $\mathcal{B}\colon \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{R}$ is an $\mathrm{Ad}(K)$-invariant symmetric bilinear form, then $\mathcal{B}$ has the form (3.1) for some $a, b, c \in \mathbb{R}$.*

Proof. The result is straightforward when K is 1-dimensional, so we focus on the case that K is simple. We use the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$ (cf. (3.1)) as a reference; there is then some endomorphism $M\colon \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ such that

$$\mathcal{B}(Z, W) = \langle Z, M(W)\rangle_{1,1,0} \qquad \forall\, Z, W \in \mathfrak{k}_{\mathbb{C}}.$$

The symmetry of $\mathcal{B}$ forces M to be self-adjoint. We identify $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus J\mathfrak{k}$ with $\mathfrak{k} \oplus \mathfrak{k}$. Thus we can decompose the endomorphism M in block diagonal form

$$M = \begin{bmatrix} A & C \\ C^{\top} & B \end{bmatrix} \tag{3.3}$$

where A and B are symmetric matrices.

Since the adjoint representation of K commutes with J, it follows that, under the isomorphism $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{k} \oplus \mathfrak{k}$, Ad_k acts diagonally for all $k \in K$. Using the fact that both the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$ and the bilinear form $\mathcal{B}$ are Ad_k-invariant, it is straightforward to compute that the matrices A, B, C, and $C^{\top}$ all commute with Ad_k for each $k \in K$. The same therefore applies to the complex-linear extensions of these endomorphisms to $\mathfrak{k}_{\mathbb{C}}$. It then follows from Lemma 3.5 and Schur's lemma that there are constants $a, b, c \in \mathbb{C}$ with $A = aI$, $B = bI$, and $C = C^{\top} = cI$. Since each of the endomorphisms preserves the real subspace $\mathfrak{k}$, it follows that $a, b, c \in \mathbb{R}$.

Hence, for $Z = X + JY \in \mathfrak{k}_{\mathbb{C}}$, (3.3) yields $M(Z) = (aX + cY) + J(cX + bY)$. From the definition of the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$, we therefore have

$$\begin{aligned}
\mathcal{B}(X_1 + JY_1, X_2 + JY_2) &= \langle X_1 + JY_1, aX_2 + cY_2 + J(cX_2 + bY_2)\rangle_{1,1,0} \\
&= \langle X_1, aX_2 + cY_2\rangle_{\mathfrak{k}} + \langle Y_1, cX_2 + bY_2\rangle_{\mathfrak{k}} \\
&= a\langle X_1, X_2\rangle_{\mathfrak{k}} + c\langle X_1, Y_2\rangle_{\mathfrak{k}} + c\langle Y_1, X_2\rangle_{\mathfrak{k}} + b\langle Y_1, Y_2\rangle_{\mathfrak{k}} \\
&= \langle X_1 + JY_1, X_2 + JY_2\rangle_{a,b,c}
\end{aligned}$$

concluding the proof. □

The proof of Theorem 3.2 now follows quite easily.

Proof of Theorem 3.2. Let $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ and $\langle\cdot,\cdot\rangle'_{\mathfrak{k}}$ denote two Ad-invariant inner products on K. We may view the second inner product as a symmetric (degenerate) bilinear form on $\mathfrak{k}_{\mathbb{C}}$, which is $\mathrm{Ad}(K)$-invariant. By Proposition 3.6, it follows that $\langle\cdot,\cdot\rangle'_{\mathfrak{k}} = a\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ for some $a \in \mathbb{R}$ (the other terms in (3.1) are 0); the fact that both are inner products forces $a > 0$. This proves the uniqueness, up to scale, of the Ad-invariant inner product on K.

Now, any real inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}_{\mathbb{C}}$ is a symmetric bilinear form on $\mathfrak{k}_{\mathbb{C}}$, and so by $\mathrm{Ad}(K)$-invariance, Proposition 3.6 shows that it has the form (3.1) for some $a, b, c \in \mathbb{R}$. Since it is an inner product, it follows that the matrix M of (3.3) is positive definite, and given its block diagonal form, this is equivalent to $a, b > 0$ and $ab - c^2 > 0$. This concludes the proof. □

3.2. Laplacians

We use the notation $\tilde{X}$ for the left-invariant vector field associated to a Lie algebra element X, as in (2.2). We fix an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on K. Then if $\{X_j\}_{j=1}^{\dim\mathfrak{k}}$ is an orthonormal basis for $\mathfrak{k}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, we define Δ_K to be the operator given by

$$\Delta_K = \sum_{j=1}^{\dim\mathfrak{k}} \tilde{X}_j^2. \tag{3.4}$$

The operator is easily seen to be independent of the choice of orthonormal basis. Since K is unimodular, this operator is the Laplace–Beltrami operator for the left-invariant metric determined by $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ [12, Remark 2.2]. Since $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is $\mathrm{Ad}(K)$-invariant, the metric on K is actually bi-K-invariant and thus Δ_K is bi-K-invariant.

We now fix real numbers a, b, and c with $a, b > 0$ and $c^2 < ab$, as in Section 3.1, and let $\langle\cdot,\cdot\rangle_{a,b,c}$ be the associated $\mathrm{Ad}(K)$-invariant inner product. We then choose an orthonormal basis $\{Z_j\}_{j=1}^{2\dim\mathfrak{k}}$ for $\mathfrak{k}_{\mathbb{C}}$ with respect to this inner product and define the Laplacian $L_{a,b,c}$ by

$$L_{a,b,c} = \sum_{j=1}^{2\dim\mathfrak{k}} Z_j^2, \tag{3.5}$$

similarly to (3.4).

Proposition 3.7. *Let $L_{a,b,c}$ denote the Laplacian in (3.5). Fix any basis $\{X_j\}_{j=1}^d$ of $\mathfrak{k}$ orthonormal with respect to the given $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$, and let $Y_j = JX_j$. Then*

$$L_{a,b,c} = \frac{1}{ab-c^2}\sum_{j=1}^{d}\left[b\tilde{X}_j^2 + a\tilde{Y}_j^2 - 2c\tilde{X}_j\tilde{Y}_j\right]. \tag{3.6}$$

Proof. We use the basis $\{Z_j\}_{j=1}^{2\dim\mathfrak{k}}$ consisting of $X_1, Y_1, \ldots, X_{\dim\mathfrak{k}}, Y_{\dim\mathfrak{k}}$ (in that order). We let $\{q_{lm}\}_{l,m=1}^{2\dim\mathfrak{k}}$ be the associated Gram matrix; that is, the matrix of inner products of these basis elements with respect to the inner product $\langle\cdot,\cdot\rangle_{a,b,c}$. If q^{-1} is the inverse matrix to q, it is an elementary computation to verify that

$$L_{a,b,c} = \sum_{l,m=1}^{2\dim\mathfrak{k}} (q^{-1})_{lm}\tilde{Z}_l\tilde{Z}_m. \tag{3.7}$$

Now, we can compute directly from (3.1) and the orthonormality of $\{X_j\}_{j=1}^d$ that

$$\langle X_i, X_j\rangle_{a,b,c} = a\delta_{ij}, \quad \langle Y_i, Y_j\rangle_{a,b,c} = b\delta_{ij}, \quad \langle X_i, Y_j\rangle_{a,b,c} = \langle Y_i, X_j\rangle_{a,b,c} = c\delta_{ij}.$$

It follows that the matrix q is block diagonal with 2×2 diagonal blocks all equal to the matrix B (below). Thus q^{-1} is also block diagonal with 2×2 diagonal blocks all equal to B^{-1} (below):

$$B = \begin{bmatrix} a & c \\ c & b \end{bmatrix}, \qquad B^{-1} = \frac{1}{ab - c^2} \begin{bmatrix} b & -c \\ -c & a \end{bmatrix}.$$

Combining this with (3.7) yields (3.6). □

To dispense with the cumbersome determinant in the denominator in (3.6), and to match the parametrization relevant to the Segal–Bargmann transform, we make the following change of parametrization:

$$(s, t, u) = \Phi(a, b, c) := \frac{1}{ab - c^2}(a + b, 2a, 2c). \tag{3.8}$$

It is straightforward to verify that Φ is a diffeomorphism

$$\Phi\colon \{(a, b, c)\colon a, b > 0, c^2 < ab\} \to \{(s, t, u)\colon t > 0, u \in \mathbb{R}, 2s > t + u^2/t\}$$

with inverse

$$(a, b, c) = \Phi^{-1}(s, t, u) = \frac{4}{2st - t^2 - u^2}(\tfrac{t}{2}, s - \tfrac{t}{2}, \tfrac{u}{2}) = \tfrac{1}{\alpha}(\tfrac{t}{2}, s - \tfrac{t}{2}, \tfrac{u}{2}) \tag{3.9}$$

referring to the constant α of (1.8), which is positive precisely in range of Φ. From here on, we use the parameters (s, t, u) which leads to the notation used in Definition 1.4 of $\Delta_{s,\tau}$ on $K_{\mathbb{C}}$ in the introduction. In particular, this means that the Laplacian $\Delta_{s,\tau}$ corresponds to the inner product $\langle \cdot, \cdot \rangle_{a,b,c}$ where (a, b, c) are given as in (3.9). The fact that Φ is a bijection shows that there is a one-to-one correspondence between the Laplacians $\Delta_{s,\tau}$ and the inner products $\langle \cdot, \cdot \rangle_{a,b,c}$.

4. Heat kernels and matrix entries

We refer the reader to [45] or [54] for the general theory of heat kernels on Lie groups.

4.1. Heat kernels on K and $K_{\mathbb{C}}$

We now fix a connected Lie group K of compact type, together with an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$. We let Δ_K be the associated Laplacian on K, as in Section 3.2. We then let ρ_t be the associated **heat kernel** on K, i.e., the fundamental solution at the identity to the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_K u.$$

Then the heat operator may be computed as

$$(e^{t\Delta/2}f)(x) = \int_K \rho_t(xy^{-1})f(y)\ dy, \tag{4.1}$$

where dy is the Riemannian volume measure associated to the left-invariant Riemannian metric on K induced by the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k}$.

Remark 4.1. For a general left-invariant metric on K, the right-hand side of (4.1) should have $\rho_t(y^{-1}x)$ rather than $\rho_t(xy^{-1})$. Since, however, our metric is $\mathrm{Ad}(K)$-invariant, the heat kernel ρ_t is a class function, so that $\rho_t(y^{-1}x) = \rho_t(xy^{-1})$. We write $\rho_t(xy^{-1})$ to maintain consistency with [22].

We fix $s > 0$ and $\tau \in \mathbb{C}$ with $\tau \in \mathbb{D}(s,s)$ (the disk of radius s, centered at s). We consider a left-invariant metric on $K_{\mathbb{C}}$ whose value at the identity is one of the inner products considered in Section 3.1. The associated Laplacian, denoted $\Delta_{s,\tau}$, is the one considered in Definition 1.4. We emphasize that, although τ is a complex number, the Laplacian $\Delta_{s,\tau}$ is a real elliptic operator on $K_{\mathbb{C}}$. We then let $\mu_{s,\tau,r}$ be the associated heat kernel, i.e., the fundamental solution at the identity to the heat equation

$$\frac{\partial u}{\partial r} = \frac{1}{2}\Delta_{s,\tau}u,$$

with r being the time-variable in the heat equation. We will mainly be interested in the value of this heat kernel at $r = 1$:

$$\mu_{s,\tau} := \mu_{s,t,1}.$$

That is to say, formally,

$$\mu_{s,\tau} = e^{\Delta_{s,\tau}/2}(\delta),$$

where δ is a δ-function at the identity.

Lemma 4.2 *(Averaging lemma). Assume K is compact. For each s and τ with $\tau \in \mathbb{D}(s,s)$, let $\nu_{s,\tau}$ be the associated K-**averaged heat kernel**, given in Definition 1.8:*

$$\nu_{s,\tau}(g) = \int_K \mu_{s,\tau}(gk)\ dk.$$

Then there exist constants $a_{s,\tau}$ and $b_{s,\tau}$ such that

$$a_{s,\tau}\nu_{s,\tau}(g) \le \mu_{s,\tau}(g) \le b_{s,\tau}\nu_{s,\tau}(g)$$

for all $g \in K_{\mathbb{C}}$. Specifically, for each s and τ, let σ be any positive number such that $\tau \in \mathbb{D}(s-\sigma, s-\sigma)$. Then we may take

$$a_{s,\tau} = \min_{k\in K} \rho_\sigma(k); \quad b_{s,\tau} = \max_{k\in K} \rho_\sigma(k).$$

Proof. We write the operator $\Delta_{s,\tau}$, as defined in (1.7), in the form

$$\Delta_{s,\tau} = \sigma \sum_{j=1}^{\dim \mathfrak{k}} \tilde{X}_j^2 + \Delta_{s-\sigma,\tau}. \tag{4.2}$$

Now, the operator $\Delta_{s-\sigma,\tau}$ is constructed from left-invariant vector fields and is therefore a left-invariant operator on $K_{\mathbb{C}}$. Since the inner product in the construction of $\Delta_{s,\tau}$ is $\mathrm{Ad}(K)$-invariant, $\Delta_{s,\tau}$ is also invariant under the *right* action of K. It follows that $\Delta_{s,\tau}$ commutes with the left-invariant vector field $\tilde{X}$ on $K_{\mathbb{C}}$, with $X \in \mathfrak{k}$, since $\tilde{X}$ is an infinitesimal right translation. We conclude that the two terms on the right-hand side of (4.2) commute. Once this observation has been made, the proof of the averaging lemma from [22, Lemma 11] tells us that

$$\mu_{s,\tau}(g) = \int_K \mu_{s-\sigma,\tau}(gk^{-1})\rho_\sigma(k)\, dk.$$

Since $\rho_s(k)\, dk$ is a probability measure, the integral of $\mu_{s,\tau}$ over each K-orbit is the same as the corresponding integral of $\mu_{s-\sigma,\tau}$. Thus, we obtain

$$\mu_{s,\tau}(g) \le \max_{k\in K} \rho_\sigma(k) \int_K \mu_{s-\sigma,\tau}(gk^{-1})\, dk = \max_{k\in K} \rho_\sigma(k) \nu_{s,\tau}(g),$$

as claimed, and similarly for the lower bound. □

4.2. Matrix entries

In the case $K = \mathbb{R}^d$, it is convenient to do computations with the heat operator on polynomials. Although these functions are not in $L^2(\mathbb{R}^d)$, one can naïvely make sense of $e^{\frac{t}{2}\Delta_{\mathbb{R}^d}} f$ as a terminating power series for any polynomial f. It is then an easy matter to verify that the integral formula for the heat operator coincides with its Taylor series. That is to say, if f is a polynomial on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \rho_t(x-y) f(y)\, dy = \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} (\Delta_{\mathbb{R}^d})^n f(x). \tag{4.3}$$

Equation (4.3) is easy to prove directly; the result is also a special case of Proposition 4.7 below.

We will need a counterpart of polynomial functions on a general (compact-type) Lie group; these are *matrix entries*, which we define as follows.

Definition 4.3. Let G be a Lie group. Let (π, V_π) be a finite-dimensional complex representation of G, and let $A \in \mathrm{End}(V_\pi)$ be a fixed endomorphism. The associated **matrix entry** function $f_{\pi,A}$ on G is the function

$$f_{\pi,A}(x) = \mathrm{Tr}(\pi(x)A).$$

If G is a complex Lie group and the representation $\pi : G \to GL(V_\pi)$ is holomorphic, then we refer to $f_{\pi,A}$ as a **holomorphic matrix entry**. In particular, every holomorphic matrix entry on a complex Lie group is a holomorphic function.

Remark 4.4. A number of comments on matrix entries are in order.

(1) Although some authors might require π to be irreducible in order to call $f_{\pi,A}$ a *matrix entry*, we make no irreducibility assumption in our definition. If G is compact, every finite-dimensional representation of G decomposes as a direct sum of irreducibles, in which case every matrix entry is a linear combination of matrix entries for irreducible representations. In general, not every matrix entry (in the sense of Definition 4.3) will decompose as a sum of matrix entries of irreducible representations.
(2) Some authors require a matrix entry to be of the form $f(x) = \xi(\pi(x)v)$ for some $v \in V$ and $\xi \in V^*$. This is a special case of Definition 4.3 with $f = f_{\pi,A}$ where $A(w) = \xi(w)v$, i.e., $A = \xi \otimes v$. The more general matrix entries of Definition 4.3 are linear combinations of these more restricted "rank-1" entries.
(3) Matrix entries are smooth functions on G.
(4) If $G = \mathbb{R}^d$, all polynomials are matrix entries. Indeed: if q is a polynomial of degree $\leq n$, take the representation space V to be all polynomials p of degree $\leq n$, where $\pi(x)p = p(\,\cdot + x)$. If $\xi_0(p) = p(0)$ is the evaluation linear functional, then $\xi_0(\pi(x)q) = q(x)$, so q is a matrix entry.
(5) Even if G is complex, we will have a reason to consider matrix entries associated to representations of G that are not holomorphic.

Lemma 4.5. *For any Lie group G, the set of matrix entries on G forms a self-adjoint complex algebra.*

Proof. It is straightforward to compute that, for $\lambda \in \mathbb{C}$, $\lambda f_{\pi,A} = f_{\pi,\lambda A}$, while sums and products satisfy $f_{\pi,A} + f_{\sigma,B} = f_{\pi\oplus\sigma,A\oplus B}$ and $f_{\pi,A} f_{\sigma,B} = f_{\pi\otimes\sigma,A\otimes B}$. For complex conjugation, we must define the complex conjugate of a representation and an endomorphism. This can be done invariantly, but for our purposes there is no reason not to simply choose a basis. Given a representation (π, V_π) of dimension d, choose a complex-linear isomorphism $\varphi\colon V_\pi \to \mathbb{C}^d$, and let $[\pi(x)] = \varphi \circ \pi(x) \circ \varphi^{-1}$ and $[A] = \varphi \circ A \circ \varphi^{-1}$. As $d \times d$

complex matrices, both $[\pi(x)]$ and $[A]$ have complex conjugates $\overline{[\pi(x)]}$ and $\overline{[A]}$, defined entry-wise. Then

$$\bar{f}_{\pi,A}(x) = \overline{\mathrm{Tr}(\pi(x)A)} = \overline{\mathrm{Tr}([\pi(x)][A])} = \mathrm{Tr}(\overline{[\pi(x)]}\,\overline{[A]}). \tag{4.4}$$

The map $\overline{[\pi]}\colon G \to \mathrm{GL}(\mathbb{C}^d)$ given by $\overline{[\pi]}(x) = \overline{[\pi(x)]}$ is a representation of G on $\mathbb{C}^d$, and (4.4) shows that

$$\bar{f}_{\pi,A} = f_{\overline{[\pi]},\overline{A}}$$

is also a matrix entry of G. This concludes the proof. $\square$

We now establish two key results about matrix entries.

Theorem 4.6. *Let K be a real Lie group of compact type. For any $s > 0$, the matrix entries on K are dense in $L^2(K, \rho_s)$. If $s > 0$ and $\tau \in \mathbb{D}(s, s)$, then the holomorphic matrix entries on $K_{\mathbb{C}}$ are dense in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.*

Proof. We consider first the case that $K = \mathbb{R}^d$ and $K_{\mathbb{C}} = \mathbb{C}^d$. Then ρ_s is a Gaussian measure on K. Since every polynomial on $\mathbb{R}^d$ is a matrix entry, we may appeal to the classical result that polynomials are dense in L^2 of Gaussian measures on $\mathbb{R}^d$. (For a proof of a more general result, see [14, Theorem 3.6].) On the complex side, every holomorphic polynomial is a holomorphic matrix entry, and the measure $\mu_{s,\tau}$ on $\mathbb{C}^d$ is Gaussian. Thus, by [14, Proposition 3.5], matrix entries are dense in $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,\tau})$. (Note that, in general, the measure $\mu_{s,\tau}$ is not invariant under multiplication by $e^{i\theta}$ and monomials of different degrees are not necessarily orthogonal. Thus the proof of density of holomorphic polynomials in [1, Section 1b] does not apply.)

We consider next the case that K is compact. In that case, the heat kernel density ρ_s on K is bounded and bounded away from zero for each fixed $s > 0$. Thus, the Hilbert space $L^2(K, \rho_s)$ is the same as the Hilbert space $L^2(K)$, with a different but equivalent norm. Hence, the density of matrix entries in $L^2(K, \rho_s)$ follows from the Peter–Weyl theorem. On the complex side, we appeal to the averaging lemma (Lemma 4.2), which tells us that the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is the same as the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, with a different but equivalent norm. Thus, it suffices to establish the density of matrix entries in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$; this claim follows verbatim from the proof of the “onto” part of Theorem 2 in [22, Section 8].

We consider finally the case of a general compact-type group K. Recall (Proposition 2.2) that K is isometrically isomorphic to $K_0 \times \mathbb{R}^d$ for some compact Lie group K_0 and some $d \geq 0$. Thus, the heat kernel measure ρ_s on K factors as a product of the heat kernel measures ρ_s^0 on K_0 and ρ_s^1 on $\mathbb{R}^d$. Now, a standard result from measure theory tells us that there is a unitary map U from $L^2(K_0, \rho_s^0) \otimes L^2(\mathbb{R}^d, \rho_s^1)$ onto $L^2(K, \rho_s)$ uniquely determined by the requirement that $U(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$. If f_1 and f_2 are matrix entries on K_0 and $\mathbb{R}^d$, respectively, then $f_1(x_1)f_2(x_2)$ is a matrix entry on K

(by an argument very similar to the proof of Lemma 4.5). Using the density results for K_0 and for $\mathbb{R}^d$ and the unitary map U, we can easily show that linear combinations of matrix entries of this sort (which are again matrix entries) are dense in $L^2(K, \rho_s)$.

On the complex side, $K_{\mathbb{C}}$ is isomorphic to $(K_0)_{\mathbb{C}} \times \mathbb{C}^d$. If we restrict our Ad-invariant inner product on $\mathfrak{k}$ to the Lie algebras of K_0 and of $\mathbb{R}^d$, these restrictions will also be Ad-invariant. We may then construct left-invariant metrics on $(K_0)_{\mathbb{C}}$ and $\mathbb{C}^d$ by the same procedure as for $K_{\mathbb{C}}$. In that case, it is easily verified that the isomorphism $K_{\mathbb{C}} \cong (K_0)_{\mathbb{C}} \times \mathbb{C}^d$ is isometric. Thus, the heat kernel measure $\mu_{s,\tau}$ on $K_{\mathbb{C}}$ is a product of the associated heat kernel measures $\mu^0_{s,\tau}$ on $(K_0)_{\mathbb{C}}$ and $\mu^1_{s,\tau}$ on $\mathbb{C}^d$.

Then, as on the real side, we have a unitary map V from $L^2((K_0)_{\mathbb{C}}, \mu^0_{s,\tau}) \otimes L^2(\mathbb{C}^d, \mu^1_{s,\tau})$ onto $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. According to the Appendix of [23], the restriction of V to the tensor product of the two $\mathcal{H}L^2$ spaces maps *onto* $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. (It is easy to see that V maps the tensor product of the two $\mathcal{H}L^2$ spaces *into* $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$; it requires some small argument to show that it maps onto.) Thus, as on the real side, the density result for $K_{\mathbb{C}}$ reduces to the previously established results for $(K_0)_{\mathbb{C}}$ and for $\mathbb{C}^d$. $\square$

Proposition 4.7. *Let $f_{\pi,A}$ be a matrix entry on K. Then*

$$\begin{aligned} \int_K \rho_t(xy^{-1}) f_{\pi,A}(x)\, dx &= \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} (\Delta_K)^n f_{\pi,A}(x) \\ &= \mathrm{Tr}(\pi(x) e^{tC_\pi/2} A) \end{aligned} \tag{4.5}$$

with absolute convergence of the integral on the left-hand side and locally uniform convergence of the sums on the right-hand side. Here $C_\pi = \sum_{j=1}^{\dim \mathfrak{k}} \pi_*(X_j)^2$, where π_* is the Lie algebra representation associated to the Lie group representation π.

Let $f_{\pi,A}$ be a matrix entry on $K_{\mathbb{C}}$. Then

$$\begin{aligned} \int_{K_{\mathbb{C}}} f_{\pi,A}(g) \mu_{s,\tau}(g)\, dg &= \sum_{n=0}^{\infty} \frac{1}{2^n n!} (\Delta_{s,\tau})^n f_{\pi,A}(e), \\ &= \mathrm{Tr}(\pi(x) e^{D_{\pi,s,\tau}/2} A) \end{aligned} \tag{4.6}$$

with absolute convergence of the integral on the left-hand side and locally uniform convergence of the sum on the right-hand side. Here

$$D_{\pi,s,\tau} = \sum_{j=1}^{\dim \mathfrak{k}} \left[\left(s - \frac{t}{2} \right) \pi_*(X_j)^2 + \frac{t}{2} \pi_*(Y_j)^2 - u\, \pi_*(X_j) \pi_*(Y_j) \right]$$

where π_* is the Lie algebra representation associated to the Lie group representation π.

We note that unless K is compact (as opposed to merely being of compact type), matrix entries on K are typically not in $L^2(K, dx)$ and thus not in the usual domain of definition of the heat operator $e^{t\Delta_K/2}$. Similarly, matrix entries on $K_{\mathbb{C}}$ are typically not in the usual domain of the heat operator $e^{\Delta_{s,\tau}/2}$.

Proof. The proposition is an immediate consequence of Langland's theorem (cf. [45, Theorem 2.1]). See also [22, Lemma 8]. If one assumes it is valid to differentiate under the integral and to integrate by parts, one can prove the proposition easily; see the proof of [10, Theorem 2.13]. □

Remark 4.8. If f is a matrix entry on K or $K_{\mathbb{C}}$, then by Lemma 4.5, $|f|^2$ is also a matrix entry. Thus, the absolute convergence of the integral in Proposition 4.7 tells us that f is in $L^2(K, \rho_t)$ or $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.

5. The Segal–Bargmann transform

We analyze the complex-time Segal–Bargmann transform for a connected Lie group of compact type in two stages. In the first stage, we consider a transform M_τ (see Definition 5.1 below) defined on matrix entries using a power-series definition of the heat operator. Using the strategy outlined in Section 1.4 along with density results in Theorem 4.6, we show that M_τ maps a dense subspace of $L^2(K, \rho_s)$ isometrically onto a dense subspace of $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. Thus, M_τ extends to a unitary map $\overline{M}_{s,\tau}$ of $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.

In the second stage, we show that the heat kernel $\rho_t(x)$ on K has a holomorphic extension in both t and x, denoted $\rho_{\mathbb{C}}(\cdot,\cdot)$. We then prove that the unitary map $\overline{M}_{s,\tau}$ may be computed by "convolution" with the holomorphically extended heat kernel. That is to say,

$$(\overline{M}_{s,\tau} f)(z) = \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk$$

for all $s > 0$, $f \in L^2(K, \rho_s)$, $\tau \in \mathbb{D}(s, s)$, and $z \in K_{\mathbb{C}}$.

The advantage of the two-stage approach to the proof is that we can use the unitary map $\overline{M}_{s,\tau}$ to establish the existence of the holomorphic extension of the heat kernel, thus avoiding the representation-theoretic estimates used in [22, Section 4]. Although this approach was used already in [10], a number of details are different in the complex-time case. We therefore provide full proofs here.

5.1. Constructing a unitary map

As usual, we work on a connected Lie group K of compact type, with a fixed $\mathrm{Ad}(K)$-invariant inner product on its Lie algebra $\mathfrak{k}$. According to Theorem 4.6, the space of matrix entries is dense in $L^2(K, \rho_s)$ and the space of holomorphic matrix entries is dense in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.

We now define a transform M_τ directly by its action on matrix entries. Let $f_{\pi,A}$ be a matrix entry on K acting on a complex vector space V_π. By the universal property of complexifications, the representation π extends uniquely to a holomorphic representation $\pi_{\mathbb{C}}$ of $K_{\mathbb{C}}$ on V_π. Hence, the matrix entry $f_{\pi,A}$ has an analytic continuation as well,

$$(f_{\pi,A})_{\mathbb{C}}(g) = \mathrm{Tr}(\pi_{\mathbb{C}}(g)A) = f_{\pi_{\mathbb{C}},A}(g), \qquad g \in K_{\mathbb{C}}.$$

Definition 5.1. For $\tau \in \mathbb{C}_+$, define M_τ on matrix entries on K as

$$M_\tau f_{\pi,A} = \left[\sum_{n=0}^{\infty} \frac{(\tau/2)^n}{n!} (\Delta_K)^n f_{\pi,A}\right]_{\mathbb{C}}.$$

Note that, by (4.5), $M_\tau f_{\pi,A}$ is again a matrix entry, and thus has a holomorphic extension.

5.1.1. Complex vector fields and commutation relations

We would now like to emulate the proof of the Segal–Bargmann isometry for the $\mathbb{R}^d$ case outlined in Section 1.4. To that end, we must introduce the complex vector fields generalizing the complex derivatives $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ in the Euclidean context.

Definition 5.2. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and let X be an element of $\mathfrak{g}$. The **holomorphic** and **antiholomorphic vector fields** associated to X are complex vector fields ∂_X and $\bar{\partial}_X$ on G defined by

$$\partial_X \equiv \frac{1}{2}\left(\tilde{X} - i\,\widetilde{JX}\right) \quad \text{and} \quad \bar{\partial}_X \equiv \frac{1}{2}\left(\tilde{X} + i\,\widetilde{JX}\right). \tag{5.1}$$

In the special case $G = \mathbb{C}^d$, if $X = \partial/\partial x_j$ then $\partial_X = \partial/\partial z_j$ and $\bar{\partial}_X = \partial/\partial \bar{z}_j$. By (2.3), if $X \in \mathfrak{g}$ and F is holomorphic on G then

$$\partial_X F = \tilde{X}F, \qquad \bar{\partial}_X F = 0 \tag{5.2}$$

$$\partial_X \bar{F} = 0, \qquad \bar{\partial}_X F = \tilde{X}F. \tag{5.3}$$

Lemma 5.3. *If $X, V \in \mathfrak{g}$, then*

$$[\partial_V, \widetilde{JX}] = i[\partial_V, \tilde{X}], \qquad \textit{and} \qquad [\bar{\partial}_V, \widetilde{JX}] = -i[\bar{\partial}_V, \tilde{X}].$$

Proof. By (2.1), for any $W_1, W_2 \in \mathfrak{g}$, $[JW_1, W_2] = J[W_1, W_2] = [W_1, JW_2]$ and therefore by the definition of the Lie bracket,

$$[\widetilde{JW_1}, \tilde{W}_2] = [\widetilde{JW_1, W_2}] = J[\widetilde{W_1, W_2}] = [\widetilde{W_1, JW_2}] = [\tilde{W}_1, \widetilde{JW_2}].$$

We can then compute from the definition that

$$[\partial_V, \widetilde{JX}] = \frac{1}{2}[\tilde{V} - i\widetilde{JV}, \widetilde{JX}] = \frac{1}{2}[\widetilde{JV} - i\widetilde{JJV}, \tilde{X}] = \frac{1}{2}[\widetilde{JV} + i\tilde{V}, \tilde{X}] = i[\partial_V, \tilde{X}].$$

The calculation for $\bar{\partial}_V$ is similar. □

We now specialize to the case $G = K_{\mathbb{C}}$ for a compact-type Lie group K.

Definition 5.4. Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ for $\mathfrak{k}$, and let $\partial_j := \partial_{X_j}$ as in (5.1). Then set

$$\partial^2 \equiv \sum_{j=1}^{d} \partial_j^2, \qquad \text{and} \qquad \bar{\partial}^2 \equiv \sum_{j=1}^{d} \bar{\partial}_j^2. \tag{5.4}$$

A routine calculation shows that the operators ∂^2 and $\bar{\partial}^2$ are well-defined, independent of the choice of orthonormal basis.

Lemma 5.5. *The operators ∂^2 and $\bar{\partial}^2$ commute with the right action of K on $K_{\mathbb{C}}$.*

This is a routine computation and is left to the reader.

This brings us to the main commutator result of this section.

Proposition 5.6. *For any $A \in \mathfrak{k}_{\mathbb{C}}$,*

$$[\partial^2, \widetilde{A}] = [\bar{\partial}^2, \widetilde{A}] = 0.$$

Proof. As any $A \in \mathfrak{k}_{\mathbb{C}}$ has the form $A = V + JW$ for some $V, W \in \mathfrak{k}$, it suffices by linearity to prove that ∂^2 and $\bar{\partial}^2$ commute with $\widetilde{V}$ and $\widetilde{JV}$ for any $V \in \mathfrak{k}$. For the former statement, apply Lemma 5.5 to the right action of $k = e^{tV}$, and differentiate at $t = 0$ to yield the result. For the second statement, we employ Lemma 5.3 and compute as follows.

$$\begin{aligned}
[\partial^2, \widetilde{JV}] = \sum_{j=1}^{d} [\partial_j \partial_j, \widetilde{JV}] &= \sum_{j=1}^{d} \Big(\partial_j [\partial_j, \widetilde{JV}] + [\partial_j, \widetilde{JV}] \partial_j \Big) \\
&= i \sum_{j=1}^{d} \big(\partial_j [\partial_j, \tilde{V}] + [\partial_j, \tilde{V}] \partial_j \big) = i \sum_{j=1}^{d} [\partial_j \partial_j, \tilde{V}] = i[\partial^2, \tilde{V}]
\end{aligned}$$

and we already showed that $[\partial^2, \tilde{V}] = 0$. A similar calculation proves the result for $\bar{\partial}^2$. □

Corollary 5.7. *The operators ∂^2, $\bar{\partial}^2$, Δ_K, and $\Delta_{s,\tau}$ all mutually commute.*

Here we regard Δ_K as a left-invariant operator on $K_{\mathbb{C}}$.

Proof. Since Δ_K and $\Delta_{s,\tau}$ are linear combinations of squares of left-invariant vector fields on $K_{\mathbb{C}}$, Proposition 5.6 shows that they both commute with ∂^2 and $\bar{\partial}^2$. Similarly, letting $Y_j = JX_j$, since ∂_j^2 and $\bar{\partial}_j^2$ are linear combinations of $\tilde{X}_j^2$, $\tilde{Y}_j^2$, and $\tilde{X}_j\tilde{Y}_j = \tilde{Y}_j\tilde{X}_j$ (cf. (2.1)), the commutator $[\partial^2, \bar{\partial}^2] = 0$ also follows from Proposition 5.6. Now, since $\Delta_{s,\tau}$ is the Laplacian associated to an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}_{\mathbb{C}}$, it commutes with the right action of K. Thus, $\Delta_{s,\tau}$ commutes with each $\tilde{X}_j$ and thus with Δ_K. $\square$

Remark 5.8. The fact that $[\partial^2, \bar{\partial}^2] = 0$ holds quite generally. Indeed, on any complex manifold, if $Z = \sum_j a_j(z)\frac{\partial}{\partial z_j}$ and $W = \sum_j b_j(z)\frac{\partial}{\partial z_j}$ are two holomorphic vector fields, than a simple computation shows that $[Z, \overline{W}] = 0$.

5.1.2. The transform M_τ, and the isomorphism $\overline{M}_{s,\tau}$

The usefulness of the ∂^2 and $\bar{\partial}^2$ operators and the commutation result in Corollary 5.7 in the present context lies in the following result.

Lemma 5.9. *Let $s > 0$ and $\tau \in \mathbb{D}(s,s)$. Let $\Delta_{s,\tau}$ denote the $K_{\mathbb{C}}$ Laplacian of Definition 1.4, and let Δ_K denote the Laplacian of K acting on $C^\infty(K_{\mathbb{C}})$ as usual. Then*

$$s\Delta_K = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2$$

where all operators appearing in this identity are mutually commuting.

Proof. Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{k}$. For ease of reading, let $Y_j = JX_j$. To begin, we compute that, for each j,

$$\partial_j^2 + \bar{\partial}_j^2 = \frac{1}{4}(\tilde{X}_j - i\tilde{Y}_j)^2 + \frac{1}{4}(\tilde{X}_j + i\tilde{Y}_j)^2 = \frac{1}{2}(\tilde{X}_j^2 - \tilde{Y}_j^2), \tag{5.5}$$

$$\partial_j^2 - \bar{\partial}_j^2 = \frac{1}{4}(\tilde{X}_j - i\tilde{Y}_j)^2 - \frac{1}{4}(\tilde{X}_j + i\tilde{Y}_j)^2 = -i\tilde{X}_j\tilde{Y}_j \tag{5.6}$$

where we have used the fact that $[\tilde{X}_j, \tilde{Y}_j] = 0$ (cf. (2.1)).

Now, let $\tau = t + iu$. Then for each j,

$$\tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = t(\partial_j^2 + \bar{\partial}_j^2) + iu(\partial_j^2 - \bar{\partial}_j^2) = \frac{t}{2}(\tilde{X}_j^2 - \tilde{Y}_j^2) + u\tilde{X}_j\tilde{Y}_j.$$

Thus, we have

$$\left[\left(s - \frac{t}{2}\right)\tilde{X}_j^2 + \frac{t}{2}\tilde{Y}_j^2 - u\tilde{X}_j\tilde{Y}_j\right] + \tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = s\tilde{X}_j^2. \tag{5.7}$$

Summing (5.7) on j proves the lemma. $\square$

We can now prove that M_τ is a bijection from the space of matrix entries on K to the space of holomorphic matrix entries on $K_{\mathbb{C}}$, isometric from $L^2(K, \rho_s)$ into $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.

Theorem 5.10. *Let f be a matrix entry function on K. Then for $s > 0$ and $\tau \in \mathbb{D}(s, s)$,*

$$\|M_\tau f\|_{L^2(K_\mathbb{C},\mu_{s,\tau})} = \|f\|_{L^2(K,\rho_s)}. \tag{5.8}$$

Moreover, every holomorphic matrix entry F on $K_\mathbb{C}$ has the form $F = M_\tau f$ for some matrix entry f on K.

The proof of (5.8) follows the strategy outlined in Section 1.4, using left-invariant vector fields in place of the partial derivatives in the Euclidean case. A key step in the argument requires us to combine exponentials, which is possible only if the operators in the exponent commute. It is at this point that we use the commutativity result in Corollary 5.7.

Proof. Let $F = M_\tau f$. The matrix entry f on K has a holomorphic extension $f_\mathbb{C}$ to $K_\mathbb{C}$. Now, Δ_K, viewed as a left-invariant differential operator on $K_\mathbb{C}$, is a sum of squares of left-invariant vector fields; thus, it preserves the space of holomorphic functions. Thus, we have that $((\Delta_K)^n f)_\mathbb{C} = (\Delta_K)^n(f_\mathbb{C})$ for all $n \geq 0$. It follows that F may be computed as $F = e^{\tau\Delta_K/2}(f_\mathbb{C})$. Since $f_\mathbb{C}$ is holomorphic, we may use (5.2) to rewrite this relation as

$$F = e^{\tau\partial^2/2}(f_\mathbb{C}).$$

It is then straightforward, using (5.2) and (5.3), to see that

$$|F|^2 = e^{\tau\partial^2/2}e^{\bar{\tau}\bar{\partial}^2/2}(f_\mathbb{C}\bar{f}_\mathbb{C}).$$

Thus, using Proposition 4.7, we may compute the norm of F as

$$\begin{aligned}\|F\|^2_{L^2(K_\mathbb{C},\mu_{s,\tau})} &= \left(e^{\Delta_{s,\tau}/2}|F|^2\right)(e) \\ &= \left(e^{\Delta_{s,\tau}/2}e^{\tau\partial^2/2}e^{\bar{\tau}\bar{\partial}^2/2}(f_\mathbb{C}\bar{f}_\mathbb{C})\right)(e). \end{aligned} \tag{5.9}$$

By the commutativity result in Corollary 5.7, we may combine the exponents in the last expression in (5.9). Note that there are no domain issues to worry about here: All the exponentials in (5.9) are defined by power series and since $f_\mathbb{C}\bar{f}_\mathbb{C}$ is a matrix entry (cf. Lemma 4.5), all exponentials are acting in a fixed finite-dimensional subspace of functions on $K_\mathbb{C}$. Using Lemma 5.9, (5.9) therefore becomes

$$\|F\|^2_{L^2(K_\mathbb{C},\mu_{s,\tau})} = (e^{s\Delta_K/2}\,|f_\mathbb{C}|^2)(e) = (e^{s\Delta_K/2}\,|f|^2)(e).$$

The last equality holds because e belongs to K and Δ_K is a sum of squares of left-invariant vector fields associated to elements of $\mathfrak{k}$. Using Proposition 4.7 again, we finally conclude that

$$\|F\|^2_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} = \|f\|^2_{L^2(K,\rho_s)}$$

establishing (5.8).

Suppose now that F is a holomorphic matrix entry on $K_{\mathbb{C}}$; that is, $F = f_{\pi_{\mathbb{C}},A}$ for some finite-dimensional holomorphic representation $\pi_{\mathbb{C}}$ of $K_{\mathbb{C}}$. Then $F|_K = f_{\pi,A}$, where π is the restriction of $\pi_{\mathbb{C}}$ to K. We may then define

$$f = e^{-\frac{\tau}{2}\Delta_K}(F|_K) = f_{\pi, e^{-\frac{\tau}{2}C_\pi}A}.$$

Then f is a matrix entry and we have $M_\tau f = (e^{\frac{\tau}{2}\Delta_K} f)_{\mathbb{C}} = F$. $\square$

Theorem 5.11. *The map M_τ has a unique continuous extension to $L^2(K, \rho_s)$, denoted $\overline{M}_{s,\tau}$, and this extension is a unitary map from $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.*

Proof. Theorem 4.6 tells us that M_τ is defined on a dense subspace of $L^2(K, \rho_s)$. Since M_τ is isometric, the bounded linear transformation theorem (e.g., Theorem I.7 in [44]) tells us that M_τ has a unique continuous extension to a map $\overline{M}_{s,\tau}$ of $L^2(K, \rho_s)$ into $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. This extension is easily seen to be isometric, and since (by Theorem 4.6 again) the image of M_τ is dense, the extension is actually a unitary map. $\square$

For a general $f \in L^2(K, \rho_s)$, the value of $\overline{M}_{s,\tau}$ may be computed by approximating f by a sequence f_n of matrix entries and setting

$$\overline{M}_{s,\tau} f = \lim_{n\to\infty} M_\tau f_n. \tag{5.10}$$

(The bounded linear transformation theorem guarantees that the limit exists and that the value of $\overline{M}_{s,\tau}$ is independent of the choice of approximating sequence.) Now, (5.10) is not a very convenient way to compute. In the next section, we will seek a direct way of computing $\overline{M}_{s,\tau}$, which will also demonstrate that $\overline{M}_{s,\tau}$ coincides with the way we defined the complex-time Segal–Bargmann transform in the introduction; cf. (1.5). A first step in that direction is proving that $(\overline{M}_{s,\tau} f)(z)$ is holomorphic in both τ and z.

Lemma 5.12. *Fix $s > 0$. For each $f \in L^2(K, \rho_s)$, the function $(\tau, z) \mapsto (\overline{M}_{s,\tau} f)(z)$ is a holomorphic function on $\mathbb{D}(s, s) \times K_{\mathbb{C}}$.*

Proof. If $f = f_{\pi,A}$ is a matrix entry, then

$$(\overline{M}_{s,\tau} f_{\pi,A})(z) = (M_\tau f_{\pi,A})(z) = \mathrm{Tr}(\pi_{\mathbb{C}}(g) e^{\tau C_\pi/2} A)$$

which is easily seen to depend holomorphically on τ and z.

We then approximate an arbitrary $f \in L^2(K, \rho_s)$ by a sequence f_n of matrix entries. Then $M_\tau f_n = \overline{M}_{s,\tau} f_n$ will converge to $\overline{M}_{s,\tau} f$ in $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. It is well known that the evaluation map $F \mapsto F(z)$ on $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is a bounded linear functional; this is due to the ubiquitous pointwise L^2 estimates in this holomorphic space (cf. [11,24]). We claim that we can actually find locally uniform bounds on this functional. That is to say: for each precompact open subset U of $K_{\mathbb{C}}$ and $r \in (0,s)$, there exists $C = C(r,U) < \infty$ such that, for all $\tau \in \mathbb{D}(s,r)$ and $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$,

$$\sup_{z\in U} |F(z)| \le C(r,U)\, \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} . \tag{5.11}$$

Assuming this result for the moment, we can conclude that the convergence of $(\overline{M}_{s,\tau} f_n)(z)$ to $(\overline{M}_{s,\tau} f)(z)$ is locally uniform jointly in (τ, z), and since each function in the sequence is holomorphic, it follows that the limit $(\overline{M}_{s,\tau} f)(z)$ is jointly holomorphic in (τ, z) as claimed.

To establish the bound in (5.11), we observe that the norm of the pointwise evaluation functional can be estimated in terms of lower bounds on the density $\mu_{s,\tau}$. For example, [11, Theorem 3.6] shows (in our context) that, for any precompact neighborhood V of the identity e, there is a constant $C(V)$ so that, for all holomorphic F and $z \in K_{\mathbb{C}}$,

$$|F(z)| \le \frac{C(V)}{\inf_{v\in V} \sqrt{\mu_{s,\tau}(vz)}} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,\tau})}.$$

The constant $C(V)$ is determined only by the holomorphic structure of the group (given by averaging a symmetrized bump function on V, applying the Cauchy integral formula); hence, $C(V)$ is independent of s and τ. Hence, it suffices to show that $\mu_{s,\tau}(z)$ is bounded strictly above 0 locally uniformly in τ and z.

Since $K_{\mathbb{C}}$ factors as $(K_0)_{\mathbb{C}} \times \mathbb{C}^d$ (recall Proposition 2.2), the heat kernel $\mu_{s,\tau}$ also factors over this product. On the $\mathbb{C}^d$ side, there is an explicit formula for $\mu_{s,\tau}(z)$ (given in (1.15)) which is manifestly bounded away from zero locally in both τ and z. Thus, it suffices to assume that K is compact, which we do from now on.

Denote $t = \operatorname{Re}\tau$. From the averaging lemma (Lemma 4.2) and Proposition 5.15, we see that there is a strictly positive constant $C'(s,\tau)$ such that $\mu_{s,\tau} \asymp_{C'(s,\tau)} \mu_{t,t}$, and the constants can be chosen to depend continuously on (s,τ). Note that $\mu_{t,t}$ is the heat kernel for a single metric, which is therefore a continuous positive function of $(t,z) \in (0,\infty) \times K_{\mathbb{C}}$. In particular, $\mu_{t,t}(z)$ is bounded strictly away from 0 for (t,z) in compact subsets of $(0,\infty) \times K_{\mathbb{C}}$. It follows from the continuity of the function $(s,\tau) \mapsto C'(s,\tau)$ that the same holds true for $\mu_{s,\tau}(z)$, establishing (5.11) and completing the proof. □

5.2. *The analytic continuation of the heat kernel*

In this section, we show that the unitary map $\overline{M}_{s,\tau} \colon L^2(K,\rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ constructed in Section 5.1 may be computed as a "convolution" against a holomorphic extension of the heat kernel ρ_t on K. The following theorem makes this precise.

Theorem 5.13. *Let K be a compact-type Lie group.*

(1) *There exists a unique holomorphic function* $\rho_{\mathbb{C}} : \mathbb{C}_+ \times K_{\mathbb{C}} \to \mathbb{C}$ *such that for* $t > 0$ *and* $x \in K$ *we have*

$$\rho_{\mathbb{C}}(t, x) = \rho_t(x).$$

(2) *If* $s > 0$ *and* $\tau \in \mathbb{D}(s, s)$*, then for each* $z \in K_{\mathbb{C}}$*, the function*

$$x \mapsto \frac{\rho_{\mathbb{C}}(\tau, zx^{-1})}{\rho_s(x)}$$

belongs to $L^2(K, \rho_s)$*.*

(3) *The unitary map* $\overline{M}_{s,\tau}$ *may be computed as*

$$(\overline{M}_{s,\tau} f)(z) = \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk$$

for all $f \in L^2(K, \rho_s)$ *and all* $z \in K_{\mathbb{C}}$*.*

Since

$$\rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk = \frac{\rho_{\mathbb{C}}(\tau, zk^{-1})}{\rho_s(k)} f(k)\ \rho_s(k) dk$$

it follows by the Cauchy–Schwarz inequality and Theorem 5.13(2) that the function $k \mapsto \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)$ is integrable. Using the decomposition of K as $K_0 \times \mathbb{R}^d$, where K_0 is compact (Proposition 2.2), we may easily reduce the general case to the compact case and the Euclidean case, which we now address separately.

5.2.1. The compact case

It is possible to construct the holomorphic extension of the heat kernel on K using the method of [22, Section 4], which is based on a term-by-term analytic continuation of the expansion of the heat kernel in terms of characters. Indeed, replacing t by $t + iu$ in the heat kernel makes no change to the (absolute) convergence estimates in [22]. (The time-parameter occurs only linearly in the exponent there, so the absolute value of each term would be independent of u.) On the other hand, the argument in [22] requires detailed knowledge of the representation theory of K. We present here a different argument (similar to the proof of Corollary 4.6 in [10]) that uses the unitary map $\overline{M}_{s,\tau}$ of Theorem 5.11 to construct the desired analytic continuation.

Lemma 5.14. *If* K *is compact,* $s > 0$*,* $0 < t < 2s$*, and* $\overline{M}_{s,t}$ *is the unitary map as in Theorem 5.11, then for any* $f \in L^2(K, \rho_s)$*,*

$$(\overline{M}_{s,t} f)(x) = (\rho_t * f)(x) = \int_K \rho_t(xk^{-1}) f(k)\, dk \ \ \forall x \in K \subset K_{\mathbb{C}}. \tag{5.12}$$

(Note: for K compact, $L^2(K) = L^2(K, \rho_s)$ independent of $s > 0$ and hence $\overline{M}_{s,t} f$ does not really depend on s.)

Proof. By Definition 5.1, we have that for any matrix entry $f_{\pi,A}$ on K,

$$(\overline{M}_{s,t} f_{\pi,A})(g) = (M_t f_{\pi,A})(g) = \operatorname{Tr}(\pi_{\mathbb{C}}(g) e^{tC_\pi/2} A),$$

where $\pi_{\mathbb{C}}$ is the holomorphic extension of π from K to $K_{\mathbb{C}}$. Thus, by (4.5), we have

$$\left(\overline{M}_{s,t} f_{\pi,A}\right)|_K(x) = \operatorname{Tr}(\pi(x) e^{tC_\pi/2} A) = (\rho_t * f_{\pi,A})(x).$$

This suffices to complete the proof as matrix entries are dense in $L^2(K)$ and both $L^2(K) \ni f \to \left(\overline{M}_{s,t} f\right)(x) \in \mathbb{C}$ and $L^2(K) \ni f \to (\rho_t * f)(x) \in \mathbb{C}$ are continuous linear functionals on $L^2(K)$ for each fixed $x \in K$. The first assertion holds since $\overline{M}_{s,\tau} : L^2(K, \rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is unitary and pointwise evaluation on $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is continuous and the second follows by Hölder's inequality. $\square$

Proof of Theorem 5.13 in the compact group case. We begin with point (1): the space-time analytic continuation of the heat kernel. Let $0 < \delta < r < \infty$, and consider the vertically symmetric rectangle $U_{\delta,r} = \{\tau \in \mathbb{C}_+ : \delta < \operatorname{Re}\tau < r,\ |\operatorname{Im}\tau| < r\}$. Let $0 < \epsilon < \delta$, and fix $s > 0$ large enough that $U_{\delta,r} - \epsilon \subset \mathbb{D}(s,s)$. The function ρ_ϵ is continuous and hence in $L^2(K, \rho_s)$. We then define $\rho_{\mathbb{C}} : U_{\delta,r} \times K_{\mathbb{C}} \to \mathbb{C}$ by

$$\rho_{\mathbb{C}}(\tau, z) = \left(\overline{M}_{s,\tau-\epsilon} \rho_\epsilon\right)(z). \tag{5.13}$$

By Lemma 5.12, $\rho_{\mathbb{C}}$ is analytic in both variables so long as $\tau - \epsilon \in \mathbb{D}(s,s)$; in particular, $\rho_{\mathbb{C}}$ is analytic on $U_{\delta,r} \times K_{\mathbb{C}}$. For the moment, it appears a priori that the value of $\rho_{\mathbb{C}}$ depends on s and ϵ.

Now consider the restriction of $\rho_{\mathbb{C}}$ to $(t, x) \in (U_{\delta,r} \cap \mathbb{R}) \times K$. By Lemma 5.14 and the semigroup property of the heat kernel,

$$\rho_{\mathbb{C}}(t, x) = \left(\overline{M}_{s,t-\epsilon} \rho_\epsilon\right)(x) = (\rho_{t-\epsilon} * \rho_\epsilon)(x) = \rho_t(x) \ \ \forall x \in K. \tag{5.14}$$

Thus, $\rho_{\mathbb{C}}$ is a holomorphic extension of the heat kernel $\rho_t(x)$ in t and x. Analytic continuation from K to $K_{\mathbb{C}}$ is unique (cf. [53, Lemma 4.11.13]), and also from $U_{\delta,r} \cap \mathbb{R}$ to $U_{\delta,r}$ by elementary complex analysis. In particular, since $\rho_t(x)$ does not depend on s or ϵ, neither does the function $\rho_{\mathbb{C}}$.

Thus, for each rectangle $U_{\delta,r}$, there is a unique analytic continuation of the heat kernel to a holomorphic function $\rho_{\mathbb{C}}$ on $U_{\delta,r} \times K_{\mathbb{C}}$. Let δ_n and r_n be sequences with $\delta_n \downarrow 0$ and $r_n \uparrow \infty$, let $U_n = U_{\delta_n, r_n}$, and let $\rho_{\mathbb{C}}^n$ be the analytic continuation of $\rho_t(x)$ to U_n. The rectangles U_n are nested with union $\mathbb{C}_+$; since $\rho_{\mathbb{C}}^n$ and $\rho_{\mathbb{C}}^m$ agree on $(U_{n\wedge m} \cap \mathbb{R}) \times K$, uniqueness of analytic continuation shows that they agree on their common domain $U_{n\wedge m} \times K_{\mathbb{C}}$. Thus, there is a globally defined holomorphic function $\rho_{\mathbb{C}}$ whose value in

$U_n \times K_{\mathbb{C}}$ is $\rho_{\mathbb{C}}^n$, and thus restricts to $\rho_t(x)$ on $(U_n \cap \mathbb{R}) \times K$; ergo $\rho_{\mathbb{C}}(t,x) = \rho_t(x)$ for $t > 0$ and $x \in K$, as desired. Uniqueness again follows from [53, Lemma 4.11.13]. This establishes point (1).

Point (2) is immediate since K is compact and the function in question is continuous. For point (3), we first note that, by Lemma 5.12, $(\overline{M}_{s,\tau}f)(z)$ is holomorphic in τ and z. Meanwhile, since $\rho_{\mathbb{C}}(\tau, zk^{-1})$ is holomorphic in τ and z for each fixed $k \in K$, we may use Fubini's theorem and Morera's theorem to verify that $\int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dx$ is also holomorphic in τ and z. Since both sides of the desired equality are holomorphic in τ and z, it suffices by uniqueness of analytic continuation to verify the result when $\tau = t \in (0, 2s)$ and $z = x$ belongs to K. Using Lemma 5.14 and the defining property of $\rho_{\mathbb{C}}$, the desired equality thus becomes

$$(e^{t\Delta_K/2} f)(x) = \int_K \rho_t(xk^{-1}) f(k)\, dk,$$

which is true. This concludes the proof. $\square$

5.2.2. The Euclidean case

The heat kernel ρ_s on $\mathbb{R}^d$ is explicitly known to be the Gaussian density mentioned in the introduction:

$$\rho_s(x) = (2\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{2s}\right)$$

and the density $\mu_{s,\tau}(z)$ in this case has been described in (1.15) in the introduction.

Proof of Theorem 5.13 in the Euclidean case. For point (1), the desired holomorphic extension is given by

$$\rho_{\mathbb{C}}(\tau, z) := \left(\sqrt{2\pi\tau}\right)^{-d} \exp\left(-\frac{z \cdot z}{2\tau}\right) \tag{5.15}$$

where $z \cdot z = \sum_{j=1}^d z_j^2$ and where $\sqrt{2\pi\tau}$ is defined by the standard branch of the square root (with branch cut along the negative real axis).

Point (2) of the theorem is an elementary computation. Using additive notation for the group operation, we need to verify that

$$\int_{\mathbb{R}^d} \frac{|\rho_{\mathbb{C}}(\tau, z-x)|^2}{\rho_s(x)^2} \rho_s(x)\, dx < \infty \tag{5.16}$$

for all $z \in \mathbb{C}^d$, provided that $s > 0$ and $\tau \in \mathbb{D}(s,s)$ (or, equivalently, provided that $\alpha > 0$; cf. (1.8)). Equation (5.16) is a Gaussian integral whose computation is tedious but straightforward. (The integral factors into separate integrals over each copy of $\mathbb{R}$, which may then be evaluated in a computer algebra system.) We record the result here: if $z = \xi + i\eta$ and $\tau = t + iu$, then

$$\int_{\mathbb{R}^d} \frac{|\rho_{\mathbb{C}}(\tau, z-x)|^2}{\rho_s(x)^2}\rho_s(x)\,dx = \left(\frac{\pi s}{\sqrt{\alpha}}\right)^d \exp\left(\frac{t/2}{2\alpha}|\xi|^2 + \frac{s-t/2}{2\alpha}|\eta|^2 + \frac{u}{2\alpha}\xi\cdot\eta\right) \tag{5.17}$$

where, as in (1.8), $\alpha = (2st - t^2 - u^2)/4$.

For point (3), we must show that $(\overline{M}_{s,\tau}f)(z)$ may be computed as

$$(\overline{M}_{s,\tau}f)(z) = \int_{\mathbb{R}^d} \rho_{\mathbb{C}}(\tau, z-x)f(x)\,dx \tag{5.18}$$

for all $f \in L^2(\mathbb{R}^d, \rho_s)$. If f is a polynomial (and thus a matrix entry) and $\tau \in \mathbb{R}$ and $z \in \mathbb{R}^d$, (5.18) follows from Proposition 4.7. Furthermore, when f is a polynomial, both sides of (5.18) are holomorphic in τ and z, so the result continues to hold when $\tau \in \mathbb{C}_+$ and $z \in \mathbb{C}^d$. Now, both sides of (5.18) depend continuously on $f \in L^2(\mathbb{R}^d, \rho_s)$—the left-hand side by the unitarity of $\overline{M}_{s,\tau}$ and the continuity of pointwise evaluation, and the right-hand side by the fact that $\rho_{\mathbb{C}}(t, z-x)$ is square-integrable in x. Thus, we may pass to the limit starting from polynomials to obtain the result for all $f \in L^2(\mathbb{R}^d, \rho_s)$, thus completing the proof of Theorem 5.13 in the $\mathbb{R}^d$ case. $\square$

We note that, by (5.17), we have bounds on the value of $(\overline{M}_{s,\tau}f)(z)$ in terms of the L^2 norm of f. Since $\overline{M}_{s,\tau}$ maps isometrically onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,\tau})$, these bounds translate into pointwise bounds in $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,\tau})$ as follows:

$$|F(\xi+i\eta)|^2 \leq \left(\frac{\pi s}{\sqrt{\alpha}}\right)^d \exp\left(\frac{t/2}{2\alpha}|\xi|^2 + \frac{s-t/2}{2\alpha}|\eta|^2 + \frac{u}{2\alpha}\xi\cdot\eta\right)\|F\|^2_{L^2(\mathbb{C}^d,\mu_{s,\tau})}, \tag{5.19}$$

where $\mu_{s,\tau}$ is given as in (1.15). Note that the bounds on $|F(z)|^2$ are, up to a constant, just the reciprocal of the density $\mu_{s,\tau}$. This is typical behavior for $\mathcal{H}L^2$ spaces over $\mathbb{C}^d$ with respect to a Gaussian measure.

5.3. *The $s \to \infty$ limit*

Throughout this section, we assume that the compact-type group K is actually compact and we normalize the Haar measure dk on K to be a probability measure. Recall that $\nu_t \in C^\infty(K_{\mathbb{C}}, (0,\infty))$ is the K-averaged heat kernel measure, as in Definition 1.8.

Proposition 5.15. *For all $s > 0$ and $\tau = t + iu$ with $\tau \in \mathbb{D}(s,s)$, we have*

$$\int_K \mu_{s,\tau}(gk)\,dk = \nu_t(g). \tag{5.20}$$

That is to say, the integral on the left-hand side of (5.20) is independent of s and u and therefore equals its value when $u = 0$ and $s = t$, which is ν_t.

For the moment, we give only a heuristic argument for Proposition 5.15; a full proof requires some functional-analytic technicalities, which will be provided in Appendix A. By Corollary 5.7, the three terms in the definition (1.7) of $\Delta_{s,\tau}$ all commute with one another. Thus, *formally*, we can differentiate in the naive way, as if the terms in the exponents were scalars rather than operators. Assuming this approach is valid, we would get

$$\frac{\partial \mu_{s,\tau}}{\partial s} = \sum_{j=1}^{\dim \mathfrak{k}} \tilde{X}_j^2 \mu_{s,\tau}; \quad \frac{\partial \mu_{s,\tau}}{\partial u} = \sum_{j=1}^{\dim \mathfrak{k}} \tilde{X}_j \tilde{Y}_j \mu_{s,\tau}. \tag{5.21}$$

We now denote the integral on the left-hand side of (5.20) by $\nu_{s,\tau}$. Then (5.21) would tell us that

$$\frac{\partial \nu_{s,\tau}}{\partial s} = \sum_{j=1}^{\dim \mathfrak{k}} \tilde{X}_j^2 \nu_{s,\tau}; \quad \frac{\partial \nu_{s,\tau}}{\partial u} = \sum_{j=1}^{\dim \mathfrak{k}} \tilde{X}_j \tilde{Y}_j \nu_{s,\tau}.$$

But $\nu_{s,\tau}$ is by construction invariant under the right action of K, so that $\tilde{X}_j \nu_{s,\tau} = 0$. Since $\tilde{X}_j$ commutes with $\tilde{Y}_j = \widetilde{JX}_j$, we would find that $\nu_{s,\tau}$ is independent of s and u, as claimed.

We will use the following well-known result for the heat kernel measure on a compact Lie group at large time.

Lemma 5.16. *If K is a compact Lie group, the heat kernel ρ_s converges to the constant 1 uniformly over K as $s \to \infty$.*

This result holds more generally on compact Riemannian manifolds. (Apply Theorem 2 on p. 141 of [7] to the heat kernel ρ_ε, for $\varepsilon > 0$.) In the case of a compact Lie group, the result follows easily from the expansion of the heat kernel in terms of characters (e.g., Eq. (15) in [22]).

With these results in hand, we may now prove Theorem 1.9, describing the large-s limit of the transform $B_{s,\tau}$.

Proof of Theorem 1.9. Since K is compact, the function ρ_s is bounded and bounded away from zero, showing that $L^2(K) = L^2(K, \rho_s)$ as sets. The equality of $L^2(K_{\mathbb{C}}, \nu_t)$ and $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ as sets follows from the averaging lemma (Lemma 4.2) and Proposition 5.15. We then note that as s tends to infinity with τ fixed, the parameter σ in the averaging lemma can be chosen to tend to infinity. Thus, by Lemma 5.16, the constants in the averaging lemma tend to 1 as s tend to infinity, from which the claimed convergence of norms follows. The equalities of the various Hilbert spaces as sets and the convergence of the norms allows us to deduce the unitarity of $B_{\infty,\tau}$ from the unitarity of the maps $B_{s,\tau}$. $\square$

Acknowledgments

This project began as the result of a conversation between the third author and Thierry Lévy at Oberwolfach in June, 2015, regarding the idea of classifying all $\mathrm{Ad}(\mathrm{U}(n))$-invariant inner products on $\mathrm{GL}(n)$ (and studying their large-n limits). This led the third author to prove Theorem 3.2, and consequently to wonder if this extension of the two-parameter family of inner products studied in [36] was associated to some kind of "twisted Segal–Bargmann transform" extending the one in [14] and [23].

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Appendix A. Proof of Proposition 5.15

In this section, we provide a proof of Proposition 5.15, which we argued for heuristically in Section 5.3.

Theorem A.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and fix an inner product on $\mathfrak{g}$. For any subspace $V \subseteq \mathfrak{g}$, define*

$$\Delta_V = \sum_j \tilde{X}_j^2,$$

where $\{X_j\}$ is an orthonormal basis for V, with domain $\mathcal{D}(\Delta_V) = C_c^\infty(G)$. Then Δ_V is essentially self-adjoint as an unbounded operator on $L^2(G, dg)$, where dg is a right Haar measure. Moreover, its closure $\bar{\Delta}_V$ is non-positive, and the associated heat operators $e^{\frac{t}{2}\bar{\Delta}_V}$ are left-invariant for each $t > 0$.

We give here a proof based on work of Jørgensen; a brief outline of a more elementary argument was given in [13, p. 950], based on a method communicated to the first author by L. Gross. We emphasize that we do not assume that the smallest Lie algebra containing the X_j's is all of $\mathfrak{g}$; thus, Hörmander's criterion for hypoellipticity need not apply.

Proof. We fix a left Haar measure m in addition to the right Haar measure λ on G. Let R be the unitary right regular representation on $L^2(G, \lambda)$, i.e. for $x \in G$ and $\varphi \in L^2(G, \lambda)$ let

$$(R(x)\varphi)(y) = \varphi(yx) \quad \text{for all } y \in G.$$

For $f \in C_c^\infty(G)$ and $\varphi \in L^2(G, \lambda)$ we associate a "Gårding vector", $g := R(f)\varphi \in L^2(G, \lambda)$, defined by

$$(R(f)\varphi)(y) := \int_G f(x)(R(x)\varphi)(y)\,dm(x)$$
$$= \int_G f(x)\varphi(yx)\,dm(x) = \int_G f(y^{-1}x)\varphi(x)\,dm(x). \tag{A.1}$$

(According to a result of Malliavin and Dixmier [8], the space of Gårding vectors coincides with the space of "C^∞ vectors.")

For $X \in \mathfrak{g}$ let $\hat{X}$ denote the right-invariant vector field on G which agrees with X at the identity (as compared with the left-invariant vector field $\tilde{X}$). By general theory in [35, Theorem 1.1] or by direct computation, $R(f)\varphi \in C^\infty(G) \cap L^2(G, \lambda)$ and

$$\widetilde{X}g = \widetilde{X}R(f)\varphi = R\left(-\hat{X}f\right)\varphi \in C^\infty(G) \cap L^2(G, \lambda), \quad \forall\, X \in \mathfrak{g}. \tag{A.2}$$

Let $\mathcal{D}(L_1)$ denote the span of the Gårding vectors and $L_1 := L_0|_{\mathcal{D}(L_1)}$. According to [35, Theorem 1.1] with $U = R$, the operator L_1 is essentially self-adjoint. To complete the proof it suffices to show $\bar{L} = \bar{L}_1$ and for this it suffices to show $L_1 \subset \bar{L}$ and $L \subset \bar{L}_1$. We now verify the two desired operator inclusions.

- ($L_1 \subset \bar{L}$) Let $g := R(f)\varphi \in \mathcal{D}(L_1)$ be a Gårding vector as above. Choose a sequence $\{h_n\}_{n=1}^\infty \subset C_c^\infty(G, [0, 1])$ as in [12, Lemma 3.6] such that $h_n = 1$ on a Riemannian ball of radius n relative to the left-invariant Riemannian metric on G, and so $\sup_{x\in G} |Sh_n(x)| < \infty$ whenever S is any left-invariant differential operator on G. By the dominated convergence theorem, the fact that $Sg \in C^\infty(G) \cap L^2(G, \lambda)$ for any left-invariant differential operator S on G (see (A.2)), and the stated properties of $\{h_n\}_{n=1}^\infty$, it is easily shown that $h_n g \to g$ and $L(h_n g) \to L_1 g$ in $L^2(G, \lambda)$ as $n \to \infty$. This shows that $g \in \mathcal{D}(\bar{L})$ and $\bar{L}g = L_1 g$, i.e., $L_1 \subset \bar{L}$.
- ($L \subset \bar{L}_1$) Choose $\delta_n \in C_c^\infty(G, [0, \infty))$ such that $\int_G \delta_n(x)\,dm(x) = 1$ for each n and $\mathrm{supp}(\delta_n) \downarrow \{e\}$ as $n \to \infty$. Let $\iota\colon G \to G$ denote the inversion map, i.e. $\iota(x) = x^{-1}$ for all $x \in G$. If $f \in C_c^\infty(G)$, then $g_n := R(f \circ \iota))\delta_n \to f$ in $L^2(G, \lambda)$ as $n \to \infty$ (see (A.1)). Moreover, $g_n \in \mathcal{D}(L_1) \cap \mathcal{D}(L)$ and

$$L_1 g_n = R\left\{\sum_{j=1}^k \hat{X}_j^2 (f \circ \iota)\right\}\delta_n = R\left((Lf) \circ \iota\right)\delta_n \to Lf, \quad \text{as } n \to \infty$$

  where the convergence is in $L^2(G, \lambda)$. Thus, it follows that $f \in \mathcal{D}(\bar{L}_1)$ and $\bar{L}_1 f = Lf$, i.e., $L \subset \bar{L}_1$.

This concludes the proof of self-adjointness. The non-positivity of the self-adjoint extension $\bar{L}$ and the left invariance of the operators $e^{t\bar{L}}$ are now standard exercises. $\square$

Lemma A.2. *Let H be a separable Hilbert space, let A and B be two essentially self-adjoint non-positive operators on H, and suppose $Q\colon H \to H$ is a bounded operator such $QB \subseteq AQ$; i.e., $Q(\mathcal{D}(B)) \subseteq \mathcal{D}(A)$ and $QB = AQ$ on $\mathcal{D}(B)$. Then $Qe^{t\bar{B}} = e^{t\bar{A}}Q$ for all $t \geq 0$.*

Proof. If $f \in \mathcal{D}(\bar{B})$ and $f_n \in \mathcal{D}(B)$ such that $f_n \to f$ and $Bf_n \to \bar{B}f$, then $Qf_n \to Qf$ and $AQf_n = QBf_n \to Q\bar{B}f$ as $n \to \infty$. Therefore it follows that $Qf \in \mathcal{D}(\bar{A})$ and $\bar{A}Qf = Q\bar{B}f$ for all $f \in \mathcal{D}(\bar{B})$; i.e., $Q\bar{B} \subseteq \bar{A}Q$. So for any $\lambda \in \mathbb{C}$ we may conclude that $(\lambda I - \bar{A})Qf = Q(\lambda I - \bar{B})f$ for all $f \in \mathcal{D}(\bar{B})$. If we assume $\lambda > 0$ and $g \in H$, we may take $f = (\lambda I - \bar{B})^{-1}g \in \mathcal{D}(\bar{B})$ in the previous identity to find

$$(\lambda I - \bar{A})Q(\lambda I - \bar{B})^{-1}g = Qg.$$

Multiplying this equation by $(\lambda I - \bar{A})^{-1}$ and using the fact that g was arbitrary shows that $Q(\lambda I - \bar{B})^{-1} = (\lambda I - \bar{A})^{-1}Q$ or, equivalently,

$$Q(I - \lambda^{-1}\bar{B})^{-1} = (I - \lambda^{-1}\bar{A})^{-1}Q \quad \text{for all } \lambda > 0.$$

A simple induction argument then shows that

$$Q(I - \lambda^{-1}\bar{B})^{-n} = (I - \lambda^{-1}\bar{A})^{-n}Q \quad \text{for all } \lambda > 0. \tag{A.3}$$

Now, note that $\lim_{n\to\infty}(1 - \frac{y}{n})^{-n} = e^y$ and $0 \leq (1 - \frac{y}{n})^{-n} \leq 1$ for $y \leq 0$. We thus obtain the following strong operator limits, using the spectral theorem and the dominated convergence theorem:

$$e^{t\bar{B}} = \lim_{n\to\infty}\left(I - \frac{t}{n}\bar{B}\right)^{-n} \quad \text{and} \quad e^{t\bar{A}} = \lim_{n\to\infty}\left(I - \frac{t}{n}\bar{A}\right)^{-n}.$$

Therefore, taking $\lambda = n/t$ in (A.3) and then letting $n \to \infty$ shows $Qe^{t\bar{B}} = e^{t\bar{A}}Q$ for all $t > 0$. This completes the proof for $t > 0$, and the $t = 0$ case is immediate. □

Corollary A.3. *If K is a Lie subgroup of G, $V \subseteq \mathfrak{g}$ is an $\mathrm{Ad}(K)$-invariant subspace, and $\langle\cdot,\cdot\rangle_V$ is an $\mathrm{Ad}(K)$-invariant inner product on V, then $e^{\frac{t}{2}\bar{\Delta}_V}$ commutes with right translations by elements of K.*

Proof. If Q is a right-translation by an element of K and $A = B = \Delta_V$ with $\mathcal{D}(\Delta_V) = C_c^\infty(G)$, then $QB = AQ$, and Q preserves $\mathcal{D}(\Delta_V)$ in this case. The result now follow by an application of Lemma A.2. □

Definition A.4 *(K-averaging).* Let P be the **K-averaging operator** defined on $L^1_{\mathrm{loc}}(K_{\mathbb{C}})$ by

$$(Pf)(z) = \int_K f(zk)\,dk$$

where dk denotes the Haar probability measure on K.

Since the Haar measure on K is invariant under inversion and the convolution with itself is still Haar measure, we can easily check that $P\colon L^2(K_{\mathbb{C}}) \to L^2(K_{\mathbb{C}})$ is an orthogonal projection. The operator P also preserves the subspaces $C^\infty(K_{\mathbb{C}})$ and $C_c^\infty(K_{\mathbb{C}})$ and if $f \in C(K_{\mathbb{C}})$ we have $Pf(zk) = Pf(z)$ for all $k \in K$ and $z \in K_{\mathbb{C}}$. Proposition 5.15 states, in this language, that $P\mu_{s,\tau} = \nu_t$, where $t = \operatorname{Re}\tau$.

Proof of Proposition 5.15. If $X \in \mathfrak{k}$ and $f \in C_c^\infty(K_{\mathbb{C}})$, then $(Pf)(ze^{rX}) = (Pf)(z)$ for all $z \in K_{\mathbb{C}}$ and $r \in \mathbb{R}$. Differentiating at $r = 0$ shows that $\widetilde{X}Pf = 0$ for any $X \in \mathfrak{k}$. Using the fact that $\tilde{X}_j\tilde{Y}_j = \tilde{Y}_j\tilde{X}_j$, which follows from the definition $Y_j = JX_j$ and (2.1), it follows from Definition 1.4 that

$$\Delta_{s,\tau}P = \frac{t}{2}\Delta_{J\mathfrak{k}}P = P\frac{t}{2}\Delta_{J\mathfrak{k}} \text{ on } C_c^\infty(K_{\mathbb{C}}), \quad \text{where } \Delta_{J\mathfrak{k}} := \sum_{j=1}^d \tilde{Y}_j^2. \tag{A.4}$$

For the last equality, we have used that $\Delta_{J\mathfrak{k}}$ commutes with right translations by elements of K and therefore with P. An application of Lemma A.2 with $Q = P$, $A = \Delta_{s,\tau}$, and $B = \frac{t}{2}\Delta_{J\mathfrak{k}}$ gives $Pe^{\frac{t}{2}\bar{\Delta}_{J\mathfrak{k}}} = e^{\bar{\Delta}_{s,\tau}}P$ for all $\tau \in \mathbb{D}(s,s)$ with $\operatorname{Re}\tau = t$. In particular we may conclude that

$$e^{\bar{\Delta}_{s,\tau}}P = e^{\bar{\Delta}_{s,t}}P \qquad \forall\, \tau = t + iu \in \mathbb{D}(s,s) \tag{A.5}$$

or equivalently that

$$\langle e^{\bar{\Delta}_{s,\tau}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} = \langle e^{\bar{\Delta}_{t,t}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} \qquad \forall\, u, v \in C_c(K_{\mathbb{C}}, \mathbb{R}). \tag{A.6}$$

For the rest of the proof let $\bar{\mu}_{s,\tau} = P\mu_{s,\tau}$ be the K-average of $\mu_{s,\tau}$. We may rewrite the left-hand-side of (A.6) as

$$\begin{aligned}
\langle e^{\bar{\Delta}_{s,\tau}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} &= \int_{K_{\mathbb{C}}^2} \mu_{s,\tau}(g)(Pv)(zg)w(z)\,dg\,dz \\
&= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(g)v(zgk)w(z)\,dg\,dz\,dk \\
&= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(gk^{-1})v(zg)w(z)\,dg\,dz\,dk \\
&= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(gk)v(zg)w(z)\,dg\,dz\,dk
\end{aligned}$$

$$= \int_{K_{\mathbb{C}}^2} \bar{\mu}_{s,\tau}(g)v(zg)w(z)\,dg\,dz.$$

This equation with $\tau = t$ also shows the right-hand-side of (A.6) is given by

$$\langle e^{\bar{\Delta}_{t,t}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} = \int_{K_{\mathbb{C}}^2} \nu_t(g)v(zg)w(z)\,dg\,dz.$$

Comparing the last two identities shows, for all $v, w \in C_c(K_{\mathbb{C}})$,

$$\int_{K_{\mathbb{C}}^2} \bar{\mu}_{s,\tau}(g)v(zg)w(z)\,dg\,dz = \int_{K_{\mathbb{C}}^2} \nu_t(g)v(zg)w(z)\,dg\,dz.$$

As $C_c(K_{\mathbb{C}})$ is dense in $L^2(K_{\mathbb{C}})$, we may conclude that, for all $v \in C_c(K_{\mathbb{C}})$,

$$\int_{K_{\mathbb{C}}} \bar{\mu}_{s,\tau}(g)v(zg)\,dg = \int_{K_{\mathbb{C}}} \nu_t(g)v(zg)\,dg \quad \text{for a.e. } z$$

and hence for every $z \in K_{\mathbb{C}}$ as both sides of the previous equation are continuous in z. Thus, taking $z = e$, it follows that,

$$\int_{K_{\mathbb{C}}} \bar{\mu}_{s,\tau}(g)v(g)\,dg = \int_{K_{\mathbb{C}}} \nu_t(g)v(g)\,dg \;\forall\; v \in C_c(K_{\mathbb{C}}, \mathbb{R}).$$

So as above, the density of $C_c(K_{\mathbb{C}})$ in $L^2(K_{\mathbb{C}})$ along with the continuity of both $\bar{\mu}_{s,\tau}$ and ν_t, allows us to conclude that $\bar{\mu}_{s,\tau}(g) = \nu_t(g)$ for all $g \in K_{\mathbb{C}}$. $\square$

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