E. A. Carlen *et al.* (2021) "On the Convolution Inequality $f \geqslant f \star f$,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–9
doi:10.1093/imrn/rnaa350

On the Convolution Inequality $f \geqslant f \star f$

Eric A. Carlen[1,*], Ian Jauslin[2], Elliott H. Lieb[3] and Michael P. Loss[4]
[1]Department of Mathematics, Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA, [2]Department of Physics, Princeton University, Washington Road, Princeton, NJ 08544, USA, [3]Departments of Mathematics and Physics, Princeton University, Washington Road, Princeton, NJ 08544, USA, [4]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

[*]*Correspondence to be sent to: e-mail: carlen@math.rutgers.edu*

We consider the inequality $f \geqslant f \star f$ for real functions in $L^1(\mathbb{R}^d)$ where $f \star f$ denotes the convolution of f with itself. We show that all such functions f are nonnegative, which is not the case for the same inequality in L^p for any $1 < p \leqslant 2$, for which the convolution is defined. We also show that all solutions in $L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \leqslant \frac{1}{2}$. Moreover, if $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \frac{1}{2}$, then f must decay fairly slowly: $\int_{\mathbb{R}^d} |x| f(x)\,\mathrm{d}x = \infty$, and this is sharp since for all $r < 1$, there are solutions with $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \frac{1}{2}$ and $\int_{\mathbb{R}^d} |x|^r f(x)\,\mathrm{d}x < \infty$. However, if $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x =: a < \frac{1}{2}$, the decay at infinity can be much more rapid: we show that for all $a < \frac{1}{2}$, there are solutions such that for some $\varepsilon > 0$, $\int_{\mathbb{R}^d} e^{\varepsilon |x|} f(x)\,\mathrm{d}x < \infty$.

1 Introduction

Our subject is the set of real, integrable solutions of the inequality

$$f(x) \geqslant f \star f(x), \quad \forall x \in \mathbb{R}^d, \tag{1}$$

where $f \star f(x)$ denotes the convolution $f \star f(x) = \int_{\mathbb{R}^d} f(x-y) f(y)\,\mathrm{d}y$. By Young's inequality [6, Theorem 4.2], for all $1 \leqslant p \leqslant 2$ and all $f \in L^p(\mathbb{R}^d)$, $f \star f$ is well defined as an element

Received February 26, 2020; Revised November 04, 2020; Accepted November 09, 2020
Communicated by Prof. Shiri Artstein-Avidan

of $L^{p/(2-p)}(\mathbb{R}^d)$. Thus, one may consider the inequality (1) in $L^p(\mathbb{R}^d)$ for all $1 \leqslant p \leqslant 2$, but the case $p = 1$ is special: the solution set of (1) is restricted in a number of surprising ways. Integrating both sides of (1), one sees immediately that $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \leqslant 1$. We prove that, in fact, all integrable solutions satisfy $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \leqslant \frac{1}{2}$, and this upper bound is sharp.

Perhaps even more surprising, we prove that all integrable solutions of (1) are nonnegative. This is *not true* for the solutions in $L^p(\mathbb{R}^d)$, $1 < p \leqslant 2$. For $f \in L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2$, the Fourier transform $\widehat{f}(k) = \int_{\mathbb{R}^d} e^{-i2\pi k\cdot x} f(x)\,\mathrm{d}x$ is well defined as an element of $L^{p/(p-1)}(\mathbb{R}^d)$. If f solves the equation $f = f \star f$, then $\widehat{f} = \widehat{f}^2$, and hence $\widehat{f}$ is the indicator function of a measurable set. By the Riemann–Lebesgue theorem, if $f \in L^1(\mathbb{R}^d)$, then $\widehat{f}$ is continuous and vanishes at infinity and the only such indicator function is the indicator function of the empty set. Hence, the only integrable solution of $f = f \star f$ is the trivial solution $f = 0$. However, for $1 < p \leqslant 2$, solutions abound: take $d = 1$, and define g to be the indicator function of the interval $[-a, a]$. Define

$$f(x) = \int_{\mathbb{R}} e^{-i2\pi kx} g(k)\,\mathrm{d}k = \frac{\sin 2\pi xa}{\pi x} \tag{2}$$

which is not integrable but belongs to $L^p(\mathbb{R})$ for all $p > 1$. By the Fourier inversion theorem $\widehat{f} = g$. Taking products, one gets examples in any dimension.

To construct a family of solutions to (1), fix $a, t > 0$, and define $g_{a,t}(k) = ae^{-2\pi|k|t}$. By [9, Theorem 1.14],

$$f_{a,t}(x) = \int_{\mathbb{R}^d} e^{-i2\pi kx} g_{a,t}(k)\,\mathrm{d}k = a\Gamma((d+1)/2)\pi^{-(d+1)/2}\frac{t}{(t^2+x^2)^{(d+1)/2}}.$$

Since $g^2_{a,t}(k) = g_{a^2,2t}$, $f_{a,t} \star f_{a,t} = f_{a^2,2t}$, Thus, $f_{a,t} \geqslant f_{a,t} \star f_{a,t}$ reduces to

$$\frac{t}{(t^2+x^2)^{(d+1)/2}} \geqslant \frac{2at}{(4t^2+x^2)^{(d+1)/2}},$$

which is satisfied for all $a \leqslant 1/2$. Since $\int_{\mathbb{R}^d} f_{a,t}(x)\,\mathrm{d}x = a$, this provides a class of solutions of (1) that are nonnegative and satisfy

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \leqslant \frac{1}{2} \tag{3}$$

all of which have fairly slow decay at infinity, so that in every case,

$$\int_{\mathbb{R}^d} |x| f(x)\,\mathrm{d}x = \infty. \tag{4}$$

Our results show that this class of examples of integrable solutions of (1) is surprisingly typical of *all* integrable solutions: every real integrable solution f of (1) is positive and satisfies (3), and if there is equality in (3), f also satisfies (4). The positivity of all real solutions of (1) in $L^1(\mathbb{R}^d)$ may be considered surprising since it is false in $L^p(\mathbb{R}^d)$ for all $p > 1$, as example (2) shows. We also show that when strict inequality holds in (3) for a solution f of (1), it is possible for f to have a rather fast decay; we construct examples such that $\int_{\mathbb{R}^d} e^{\varepsilon|x|} f(x)\,\mathrm{d}x < \infty$ for some $\varepsilon > 0$. The conjecture that integrable solutions of (1) are necessarily positive was motivated by recent works [3, 4] on a partial differential equation involving a quadratic nonlinearity of $f \star f$ type, and the result proved here is the key to the proof of positivity for the solutions of this partial differential equation; see [3]. Autoconvolutions $f \star f$ have been studied extensively; see [7] and the work quoted there. However, the questions investigated by these authors are quite different from those considered here.

2 Theorems and proofs

Theorem 1. Let f be a real-valued function in $L^1(\mathbb{R}^d)$ such that

$$f(x) - f \star f(x) =: u(x) \geqslant 0 \tag{5}$$

for all x. Then, $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \leqslant \frac{1}{2}$, and f is given by the series

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x), \tag{6}$$

which converges in $L^1(\mathbb{R}^d)$ and where the $c_n \geqslant 0$ are the Taylor coefficients in the expansion of $\sqrt{1-x}$

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n, \quad c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2}. \tag{7}$$

In particular, f is positive. Moreover, if $u \geqslant 0$ is any integrable function with $\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x \leqslant \frac{1}{4}$, then the sum on the right in (6) defines an integrable function f that satisfies (5), and $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \frac{1}{2}$ if and only if $\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = \frac{1}{4}$.

Proof. Note that u is integrable. Let $a := \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x$ and $b := \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x \geqslant 0$. Fourier transforming, (5) becomes

$$\widehat{f}(k) = \widehat{f}(k)^2 + \widehat{u}(k). \tag{8}$$

At $k = 0$, $a^2 - a = -b$, so that $\left(a - \frac{1}{2}\right)^2 = \frac{1}{4} - b$. Thus, $0 \leqslant b \leqslant \frac{1}{4}$. Furthermore, since $u \geqslant 0$,

$$|\widehat{u}(k)| \leqslant \widehat{u}(0) \leqslant \tfrac{1}{4}, \tag{9}$$

and the 1st inequality is strict for $k \neq 0$. Hence, for $k \neq 0$, $\sqrt{1 - 4\widehat{u}(k)} \neq 0$. By the Riemann–Lebesgue theorem, $\widehat{f}(k)$ and $\widehat{u}(k)$ are both continuous and vanish at infinity, and hence, we must have that

$$\widehat{f}(k) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4\widehat{u}(k)} \tag{10}$$

for all sufficiently large k, and in any case $\widehat{f}(k) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$. But by continuity and the fact that $\sqrt{1 - 4\widehat{u}(k)} \neq 0$ for any $k \neq 0$, the sign cannot switch. Hence, (10) is valid for all k, including $k = 0$, again by continuity. At $k = 0$, $a = \frac{1}{2} - \sqrt{1 - 4b}$, which proves (3). The fact that c_n as specified in (7) satisfies $c_n \sim n^{-3/2}$ is a simple application of Stirling's formula, and it shows that the power series for $\sqrt{1 - z}$ converges absolutely and uniformly everywhere on the closed unit disc. Since $|4\widehat{u}(k)| \leqslant 1$, $\sqrt{1 - 4\widehat{u}(k)} = 1 - \sum_{n=1}^{\infty} c_n (4\widehat{u}(k))^n$. Inverting the Fourier transform yields (6), and since $\int_{\mathbb{R}^d} 4^n \star^n u(x)\,\mathrm{d}x \leqslant 1$, the convergence of the sum in $L^1(\mathbb{R}^d)$ follows from the convergence of $\sum_{n=1}^{\infty} c_n$. The final statement follows from the fact that if f is defined in terms of u in this manner, then (10) is valid, and then (8) and (5) are satisfied. ■

Theorem 2. Let $f \in L^1(\mathbb{R}^d)$ satisfy (1) and $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \frac{1}{2}$. Then, $\int_{\mathbb{R}^d} |x| f(x)\,\mathrm{d}x = \infty$.

Proof. If $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \frac{1}{2}$, $\int_{\mathbb{R}^d} 4u(x)\,\mathrm{d}x = 1$, then $w(x) = 4u(x)$ is a probability density, and we can write $f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n \star^n w$. Aiming for a contradiction, suppose that $|x| f(x)$ is integrable. Then, $|x| w(x)$ is integrable. Let $m := \int_{\mathbb{R}^d} x w(x)\,\mathrm{d}x$. Since the 1st moments add under convolution, the trivial inequality $|m||x| \geqslant m \cdot x$ yields

$$|m| \int_{\mathbb{R}^d} |x| \star^n w(x)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^d} m \cdot x \star^n w(x)\,\mathrm{d}x = n|m|^2.$$

It follows that $\int_{\mathbb{R}^d} |x| f(x)\,\mathrm{d}x \geqslant \frac{|m|}{2}\sum_{n=1}^{\infty} n c_n = \infty$. Hence, $m = 0$.

Suppose temporarily that in addition, $|x|^2 w(x)$ is integrable. Let σ^2 be the variance of w that is, $\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x) \mathrm{d}x$. Define the function $\varphi(x) = \min\{1, |x|\}$. Then,

$$\int_{\mathbb{R}^d} |x| \star^n w(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x) n^{d/2}\, \mathrm{d}x \geqslant n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x) n^{d/2}\, \mathrm{d}x.$$

By the central limit theorem, since φ is bounded and continuous,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x) n^{d/2}\, \mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\gamma(x)\, \mathrm{d}x =: C > 0, \tag{11}$$

where $\gamma(x)$ is a centered Gaussian probability density with variance σ^2.

This shows that there is a $\delta > 0$ such that for all sufficiently large n, $\int_{\mathbb{R}^d} |x| \star^n w(x)\, \mathrm{d}x \geqslant \sqrt{n}\delta$, and then since $c_n \sim n^{-3/2}$, $\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^d} |x| \star^n w(x)\, \mathrm{d}x = \infty$.

To remove the hypothesis that w has finite variance, note that if w is a probability density with zero mean and infinite variance, $\star^n w(n^{1/2}x) n^{d/2}$ is "trying" to converge to a "Gaussian of infinite variance". In particular, one would expect that for all $R > 0$,

$$\lim_{n\to\infty} \int_{|x|\leqslant R} \star^n w(n^{1/2}x) n^{d/2}\, \mathrm{d}x = 0 \tag{12}$$

so that the limit in (11) has the value 1. The proof then proceeds as above. The fact that (12) is valid is a consequence of Lemma 6 below, which is closely based on the proof of [2, Corollary 1]. ■

Theorem 3. Let $f \in L^1(\mathbb{R}^d)$ satisfy (5), $\int_{\mathbb{R}^d} x u(x)\, \mathrm{d}x = 0$ and $\int_{\mathbb{R}^d} |x|^2 u(x)\, \mathrm{d}x < \infty$. Then, for all $0 \leqslant p < 1$,

$$\int_{\mathbb{R}^d} |x|^p f(x)\ \mathrm{d}x < \infty. \tag{13}$$

Proof. We may suppose that f is not identically 0. Let $t := 4\int_{\mathbb{R}^d} u(x)\, \mathrm{d}x \leqslant 1$. Then, $t > 0$. Define $w := t^{-1}4u$; w is a probability density and

$$f(x) = \tfrac{1}{2} \sum_{n=1}^{\infty} c_n t^n \star^n w(x). \tag{14}$$

By hypothesis, w has a zero mean and variance $\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x)\,\mathrm{d}x < \infty$. Since variance is additive under convolution,

$$\int_{\mathbb{R}^d} |x|^2 \star^n w(x)\,\mathrm{d}x = n\sigma^2.$$

By Hölder's inequality, for all $0 < p < 2$, $\int_{\mathbb{R}^d} |x|^p \star^n w(x)\,\mathrm{d}x \leqslant (n\sigma^2)^{p/2}$. It follows that for $0 < p < 1$,

$$\int_{\mathbb{R}^d} |x|^p f(x)\,\mathrm{d}x \leqslant \frac{1}{2}(\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n < \infty,$$

again using the fact that $c_n \sim n^{-3/2}$. ∎

Remark 4. In the subcritical case $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x < \frac{1}{2}$, the hypothesis that $\int_{\mathbb{R}^d} xu(x)\,\mathrm{d}x = 0$ is superfluous, and one can conclude more. In this case, the quantity t in (14) satisfies $0 < t < 1$, and if we let m denote the mean of w, $\int_{\mathbb{R}^d} |x|^2 \star^n w(x)\,\mathrm{d}x = n^2|m|^2 + n\sigma^2$. For $0 < t < 1$, $\sum_{n=1}^{\infty} n^2 c_n t^n < \infty$ and we conclude that $\int_{\mathbb{R}^d} |x|^2 f(x)\,\mathrm{d}x < \infty$. Finally, the final statement of Theorem 1 shows that critical case functions f satisfying the hypotheses of Theorem 2 are readily constructed.

Theorem 2 implies that when $\int f = \frac{1}{2}$, f cannot decay faster than $|x|^{-(d+1)}$. However, integrable solutions f of (1) such that $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x < \frac{1}{2}$ can decay more rapidly, as indicated in the previous remark. In fact, they may even have finite exponential moments, as we now show.

Consider a nonnegative, integrable function u, which integrates to $r < \frac{1}{4}$ and satisfies

$$\int_{\mathbb{R}^d} u(x)e^{\lambda|x|}\,\mathrm{d}x < \infty \tag{15}$$

for some $\lambda > 0$. The Laplace transform of u is $\widetilde{u}(p) := \int e^{-px}u(x)\,\mathrm{d}x$, which is analytic for $|p| < \lambda$, and $\widetilde{u}(0) < \frac{1}{4}$. Therefore, there exists $0 < \lambda_0 \leqslant \lambda$ such that, for all $|p| \leqslant \lambda_0$, $\widetilde{u}(p) < \frac{1}{4}$. By Theorem 1, $f(x) := \frac{1}{2}\sum_{n=1}^{\infty} 4^n c_n (\star^n u)(x)$ is an integrable solution of (1). For $|p| \leqslant \lambda_0$, it has a well-defined Laplace transform $\widetilde{f}(p)$ given by

$$\widetilde{f}(p) = \int e^{-px} f(x)\,\mathrm{d}x = \frac{1}{2}(1 - \sqrt{1 - 4\widetilde{u}(p)}), \tag{16}$$

which is analytic for $|p| \leqslant \lambda_0$. Note that $e^{s|x|} \leqslant \prod_{j=1}^{d} e^{|sx_j|} \leqslant \frac{1}{d}\sum_{j=1}^{d} e^{d|sx_j|} \leqslant \frac{2}{d}\sum_{j=1}^{d}\cosh(dsx_j)$. Thus, for $|s| < \delta := \lambda_0/d$, $\int_{\mathbb{R}^d}\cosh(\mathrm{d}sx_j)f(x)\,\mathrm{d}x < \infty$ for each j, and hence $|s| < \delta$, $\int_{\mathbb{R}^d} e^{s|x|}f(x)\,\mathrm{d}x < \infty$.

However, there are no integrable solutions of (1) that have compact support: we have seen that all solutions of (1) are nonnegative, and if A is the support of a nonnegative integrable function, the Minkowski sum $A + A$ is the support of $f \star f$.

Remark 5. One might also consider the inequality $f \leqslant f \star f$ in $L^1(\mathbb{R}^d)$, but it is simple to construct solutions that have both signs. Consider any radial Gaussian probability density g. Then, $g \star g(x) \geqslant g(x)$ for all sufficiently large $|x|$, and taking $f := ag$ for a sufficiently large, we obtain $f < f \star f$ everywhere. Now, on a small neighborhood of the origin, replace the value of f by -1. If the region is taken small enough, the new function f will still satisfy $f < f \star f$ everywhere.

We close with a lemma validating (12) that is closely based on a construction in [2].

Lemma 6. Let w be a mean zero, infinite variance probability density on $\mathbb{R}^d$. Then, for all $R > 0$, (12) is valid.

Proof. Let $X_1,\dots,X_n$ be n independent samples from the density w, and let B_R denote the centered ball of radius R. The quantity in (12) is $p_{n,R} := \mathbb{P}(n^{-1/2}\sum_{j=1}^{n} X_j \in B_R)$. Let $\widetilde{X}_1,\dots,\widetilde{X}_n$ be another n independent samples from the density w, independent of the 1st n. Then, also $p_{n,R} := \mathbb{P}(-n^{-1/2}\sum_{j=1}^{n}\widetilde{X}_j \in B_R)$. By the independence and the triangle inequality,

$$p_{n,R}^2 \leq \mathbb{P}(n^{-1/2}\sum_{j=1}^{n}(X_j - \widetilde{X}_j) \in B_{2R}).$$

The random variable $X_1 - \widetilde{X}_1$ has a zero mean, an infinite variance, and an even density. Therefore, without loss of generality, we may assume that $w(x) = w(-x)$ for all x.

Pick $\varepsilon > 0$, and choose a large value σ_0 such that $(2\pi\sigma_0^2)^{-d/2}R^d|B| < \varepsilon/3$, where $|B|$ denotes the volume of the unit ball B. The point of this is that if G is a centered Gaussian random variable with variance *at least* σ_0^2, the probability that G lies in *any* particular translate $B_R + y$ of the ball of radius R is no more than $\varepsilon/3$. Let $A \subset \mathbb{R}^d$ be a centered cube such that

$$\int_A |x|^2 w(x)\,\mathrm{d}x =: \sigma^2 \geqslant 2\sigma_0^2 \quad \text{and} \quad \int_A w(x)\,\mathrm{d}x > \tfrac{3}{4}$$

and note that since A and w are even, $\int_A xw(x)\,\mathrm{d}x = 0$.

It is then easy to find mutually independent random variables X, Y, and α such that X takes values in A, has zero mean and variance σ^2 and α is a Bernoulli variable with success probability $\int_A w(x)\,\mathrm{d}x$, and finally, such that $\alpha X + (1-\alpha)Y$ has the probability density w. Taking independent identically distributed (i.i.d.) sequences of such random variables, $w(n^{1/2}x)n^{d/2}$ is the probability density of $W_n := n^{-1/2}\sum_{j=1}^n \alpha_j X_j + n^{-1/2}\sum_{j=1}^n (1-\alpha_j)Y_j$, and we seek to estimate the expectation of $1_{B_R}(W_n)$. We first take the conditional expectation, given the values of the αs and the Ys, and we define $\hat{n} = \sum_{j=1}^n \alpha_j$. These conditional expectations have the form $\mathbb{E}\left[1_{B_R+y}\left(\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right]$ for some translate $B_R + y$ of B_R, the ball of radius R. The sum $n^{-1/2}\sum_{j=1}^n \alpha_j X_j$ is actually the sum of $\hat{n}$ i.i.d. random variables with zero mean and variance σ^2/n. The probability that $\hat{n}$ is significantly less than $\frac{3}{4}n$ is negligible for large n; by classical estimates associated with the law of large numbers, for all n large enough, the probability that $\hat{n} < n/2$ is no more than $\varepsilon/3$. Now, let Z be a Gaussian random variable with mean zero and variance $\sigma^2\hat{n}/n$, which is at least σ_0^2 when $\hat{n} \geqslant n/2$. Then, by the multivariate version [8] of the Berry–Esseen theorem [1, 5], a version of the central limit theorem with rate information, there is a constant K_d depending only on d such that

$$\left|\mathbb{E}\left[1_{B_R+y}\left(\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right] - \mathbb{P}\{Z \in B_R + y\}\right| \leqslant K_d \hat{n}\frac{\mathbb{E}|X_1|^3}{n^{3/2}} \leqslant K_d\frac{\mathbb{E}|X_1|^3}{n^{1/2}}.$$

Since A is bounded, $\mathbb{E}|X_1|^3 < \infty$, and hence for all sufficiently large n, when $\hat{n} \geqslant n/2$,

$$\mathbb{E}\left[1_{B_R+y}\left(\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right] \leqslant \frac{2}{3}\varepsilon.$$

Since this is uniform in y, we finally obtain $\mathbb{P}(W_n \in B_R) \leq \varepsilon$ for all sufficiently large n. Since $\varepsilon > 0$ is arbitrary, (12) is proved. ■

Funding

This work was supported by the US National Science Foundation [DMS-1764254 to E.A.C., DMS-1802170 to I.J., DMS-1856645 to M.P.L.].

Acknowledgments

We close by thanking the anonymous referee for useful suggestions.

References

[1] Berry, A. "The accuracy of the Gaussian approximation to the sum of independent Variates." *Trans. Amer. Math. Soc.* 49 (1941): 122–36.

[2] Carlen, E. A., E. Gabetta, and E. Regazzini. "Probabilistic investigation on explosion of solutions of the KAC equation with infinite initial energy." *J. Appl. Probab.* 45 (2008): 95–106.

[3] Carlen, E. A., I. Jauslin, and E. H. Lieb. "Analysis of a simple equation for the ground state energy of the Bose gas." *Pure and Applied Analysis* 2, no. 3 (2020): 659–84.

[4] Carlen, E. A., I. Jauslin, and E. H. Lieb. "Analysis of a simple equation for the ground state of the Bose gas II: monotonicity, convexity and condensate fraction." Preprint arXiv:2010.13882.

[5] Esseen, C.-G. "A moment inequality with an application to the central limit theorem." *Skand. Aktuarietidskr.* 39 (1942): 160–70.

[6] Lieb, E. H. and M. Loss. *Analysis*, vol. 14. Graduate Studies in Mathematics. Providence RI: AMS, 1996.

[7] Matolcsi, M. and C. Vinuesa. "Improved bounds on the supremum of autoconvolutions." *J. Math. Anal. Appl.* 372 (2010): 439–47.

[8] Raič, M. "A multivariate Berry–Esseen theorem with explicit constants." *Bernoulli* 25 (2019): 2824–53.

[9] Stein, E. and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton NJ: Princeton University Press, 1971.