Is this Game Fair? Deciding with Simulation Data and Organized Lists

Interest in games of chance motivated mathematicians such as Cardano, Pascal, and Fermat to develop mathematical probability theory in the sixteenth and seventeenth centuries (Jardine, 2000). Games of chance continue to motivate students to explore probability today (e.g., Degner, 2015; Nelson & Williams, 2008). In the present article, we explain how playing, simulating, and analyzing games of chance to judge their fairness helped a particular group of students develop stronger understanding of concepts related to simple probability, compound probability, sample space, theoretical probability, and experimental probability. These areas are included in the Virginia Standards of Learning for the middle grades.

The activities described in this article were implemented during a summer mathematics program with a small group of middle school students: Tom, Laura, Aiden, and Emilia (pseudonyms). We interviewed each student at the start of the program to assess their existing knowledge of probability. When we asked students about the fairness of games of chance using coins, spinners, and dice during the interviews, they frequently claimed that just allowing each player the same number of turns made games fair. We wanted to help students expand their ideas of fairness by considering each player's probability of winning rather than just the number of turns. So, during our first lesson, we had the students play a game in which each player had the same number of turns but different chances of winning.

Drawing Cubes from a Bag

The first game students played involved drawing cubes from a paper bag, with replacement. We filled a brown paper bag with 8 red cubes and 2 blue ones and did not tell students how many of each color were in the bag. They played against each other in pairs, with one player earning a point for drawing red and the other earning a point for drawing blue. After doing 20 draws per pair, with 10 draws per partner, one pair had 18 red and 2 blue, and the other pair had 17 red and 3 blue. When asked if the game was fair, all students said it was not, even though each player took the same number of turns. Laura explained, "Because I kept getting reds, I said to you that I didn't think there was any blue in it [the bag]; so I think there's more red than blue." We then showed students how many reds and blues were in the bag and asked them to create bags to make the game fair. One pair put 5 red and 5 blue cubes in their bag, and the other used 2 red and 2 blue. Hence, the activity prompted students to focus on each player's probability of winning, and not just number of turns, when deciding if a game is fair.

In the next lesson, we used one of the bags students had created to introduce compound probability. Students used the bag with 5 red cubes and 5 blue cubes. Each student took a turn that consisted of drawing a cube, replacing it, shaking the bag, and drawing another. We recorded how many blue cubes they obtained on each turn. To allow students to gather data more quickly and efficiently, we introduced *TinkerPlots* software (Konold & Miller 2011, www.tinkerplots.com) to simulate the process they had carried out with concrete materials. Readers can visit the YouTube link in the caption of Figure 1 to see how we set up and conducted the simulated draws. As draws were simulated, students kept track of the number of blue cubes obtained on each turn (Tom's graph is shown in Figure 2). As we brought students into the simulation, we found it important to ask questions to help them interpret the elements of the *TinkerPlots* document, read the graphs they produced, and connect the simulated events to the corresponding concrete situation. Helpful questions included: (i) What does each part of the graph represent?, (ii) How many times did we draw two blue cubes?; (iii) How does *TinkerPlots* show which cube was drawn first?; (iv) Why is the middle column on the graph so tall? Asking such questions throughout the lesson let us know which patterns students were seeing and helped them map the simulation to the original concrete situation.

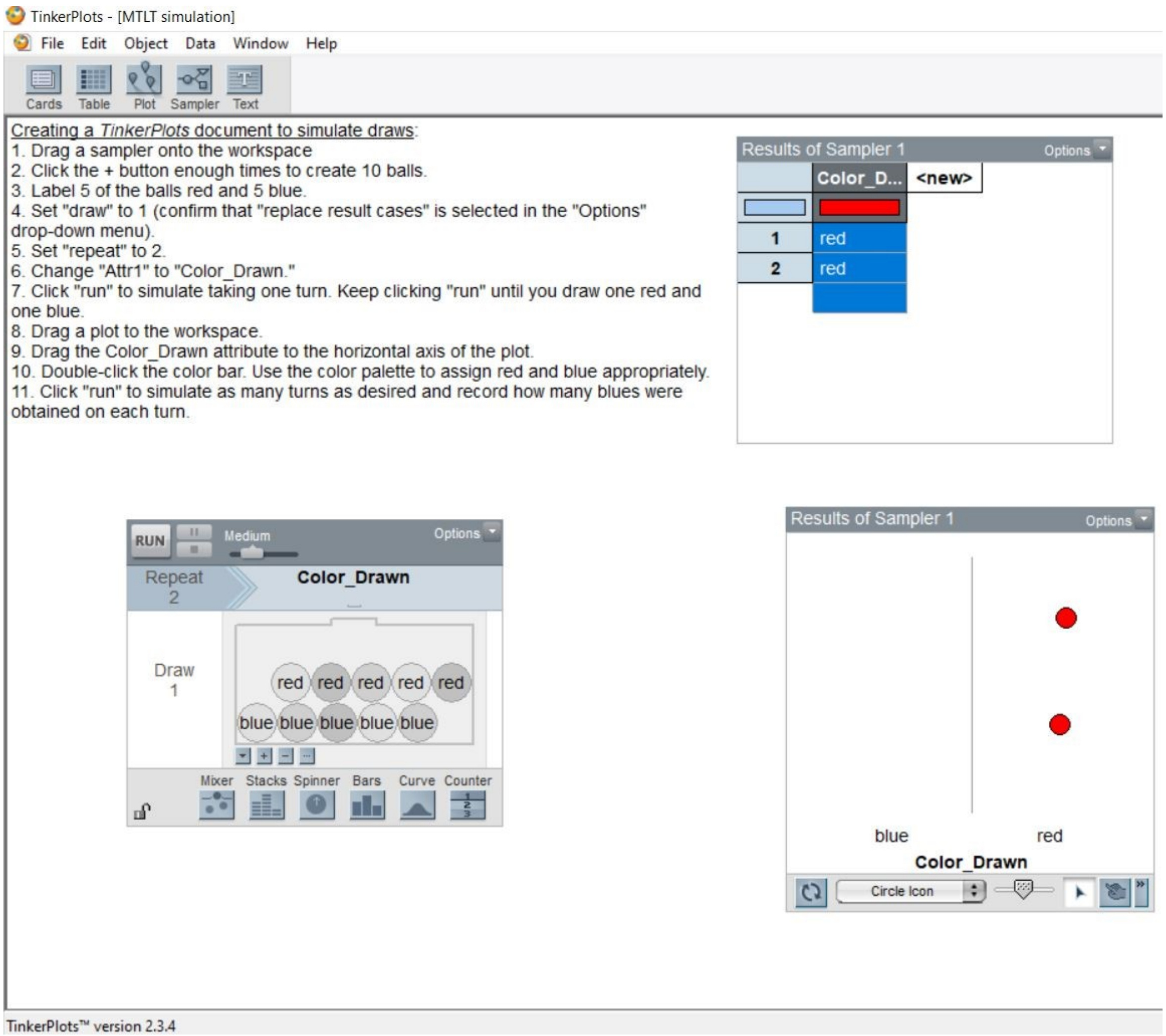

Figure 1. Using *TinkerPlots* to simulate drawing two cubes, with replacement. Visit https://youtu.be/ikrV3VgPet4 to see how to set up and run the simulation.

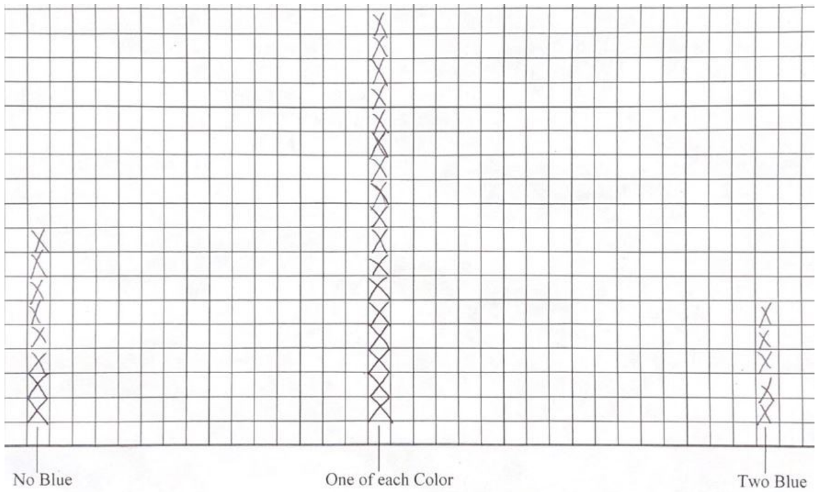

Figure 2. Tom's graph to track the results of the *TinkerPlots* simulation.

At the start of the third lesson, each student conducted the *TinkerPlots* simulation (Figure 1) on their own, graphed the results, and then compared their graphs with one another. As they compared graphs, they noticed that the middle column in each graph was consistently higher than the others (as in Figure 2). This gave us an opportunity to talk about the two different outcomes that produce a data point in the middle column (red on the first and blue on the second; blue on the first and red on the second), and we guided students to make a complete organized list of all possible outcomes for the situation: blue on each one, red on the first and blue on the second, blue on the first and red on the second, and red on each one.

After students had analyzed the situation using a simulation and an organized list, we posed the following problem in reference to the bag with 5 blue cubes and 5 red cubes: *You are playing a carnival game. You win a prize if you pick two blue cubes (pick one, replace it, shake the bag, then pick another). The carnival worker tells you this is a fair game. Do you agree? Why or why not?* This type of compound probability problem is often quite difficult for students. Only 8% of twelfth-graders were able to completely explain why an analogous game was not fair in an item on the National Assessment of Educational Progress (Shaughnessy, 2007).

One of our students, Laura, used an organized list to reason that the carnival game was not fair because only one of the four possible outcomes was favorable, giving a 25% chance of winning. Another student, Aiden, initially thought the game was fair but then questioned its fairness after seeing how common it was to get one of each color in his simulation results. The other two students, Tom and Emilia, were still not convinced that the game was unfair, so we followed this lesson up with an exploration of the parallel situation of flipping a penny and quarter simultaneously (winning a prize only if both coins come up heads).

Flipping Two Coins

We asked students to flip the two coins simultaneously, record results, and draw pictures of all possible outcomes. Drawing such pictures (Emilia's work is shown in Figure 3) helped all students develop the habit of listing all possible outcomes when analyzing the fairness of games.

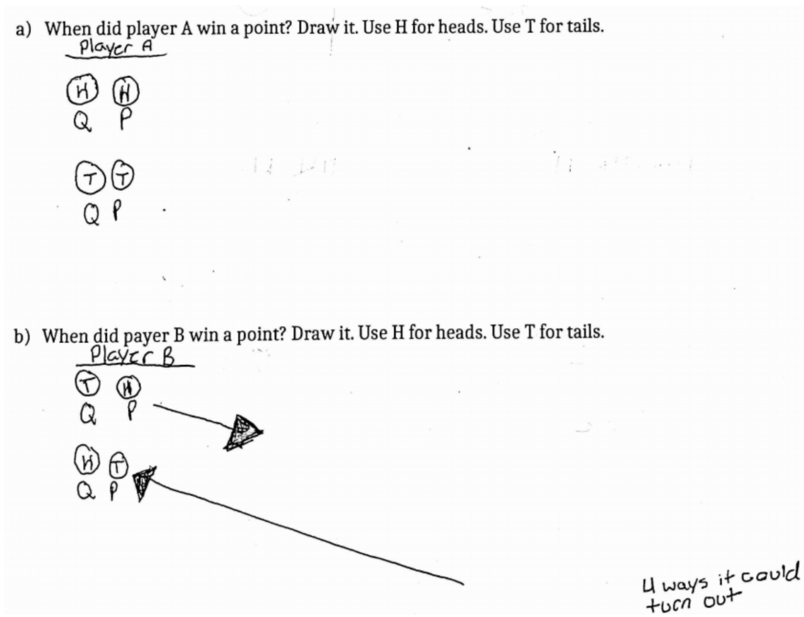

Figure 3. Emilia's picture of all possible outcomes when flipping a penny and a quarter.

Students next worked with organized lists and simulations for the situation of flipping two quarters simultaneously. This caused a great deal of discussion about whether obtaining heads on the first quarter and tails on the second was the same outcome as obtaining tails on the first and heads on the second, as in this exchange:

> Teacher: Laura thinks that tails-heads is the same thing as heads-tails. Do you think that it is the same thing or do you think that it's different?
> Aiden: It's a little bit different because it's the same swapped around.
> Teacher: Oh, so it's swapped around. Like, what do you mean swapped around?
> Aiden: Instead of heads-tails, it's tails-heads.
> Teacher: OK, so like you'd have heads on Quarter 1 and tails on Quarter 2. So you're saying no that they're not the same thing? Emilia, what do you think?
> Emilia: I think that they're the same thing.
> Teacher: You do?
> Emilia: Like tails and heads. I think they're the same thing because you're just still going to get a point or Player B's still going to get a point whether it's heads-tails or tails-heads.
> Teacher: So you're thinking of it in terms of who gets a point?
> Emilia: Yeah.
> Teacher: OK, Tom, do you agree?
> Tom: For that one?
> Teacher: Yeah, so that tails-heads is the same thing as heads-tails.
> Tom: No.
> Teacher: No, why?
> Tom: It's different because heads and tails Quarter 1 is doing the heads and Quarter 2 is doing the tails, so it's totally different because they're—because they're whole different quarters just swapped.

As the class continued to discuss the situation, students gradually adopted Tom's viewpoint, though Tom himself was still uncertain of his reasoning at some points in time.

To check students' understanding of the two-coin situation after our series of lessons, we interviewed them individually at the end of the summer and posed the following task:

> Two fair coins are part of a carnival game. A player wins a prize only when both coins come up heads. After each coin has been flipped once, James thinks he has a 50 50 chance of winning. Do you agree?"

Each of our students recognized it was not fair for James to have to flip two heads to win a carnival game and lose otherwise. Aiden, for example, said such a game was not fair because, "You only get a win if both coins are heads and heads, and there's more outcomes than just heads and heads; there's heads-tails, tails-heads or tails-tails." Likewise, during her post-interview, Emilia listed the possible outcomes when flipping two quarters and then reasoned,

> OK, so if both coins come up as heads that's when you win. So then you have heads, you have, heads-tails and tails-tails and then we said in class that tails and heads aren't the same thing so tails-heads. And then you want to have heads-heads. So then you have, you want this one (circles heads-heads) so that would be 1⁄4 which equals 25% chance of winning.

Such responses indicated that students had developed more sophisticated strategies for analyzing probability games than just checking to see if each player received the same number of turns, as they had done prior to participating in our lessons.

Conclusion

Our lesson sequence illustrates how simulations and organized lists can be used in tandem to address challenging probability concepts. Simulating games with concrete materials and technology helped students re-examine their existing notions about fairness of games. In the process, they began to suspect that games they thought were fair actually were not. Systematically listing the possible outcomes in such situations helped them confirm their suspicions. We encourage other teachers to engage students in the same type of back-and-forth interplay between concrete experiences and simulations, offering our experience as a starting point in this endeavor.

References

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