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SPECTRAL GAPS FOR REVERSIBLE MARKOV PROCESSES WITH CHAOTIC INVARIANT MEASURES: THE KAC PROCESS WITH HARD SPHERE COLLISIONS IN THREE DIMENSIONS

BY ERIC CARLEN[1], MARIA CARVALHO[2] AND MICHAEL LOSS[3]

[1]*Department of Mathematics, Rutgers University, carlen@math.rutgers.edu*
[2]*CMAFcIO, Departamento de Mathemática, Faculdade de Ciênces, Universidade de Lisboa, mcarvalh@math.rutgers.edu*
[3]*School of Mathematics, Georgia Institute of Technology, loss@math.gatech.edu*

We develop a method for producing estimates on the spectral gaps of reversible Markov jump processes with chaotic invariant measures, that is effective in the case of degenerate jump rates, and we apply it to prove the Kac conjecture for hard sphere collision in three dimensions.

1. Introduction. In a seminal paper of 1956, Mark Kac [12] introduced a family of continuous time reversible Markov jump processes on the sphere $S^{N-1}(\sqrt{N})$ of radius $\sqrt{N}$ in $\mathbb{R}^N$. This family of processes, and its generalizations, have drawn the attention of many researchers. Kac was motivated by a connection, in the large N limit, to the nonlinear Boltzmann equation. The connection arises through a particular "asymptotic independence" property of sequences $\{\mathrm{d}\mu_N\}$, where $\mathrm{d}\mu_N$ is a probability measure on S^{N-1}. This property is possessed, in particular, by the sequence $\{\mathrm{d}\sigma_N\}$ of uniform probability measures on $S^{N-1}(\sqrt{N})$. Let $\vec{v} = (v_1, \dots, v_N)$ denote a generic point on $S^{N-1}(\sqrt{N})$ of radius $\sqrt{N}$. Let ϕ be any bounded continuous function on $\mathbb{R}^k$ and $\mathrm{d}\gamma = (2\pi)^{-1/2} e^{-v^2/2}\,\mathrm{d}v$ be the unit Gaussian probability measure on $\mathbb{R}$. As is well known, going back at least to Mehler [14],

$$\lim_{N\to\infty} \int_{S^{N-1}(\sqrt{N})} \phi(v_1, \dots, v_k)\,\mathrm{d}\sigma_N = \int_{\mathbb{R}^k} \phi(v_1, \dots, v_k)\,\mathrm{d}\gamma^{\otimes k}.$$

As long as one only looks at coordinates belonging to a fixed, finite set, in the large N limit, the coordinates in this set are asymptotically independent. The main result of [12] concerned sequences of probability measures $\{\mathrm{d}\mu_N\}$ on $S^{N-1}(\sqrt{N})$ with the property that, for some probability density f on $\mathbb{R}$ with zero mean and unit variance,

$$\lim_{N\to\infty} \int_{S^{N-1}(\sqrt{N})} \phi(v_1, \dots, v_k)\,\mathrm{d}\mu_N = \int_{\mathbb{R}^k} \phi(v_1, \dots, v_k) \prod_{j=1}^{k} f(v_j)\,\mathrm{d}v_j,$$

in which case the sequence $\{\mathrm{d}\mu_N\}$ was said by Kac to be $f(v)\,\mathrm{d}v$ *chaotic*. He proved that chaoticity was propagated in time by solutions of the forward Kolmogorov equations associated to the Kac processes. Moreover, if $\{\mathrm{d}\mu_N(t)\}$ is the sequence of laws at time t starting from an $f(v)\,\mathrm{d}v$ chaotic sequence, $\{\mathrm{d}\mu_N(t)\}$ is $f(t,v)\,\mathrm{d}v$ chaotic where $f(t,v)$ is the solution of the *Kac–Boltzmann equation* with initial data $f(v)$. (The Kac–Boltzmann equation is a simple model of the Boltzmann equation for a gas in one dimension.) He also made a conjecture, that went unsolved for a long time, concerning the spectral gap of the generator of this family of processes. Since the processes are reversible, their generators are self-adjoint, and it is not hard to see that the null space is spanned by the constants. Kac conjectured a gap Δ_N separating 0 from the rest of the spectrum that is bounded below uniformly in N. That is,

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$\lim_{N\to\infty} \Delta_N > 0$. This was finally proved by Janvresse in 2000 [11], and shortly afterwards the exact value of Δ_N for all N was determined in [3].

A few years after his original work, Kac returned to these problems [13], but this time for a physically realistic model of a gas in three dimensions undergoing "hard sphere" collisions that conserve energy and momentum. As he showed, this physical model would have, through propagation of chaos, a direct connection to the actual Boltzmann equation for hard sphere collisions, and not only a toy model of it. However, in the physical model, the rates at which different pairs of molecules collide depend on their velocities: The rates are not bounded away from 0, and there is no bound from above that is uniform in N. It is much harder to estimate spectral gaps for the generators of jump processes with rates that are not bounded from below, and the lack of an upper bound that is uniform in N makes it much harder to prove propagation of chaos.

In this paper, we prove the Kac conjecture for the Kac model with hard sphere collisions in $\mathbb{R}^3$. We do so using a method that has three essential components. These are:

(1) The introduction of a *conjugate process*, in which at each step all but one of the velocities are updated. The rates in this process are still not bounded below, but they depend only on the one velocity that is left fixed during the jump. There is also a simple connection between the spectral gaps of the original process and the conjugate process, and the central problem becomes the determination of the spectral gap for the conjugate process.

(2) Quantitative estimates on the chaoticity of the sequence of invariant measures: We prove and apply estimates quantitatively expressing the near independence of any finite set of coordinates for large N.

(3) A trial function decomposition: We decompose any trial function f for the spectral gap problem into 3 pieces, $f = s + g + h$ that are mutually orthogonal in the L^2 space for the invariant measure, and due to quantitative chaos estimates, are nearly orthogonal with respect to the inner product given by the Dirichlet form of the conjugate process. Each of these pieces has a particular special structure that facilitates the proof of estimates of the type we seek.

The first two components have been present in our work on Kac-type models since our early papers [3, 5] on the models (as in [12]) with uniform jump rates, though in the early papers, the conjugate process is not considered explicitly as a process. However, the connection between its spectral gap and the spectral gap for the Kac process has been central to the approach from the beginning. Work by two of us and Jeff Geronimo [7] dealt with the quantitative chaos estimates needed for the three-dimensional energy and momentum conserving collision considered here, but applied them to "Maxwellian molecules" models which, unlike the hard sphere model, has rates that are bounded below. There, too, the approach yielded the exact value of the the spectral gap for a wide class of "Maxwellian molecules" models.

Finally, in [6], we proved the Kac conjecture for a "hard sphere" model with one-dimensional velocities, and introduced a somewhat simpler version of component (3), the trial function decomposition. In application to kinetic theory, as explained in [6], the spectral gap in the symmetric sector, that is, for functions that are invariant under permutations of coordinates, is especially important. It is this quantity that can be related to the spectral gap for the linearized Boltzmann equation, and one would like to have explicit estimates on this gap. Therefore, in [6], we worked hard to render all estimates as sharp and explicit as possible, and to treat only the symmetric sector for which fewer estimates were required.

It was clear to us at the time we wrote [6] that we had a general method that would prove the existence of a spectral gap, uniformly in N, for the physical three-dimensional hard sphere Kac model, and we announced this in several lectures. The result is quoted in reference (9) of [15], as a personal communication, and used in the development of the quantitative treatment of propagation of chaos that is provided there. After our paper [6] appeared with the details provided only for the symmetric sector and the one-dimensional model, Stéphane Mischler and Clément Mouhot asked us several times to provide the details. This paper answers their request, and moreover, in the course of preparing this answer, it has provided a clearer picture of how the method explained in [6] can be extended and applied to more complicated models, such as the main example treated here.

The method to be explained here may be applied to a wide class of sequences of reversible Markov jump processes whose sequence of invariant measures satisfies certain “quantitative chaos” estimates that are specified here. The method is not at all restricted to the treatment of the symmetric sector, and perhaps had we explained the method in [6] without obscuring it behind the details of so many explicit computations, necessary for the precise quantitative estimates obtained there; this would have been clear some years ago.

Therefore, in the present paper, we prove the Kac conjecture for hard sphere collisions in three dimensions without any symmetry condition in as simple a manner as possible to provide a clear view of the method. To do this, we make use of constants C that change from line to line but are independent of N that are not explicitly evaluated here, but easily could be—at the expense of more pages and less clarity.

In addition to the applications to quantitative propagation of chaos developed in [15], uniform bounds on the spectral gap are important in certain problems concerning the hydrodynamic limits of certain kinetic models, as explained in [10]. These authors considered a one-dimensional model [9] essentially equivalent to the one considered in [6], and asked for the spectral gap. Sasada [17] provided the answer to the question they raised, noting that she could not simply apply the result of [6] as it applied to the symmetric sector only. This is true, but as shown here, the method used in [6] may readily extended to answer a much broader range of questions. Much beautiful work has been done on the question of estimating spectral gaps for Kac-type processes, and we refer to the papers of Sasada [16] and Caputo [1, 2], in addition to our own papers cited here, for significant contributions. However, it is not clear to us that any of the other methods that have been developed for this class of models applies to the main example at hand which is considerably more complex than the models considered in most other work.

1.1. *The Kac collision process.* For $N \in \mathbb{N}$, $p \in \mathbb{R}^3$ and $E > |p|^2$, let $\mathcal{S}_{N,E,p}$ be the set consisting of N-tuples $\vec{v} = (v_1, \dots, v_N)$ of vectors v_j in $\mathbb{R}^3$ with $\frac{1}{N}\sum_{j=1}^N |v_j|^2 = E$ and $\frac{1}{N}\sum_{j=1}^N v_j = p$. In what follows, a point $\vec{v} \in \mathcal{S}_{N,E,p}$ specifies the velocities of a collection of N particles with mass 2, so that E is the kinetic energy per particle, and p is one-half the momentum per particle. The Markov jump process introduced by Mark Kac [13] describes a random binary collision process for the N particles, in which the collisions conserve both energy and momentum, and thus if the process starts on $\mathcal{S}_{N,E,p}$, it will remain on $\mathcal{S}_{N,E,p}$ for all time.

Recall that a random variable T with values in $(0, \infty)$ is *exponential with parameter* λ in case $\Pr(T \geq t) = e^{-\lambda t}$. When the collision process begins, associated to each pair (v_i, v_j), $i < j$, is an exponential random variable $T_{i,j}$ with parameter

$$\lambda_{i,j} = N\binom{N}{2}^{-1} |v_i - v_j|^{\alpha}, \tag{1}$$

where $0 \leq \alpha \leq 2$, and $\alpha = 1$ is the case of main interest: As explained in [13], (1) is motivated by a connection between the Kac process and the Boltzmann equation, and $\alpha = 1$ corresponds to “hard-sphere collisions.”

$T_{i,j}$ represents the waiting time for particles i and j to collide, and the set of these random times is taken to be independent. The first collision occurs at time

$$T = \min_{i<j}\{T_{i,j}\}. \tag{2}$$

As is well known [8], the minimum of an independent set of exponential random variables is itself exponential, and the parameter of the minimum is the sum of the parameters of the random variables in the set. In particular, if $\alpha = 0$, T is exponential with parameter N, and the expected waiting time for the first collision of some pair to occur is $1/N$.

At the time T, the pair (i, j) furnishing the minimum collide: The state of the process "jumps" from $(v_1, \dots, v_N)$ to $(v_1, v_2, \dots, v_i^*, \dots, v_j^*, \dots, v_N)$, where only v_i and v_j have changed. Since the process is conceived to model momentum and energy conserving collisions, we require that

$$v_i^* + v_j^* = v_i + v_j \quad \text{and} \quad |v_i^*|^2 + |v_j^*|^2 = |v_i|^2 + |v_j|^2. \tag{3}$$

Then, by the parallelogram law, it follows that

$$|v_i^* - v_j^*| = |v_i - v_j|. \tag{4}$$

Given v_i and v_j, the kinematically possible collisions of particles i and j; that is, those satisfying (3), may be parameterized in term of a unit vector $\sigma \in S^2$, the unit sphere in $\mathbb{R}^3$ as follows:

$$\begin{aligned} v_i^*(\sigma) &= \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2}\sigma, \\ v_j^*(\sigma) &= \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2}\sigma. \end{aligned} \tag{5}$$

The particular kinematically possible collision that occurs at time T is selected according to the following rule: There is given, in the specification of the process, a nonnegative, even function b on $[-1, 1]$ such that for any fixed $\sigma' \in S^2$, with $\mathrm{d}\sigma$ denoting the uniform probability measure on S^2,

$$\int_{S^2} b(\sigma \cdot \sigma')\,\mathrm{d}\sigma = 1 \quad \text{or, equivalently,} \quad \frac{1}{2}\int_{-1}^{1} b(t)\,\mathrm{d}t = 1. \tag{6}$$

The example of main interest turns out to be

$$b(x) = 1. \tag{7}$$

When $\alpha = 1$ and b is given by (7), the Kac process models "hard sphere" or "billiard ball" collisions [13]. (There are two standard parameterizations of the set of energy and momentum conserving collisions, the "σ parameterization" given by (5), and the "n parameterization." While the latter is often used in physics texts and is used in [13], the former, used here, has advantages. One is that in this parameterization, b is constant, while in the other it is not due to a nonconstant Jacobian relating the two parameterizations. See Appendix A.1 of [4] for more information; equation (A.18) of [4] is the formula relating the b functions for the two representations.)

In any case, as long as $v_i \neq v_j$, $b(\sigma \cdot (v_i - v_j)/|v_i - v_j|)$ is a probability density on S^2. At time T, σ is selected from the law $b(\sigma \cdot (v_i - v_j)/|v_i - v_j|)\,\mathrm{d}\sigma$, and then the process executes the collision step in which v_i^* and v_j^* are given by (5). (If $v_i = v_j$, no jump is made.) Then, all of the waiting times are "reset" and the process begins afresh. This completes the probabilistic description of the one parameter family of Kac collision process.

This one parameter family of Kac collision processes is a little more general than the one considered by Kac: There is an extra parameter α that ranges between 0 and 2. The case $\alpha = 0$ corresponds to Maxwellian molecules as in [12] or [7]. The case $\alpha = 1$ is the hard sphere case that is our main focus. The case $\alpha = 2$ is the case of "super hard spheres" and estimates for this case will be useful in our study of $\alpha = 1$. Villani [19] discovered in the context of entropy production estimates that analysis of the nonphysical case $\alpha = 2$ could provide very helpful information on the physical cases $\alpha \leq 1$, and we make essential use of this insight in our analysis of spectral gaps.

1.2. *The generator of the Kac process*. The object of our investigation is the spectral gap for the generator of the Markov semigroup associated to this process. For *any* continuous function f on $\mathcal{S}_{N,E,p}$, in particular without any symmetry assumption, define

$$L_{N,\alpha} f(\vec{v}) = \lim_{h\to 0} \frac{1}{h}\big[\mathbb{E}\{f(\vec{v}(h)) - f(\vec{v}) \mid \vec{v}(0) = \vec{v}\}\big].$$

We can write this more explicitly as

$$L_{N,\alpha} f(\vec{v}) = -N\binom{N}{2}^{-1} \sum_{i<j} |v_i - v_j|^\alpha \big[f(\vec{v}) - [f]^{(i,j)}(\vec{v})\big], \tag{8}$$

where

$$[f]^{(i,j)}(\vec{v}) = \int_{S^2} b\left(\sigma \cdot \frac{v_i - v_j}{|v_i - v_j|}\right) f(R_{i,j,\sigma}\vec{v})\,\mathrm{d}\sigma \tag{9}$$

and

$$(R_{i,j,\sigma}\vec{v})_k = \begin{cases} v_i^*(\sigma) & k = i, \\ v_j^*(\sigma) & k = j, \\ v_k & k \neq i, j. \end{cases}$$

By (4) and (5),

$$\cos\theta := \sigma \cdot \frac{v_i - v_j}{|v_i - v_j|} = \frac{v_i^* - v_j^*}{|v_i^* - v_j^*|} \cdot \frac{v_i - v_j}{|v_i - v_j|}.$$

By this and (4) once again, rates for the jump from $\vec{v}$ to $R_{i,j,\sigma}\vec{v}$ and from $R_{i,j,\sigma}\vec{v}$ to $\vec{v}$ are equal. This is the property of "detailed balance" or "microscopic reversibility." The analytic expression of this is self-adjointness of the generator $L_{N,\alpha}$.

Let $\mathrm{d}\sigma_N$ denote the uniform probability measure on $\mathcal{S}_{N,E,p}$. (Note that $\mathcal{S}_{N,E,p}$ is isometric to a sphere of radius $\sqrt{N(E - |p|^2)}$ in $\mathbb{R}^{3N-4}$, and by uniform, we mean uniform with respect to the symmetries of this sphere.)

For any two unit vectors σ and ω, one sees from (5) that

$$R_{i,j,\sigma}(R_{i,j,\omega}\vec{v}) = R_{i,j,\sigma}\vec{v}. \tag{10}$$

From this and the fact that the measure $\mathrm{d}\sigma_N \otimes \mathrm{d}\sigma$ is invariant under

$$(\vec{v}, \sigma) \mapsto \big(R_{i,j,\sigma}\vec{v}, (v_i - v_j)/|v_i - v_j|\big),$$

it follows that for any two continuous functions f and g on $\mathcal{S}_{N,E,p}$,

$$\langle g, L_{N,\alpha} f\rangle_{L^2(\sigma_N)} = \langle L_{N,\alpha} g, f\rangle_{L^2(\sigma_N)},$$

where $\langle\cdot,\cdot\rangle_{L^2(\sigma_N)}$ denotes the inner product on $L^2(\mathcal{S}_{N,E,p}, \sigma_N)$. Thus, $L_{N,\alpha}$ is a self-adjoint operator on $L^2(\mathcal{S}_{N,E,p}, \sigma_N)$. Notice that the formulas (8) and (9) do not involve the parameters E and p, and hence our notation references only N and α.

Define the quadratic form $\mathcal{E}_{N,\alpha}$ by $\mathcal{E}_{N,\alpha}(f, f) = -\langle f, L_{N,\alpha} f\rangle_{L^2(\sigma_N)}$. A simple computation using (10) shows that

$$\mathcal{E}_{N,\alpha}(f, f) = \frac{N}{2}\binom{N}{2}^{-1} \sum_{i<j} \int_{\mathcal{S}_{N,E,p}} \int_{S^2} |v_i - v_j|^\alpha b\left(\sigma \cdot \frac{v_i - v_j}{|v_i - v_j|}\right) \times \big[f(\vec{v}) - f(R_{i,j,\sigma}\vec{v})\big]^2 \,\mathrm{d}\sigma\,\mathrm{d}\sigma_N. \tag{11}$$

One sees from this expression that $L_{N,\alpha}$ is a negative semidefinite operator, and that provided b is continuous at 1, $L_{N,\alpha} f = 0$ if and only if f is constant. We are interested in the *spectral gap* of the operator $L_{N,\alpha}$ on $L^2(\mathcal{S}_{N,E,p}, \sigma_N)$:

$$\Delta_{N,\alpha}(E, p) = \inf\{\mathcal{E}_{N,\alpha}(f, f) : \langle f, 1\rangle_{L^2(\sigma_N)} = 0 \text{ and } \|f\|^2_{L^2(\sigma_N)} = 1\}. \tag{12}$$

For fixed N, the dependence of $\Delta_{N,E,p}$ on E and p is quite simple: Consider the point transformation

$$\phi_{E,p}(v_1, \dots, v_N) := \frac{1}{\sqrt{E - |p|^2}}(v_1 - p, \dots, v_N - p)$$

that identifies $\mathcal{S}_{N,E,p}$ with $\mathcal{S}_{N,1,0}$. The induced transformation $U_{E,p}$ from $L^2(\mathcal{S}_{N,1,0}, \sigma_N)$ to $L^2(\mathcal{S}_{N,E,p}, \sigma_N)$ given by $U_{E,p} f = f \circ \phi_{E,p}$ is evidently unitary. A simple computation then shows that

$$\mathcal{E}_{N,\alpha}(U_{E,p} f, U_{E,p} f) = (E - |p|^2)^{\alpha/2} \mathcal{E}_{N,\alpha}(f, f,). \tag{13}$$

As an immediate consequence,

$$\Delta_{N,\alpha}(E, p) = (E - |p|^2)^{\alpha/2} \Delta_{N,\alpha}(1, 0). \tag{14}$$

The dependence of $\Delta_{N,\alpha}(E, p)$ on N is not so simple. Nonetheless, we have seen that the problem of estimating the quantity $\Delta_{N,\alpha}(E, p)$ is essentially the same as the problem of estimating $\Delta_{N,\alpha}(1, 0)$. We therefore simplify our notation.

DEFINITION 1.1 (Spectral gap). The *spectral gap for the N particle Kac model* is the quantity

$$\Delta_{N,\alpha} := \Delta_{N,\alpha}(1, 0). \tag{15}$$

In what follows, we write $\mathcal{S}_N$ to denote $\mathcal{S}_{N,1,0}$, and consider the Kac process on $\mathcal{S}_N$ unless other values of E and p are explicitly specified. The *Kac conjecture for hard sphere collisions* [13] is that $\liminf_{N\to\infty} \Delta_{N,1} > 0$. Our main result shows somewhat more.

THEOREM 1.2 (Spectral gap for the Kac model with $0 \le \alpha \le 2$). *For each continuous nonnegative even function b on $[-1, 1]$ satisfying* (6), *and for each $\alpha \in [0, 2]$, there is a strictly positive constant K depending only on b and α, and explicitly computable, such that*

$$\Delta_{N,\alpha} \ge K > 0$$

for all N. In particular, this is true with b given by (7) *and $\alpha = 1$, the 3-dimensional hard sphere Kac model.*

1.3. *The conjugate Kac process and its generator.* Our method involves the introduction of another family of reversible Markov jump processes on $\mathcal{S}_N$ that are *conjugate* to the Kac process. For fixed N and α, this process is described as follows: Given $\vec{v} \in \mathcal{S}_N$, Let $\{\widehat{T}_1, \dots, \widehat{T}_N\}$ be N independent exponential variables such that the parameter $\lambda_k(\vec{v})$ of $\widehat{T}_k$ is

$$\lambda_k(\vec{v}) = \frac{1}{N}\left[\frac{N^2 - (1 + |v_k|^2)N}{(N - 1)^2}\right]^{\alpha/2}.$$

Since the total energy is N and the total momentum is zero, the maximum possible value of $|v_k|^2$ on $\mathcal{S}_N$ is $N - 1$, and thus $\lambda_k \ge 0$, with equality only when $|v_k|^2$ takes on its maximal value.

The first jump time is $\widehat{T} = \min\{\widehat{T}_1, \dots, \widehat{T}_N\}$. At the jump time, if k is the index furnishing the minimum, $\vec{v}$ jumps to a new point on $\mathcal{S}_N$ such that v_k is unchanged, but conditional on v_k, the other coordinates are redistributed uniformly. That is, the process makes a conditional jump to uniform, conditional on v_k which is held fixed. After the jump, the process starts afresh. This completes the description of the *conjugate Kac process*.

Note that the conjugate process is trivial for $N = 2$, since then $v_2 = -v_1$, so that given one velocity, the other is known exactly, and the "conditional jump to uniform" is no jump at all in this case. However, already for $N = 3$, the process is far from trivial.

REMARK 1.3. If $|v_k|^2$ is close to its expected value of 1, then $\lambda_k(\vec{v}) \approx \frac{1}{N}$, which is exact for $\alpha = 0$. In this case, we have N independent Poisson clocks with rate $\frac{1}{N}$ each, so that the mean waiting time for *some* jump is 1.

For $\alpha > 0$, the rates $\lambda_k(\vec{v})$ are not bounded away from 0. However, *at most one of them can be very close to zero for any given state* $\vec{v}$. This is because, $\lambda_k(\vec{v}) = 0$ if and only if $|v_k|^2$ takes on its maximum value, $N - 1$. For at most one value of k, is it possible that $|v_k|^2 > \frac{1}{2}N$, and for $|v_j|^2 \le \frac{1}{2}N$, $\lambda_j(\vec{v}) = \frac{1}{2N} + \mathcal{O}(\frac{1}{N^2})$. Thus, for all $\alpha \in [0, 2]$, for large N, the expected waiting time for a jump is very close to $1/N$, and this one jump will bring $N - 1$ of the particles very close to equilibrium. If the expected waiting time were exactly $1/N$ and the jump took all N particles to equilibrium, the spectral gap would be exactly $1 - 1/N$. This is not misleading; we shall show that for the conjugate Kac process, the spectral gap is indeed $1 - 1/N$ plus lower order corrections.

To write down the generator, introduce the conditional expectation operators P_k, $k = 1, \dots, N$, defined as follows.

For any function ϕ in $L^2(\sigma_N)$, and any k with $1 \le k \le N$, define $P_k(\phi)$ to be the orthogonal projection of ϕ onto the subspace of $L^2(\sigma_N)$ consisting of square integrable functions that depend on $\vec{v}$ through v_k alone. That is, $P_k(\phi)$ is the unique element of $L^2(\sigma_N)$ of the form $f(v_k)$ such that

$$\int_{\mathcal{S}_N} \phi(\vec{v}) g(v_k)\, \mathrm{d}\sigma_N = \int_{\mathcal{S}_N} f(v_k) g(v_k)\, \mathrm{d}\sigma_N \tag{16}$$

for all continuous functions g on $\mathbb{R}^3$. In probabilistic language, $P_k\phi$ is the conditional expectation of ϕ given v_k:

$$P_k\phi(v) = \mathbb{E}\{\phi \mid v_k = v\}. \tag{17}$$

The generator of the conjugate Kac process is then given by

$$\widehat{L}_{N,\alpha} f = -\frac{1}{N} \sum_{k=1}^{N} \left[\frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2}\right]^{\alpha/2} [f - P_k f], \tag{18}$$

which is the analog of (8). Define the quadratic form $\mathcal{D}_{N,\alpha}$ by

$$\mathcal{D}_{N,\alpha}(f, f) = -\langle f, \widehat{L}_{N,\alpha} f\rangle_{L^2(\sigma_N)}.$$

A simple computation using (10) shows that

$$\mathcal{D}_{N,\alpha}(f, f) = \frac{1}{N} \sum_{k=1}^{N} \int_{\mathcal{S}_N} \left[\frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2}\right]^{\alpha/2} [f^2 - f P_k f]\, \mathrm{d}\sigma_N. \tag{19}$$

The spectral gap for the conjugate Kac process is the quantity defined by

$$\widehat{\Delta}_{N,\alpha} = \inf\{\mathcal{D}_{N,\alpha}(f, f) : \langle f, 1\rangle_{L^2(\sigma_N)} = 0 \text{ and } \|f\|^2_{L^2(\sigma_N)} = 1\}. \tag{20}$$

The following theorem bears out the heuristic discussion in Remark 1.3.

THEOREM 1.4. *For all $N \geq 3$, and all $\alpha \in [0,2]$, $\widehat{\Delta}_{N,\alpha} > 0$. Moreover, there is a constant C independent of N such that*

$$\widehat{\Delta}_{N,\alpha} \geq 1 - \frac{1}{N} - \frac{C}{N^{3/2}}. \tag{21}$$

REMARK 1.5. The constant C is large enough that the first statement does not follow from (21) which is only a meaningful bound when N is large enough that the right-hand side is positive.

1.4. *The link between the Kac process and its conjugate.* The following theorem provides the link between the Kac process and its conjugate.

THEOREM 1.6. *For all $N \geq 3$,*

$$\Delta_{N,\alpha} \geq \frac{N}{N-1} \Delta_{N-1,\alpha} \widehat{\Delta}_{N,\alpha}. \tag{22}$$

Before proving Theorem 1.6, we recall some explicit formulas that will be useful here and elsewhere. The proof of Theorem 1.6 uses the methods introduced in [3, 5, 6]. The estimation of $\Delta_{N,\alpha}$ in terms of $\Delta_{N-1,\alpha}$ is based on a parameterization of $\mathcal{S}_N$, for $N \geq 3$, in terms of $\mathcal{S}_{N-1} \times B$ where B is the unit ball. For each $k = 1, \dots, N$, define $\pi_k : \mathcal{S}_N \to B$ by

$$\pi_k(\vec{v}) = \frac{1}{\sqrt{N-1}} v_k. \tag{23}$$

(Note that because of the constraints $\sum_{j=1}^N v_j = 0$ and $\sum_{j=1}^N |v_j|^2 = N$, the largest value of $|v_k|$ on $\mathcal{S}_N$ is $\sqrt{N-1}$.)

Define a map $T_1 : \mathcal{S}_{N-1} \times B \to \mathcal{S}_N$ as follows:

$$T_1(\vec{y}, v) = \left(\sqrt{N-1}v, \beta(v)y_1 - \frac{1}{\sqrt{N-1}}v, \dots, \beta(v)y_{N-1} - \frac{1}{\sqrt{N-1}}v\right), \tag{24}$$

where

$$\beta^2(v) = \frac{N}{N-1}(1 - |v|^2). \tag{25}$$

The subscript 1 in T_1 indicates that the vector v from B went into the first place. We likewise define $T_2, \dots, T_N$ by placing this coordinate in the corresponding position.

In the coordinates $(\vec{y}, v)$ on $\mathcal{S}_N$ induced by any of the maps T_k, one has the integral factorization formula

$$\int_{\mathcal{S}_N} \phi(\vec{v})\, \mathrm{d}\sigma_N = \int_B \left[\int_{\mathcal{S}_{N-1}} \phi\big(T_k(\vec{y}, v)\big)\, \mathrm{d}\sigma_{N-1}\right] \mathrm{d}\nu_N(v), \tag{26}$$

where for all $N \geq 3$,

$$\mathrm{d}\nu_N(v) = \frac{|S^{3N-7}|}{|S^{3N-4}|}\big(1 - |v|^2\big)^{(3N-8)/2}\, \mathrm{d}v. \tag{27}$$

Also, note that for $i \neq k$, $j \neq k$,

$$R_{i,j,\sigma}\big(T_k(\vec{y}, v)\big) = T_k\big(R_{i,j,\sigma}(\vec{y}), v\big). \tag{28}$$

We now have the means to relate $\mathcal{E}_{N,\alpha}$ to $\mathcal{E}_{N-1,\alpha}$.

For each $k = 1, \dots, N$, define the *conditional Dirichlet form* $\mathcal{E}_{N,\alpha}(f, f|v_k)$ on $L^2(\mathcal{S}_{N-1}, \sigma_{N-1})$ by

$$\mathcal{E}_{N,\alpha}(f, f|v_k) = (N-1)\binom{N-1}{2}^{-1} \sum_{i<j;i,j\neq k} \int_{\mathcal{S}_{N-1}} \int_{S^2} |y_i - y_j|^\alpha \times b\left(\sigma \cdot \frac{y_i - y_j}{|y_i - y_j|}\right) F^2(\vec{v}, y)\, \mathrm{d}\sigma\, \mathrm{d}\sigma_{N-1}(y), \tag{29}$$

where $F(\vec{v}, y) := [f(T_k(\pi_k(\vec{v}), y) - f(R_{i,j,\sigma} T_k(\pi_k(\vec{v}), y))]$.

As the integration on the right-hand side is only over the "slices" of $\mathcal{S}_N$ at constant values of v_k, the result is still a nontrivial function of v_k. For each fixed v_k, the conditional Dirichlet form is simply the $N-1$ particle Dirichlet form acting in the $\vec{y}$ variables.

Note that by (24) and (25), when $\vec{v} = T_k(\vec{y}, v)$ and $i, j \neq k$,

$$|v_i - v_j|^2 = \beta^2(\pi_k(\vec{v}))|y_i - y_j|^2 = \frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2}|y_i - y_j|^2.$$

We define, for $v \in \mathbb{R}^3$, $|v|^2 \leq N - 1$, the following.

DEFINITION 1.7.

$$w_N(v) := \frac{N^2 - (1 + |v|^2)N}{(N-1)^2}. \tag{30}$$

We therefore have that

$$|v_i - v_j|^\alpha b\left(\sigma \cdot \frac{v_i - v_j}{|v_i - v_j|}\right) = w_N^{\alpha/2}(v_k)|y_i - y_j|^\alpha b\left(\sigma \cdot \frac{y_i - y_j}{|y_i - y_j|}\right).$$

Then, using (28), one easily checks that

$$\mathcal{E}_{N,\alpha}(f, f) = \frac{N}{N-1}\left(\frac{1}{N}\sum_{k=1}^N \int_B w_N^{\alpha/2}(v_k)\mathcal{E}_{N,\alpha}(f, f|v_k)\, \mathrm{d}\nu_N(v_k/\sqrt{N-1})\right). \tag{31}$$

1.5. *Proof of Theorem* 1.6.

PROOF OF THEOREM 1.6. To estimate the right-hand side of (31) in terms of $\Delta_{N-1,\alpha}$, we must take into account that for fixed v_k, f need not be orthogonal to the constants as a function of the remaining variables $\vec{y}$. To take this into account, we use the projection operators already introduced in (16) and (17). Using the factorization formula (26), we have an explicit formula:

$$P_k\phi(\vec{v}) = \int_{\mathcal{S}_{N-1}} \phi\left(T_k(\vec{y}, v_k/\sqrt{N-1})\right) \mathrm{d}\sigma_{N-1}.$$

Now note that $\mathcal{E}_{N,\alpha}(f, f|v_k) = \mathcal{E}_{N,\alpha}(f - P_k f, f - P_k f|v_k)$, and then using the spectral gap for $N-1$ particles and (31), one has

$$\begin{aligned}\mathcal{E}_{N,\alpha}(f, f) &\geq \frac{N}{N-1}\Delta_{N-1,\alpha}\left(\frac{1}{N}\sum_{k=1}^N \int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)[f - P_k f]^2\, \mathrm{d}\sigma_N\right) \\ &= \frac{N}{N-1}\Delta_{N-1,\alpha}\mathcal{D}_{N,\alpha}(f, f)\end{aligned} \tag{32}$$

since

$$\frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)[f - P_k f]^2\,\mathrm{d}\sigma_N$$
$$= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)[f^2 - f P_k f]\,\mathrm{d}\sigma_N = \mathcal{D}_{N,\alpha}(f, f).$$

The theorem follows directly from (32) and the variational characterizations of $\Delta_{N,\alpha}$ and $\widehat{\Delta}_{N,\alpha}$ □

1.6. *Proof of the main theorem.* Combining Theorem 1.6 and Theorem 1.4 yields, for a constant C, independent of $N > 3$,

$$\Delta_{N,\alpha} \geq \left(1 - \frac{C}{N^{3/2}}\right)\Delta_{N-1,\alpha}. \tag{33}$$

The main result will follow easily from this, and a bound on $\Delta_{2,\alpha}$, and our next task is to prove such a bound.

Because $|v_i - v_j|$ can be arbitrarily small on $\mathcal{S}_N$ for any $N > 2$, for any given $C > 0$, there will be functions $f \in L^2(\sigma_N)$ that satisfy $\langle f, 1\rangle_{L^2(\sigma_N)} = 0$ and $\|f\|^2_{L^2(\sigma_N)} = 1$ such that $f(\vec{v})L_{N,\alpha}f(\vec{v}) < Cf(\vec{v})L_{N,0}f(\vec{v})$ for some $\vec{v} \in \mathcal{S}_N$. This precludes a simple and direct comparison of the Dirichlet forms $\mathcal{E}_{N,\alpha}$ and $\mathcal{E}_{N,0}$.

For $N = 2$, things are much better: Then, by definition of $\mathcal{S}_2 := \mathcal{S}_{2,1,0}$, for all $(v_1, v_2) \in \mathcal{S}_2$, $v_2 = -v_1$, and $|v_1| = |v_2| = 1$, so that $|v_1 - v_2| = 2$ everywhere on $\mathcal{S}_2$. That is, for $N = 2$, there is no significant difference between $\alpha = 0$ and $\alpha > 0$. For $\alpha = 0$ and a number of choices of b, Δ_2 has been computed in [7]. The following is proved in Lemma 2.1 of [7].

LEMMA 1.8 (Spectral gap for $N = 2$ and hard sphere collisions). *With* $b(x) = 1$,

$$\Delta_{2,1} = 2. \tag{34}$$

The proof given in [7] is fairly simple, and it is easy to apply the formulas there to other choices for the probability density b, and to show that as long as b is even and continuous on $[-1, 1]$, $\Delta_{2,\alpha} > 0$ for all $\alpha \in [0, 2]$.

We are now ready to prove the main theorem.

PROOF OF THEOREM 1.2. Since $\Delta_{2,\alpha} > 0$ by Lemma 1.8, Theorem 1.6 and the first part of Theorem 1.4 yield

$$\Delta_{3,\alpha} \geq \frac{3}{2}\Delta_{2,\alpha}\widehat{\Delta}_{3,\alpha} > 0,$$

and then the obvious iteration yields $\Delta_{N,\alpha} > 0$ for all $N \geq 2$. To go further and prove that $\inf_{N\geq 2}\Delta_{N,\alpha} > 0$, we use the second part of Theorem 1.4.

Let N_0 be such that $1 - CN_0^{-3/2} > 0$. Then $K_0 := \prod_{j=N_0}^{\infty}(1 - \frac{C}{j^{3/2}}) > 0$ and for all $N \geq N_0$, $\Delta_{N,\alpha} \geq K_0\Delta_{N_0,\alpha}$. □

REMARK 1.9. As we shall see, it is possible to explicitly compute the constant C in Theorem 1.4. To keep the presentation free of clutter, we have not carried this through here, but it would be a simple, if tedious, exercise to track the constants step by step. As for the first part of Theorem 1.4, it is easy to give an explicit lower bound on $\widehat{\Delta}_{N,\alpha}$ for all $N \geq 4$, and we do so below. The case $N = 3$ is more difficult, and we use a simple compactness argument to prove $\widehat{\Delta}_{3,\alpha} > 0$. However, we do sketch a method for explicitly estimating $\widehat{\Delta}_{3,\alpha}$. Thus, the method we employ to prove Theorem 1.2 can be used to prove explicit bounds.

It remains to prove Theorem 1.4, and we prepare the way for this in the next section. Throughout the rest of the paper, we are concerned solely with the conjugate Kac process. All of the analysis that directly involves the Kac process itself is complete at this point.

2. Estimates for the conjugate process. It is in the proof of Theorem 1.4 that new ideas are required to deal with the nonuniform jump rates of the conjugate process, and we begin with a heuristic discussion of these ideas.

As in the case $\alpha = 0$, we rely in part on the fact that the invariant measure σ_N (of both processes) is *chaotic* in the sense of Kac. More specifically, it is γ chaotic where

$$\mathrm{d}\gamma = (2\pi/3)^{-3/2} e^{-3|v|^2/2}\,\mathrm{d}v$$

is the isotropic Gaussian distribution on $\mathbb{R}^3$ with unit variance. This means that for any $k \in \mathbb{N}$ and any bounded continuous function $\psi(v_1, \dots, v_k)$ on $\mathbb{R}^{3k}$,

$$\lim_{N\to\infty} \int_{\mathcal{S}_N} \psi(v_1, \dots, v_k)\,\mathrm{d}\sigma_N = \int_{\mathbb{R}^{3k}} \psi(v_1, \dots, v_k)\,\mathrm{d}\gamma^{\otimes k}.$$

That is, as long as k is much less than N, the random variables $v_1, \dots, v_k$ are nearly independent, and by symmetry this is true of any set of k distinct coordinate functions on $\mathcal{S}_N$. The notion of chaos was also introduced by Kac in [12], and the main result of that paper was that for the model with one-dimensional velocities and $\alpha = 0$, chaos is propagated by the dynamics. Propagation of chaos for $\alpha > 0$ is much harder, and this was only proved later by Sznitman [18], also in 3 dimensions.

In case $\alpha = 0$, the range of $I - \widehat{L}_{N,0}$ has a special structure that facilitates the study of the spectral gap for $\widehat{L}_{N,0}$. The subspace of the range that is orthogonal to the constants consists of functions f of the form: $f(\vec{v}) = \sum_{j=1}^N \varphi_j(v_j)$ such that each $\varphi_j(v_j)$ is square integrable and such that f is orthogonal to the constants. One choice for the φ_j's is $\varphi_j = P_j f$, but there are other choices: Since for any fixed $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$,

$$\sum_{j=1}^{N} \bigl(a \cdot v_j + b(|v_j|^2 - 1)\bigr) = 0, \tag{35}$$

we may make the replacement $\varphi_j(v_j) \longrightarrow \varphi_j(v_j) + a \cdot v_j + b(|v_j|^2 - 1)$ without changing $f(\vec{v})$. There is, however, a preferred choice of the functions φ_j that plays an important role in what follows. As we shall show, there is a unique choice that minimizes $\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2$ which has a number of useful properties.

DEFINITION 2.1. Let $\mathcal{A}_N$ denote the subspace of $L^2(\sigma_N)$ that is the closure of the span of functions of the form

$$f(\vec{v}) = \sum_{j=1}^{N} \varphi_j(v_j) \tag{36}$$

for bounded continuous functions $\varphi_1, \dots, \varphi_N$ in $\mathbb{R}^3$ such that $\int_{\mathcal{S}_N} f\,\mathrm{d}\sigma_N = 0$.

When $\alpha \neq 0$, $\mathcal{A}_N$ is not an invariant subspace of $\widehat{L}_{N,\alpha}$. Nonetheless, as we explain, the gap may be bounded using a trial function decomposition based on $\mathcal{A}_N$, and for this the approximate independence that comes along with the chaoticity of σ_N is essential.

To see how this works, suppose that one replaces the state space $\mathcal{S}_N$ with $\mathbb{R}^{3N}$, and replaces the conjugate Kac process with the "conditional jump to uniform" process with respect to $\mathrm{d}\gamma^{\otimes N}$. In this case, with the invariant measure being a product measure, the corresponding conditional expectation operators P_k will all commute. One might therefore expect that the operators P_k figuring in the definition (18) of $\widehat{L}_{N,\alpha}$ almost commute for large N. Suppose that they *exactly* commute, or, what is the same thing, that the coordinate functions $v_1, \dots, v_N$ are *exactly* independent.

Since $0 = \int_{\mathcal{S}_N} f \, \mathrm{d}\sigma_N = \sum_{j=1}^N \int_{\mathcal{S}_N} \varphi_j(v_j) \, \mathrm{d}\sigma_N = 0$, replacing $\varphi_j(v_j)$ by $\varphi_j(v_j) - \int_{\mathcal{S}_N} \varphi_j(v_j) \, \mathrm{d}\sigma_N$, we may assume without loss of generality in (36) that $\int_{\mathcal{S}_N} \varphi_j(v_j) \, \mathrm{d}\sigma_N = 0$ for each j. Granted the exact independence, we would then have that for $k \neq j$, $P_k \varphi_j(v_k) = 0$, while $P_k \varphi_k(v_k) = \varphi_k(v_k)$. Thus, for $f \in \mathcal{A}_N$, $f - P_k f = \sum_{j \neq k} \varphi_j(v_j)$, and then, again using the independence,

$$
\begin{aligned}
\mathcal{D}_{N,\alpha}(f, f) &= \frac{1}{N} \sum_{k=1}^N \int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k) \sum_{j \neq k} \varphi_j^2(v_j) \, \mathrm{d}\sigma_N \\
&= \frac{1}{N} \sum_{k=1}^N \left(\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k) \, \mathrm{d}\sigma_N \right) \left(\int_{\mathcal{S}_N} \sum_{j \neq k} \varphi_j^2(v_j) \, \mathrm{d}\sigma_N \right).
\end{aligned} \tag{37}
$$

As is shown below, the integral over the rate, which is evidently independent of k, is bounded below by $1 - C/N^2$ for some constant C that is independent of N. Thus, we would have

$$
\begin{aligned}
\mathcal{D}_{N,\alpha}(f, f) &\geq \left(1 - \frac{C}{N^2}\right) \frac{N-1}{N} \sum_{j=1}^N \|\varphi_j(v_j)\|_2^2 \\
&= \left(1 - \frac{C}{N^2}\right) \frac{N-1}{N} \|f\|_2^2
\end{aligned} \tag{38}
$$

which is even better than (21).

Of course, one must consider trial functions that are not in $\mathcal{A}_N$, and for trial functions f that are in $\mathcal{A}_N^\perp$, things are better still. Such functions are shown to belong to the null space of P_k for each k. Therefore, for $f \in \mathcal{A}_N^\perp$, we would have from (19)

$$
\mathcal{D}_{N,\alpha}(f, f) = \int_{\mathcal{S}_N} \left(\frac{1}{N} \sum_{k=1}^N w_N^{\alpha/2}(v_k) \right) f^2 \, \mathrm{d}\sigma_N, \tag{39}
$$

which is a significant simplification of (19). It is shown below (see Lemma 2.12 and Remark 2.13) that for some constant C independent of N,

$$
\frac{1}{N} \sum_{k=1}^N w_N^{\alpha/2}(v_k) \geq 1 - \left(1 - \frac{\alpha}{2}\right) \frac{1}{N} - \frac{C}{N^2}.
$$

Combining this with (39) would then yield

$$
\mathcal{D}_{N,\alpha}(f, f) \geq \left(1 - \left(1 - \frac{\alpha}{2}\right) \frac{1}{N} - \frac{C}{N^2}\right) \|f\|_2^2.
$$

For $\alpha > 0$, this is much stronger than (21), and for this bound we do not even use the approximate independence.

Since $\mathcal{A}_N$ is not an invariant subspace for $\widehat{L}_{N,\alpha}$, one has to show that for $g \in \mathcal{A}_N$ and $h \in \mathcal{A}_N^\perp$, $\mathcal{D}_{N,\alpha}(g, h)$ is small. We shall show, again using the approximate independence, that

$$
|\mathcal{D}_{N,\alpha}(g, h)| \leq \frac{C}{N^{3/2}} \|g\|_2 \|h\|_2.
$$

It is the estimate in this step that is responsible for the $N^{3/2}$ term in (33). A more refined argument, like the one provided for this step in [6] for the model with one-dimensional velocities, would presumably improve $N^{3/2}$ to N^2, but since we have elected not to keep track of constants, there is no point in pursuing this here.

Our proof will follow these heuristics, but of course we must carefully control the departures from exact independence wherever it was used above. There is one significant twist, and it comes from the first part of the heuristic argument: The equality in (38) comes from the identity $\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2 = \|f\|_2^2$ which is true when there is *exact* independence of the coordinate functions. In our setting, we do not have exact independence, and must prove and use appropriate *quantitative chaos* estimates. In (55) of Theorem 2.5, it is shown that in our setting, for a particular decomposition $f(\vec{v}) = \sum_{j=1}^N \varphi(v_j)$—such decompositions are not unique, even if one requires each φ_j to be orthogonal to the constants—one has

$$\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2 \geq \left(1 - \frac{C}{N}\right)\|f\|_2^2 \tag{40}$$

and

$$\|f\|_2^2 \geq \left(1 - \frac{C}{N^2}\right)\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2 \tag{41}$$

for all $N \geq 3$ and with C independent of N. If the bound in (40) contained C/N^2 in place of C/N, as in (41), we could use this after the first inequality in (38), and we would still still obtain something better than (21). However, the estimate in (40) cannot be improved, and we must do something a little different: We must avoid passing back and forth between $\|f\|_2^2$ and $\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2$ in the main term in $\mathcal{D}_{N,\alpha}(f, f)$. We therefore define

$$\mathcal{F}_{N,\alpha}(f, f) := \|f\|_2^2 - \mathcal{D}_{N,\alpha}(f, f) \tag{42}$$

and then in order to prove the lower bound (21), we must prove the *upper bound*

$$\mathcal{F}_{N,\alpha}(f, f) \leq \left(\frac{1}{N} + \frac{C}{N^{3/2}}\right)\|f\|_2^2. \tag{43}$$

Now things are much better: If we can prove

$$\mathcal{F}_{N,\alpha}(f, f) \leq \left(\frac{1}{N} + \frac{C}{N^{3/2}}\right)\sum_{j=1}^N \|\varphi_j(v_j)\|_2^2, \tag{44}$$

we can pass from here to (43) using (41), and in fact, we do not even need the C/N^2; it would suffice to have C/N. In Section 3.3, we closely follow this outline to complete the proof of our main theorem.

Though we have the estimate (41), it is much easier to prove the weaker analog of it with N^2 replaced by N, and perhaps there are other models that are "less chaotic" in which the weaker bound is all that one has. The next subsection presents the "quantitative chaos" estimates that are used to control the weak dependence of the coordinate function for large N. It is important that some of the results turn out to be meaningful even for small N, such as $N = 3$.

2.1. *Quantitative chaos*. A number of the quantitative chaos bounds that we need may be expressed in terms of the *correlation operator* K-operator that we now define.

Let B denote the unit ball in $\mathbb{R}^3$. Let $N \geq 3$ and let ν_N be given by (27), so that for any function ψ on B, and any k,

$$\int_B \psi(v)\,\mathrm{d}\nu_N = \int_{\mathcal{S}_N} \psi(\pi_k(\vec{v}))\,\mathrm{d}\sigma_N,$$

where π_k is given by (23), and ν_N is given by (27). We define the operator K on $L^2(B, \nu_N)$ by

$$\langle \psi_1, K\psi_2\rangle_{L^2(B,\nu_N)} = \int_{\mathcal{S}_N} \psi_1^*(\pi_1(\vec{v}))\psi_2(\pi_2(\vec{v}))\,\mathrm{d}\sigma_N. \tag{45}$$

K is evidently self-adjoint.

DEFINITION 2.2. For $j = 0, \dots, 4$, define functions $\xi_j(v)$ on B by

$$\xi_0(v) = 1, \qquad \xi_j(v) = v_j, \quad j = 1, 2, 3 \quad \text{and} \quad \xi_4(v) = (|v|^2 - 1)/(N-1). \tag{46}$$

The spectrum of K is determined in [7], where the following facts are proven.

LEMMA 2.3. *Let $N \geq 3$. The operator K is compact. The function ξ_0 is an eigenfunction of K with eigenvalue 1, and it spans the corresponding eigenspace. The functions ξ_j, $j = 1, 2, 3, 4$ are eigenfunctions of K with eigenvalue $-1/(N-1)$, and they are an orthogonal basis for this eigenspace. No other eigenvalues of K are larger in absolute value than $\frac{5N-3}{3(N-1)^3}$. Therefore, for all $\psi_1, \psi_2 \in L^2(B, \nu_N)$ that are orthogonal to the constants, the three components of v and $|v|^2$,*

$$\left|\int_{\mathcal{S}_N} \psi_1^*(\pi_1(\vec{v}))\psi_2(\pi_2(\vec{v}))\,\mathrm{d}\sigma_N\right| \leq \frac{5N-3}{3(N-1)^3}\|\psi_1 \circ \pi_1\|_2 \|\psi_2 \circ \pi_2\|_2. \tag{47}$$

Equivalently, for all functions $\psi \in L^2(B, \nu_N)$, that are orthogonal to 1, the three components of v and $|v|^2$,

$$\|K\psi\|_2 \leq \frac{5N-3}{3(N-1)^3}\|\psi\|_2. \tag{48}$$

Finally, every eigenvalues κ of K, other than 1, $\frac{5N-3}{3(N-1)^3}$ and $\frac{1}{N-1}$ satisfies

$$-\frac{7N-3}{3(N-1)^4} \leq \kappa < \frac{5N-3}{3(N-1)^3}. \tag{49}$$

REMARK 2.4. The number on the left-hand side in (49) is the eigenvalue denoted by $\kappa_{1,2}$ in Section 8 of [7].

Fix some k, and let $\mathcal{H}$ be the subspace of $L^2(\sigma_N)$ spanned by functions of the form $\varphi(v_k)$ for some k. Since v_k ranges over the ball of radius $\sqrt{N-1}$ in $\mathbb{R}^3$, one may think of $\mathcal{H}$ as a Hilbert space consisting of square integrable functions on this ball, with respect to a scaled version of the measure ν_N.

It will be convenient in what follows to think of K as an operator on $\mathcal{H}$. Note that $\varphi(v_k) = \tilde{\varphi}(\pi_k(\vec{v}))$ where $\tilde{\varphi}(v) = \varphi(\sqrt{N-1}v)$. Define

$$K\varphi(v_k) = (K\tilde{\varphi})(\pi_k(\vec{v})). \tag{50}$$

The spectrum of K, including multiplicity, thought of this way is naturally the same, but the eigenfunctions change by scaling. For example, now $|v|^2 - 1$ is an eigenfunction with eigenvalue $-1/(N-1)$. In this notation, we have that for any function ξ on $\mathbb{R}^3$ so that $\xi(v_1)$ is in $L^2(\sigma_N)$,

$$E\{\xi(v_1) \mid v_2 = v\} = K\xi(v). \tag{51}$$

The K operator defined by (51) is simply a "scaled" version of the K operator defined in (45), scaled so it operates on functions on $\mathcal{H}$. For some computations, particularly in the computation of eigenvalues of K, the definition (45) is more convenient. For other computations, more directly connected the Kac process, (51) has advantages. This slight abuse of notation will simplify many formulas that follow without introducing any ambiguity.

Since K is compact, there is an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of K. This orthonormal basis is determined explicitly in [7], but all we need to know is that is can be written as $\{\eta_\iota\}_{\iota\geq 0}$ where

$$\eta_0(v) = 1, \qquad \eta_j(v) = \sqrt{3}\mathbf{e_j}\cdot v, \quad 1\leq j\leq 3 \quad \text{and} \quad \eta_4(v) = C_N\big(|v|^2-1\big), \tag{52}$$

with C_N being a normalization constant. This follows directly from Lemma 2.3 and (50).

Let κ_ι denote the eigenvalue corresponding to η_ι, so that $K\eta_\iota = \kappa_\iota\eta_\iota$. Our first application of Lemma 2.3 concerns the norm of functions in $\mathcal{A}_N$.

THEOREM 2.5. *Let $N\geq 3$, and let $f\in\mathcal{A}_N$ be orthogonal to 1. Then there is a unique choice of $\varphi_1,\dots,\varphi_N$ with $f=\sum_{j=1}^N\varphi_j(v_j)$ and each $\varphi_j(v_j)$ orthogonal to the constants that minimizes $\sum_{k=1}^N\|\varphi_k\|_2^2$ where $\|\varphi_k\|_2^2$ denotes $\int_{\mathcal{S}_N}|\varphi_k(v_k)|^2\,\mathrm{d}\sigma_N$. Let*

$$\varphi_j(v_j) = \sum_{i=1}^{\infty} a_{j,i}\eta_i(v_j) \tag{53}$$

be the expansion of φ_j in the orthonormal basis consisting of eigenfunctions of K that is specified above. Then this minimizer is characterized by

$$\sum_{j=1}^{N} a_{j,i} = 0 \quad \text{for } 1\leq i\leq 4. \tag{54}$$

For this choice,

$$\left(1-\frac{7N-3}{3(N-1)^3}\right)\sum_{k=1}^{N}\|\varphi_k\|_2^2 \leq \|f\|_2^2 \leq \left(1+\frac{5N-3}{3(N-1)^2}\right)\sum_{k=1}^{N}\|\varphi_k\|_2^2, \tag{55}$$

In particular, let $\mathcal{H}_{N,k}$ denote the subspace of $L^2(\sigma_N)$ consisting of functions of the form $\varphi(v_k)$. Define $\mathcal{B}_N$ to be the subspace of $\bigoplus_{k=1}^N\mathcal{H}_{N,k}$ consisting of $(\varphi_1,\dots,\varphi_N)$ such that (54) is satisfied. Then the operator $T:\mathcal{B}_N\to\mathcal{A}_N$ defined by

$$T\big(\varphi_1(v_1),\dots,\varphi_N(v_N)\big) = \sum_{k=1}^{N}\varphi_k(v_k)$$

is bounded with a bounded inverse.

REMARK 2.6. Define $c_N := (1-\frac{7N-3}{3(N-1)^3})$ and $C_N = (1+\frac{5N-3}{3(N-1)^2})$. Note the different exponents in the denominator, and that $c_N>0$ for all $N\geq 3$. Also note that $c_N = 1-\mathcal{O}(1/N^2)$, and $C_N = 1+\mathcal{O}(1/N)$, and then we can rewrite (55) as

$$c_N\sum_{k=1}^{N}\|\varphi_k\|_2^2 \leq \|f\|_2^2 \leq C_N\sum_{k=1}^{N}\|\varphi_k\|_2^2,$$

and of course we would have equality here with $c_N = C_N = 1$ if the coordinate functions were exactly independent. Theorem 2.5 gives a quantitative expression of the fact that for large N, the coordinate functions are approximately pairwise independent.

PROOF OF THEOREM 2.5. As noted above, we may assume that each φ_j is orthogonal to the constants. We expand each φ_j in the eigenbasis of K as follows:

$$\varphi_j(v_j) = \sum_{i=1}^{\infty} a_{j,i}\eta_i(v_j). \tag{56}$$

Then evidently $\|\varphi_j\|_2^2 = \sum_{i=1}^\infty |a_{j,i}|^2$. On account of (35), for $i = 1, 2, 3, 4$, we may replace $a_{j,i}$ by $a_{j,i} - t_i$ without changing $f(\vec{v}) = \sum_{j=1}^N \varphi_j(v_j)$. With this modification,

$$\sum_{j=1}^N \|\varphi_j\|_2^2 = \sum_{j=1}^N \sum_{i=1}^4 |a_{j,i} - t_i|^2 + \sum_{j=1}^N \sum_{i=5}^\infty |a_{j,i}|^2,$$

which is evidently minimized by taking $t_i = -\frac{1}{N}\sum_{j=1}^N a_{j,i}$. and then making this replacement, (54) is satisfied.

Next, for $j \neq k$,

$$\int_{\mathcal{S}_N} \varphi_j^*(v_j)\varphi_k(v_k)\,\mathrm{d}\sigma_N = \sum_{\iota,\iota'=1}^\infty a_{j,\iota}^* a_{k,\iota'} \langle \eta_{j,\iota}, K\eta_{k,\iota'}\rangle_{\mathcal{H}} = \sum_{\iota=1}^\infty \kappa_\iota a_{j,\iota}^* a_{k,\iota}.$$

Therefore, when f is given by (36) and (56) with (54) satisfied,

$$\|f\|_2^2 = \sum_{\iota=1}^\infty \left(\sum_{j=1}^N |a_{j,\iota}|^2 + \kappa_\iota \sum_{j\neq k, j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota} \right). \tag{57}$$

For $\iota = 1, \dots, 4$, we have, using (54), the identity

$$\sum_{j\neq k, j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota} = \left| \sum_{j=1}^N a_{j,\iota} \right|^2 - \sum_{j=1}^N |a_{i,j}|^2 = -\sum_{j=1}^N |a_{i,j}|^2,$$

and then since $\kappa_i = -\frac{1}{N-1}$ for $i = 1, \dots, 4$,

$$\sum_{\iota=1}^4 \left(\sum_{j=1}^N |a_{j,\iota}|^2 + \kappa_\iota \sum_{j\neq k, j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota} \right) = \frac{N}{N-1} \sum_{\iota=1}^4 \sum_{j=1}^N |a_{j,\iota}|^2. \tag{58}$$

For $\iota > 4$, we simply use the fact for such ι, κ_ι is $\mathcal{O}(1/N^2)$ or smaller, and this takes the place of (54), which is not satisfied for such ι, in eliminating a factor of N. Then since the $N \times N$ matrix that has 0 in every diagonal entry, and 1 elsewhere has eigenvalues $N - 1$ and -1,

$$-\sum_{j=1}^N |a_{j,i}|^2 \leq \sum_{j\neq k, j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota} \leq (N-1) \sum_{j=1}^N |a_{j,i}|^2. \tag{59}$$

Hence, for $\iota > 4$, an upper bound on $(\sum_{j=1}^N |a_{j,\iota}|^2 + \kappa_\iota \sum_{j\neq k,j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota})$ is

$$\left(1 + \max\left\{ \frac{7N-3}{3(N-1)^4}, (N-1)\frac{5N-3}{3(N-1)^3} \right\}\right) \sum_{j=1}^N |a_{j,i}|^2,$$

where we have used (49). Evidently the maximum is furnished by the second quantity in the braces. Summing on $\iota > 4$ and combining this with (58) yields the upper bound in (55).

For $\iota > 4$, a lower bound on $(\sum_{j=1}^N |a_{j,\iota}|^2 + \kappa_\iota \sum_{j\neq k,j,k=1}^N \Re a_{j,\iota}^* a_{k,\iota})$ is

$$\left(1 - \max\left\{ (N-1)\frac{7N-3}{3(N-1)^4}, \frac{5N-3}{3(N-1)^3} \right\}\right) \sum_{j=1}^N |a_{j,i}|^2,$$

where we have again used (49). Evidently, the maximum is furnished by the first quantity in the braces. Summing on $\iota > 4$ and combining this with (58) yields the lower bound in (55).

The rest is now clear, including the fact that (35) is the only source of nonuniqueness in the representation of $f \in \mathcal{A}_N$ in the form $f(\vec{v}) = \sum_{j=1}^N \varphi_j(v_j)$. $\square$

There is another type of quantitative chaos estimate that we need. Any continuous function ξ on $\mathbb{R}^3$ restricts to $\mathcal{S}_N$ and is square integrable, so we may apply the scaled K operator defined in (51) to it. Recall that this is

$$K\xi(v) := \mathbb{E}\{\xi(v_k) \mid v_j = v\} \tag{60}$$

for any $j \neq k$; by the symmetry of σ_N, the choice does not matter. If the coordinate functions were exactly independent, this would simply be the expectation of $\xi(v_k)$, which is a finite constant. It turns out that when $\xi(v_k)$ is a polynomial in $|v_k|^2$, the conditional expectation is at least bounded—not only on $\mathcal{S}_N$, which is trivial, but the bound is independent of N. Here is one such estimate.

LEMMA 2.7. *For $\psi(v) = |v|^8$, there is a constant $C < \infty$ such that $\|K\psi\|_\infty \leq C$ for all N.*

PROOF OF LEMMA 2.7. The formula (24) gives us

$$\begin{aligned} K\psi(v) &= \int_{\mathcal{S}_{N-1}} \left| \sqrt{\frac{N-|v|^2}{N-1}}\vec{y} - \frac{1}{\sqrt{N(N-1)}} v \right|^8 \mathrm{d}\sigma_{N-1} \\ &\leq 2^7 \left(\left(\frac{N-|v|^2}{N-1} \right)^4 \int_{\mathcal{S}_{N-1}} |\vec{y}|^8 \, \mathrm{d}\sigma_{N-1} + \frac{|v|^8}{N^4(N-1)^4} \right). \end{aligned}$$

It is evident that $\int_{\mathcal{S}_{N-1}} |\vec{y}|^8 \, \mathrm{d}\sigma_{N-1}$ is bounded uniformly in N, and in fact,

$$\lim_{N\to\infty} \int_{\mathcal{S}_{N-1}} |\vec{y}|^8 \, \mathrm{d}\sigma_{N-1} = (2\pi/3)^{-3/2} \int_{\mathbb{R}^3} |y|^8 e^{-3|y|^2/2} \, \mathrm{d}y. \qquad \square$$

In the remainder of this section, we collect the other estimates of this type that we need. Their proofs, which are more intricate but still largely computational, are presented in Appendix A.

LEMMA 2.8. *There is a finite constant C such that for all $N > 3$ and all v such that $v = v_N$ for some $\vec{v} \in \mathcal{S}_N$,*

$$\left| \mathbb{E}\{|v_1|^4 \mid v_N = v\} - S(v) \right| \leq \frac{C}{N}, \tag{61}$$

where

$$S(v) = \frac{N^2 + |v|^4 - 2N|v|^2}{(N-1)^2}. \tag{62}$$

LEMMA 2.9. *There is a finite constant C such that for all $N > 3$ and all (v, w) such that $(v, w) = (v_{N-1}, v_N)$ for some $\vec{v} \in \mathcal{S}_N$,*

$$\left| \mathbb{E}\{|v_1|^4 \mid (v_{N-1}, v_N) = (v, w)\} - S(v, w) \right| \leq \frac{C}{N}, \tag{63}$$

where

$$S(v, w) = \frac{N^2 + |v|^4 + |w|^4 + 2N|v|^2 + 2N|w|^2 + 2|v|^2|w|^2}{(N-2)^2}. \tag{64}$$

2.2. *The operators $W^{(\alpha)}$ and $P^{(\alpha)}$.* Let $\alpha \in [0,2]$, and define the self-adjoint operator $P^{(\alpha)}$ by

$$P^{(\alpha)} = \frac{1}{N}\sum_{k=1}^{N} w_N^{\alpha/2}(v_k)P_k, \tag{65}$$

recalling that $w_N(v)$ is defined in (30), and P_k is defined in (16) or equivalently (17). For each k, both P_k and the multiplication operator $w_N^{\alpha/2}(v_k)$ are commuting and self-adjoint, and hence $P^{(\alpha)}$ is indeed self-adjoint and nonnegative. Since each P_k is a projection,

$$\frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)|P_k f|^2\,\mathrm{d}\sigma_N = \langle f, P^{(\alpha)} f\rangle_{L^2(\mathcal{S}_N,\sigma_N)}. \tag{66}$$

Define the function $W^{(\alpha)}$ by

$$W^{(\alpha)} = \frac{1}{N}\sum_{k=1}^{N} w_N^{\alpha/2}(v_k). \tag{67}$$

Then

$$\frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k) f^2\,\mathrm{d}\sigma = \int_{\mathcal{S}_N} W^{(\alpha)} f^2\,\mathrm{d}\sigma_N, \tag{68}$$

and we can write

$$\mathcal{D}_{N,\alpha}(f,f) := \int_{\mathcal{S}_N} W^{(\alpha)} f^2\,\mathrm{d}\sigma_N - \langle f, P^{(\alpha)} f\rangle_{L^2(\mathcal{S}_N,\sigma_N)}. \tag{69}$$

Equivalently, by the computations just below (32),

$$\mathcal{D}_{N,\alpha}(f,f) = \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)[f - P_k f]^2\,\mathrm{d}\sigma_N, \tag{70}$$

and hence $\mathcal{D}_{N,\alpha}(f,f) \geq 0$ for all f since for each k, $|v_k|^2 \leq N-1$. (Recall that because of the momentum constrain, not all of the energy can reside in a single particle.) It follows that $\mathcal{D}_{N,\alpha}(f,f) = 0$ if and only if $f - P_k f = 0$ almost everywhere for each k, and then in this case

$$\|f\|_2^2 - \langle f, P^{(0)} f\rangle_{L^2(\sigma_N)} = \mathcal{D}_{N,0}(f,f) = \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} |f - P_k f|^2\,\mathrm{d}\sigma_N = 0.$$

Evidently, $P^{(0)}$ is a contraction, and 1 is an eigenvalue of multiplicity one, and the eigenspace is spanned by the constant function 1 [7]. This proves the following.

LEMMA 2.10. *For all $N \geq 2$ and all $\alpha \in [0,2]$, and all nonzero $f \in L^2(\sigma_N)$ that are orthogonal to the constants*, $\mathcal{D}_{N,\alpha}(f,f) > 0$.

We use the following lemma proved in [6], Lemma 3.5.

LEMMA 2.11. *For all* $0 < \alpha \leq 2$ *and all* $x > -1$,

$$(1+x)^{\alpha/2} \geq 1 + \frac{\alpha}{2}x - \left(1 - \frac{\alpha}{2}\right)x^2. \tag{71}$$

LEMMA 2.12. *For all N, all $0 < \alpha \le 2$, and for all $\vec{v} \in \mathcal{S}_N$,*

$$1 - \left(1 - \frac{\alpha}{2}\right)\frac{N((N-1)^2+1)}{(N-1)^4} - \frac{\alpha}{2}\frac{1}{(N-1)^2} + \left(1 - \frac{\alpha}{2}\right)\frac{N+1}{(N-1)^3} \le W^{(\alpha)}(\vec{v}) \le \left(1 - \frac{1}{(N-1)^2}\right)^{\alpha/2}. \tag{72}$$

Furthermore, for all $\vec{v} \in \mathcal{S}_N$, with $W^{(\alpha)}$ given by (67),

$$W^{(0)}(\vec{v}) = 1 \quad \textit{and} \quad W^{(2)}(\vec{v}) = 1 - \frac{1}{(N-1)^2}. \tag{73}$$

PROOF. Repeated use will be made of

$$\frac{1}{N}\sum_{k=1}^{N} |v_k|^2 = 1 \tag{74}$$

that identity is part of the definition of $\mathcal{S}_N$.

Because of (74), $\frac{1}{N}\sum_{k=1}^{N}(\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}) = 1 - \frac{1}{(N-1)^2}$. Since $x \mapsto x^{\frac{\alpha}{2}}$ is concave on $\mathbb{R}_+$ for $0 \le \alpha \le 2$, Jensen's inequality yields the upper bound.

To prove the lower bound, use the inequality (71): Writing

$$\frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2} = 1 + \frac{N(1 - |v_k|^2) - 1}{(N-1)^2}, \tag{75}$$

and applying (71) and (74) yields

$$W^{(\alpha)}(\vec{v}) \ge 1 - \frac{\alpha}{2}\frac{1}{(N-1)^2} - \left(1 - \frac{\alpha}{2}\right)\frac{1}{N}\sum_{k=1}^{N}\left(\frac{N(1-|v_k|^2)-1}{(N-1)^2}\right)^2.$$

Expanding the square on the right and applying (74) twice more, we find

$$W^{(\alpha)}(\vec{v}) \ge 1 - \frac{\alpha}{2}\frac{1}{(N-1)^2} - \frac{1-\frac{\alpha}{2}}{(N-1)^4}\left[1 - N^2 + N\sum_{k=1}^{N}|v_k|^4\right]. \tag{76}$$

The maximum of $\sum_{k=1}^{N}|v_k|^4$ on $\mathcal{S}_N$ is no greater than the maximum of the convex function $\sum_{k=1}^{N} x_k^2$ on the convex set of $(x_1, \dots, x_N)$ satisfying

$$0 \le x_j \le N - 1 \quad \text{for all } j = 1, \dots, N \quad \text{and} \quad \sum_{j=1}^{N} x_j = N. \tag{77}$$

The extreme points are obtained by permuting the coordinates of $(N-1, 1, 0, \dots, 0)$. Evaluating the sum at such a point yields the stated bound, The final statement is obvious. □

REMARK 2.13. Lemma 2.12 shows that for large N,

$$W^{(\alpha)}(\vec{v}) \ge 1 - \left(1 - \frac{\alpha}{2}\right)\frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{78}$$

The fact that the coefficient of $1/N$ is no less than -1 is essential for the result that we shall prove.

We are particularly concerned with the case $\alpha = 1$, and shall provide all the details in this case only. For $\alpha = 1$, the lower bound simplifies further to

$$W^{(1)}(\vec{v}) \ge C_N := 1 - \frac{1}{2}\frac{1}{N-1} - \frac{1}{2}\frac{1}{(N-1)^2} + \frac{1}{2}\frac{1}{(N-1)^3} - \frac{1}{2}\frac{1}{(N-1)^4}. \tag{79}$$

It is easily seen that for all $N \geq 2$, C_N increases as N increases. For small N, we have the explicit values

$$C_3 = \frac{21}{32} \quad \text{and} \quad C_4 = \frac{64}{81}.$$

We now turn to $P^{(\alpha)}$. By (75), for each k for all v_k,

$$w_N^{\alpha/2}(v_k) = \left[\frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2}\right]^{\alpha/2} = (1 + x_N(v_k))^{\alpha/2}, \tag{80}$$

where

$$x_N(v_k) = \frac{1}{N-1} - \frac{N}{(N-1)^2}|v_k|^2. \tag{81}$$

Note that $-1 \leq x_N(v_k) \leq \frac{1}{N-1}$. Then

$$w_N^{\alpha/2}(v_k) \leq \left(\frac{N}{N-1}\right)^{\alpha/2} \tag{82}$$

and by (80) and the bounds from Lemma 2.11, $1 + \frac{\alpha}{2}x + (\frac{\alpha}{2} - 1)x^2 \leq (1+x)^{\alpha/2} \leq 1 + \frac{\alpha}{2}x$,

$$|w_N^{\alpha/2}(v_k) - 1| \leq \frac{1}{N-1} + \frac{3N}{(N-1)^2}|v_k|^2 + \frac{N^2}{(N-1)^4}|v_k|^4 \tag{83}$$

for all $\alpha \in [0, 2]$, where we have made estimates to simplify the right-hand side. Thus, while $W^{(\alpha)}$ is only constant for $\alpha = 0, 2$, it is nearly constant for all $\alpha \in (0, 2)$ when N is large. However, its range, and hence the spectrum of the multiplication operator specified by $W^{(\alpha)}$, is a closed interval of positive length. At this point, we record a simple lemma that will be useful later.

LEMMA 2.14. *For all $p \geq 1$, there a constant C depending only on p, so that for an $N \geq 3$ and all $\alpha \in [0, 2]$,*

$$\left(\int_{\mathcal{S}_N} |w_N^{\alpha/2}(v_k) - 1|^p \, d\sigma_N\right)^{1/p} \leq \frac{C}{N}. \tag{84}$$

PROOF. This is an immediate consequence of (83), the triangle inequality, and the fact that for all $m \in \mathbb{N}$,

$$\lim_{N \to \infty} \int_{\mathcal{S}_N} |v_k|^{mp} \, d\sigma_N = \int_{\mathbb{R}^3} |v|^{mp} \, d\gamma.$$

□

LEMMA 2.15. *For all $\alpha \in [0, 2]$, the null space of $P^{(\alpha)}$ is independent of α. If h belongs to the null space of $P^{(0)}$, then $P_k h = 0$ for each $k = 1, \dots, N$. For all $\alpha \in [0, 2]$, the closure of the range of $P^{(\alpha)}$ is the subspace $\mathcal{A}_N$ of $L^2(\sigma_N)$ defined in Definition 2.1.*

PROOF. Since $P^{(\alpha)} \geq 0$, h belongs to the null space of $P^{(\alpha)}$ if and only if $\langle h, P^{(\alpha)} h\rangle = 0$. But $0 = \langle h, P^{(\alpha)} h\rangle = \frac{1}{N}\sum_{k=1}^N \int_{\mathcal{S}_N} w_N^{\alpha/2}(v_k)|P_k h|^2 \, d\sigma_N$. Since $w_N^{\alpha/2}(v_k) \geq 0$ almost everywhere, it must be the case that $|P_k h|^2$ vanishes identically. Thus h is in the null space of $P^{(\alpha)}$, $P_k h = 0$ for each k, and h is in the null space of $P^{(0)}$. Conversely, if h is in the null space of $P^{(0)}$, then $P_k h = 0$ for each k, and then clearly $P^{(\alpha)} h = 0$.

Since each $P^{(\alpha)} \geq 0$, the closure of its range is the orthogonal complement of its null space. Since the null space does not depend on α, neither does the range. Evidently, $\mathcal{A}_N$ is the closure of the range of $P^{(0)}$. □

2.3. *The spectrum of* $\widehat{L}_{N,0}$. Already in our paper [5] we have proved results that specify the exact spectral gap of $\widehat{L}^{N,\alpha}$ for $\alpha = 0$. This case is especially amenable for several reasons. First, since $W^{(0)} = 1$,

$$\widehat{L}_{N,0} f = f - P^{(0)} f,$$

and hence the problem is to determine the spectrum of $P^{(0)}$. Second, there is an orthonormal basis of $L^2(\sigma_N)$ consisting of eigenfunctions of $P^{(0}$. This is the case because each P_k is an average of rotations, so the finite dimensional spaces spanned by spherical harmonics of given maximal degree are invariant under $P^{(0)}$ and, therefore, one can study the spectrum of $P^{(0)}$ by studying the eigenvalue equation $P^{(0)} f = \lambda f$. This is the approach we took in our previous work. However, this approach cannot work even for $\alpha = 2$, the next simplest case: In this case, $P^{(2)}$ has an interval of continuous spectrum, as we shall see below. Therefore, we now give another argument that determines the spectral gap of $\widehat{L}_{N,0}$ that does extend to $\alpha = 2$ at least.

LEMMA 2.16. *For all* $N \geq 3$,

$$\widehat{\Delta}_{N,0} = 1 - \frac{3N-1}{3(N-1)^2} = 1 - \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \tag{85}$$

and the second largest eigenvalue of $P^{(0)}$, $\mu^{(0)}$, *is given by*

$$\mu^{(0)} = \frac{3N-1}{3(N-1)^2}. \tag{86}$$

PROOF. The range of $P^{(0)}$ is $\mathcal{A}_N$, and it suffices to determine the spectrum of $P^{(0)}$ as an operator on $\mathcal{A}_N$. For $f(\vec{v}) = \sum_{j=1}^N \varphi_j(v_j) \in \mathcal{A}_N$, we compute

$$P^{(0)}\left(\sum_{j=1}^N \varphi_j(v_j)\right) = \frac{1}{N}\sum_{k=1}^N\left(\varphi_k(v_k) + \sum_{j\neq k, j=1}^N K\varphi_j(v_k)\right).$$

By this computation, with $T : \bigoplus_{j=1}^N \mathcal{H}_{N,j} \to \mathcal{A}_N$ defined as in Theorem 2.5,

$$T^{-1} P^{(0)} T = \mathbf{M}^{(0)},$$

where $\mathbf{M}^{(0)} = [M_{i,j}^{(0)}]$ is the $N \times N$ block matrix operator on $\bigoplus_{j=1}^N \mathcal{H}_{N,j}$ given by

$$M_{i,j}^{(0)} = \frac{1}{N} I \quad \text{if } i = j \quad \text{and} \quad M_{i,j}^{(0)} = \frac{1}{N} K \quad \text{if } i \neq j.$$

Note that $\mathbf{M}^{(0)}$ is unitarily equivalent to the block matrix operator in $\bigoplus_{j=1}^N \mathcal{H}_{N,j}$ given by

$$\frac{1}{N}\begin{bmatrix} I+(N-1)K & 0 & 0 & \cdots & 0 \\ 0 & I-K & 0 & \cdots & 0 \\ 0 & 0 & I-K & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I-K \end{bmatrix}.$$

It follows that if $\lambda \neq 0$ is an eigenvalue of $P^{(0)}$ then either λ is an eigenvalue of $\frac{1}{N}(I + (N - 1)K)$ or else λ is an eigenvalue of $\frac{1}{N}(I - K)$. Thus, the second largest eigenvalue of $\mathbf{M}^{(0)}$, and hence $P^{(0)}$, is either $1 + (N - 1)\kappa$, where κ is the second largest eigenvalue of K, or else $1 - \kappa$ where κ is the least eigenvalue of K. From the information on the spectrum of K provided in Lemma 2.3, one immediately deduces (86), and then (85) follows directly. □

2.4. *The spectrum of* $\widehat{L}_{N,\alpha}$, $\alpha \in (0, 2]$. After $\alpha = 0$, the next simplest case is $\alpha = 2$ since then at least $W^{(2)}$ is constant; as we have seen $W^{(2)} = 1 - (N-1)^{-2}$. It follows that 1 is an eigenfunction for $P^{(2)}$ with eigenvalue $1 - (N-1)^{-2}$, and it spans the eigenspace. That is, 1 spans the null space of $\widehat{L}_{N,2}$.

LEMMA 2.17. *For all* $N \geq 3$, $\widehat{\Delta}_{N,2} > 0$.

PROOF. The range of $P^{(2)}$ is $\mathcal{A}_N$, and as with $\alpha = 0$, $\mathbf{M}^{(2)} := T^{-1} P^{(2)} T$ has a simple block matrix structure:

$$
\begin{aligned}
P^{(2)}\left(\sum_{j=1}^{N} \varphi_j(v_j)\right) &= \frac{1}{N} \sum_{k=1}^{N} w_{N,2}(v_k) P_k \left(\sum_{j=1}^{N} \varphi_j(v_j)\right) \\
&= \sum_{k=1}^{N} \frac{1}{N} w_{N,2}(v_k) \left(\varphi_k(v_k) + \sum_{j \neq k, j=1}^{N} K\varphi_j(v_k)\right).
\end{aligned}
$$

By Theorem 2.5, $\mathbf{M}^{(2)} = T^{-1} P^{(2)} T = \mathbf{W}^{(2)}(\mathbf{I} + \mathbf{C})$, where

$$
\mathbf{W}^{(2)} = \frac{1}{N} \begin{bmatrix} w_{N,2}(v_1) & 0 & \cdots & 0 \\ 0 & w_{N,2}(v_2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_{N,N}(v_N) \end{bmatrix},
$$

$\mathbf{I}$ is the identity on $\bigoplus_{j=1}^{N} \mathcal{H}_{N,j}$, and $\mathbf{C}$ is given by

$$
\mathbf{C} = \begin{bmatrix} 0 & K & K & \cdots & K \\ K & 0 & K & \cdots & K \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ K & \cdots & K & 0 & K \\ K & \cdots & K & K & 0 \end{bmatrix},
$$

Since $\mathbf{M}^{(2)}$ and $P^{(2)}$ are similar, they have the same spectrum, and in particular, the spectrum of $\mathbf{M}^{(2)}$ is real. (This is also evident from the identity $\mathbf{M}^{(2)} = \mathbf{W}^{(2)}(\mathbf{I} + \mathbf{C})$, and the fact that for bounded operators A and B on any Hilbert space, AB and BA have the same spectrum.)

Since the range of $\frac{1}{N} w_{N,2}$ is $[0, (N-1)^{-1}]$, this interval is the spectrum of $\mathbf{W}^{(2)}$. Note that $\mathbf{C}$, and hence $\mathbf{W}^{(2)}\mathbf{C}$ is compact. By Weyl's lemma, the essential spectrum of $T P^{(2)} T^{-1}$, and hence of $P^{(2)}$, is the essential spectrum of $\mathbf{W}^{(2)}$, which is the interval $[0, (N-1)^{-1}]$. Hence any spectrum of $P^{(2)}$ in $(N-1)^{-1}, 1 - (N-1)^{-2})$ consists of isolated eigenvalues, and the isolated eigenvalues can only accumulate at a point in $[0, (N-1)^{-1}]$. In particular, $1 - (N-1)^{-2}$ cannot be an accumulation point, and hence $P^{(2)}$ has a spectral gap below its top eigenvalue $1 - (N-1)^{-2}$. This proves that $\widehat{\Delta}_{N,2} > 0$ for all $N \geq 3$. □

For $\alpha \in (0, 2)$, $W^{(\alpha)}$ is not constant—although for large N it is nearly constant. This means that for such α, one cannot determine the spectrum of $\widehat{L}_{N,\alpha}$ simply by determining the spectrum of $P^{(\alpha)}$, and moreover, $\mathcal{A}_N$ is not invariant under $\widehat{L}_{N,\alpha}$. However, there is a simple comparison that one can make between $\mathcal{D}_{N,\alpha}$ and $\mathcal{D}_{N,2}$ that provides the bound on $\widehat{\Delta}_{N,\alpha}$ that we seek.

LEMMA 2.18. *For all* $N \geq 3$, *and all* $\alpha \in [0, 2]$,

$$
\widehat{\Delta}_{N,\alpha} \geq \left(\frac{N-1}{N}\right)^{1-\alpha/2} \widehat{\Delta}_{N,2} > 0.
$$

PROOF. By (82), for all f and k and all $\alpha \in (0,2)$,

$$\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{\alpha/2}[f-P_kf]^2 \geq \left(\frac{N-1}{N}\right)^{1-\alpha/2}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right][f-P_kf]^2.$$

It follows immediately that $\mathcal{D}_{N,\alpha}(f,f) \geq (\frac{N-1}{N})^{1-\alpha/2}\mathcal{D}_{N,2}(f,f)$, and then that $\widehat{\Delta}_{N,\alpha} \geq (\frac{N-1}{N})^{1-\alpha/2}\widehat{\Delta}_{N,2} > 0$. $\square$

At this point, we have proved the first part of Theorem 1.4, and all that remains is to prove the second part.

3. A sharper lower bound on $\widehat{\boldsymbol{\Delta}}_{N,1}$ for large N. In this section, we obtain lower bounds on $\mathcal{D}_{N,1}(f,f)$ for f orthogonal to the constants that become sharper and sharper as N increases. To keep the computations simple, we do this explicitly for $\alpha = 1$, though the method applies to all $\alpha \in (0,2)$. We shall prove the following, which is simply a specialization of Theorem 1.4.

THEOREM 3.1. *There is a constant C independent of N such that whenever f is orthogonal to the constants*,

$$\mathcal{D}_{N,1}(f,f) \geq \left(1 - \frac{1}{N} - \frac{C}{N^{3/2}}\right)\|f\|_2^2. \tag{87}$$

The bound (87) is meaningless for N such that the right-hand side is negative. However, no matter what C is, there is an $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, the right-hand side is positive. From that point on, we have what we need for our induction. The rest of this section is devoted to the proof of Theorem 3.1

3.1. *The trial function decomposition.* We begin by specifying a trial function decomposition that we shall use. Let $\mathcal{A}_N$ be the subspace of $L^2(\sigma_N)$ defined in Definition 2.1. For any $f \in L^2(\sigma_N)$ orthogonal to the constants, define p and h to be the orthogonal projections of f onto $\mathcal{A}_N$ and $\mathcal{A}_N^{\perp}$, respectively. Then since $1 \in \mathcal{A}_N$, h is orthogonal to the constant, and then $p = f - h$ is orthogonal to the constants.

By Lemma 2.15, h is the component of f in the null space of $P^{(\alpha)}$ for each $\alpha \in [0,2]$, and hence

$$\langle f, P^{(\alpha)} f\rangle = \langle p, P^{(\alpha)} p\rangle \tag{88}$$

which yields

$$\int_{\mathcal{S}_N} W^{(\alpha)} f^2 \, d\sigma - \langle f, P^{(\alpha)} f\rangle_{L^2(\sigma_N)} = \int_{\mathcal{S}_N} W^{(\alpha)} f^2 \, d\sigma - \langle p, P^{(\alpha)} p\rangle_{L^2(\sigma_N)}. \tag{89}$$

Since $p \in \mathcal{A}_N$, there are N functions $\phi_1, \ldots, \phi_N$ of a single variable such that $\phi_j(v_j) \in L^2(\sigma_N)$ for each j, and

$$p(\vec{v}) = \sum_{j=1}^{N} \phi_j(v_j), \tag{90}$$

and we shall always choose the particular representation of this form that is specified in Theorem 2.5. That is, the eigenfunctions expansion

$$\phi_j = \sum_{i=1}^{\infty} a_{j,i}\eta_i(v_j) \tag{91}$$

given in (53) is such that (54) is satisfied; that is, $\sum_{j=1}^{N} a_{j,i} = 0$ for $i = 1, \ldots, 4$. We make a further decomposition of $\phi_j(v_j)$ as follows.

DEFINITION 3.2. Let p be a function given by a sum of the form (90) where for each j, $\phi_j(v_j)$ is orthogonal to the constants, and moreover, (91) is satisfied with $\sum_{j=1}^N a_{j,i} = 0$ for $i = 1, \dots, 4$. Define

$$\psi_j(v_j) = \sum_{i=1}^{4} a_{j,i}\eta_i(v_j) \quad \text{and} \quad \varphi_j(v_j) = \sum_{i=5}^{\infty} a_{j,i}\eta_i(v_j) \tag{92}$$

so that $\phi_j = \psi_j + \varphi_j$. Next, define

$$g(\vec{v}) = \sum_{j=1}^{N} \varphi_j(v_j) \quad \text{and} \quad s(\vec{v}) = \sum_{j=1}^{N} \psi_j(v_j). \tag{93}$$

Finally, the *trial function decomposition* of any $f \in L^2(\sigma_N)$ that is orthogonal to the constants is given by

$$f = g + s + h, \tag{94}$$

where h is the component of f in the null space of $P^{(\alpha)}$, p is the component of f in the closure of the range of $P^{(\alpha)}$ and $p = g + s$ is the decomposition of p defined in (93).

REMARK 3.3. It is easy to see that when p is symmetric under coordinate permutations, one can take the functions ϕ_j in (90) to be all the same, and thus the $a_{j,i}$ in (92) do not depend on j. Then since for $i = 1, \dots, 4$, $\sum_{j=1}^N a_{j,i} = 0$, we have $a_{j,i} = 0$ whenever $i = 1, \dots, 4$. Hence when p is symmetric, $s = 0$, and in this case the trial function decomposition simplifies to $f = g + h$, as in [6].

We have seen in Lemma 2.15 that for each k, $P_k h = 0$. The next lemma shows that each P_k also has a simple action on s.

LEMMA 3.4. *For each k, the function s in the trial function decomposition satisfies*

$$P_k s(\vec{v}) = \frac{N}{N-1}\psi_k(v_k). \tag{95}$$

PROOF. Note that $P_k s(\vec{v}) = \psi_k(v_k) - \frac{1}{N-1}\sum_{j\neq k}\psi_j(v_k)$. Writing $\psi_j = \sum_{i=1}^4 a_{j,i}\eta_i$ and recalling that $\sum_{j=1}^N a_{j,i} = 0$ for $i = 1, \dots, 4$, for any *fixed* v,

$$\sum_{j=1}^{N} \psi_j(v) = \sum_{j=1}^{N}\sum_{i=1}^{4} a_{j,i}\eta_i(v) = \sum_{i=1}^{4}\left(\sum_{j=1}^{N} a_{j,i}\right)\eta_i(v) = 0,$$

from which (95) follows. □

Each of the components g, s and h have their own special properties that we shall repeatedly use.

(1) A very useful feature of $g(\vec{v}) = \sum_{j=1}^N \varphi_j(v_j)$ is that, by Lemma 2.3 for each j,

$$\|K\varphi_j\|_2 \le \frac{5N-3}{3(N-1)^3}\|\varphi_j\|_2. \tag{96}$$

This gives us something *almost* like Lemma 3.4 for g:

$$P_k g(\vec{v}) = \varphi_k(v_k) + \sum_{j\neq k} K\varphi_j(v_k),$$

and hence

$$\|P_k g - \varphi_k(v_k)\|_2 \le \left\|\sum_{j\neq k} K\varphi_j(v_k)\right\|_2 \le \frac{C}{N^2}\sum_k \|\varphi_k\|^2,$$

where in the last inequality we have used (96).

(2) The key feature of $s(\vec{v}) = \sum_{j=1}^N \psi_j(v_j)$ is that P_k has a very simple action on s, given in Lemma 3.4.

Another is that each $\psi_j(v_j)$ belongs to $L^4(\sigma_N)$, and for a constant C independent of N, $\|\psi_j\|_4 \le C\|\psi_j\|_2$. This is essentially because the integrals $\int_{\mathcal{S}_N} |v|^{2m}\,\mathrm{d}\sigma_N$ are bounded uniformly in N for each m. In particular, if we wish to estimate the $L^2(\sigma_N)$ norm of $|v_k|^2\psi_k(v_k)$, we can apply Schwarz's inequality to bound this by $C\|\psi_k\|_4$, and then, changing C, to $C\|\psi_k\|_2$. This will be used in estimating the quantity in (105) below.

(3) A very useful feature of $h(\vec{v})$ is that, by Lemma 2.15, $P_k h = 0$ for each k, and in particular, $P^{(1)}h = 0$.

3.2. *Lower bound on $\mathcal{D}_{N,1}(f,f)$.* For $\alpha = 1$, the lower bound (76) simplifies to

$$W^{(1)}(\vec{v}) \ge \widetilde{W}^{(1)}(\vec{v}) := 1 + \frac{1}{(N-1)^3} - \frac{1}{2}\frac{N}{(N-1)^4}\sum_{k=1}^N |v_k|^4. \tag{97}$$

Define

$$\widetilde{\mathcal{D}}_{N,1}(f,f) = \int_{\mathcal{S}_N} \widetilde{W}^{(1)} f^2\,\mathrm{d}\sigma_N - \langle f, P^{(1)} f\rangle. \tag{98}$$

By (97), $\mathcal{D}_{N,1}(f,f) \ge \widetilde{\mathcal{D}}_{N,1}(f,f)$.

Now let f be orthogonal to the constants, and let $f = g + s + h$ be the trial function decomposition of f as specified above. This notation will be used throughout this subsection. Note that

$$\begin{aligned}\widetilde{\mathcal{D}}_{N,1}(f,f) &= \widetilde{\mathcal{D}}_{N,1}(g,g) + \widetilde{\mathcal{D}}_{N,1}(s,s) + \widetilde{\mathcal{D}}_{N,1}(h,h)\\ &\quad + 2\widetilde{\mathcal{D}}_{N,1}(g,h) + 2\widetilde{\mathcal{D}}_{N,1}(s,h) + 2\widetilde{\mathcal{D}}_{N,1}(g,s).\end{aligned}$$

The next lemma says that g, s and h are almost mutually orthogonal with respect to the inner product given by $\widetilde{\mathcal{D}}_{N,1}$, and hence the last three terms above make a negligible contribution. This decouples the contributions of g, s and h, which may then be analyzed separately, taking advantage of their different helpful properties.

LEMMA 3.5. *There is a constant C independent of N such that for any $f \in L^2(\sigma_N)$ that is orthogonal to the constants, if $f = g+s+h$ is the trial function decomposition as specified above, then*

$$2\big|\widetilde{\mathcal{D}}_{N,1}(g,h)\big| + 2\big|\widetilde{\mathcal{D}}_{N,1}(s,h)\big| + 2\big|\widetilde{\mathcal{D}}_{N,1}(g,s)\big| \le \frac{C}{N^{3/2}}\|f\|_2^2.$$

PROOF. Since $P^{(1)}h = 0$, and since g and h are orthogonal, recalling that we may write $g(\vec{v}) = \sum_{j=1}^{N} \varphi_j(v_j)$,

$$\widetilde{\mathcal{D}}_{N,1}(g,h) = \int_{\mathcal{S}_N} \widetilde{W}^{(1)} gh \, \mathrm{d}\sigma_N = -\frac{1}{2}\frac{N}{(N-1)^4}\sum_{k=1}^{N}\int_{\mathcal{S}_N} |v_k|^4 gh \, \mathrm{d}\sigma_N$$

$$= -\frac{1}{2}\frac{N}{(N-1)^4}\sum_{k=1}^{N}\int_{\mathcal{S}_N} |v_k|^4 \varphi_k(v_k) h \, \mathrm{d}\sigma_N \tag{99}$$

$$-\frac{1}{2}\frac{N}{(N-1)^4}\sum_{j\neq k}^{N}\int_{\mathcal{S}_N} |v_k|^4 \varphi_j(v_j) h \, \mathrm{d}\sigma_N \, \mathrm{d}\sigma_N. \tag{100}$$

The integral in (99) vanishes since $P_k h = 0$. Next consider the integral in (100). It will be convenient to introduce the notation $\xi(x) = x^8$ for the eighth power. Then, with this definition, the Schwarz inequality, and then application of the K operator,

$$\begin{aligned} \left|\int_{\mathcal{S}_N} |v_k|^4 \varphi_j(v_j) h \, \mathrm{d}\sigma_N\right| &\le \|h\|_2 \left(\int_{\mathcal{S}_N} |v_k|^8 \varphi_j^2(v_j)\, \mathrm{d}\sigma_N\right)^{1/2} \\ &= \|h\|_2 \left(\int_{\mathcal{S}_N} K\xi(v_j)\varphi_j^2(v_j)\, \mathrm{d}\sigma_N\right)^{1/2}. \end{aligned} \tag{101}$$

By Lemma 2.7, there is a constant C so that, independent of N, $\|K\xi\|_\infty \le C$. Therefore, $|\int_{\mathcal{S}_N} h|v_k|^4 \varphi_j(v_j)\, \mathrm{d}\sigma_N| \le C\|h\|_2\|\varphi_j\|_2$. Using this in (100) gives us

$$\left|\frac{N}{(N-1)^4}\int_{\mathcal{S}_N}\left(\sum_{k=1}^{N}|v_k|^4\right) gh\, \mathrm{d}\sigma\right| \le \frac{N}{(N-1)^3} C\|h\|_2 \left(\sum_{j=1}^{N}\|\varphi_j\|_2\right), \tag{102}$$

and then since Theorem 2.5 gives us

$$\sum_{j=1}^{N}\|\varphi\|_2 \le \left(1 - \frac{5N-3}{3(N-1)^2}\right)^{-1/2}\sqrt{N}\|g\|_2,$$

we have that the left-hand side of (102) is bounded by $\frac{C}{N^{3/2}}\|g\|_2\|h\|_2$ for a constant C independent of N. We conclude that $|\widetilde{\mathcal{D}}_{N,1}(s,h)| \le CN^{-3/2}$.

Finally, we consider $\widetilde{\mathcal{D}}_{N,1}(s,g)$. This time we must also estimate $\langle s, P^{(1)}g\rangle$. Because the span of $\{\eta_j(v_j) : 1 \le j \le N\}$ is invariant under $P^{(0)}$, and every function in it is orthogonal to g,

$$\langle s, P^{(1)}g\rangle = \langle s, P^{(1)}g\rangle - \langle P^{(0)}s, g\rangle = \langle s, (P^{(1)} - P^{(0)})g\rangle.$$

Introducing the short notation $\tilde{w}(v) := w_{N,1}(v) - 1$ *to be used in this proof only*, and writing

$$s(\vec{v}) = \sum_{\text{[redacted]}=1}^{N}\psi_j(v_j) \quad \text{and} \quad g(\vec{v}) = \sum_{j=1}^{N}\varphi_\ell(v_\ell),$$

we have

$$\langle s, P^{(1)}g\rangle = \sum_{j,k,\ell=1}^{N}\int_{\mathcal{S}_N}\psi_j(v_j)\tilde{w}(v_k)P_k\varphi_\ell(v_\ell)\, \mathrm{d}\sigma_N. \tag{103}$$

We now split the sum over j, k and ℓ, into five parts

(i) $j = \ell = k$ (ii) $j \neq k, \ell = k$ (iii) $j = k, \ell \neq k$ and (iv) $j = \ell, \ell \neq k$, and finally, (v) $j \neq \ell, \ell \neq k, k \neq j$

$$\langle s, P^{(1)} g\rangle = \langle s, (P^{(1)} - P^{(0)}) g\rangle \tag{104}$$

$$= \frac{1}{N}\sum_{k=1}^{N}\langle \tilde{w}(v_k)\psi_k(v_k), \varphi_k(v_k)\rangle \tag{105}$$

$$+ \frac{1}{N}\sum_{k=1}^{N}\sum_{j\neq k}^{N}\langle \tilde{w}(v_k)\psi_j(v_j), \varphi_k(v_k)\rangle \tag{106}$$

$$+ \frac{1}{N}\sum_{k=1}^{N}\sum_{\ell\neq k}^{N}\langle \tilde{w}(v_k)\psi_k(v_k), P_k\varphi_\ell(v_k)\rangle \tag{107}$$

$$+ \frac{1}{N}\sum_{k=1}^{N}\sum_{\ell\neq k}^{N}\langle \tilde{w}(v_k)\psi_\ell(v_k), P_k\varphi_\ell(v_k)\rangle \tag{108}$$

$$+ \frac{1}{N}\sum_{j\neq k, k\neq \ell, \ell\neq j}^{N}\langle \tilde{w}(v_k)\psi_j(v_j), P_k\varphi_\ell(v_k)\rangle. \tag{109}$$

We estimate (105) as follows, using Lemma 2.14 to bound $\|\tilde{w}(v_k)\|_4$:

$$\frac{1}{N}\left|\sum_{k=1}^{N}\langle \tilde{w}(v_k)\psi_k(v_k), \varphi_k(v_k)\rangle\right| \leq \frac{1}{N}\sum_{k=1}^{N}\|\tilde{w}(v_k)\|_4\|\psi_k\|_4\|\varphi_k\|_2 \leq \frac{C}{N^2}\sum_{k=1}^{N}\|\psi_k\|_2\|\varphi_k\|_2$$
$$\leq \frac{C}{N^2}\big(\|s\|_2^2 + \|g\|_2^2\big).$$

Since $P_k\psi_j(v_k) = -\frac{1}{N-1}\psi_j(v_k)$, the argument used to estimate (105) shows that the absolute value of the sum in (106) is bounded above by

$$\frac{C}{N^3}\sum_{k=1}^{N}\sum_{j\neq k}^{N}\|\psi_j\|_2\|\varphi_k\|_2 \leq \frac{C}{N^2}\big(\|s\|_2^2 + \|g\|_2^2\big),$$

as we found for (105). Since for $k \neq j$, $\|P_k\varphi_j\|_2 \leq CN^{-2}\|\varphi_j\|_2$ (by (48)), the argument used to estimate (105) shows that the absolute value of the sum in (107) is bounded above by

$$\frac{C}{N^4}\sum_{k=1}^{N}\sum_{j\neq k}^{N}\|\psi_k\|_2\|\varphi_j\|_2 \leq \frac{C}{N^3}\big(\|s\|_2^2 + \|g\|_2^2\big),$$

even better than the previous bounds. Finally, for the terms in (109),

$$\begin{aligned}\big|\langle \tilde{w}(v_k)\psi_j(v_j), P_k\varphi_\ell(v_\ell)\rangle\big| &= \big|\langle P_k\varphi_\ell(v_\ell)\tilde{w}(v_k), P_k\psi_j(v_j)\rangle\big| \\ &= \frac{1}{N-1}\big|\langle P_k\varphi_\ell(v_\ell)\tilde{w}(v_k), \psi_j(v_j)\rangle\big| \\ &\leq \frac{1}{N-1}\|K\varphi_\ell\|_2\|\psi_j\|_2 \leq \frac{C}{N^3}\|\varphi_\ell\|_2\|\psi_j\|_2,\end{aligned}$$

where in the last inequality we have used Lemma 2.3, and the fact that for each j, ψ_j is an eigenfunction of K considered as an operator on $L^2(\sigma_N)$, with eigenvalue $-\frac{1}{N-1}$. Thus,

$$\frac{C}{N^4}\sum_{j\neq k, k\neq \ell, \ell\neq j}^{N}\|\psi_j\|_2\|\varphi_\ell\|_2 \leq \frac{C}{N^2}\big(\|s\|_2^2 + \|g\|_2^2\big).$$

This proves $|\langle s, P^{(1)} g\rangle| \leq \frac{C}{N^2}(\|s\|_2^2 + \|g\|_2^2)$. $\square$

We now turn to the estimation of $\widetilde{\mathcal{D}}_{N,1}(g, g)$ and $\widetilde{\mathcal{D}}_{N,1}(s, s)$.

LEMMA 3.6. *There is a constant C independent of $N \geq 3$ such that for all g and s as above*,

$$\langle g, P^{(1)} g\rangle \leq \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}\varphi_k^2(v_k)\,d\sigma_N + \frac{C}{N^2}\|g\|_2^2 \tag{110}$$

and

$$\langle s, P^{(1)} s\rangle \leq \frac{1}{N-1}\sum_{k=1}^{N}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}\psi_k^2(v_k)\,d\sigma_N. \tag{111}$$

PROOF. Note first of all that $P_k g = \varphi_k(v_k) + \sum_{j\neq k} K\varphi_j(v_k)$, and thus

$$\begin{aligned}\langle g, P^{(1)} g\rangle &= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}|P_k g|^2\,d\sigma_N\\ &= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}\varphi_k^2(v_k)\,d\sigma_N\end{aligned}$$

$$+\frac{2}{N}\sum_{k=1}^{N}\sum_{j\neq k}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}\varphi_k(v_k)K\varphi_j(v_k)\,d\sigma_N \tag{112}$$

$$+\frac{1}{N}\sum_{k=1}^{N}\sum_{j\neq k}\sum_{\ell\neq k}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}K\varphi_j(v_k)K\varphi_\ell(v_k)\,d\sigma_N. \tag{113}$$

By the Schwarz inequality and (48), the sum of integrals in (112) is bounded above by

$$\frac{C}{N^3}\sum_{k=1}^{N}\sum_{j\neq k}\|\varphi_j\|_2\|\varphi_k\|_2 \leq \frac{C}{N^2}\|g\|_2^2.$$

Similarly, by (48) and Lemma 2.5, the sum of integrals in (113) is bounded above by

$$\frac{C}{N^4}\sum_{j,k=1}^{N}\|\varphi_j\|_2\|\varphi_k\|_2 \leq \frac{C}{N^3}\|g\|_2^2.$$

Using the two bounds we have just derived on (112) and (113), respectively, yields (110).

$$\langle s, P^{(1)} s\rangle = \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N}\left[\frac{N^2-(1+|v_k|^2)N}{(N-1)^2}\right]^{1/2}|P_k s|^2\,d\sigma_N,$$

(111) follows directly from Lemma 3.4. □

LEMMA 3.7. *There is a constant C such that for all N and all g and s as above*,

$$\int_{\mathcal{S}_N}\sum_{k=1}^{N}|v_k|^4 g^2\,d\sigma_N \leq \sum_{k=1}^{N}\int_{\mathcal{S}_N}\varphi_k(v_k)^2|v_k|^4\,d\sigma_N + CN\|g\|_2^2, \tag{114}$$

and

$$\int_{\mathcal{S}_N}\sum_{k=1}^{N}|v_k|^4 s^2\,d\sigma_N \leq \sum_{k=1}^{N}\int_{\mathcal{S}_N}\psi_k(v_k)^2|v_k|^4\,d\sigma_N + CN\|s\|_2^2. \tag{115}$$

PROOF.

$$\int_{\mathcal{S}_N} \sum_{k=1}^N |v_k|^4 g^2 \, d\sigma_N = \sum_{i,j,k=1}^N \int_{\mathcal{S}_N} \varphi_i(v_i)\varphi_j(v_j)|v_k|^4 \, d\sigma_N$$

$$= \sum_{k=1}^N \int_{\mathcal{S}_N} \varphi_k(v_k)^2 |v_k|^4 \, d\sigma_N \tag{116}$$

$$+ 2\sum_{k=1}^N \sum_{j\neq k} \int_{\mathcal{S}_N} \varphi_j(v_j)\varphi_k(v_k)|v_k|^4 \, d\sigma_N \tag{117}$$

$$+ \sum_{k=1}^N \sum_{j\neq k} \int_{\mathcal{S}_N} \varphi_j(v_j)^2 |v_k|^4 \, d\sigma_N \tag{118}$$

$$+ \sum_{i\neq j, j\neq k, k\neq i} \int_{\mathcal{S}_N} \varphi_i(v_i)\varphi_j(v_j)|v_k|^4 \, d\sigma_N. \tag{119}$$

By Lemma 3.8, Lemma 3.9 and Lemma 3.10 below the terms in (117), (118) and (119) add up to no more than $CN\|g\|_2^2$, which proves (114). The same argument using the same lemmas proves (115). □

LEMMA 3.8. *There is a constant C such that for all N and all g and s as above*,

$$2\sum_{k=1}^N \sum_{j\neq k} \int_{\mathcal{S}_N} \varphi_j(v_j)\varphi_k(v_k)|v_k|^4 \, d\sigma_N \le CN\|g\|_2^2 \tag{120}$$

and

$$2\sum_{k=1}^N \sum_{j\neq k} \int_{\mathcal{S}_N} \psi_j(v_j)\psi_k(v_k)|v_k|^4 \, d\sigma_N \le C\|s\|_2^2. \tag{121}$$

PROOF. For $j \neq k$, using the pointwise bound $|v_k|^4 \le (N-1)^2$ and then (48),

$$\begin{aligned} \int_{\mathcal{S}_N} \varphi_j(v_j)\varphi_k(v_k)|v_k|^4 \, d\sigma_N &\le (N-1)^2 \|K\varphi_j\|_2 \|\varphi_k\|_2 \\ &\le \frac{(5N-3)(N-1)^2}{3(N-1)^3} \|\varphi_j\|_2 \|\varphi_k\|_2. \end{aligned} \tag{122}$$

Then, by Theorem 2.5, (120) follows. Next,

$$\begin{aligned} \int_{\mathcal{S}_N} \psi_j(v_j)\psi_k(v_k)|v_k|^4 \, d\sigma_N &\le \|K\psi_j\|_2 \||v_k|^4\psi_k\|_2 \le \frac{1}{N-1}\|\psi_j\|_2 C\|\psi_k\|_4 \\ &\le \frac{C}{N}\|\psi_j\|_2 C\|\psi_k\|_2. \end{aligned} \tag{123}$$

Then, by Theorem 2.5 again, (121) follows. □

LEMMA 3.9. *There is a constant C such that for all N and all g and s as above*,

$$\sum_{k=1}^N \sum_{j\neq k} \int_{\mathcal{S}_N} \varphi_j(v_j)^2 |v_k|^4 \, d\sigma_N \le CN\|g\|_2^2 \tag{124}$$

and

$$\sum_{k=1}^{N}\sum_{j\neq k}\int_{\mathcal{S}_N}\psi_j(v_j)^2|v_k|^4\,\mathrm{d}\sigma_N \le CN\|s\|_2^2. \tag{125}$$

PROOF. By Lemma 2.8, there is a finite constant C independent of N such that

$$\sum_{k=1}^{N}\sum_{j\neq k}\int_{\mathcal{S}_N}\varphi_j(v_j)^2|v_k|^4\,\mathrm{d}\sigma_N$$
$$\le N\sum_{j=1}^{N}\int_{\mathcal{S}_N}\frac{N^2+|v_j|^4-2N|v_j|^2}{(N-1)^2}\varphi_j^2(v_j)\,\mathrm{d}\sigma_N + CN\sum_{j=1}^{N}\|\varphi_j\|_2^2.$$

Then, by Lemma 2.5, (124) follows. The same analysis yields (125). □

LEMMA 3.10. *There is a constant C such that for all N and all g and s as above*,

$$\sum_{i\neq j, j\neq k, k\neq i}\int_{\mathcal{S}_N}\varphi_i(v_i)\varphi_j(v_j)|v_k|^4\,\mathrm{d}\sigma_N \le CN\|g\|_2^2 \tag{126}$$

and

$$\sum_{i\neq j, j\neq k, k\neq i}\int_{\mathcal{S}_N}\psi_i(v_i)\psi_j(v_j)|v_k|^4\,\mathrm{d}\sigma_N \le CN\|s\|_2^2. \tag{127}$$

PROOF. By Lemma 2.9, with $S(v,w)$ given by (64), there is a finite constant C independent of N such that for i, j and k all different

$$\left|\int_{\mathcal{S}_N}\varphi_i(v_i)\varphi_j(v_j)|v_k|^4\,\mathrm{d}\sigma_N - \int_{\mathcal{S}_N}\varphi_i(v_i)\varphi_j(v_j)S(v_i,v_j)\,\mathrm{d}\sigma_N\right|$$
$$\le \frac{C}{N}\int_{\mathcal{S}_N}|\varphi_i(v_i)||\varphi_j(v_j)|\,\mathrm{d}\sigma_N.$$

Therefore, with C changing from line to line,

$$\sum_{i\neq j, j\neq k, k\neq i}\int_{\mathcal{S}_N}\varphi_i(v_i)\varphi_j(v_j)|v_k|^4\,\mathrm{d}\sigma_N$$
$$\le C\sum_{i\neq j}\|\varphi_i\|_2\|\varphi_j\|_2 + (N-2)$$
$$\times\sum_{i\neq j}\int_{\mathcal{S}_N}\frac{N^2+|v_i|^4+|v_j|^4+2N|v_i|^2+2N|v_j|^2+2|v_i|^2|v_j|^2}{(N-2)^2}\varphi_i(v_i)\varphi_j(v_j)\,\mathrm{d}\sigma_N.$$

Note that for $i\neq j$,

$$\int_{\mathcal{S}_N}\frac{N^2+|v_i|^4+|v_j|^4+2N|v_i|^2+2N|v_j|^2}{N-2}\varphi_i(v_i)\varphi_j(v_j)\,\mathrm{d}\sigma_N \le \frac{C}{N}\|\varphi_i\|_2\|\varphi_j\|_2$$

since in each term we may either replace φ_i by $K\varphi_i$ or φ_j by $K\varphi_j$, and this gives a factor of CN^{-2}. Then, by Theorem 2.5, $\sum_{i\neq j}\|\varphi_i\|_2\|\varphi_j\|_2 \le CN\|g\|_2^2$.

The remaining terms must be handled differently. For $j = 1, \dots, N$, let ξ_j denote the function $\xi_j(v) = |v_j|^2 \varphi_j(v_j)$, and note that ξ_j is orthogonal to the constants. Therefore,

$$\int_{\mathcal{S}_N} \frac{|v_i|^2 |v_j|^2}{N-2} \varphi_i(v_i)\varphi_j(v_j)\,\mathrm{d}\sigma_N = \frac{1}{N-2}\langle \xi_i, K\xi_j\rangle$$
$$\le \frac{1}{N-2}\frac{1}{N-1}\|\xi_i\|_2\|\xi_j\|_2 \le C\|\varphi_i\|_2\|\varphi_j\|_2.$$

Then, by Lemma 2.5, $\sum_{i\neq j}\|\varphi_i\|_2\|\varphi_j\|_2 \le CN\|g\|_2^2$, and (126) follows.

Next,

$$\sum_{i\neq j, j\neq k, k\neq i} \int_{\mathcal{S}_N} \psi_i(v_i)\psi_j(v_j)|v_k|^4\,\mathrm{d}\sigma_N$$
$$\le C\sum_{i\neq j}\|\psi_i\|_2\|\psi_j\|_2 + (N-2)$$
$$\times \sum_{i\neq j}\int_{\mathcal{S}_N} \frac{N^2 + |v_i|^4 + |v_j|^4 + 2N|v_i|^2 + 2N|v_j|^2 + 2|v_i|^2|v_j|^2}{(N-2)^2}\psi_i(v_i)\psi_j(v_j)\,\mathrm{d}\sigma_N.$$

The main term is

$$\frac{N^2}{N-2}\sum_{i\neq j}\int_{\mathcal{S}_N}\psi_i(v_i)\psi_j(v_j)\,\mathrm{d}\sigma_N = \frac{N^2}{N-2}\sum_{i\neq j}\langle\psi_i, K\psi_j\rangle$$
$$\le \frac{N^2}{(N-2)(N-1)}\sum_{i\neq j}\|\psi_i\|_2\|\psi_j\|_2,$$

and simple estimates show that all remaining terms are smaller. □

3.3. *Lower bounds on* $\widetilde{\mathcal{D}}_{N,1}(g,g)$ *and* $\widetilde{\mathcal{D}}_{N,1}(s,s)$. We are now ready to estimate $\widetilde{\mathcal{D}}_{N,1}(g,g)$ and $\widetilde{\mathcal{D}}_{N,1}(s,s)$. We first define a quadratic form $\mathcal{F}_{N-1}$ on $L^2(\sigma_N)$ as follows: For all functions r in $L^2(\sigma_N)$, define

$$\mathcal{F}_{N-1}(r,r) := \frac{1}{2}\frac{N}{(N-1)^4}\int_{\mathcal{S}_N}\sum_{k=1}^N |v_k|^4 r^2\,\mathrm{d}\sigma_N + \langle r, P^{(1)}r\rangle. \tag{128}$$

LEMMA 3.11. *For all g and s as above*,

$$\widetilde{\mathcal{D}}_{N,1}(g,g) \ge \|g\|_2^2 - \mathcal{F}_{N-1}(g,g) \quad \text{and} \quad \widetilde{\mathcal{D}}_{N,1}(s,s) \ge \|s\|_2^2 - \mathcal{F}_{N-1}(s,s). \tag{129}$$

PROOF. This is immediate from (76), (98) and the definition of $\mathcal{F}_{N-1}$. □

LEMMA 3.12. *There is a finite constant C independent of N such that for all g and s as above*, *with* $\mathcal{F}$ *defined by* (128),

$$\mathcal{F}_{N-1}(g,g) \le \left(\frac{1}{N} + \frac{C}{N^2}\right)\|g\|_2^2 \quad \text{and} \quad \mathcal{F}_{N-1}(s,s) \le \left(\frac{1}{N} + \frac{C}{N^2}\right)\|s\|_2^2. \tag{130}$$

PROOF. By Lemma 3.6 and Lemma 3.7,

$$\mathcal{F}_{N-1}(g,g) \le \frac{1}{N}\sum_{k=1}^N\int_{\mathcal{S}_N}\left[\left[\frac{N^2 - (1+|v_k|^2)N}{(N-1)^2}\right]^{1/2} + \frac{1}{2}\frac{N^2}{(N-1)^4}|v_k|^4\right]\varphi_k(v_k)^2\,\mathrm{d}\sigma_N$$
$$+ \frac{C}{N^2}\|g\|_2^2.$$

Define $y_k := \frac{N}{(N-1)^2}|v_k|^2$. Then $0 \le y_k \le N/(N-1)$, and

$$\mathcal{F}_{N-1}(g,g) \le \frac{1}{N}\sum_{k=1}^{N}\int_{\mathcal{S}_N} w(y_k)\varphi_k^2(v_k)\,\mathrm{d}\sigma_N + \frac{C}{N^2}\|g\|_2^2, \tag{131}$$

where $w(y) = (\frac{N}{N-1} - y)^{1/2} + \frac{1}{2}y^2$. Simple calculations show that $w(y) \le \sqrt{N/(N-1)}$ for all $0 \le y \le N/(N-1)$, and in fact, for $N \ge 7$, $w(y)$ is monotone decreasing on this interval. Then (131) becomes

$$\mathcal{F}_{N-1}(g,g) \le \sqrt{N/(N-1)}\frac{1}{N}\sum_{k=1}^{N}\|\varphi_k\|_2^2 + \frac{C}{N^2}\|g\|_2^2.$$

Now (130) follows directly from Theorem 2.5. The proof of (130) is the same. □

3.4. *Proof of Theorem* 3.1. By Lemma 3.5,

$$\mathcal{D}_{N,1}(f,f) \ge \widetilde{\mathcal{D}}_{N,1}(f,f) \ge \widetilde{\mathcal{D}}_{N,1}(g,g) + \widetilde{\mathcal{D}}_{N,1}(s,s) + \widetilde{\mathcal{D}}_{N,1}(h,h) - CN^{-3/2}\|f\|^2 s.$$

By Lemma 3.11 and Lemma 3.12,

$$\widetilde{\mathcal{D}}_{N,1}(g,g) + \widetilde{\mathcal{D}}_{N,1}(s,s) \ge \left(1 - \frac{1}{N} - \frac{C}{N^2}\right)(\|g\|_2^2 + \|s\|_2^2).$$

Since $P^{(1)}h = 0$, (78) yields $\widetilde{\mathcal{D}}_{N,1}(h,h) \ge (1 - \frac{1}{2N} - \frac{C}{N^2})\|h\|_2^2$, adding the estimates completes the proof since $\|f\|_2^2 = \|g\|_2^2 + \|s\|_2^2 + \|h\|_2^2$.

APPENDIX A: SOME COMPUTATIONAL PROOFS

PROOF OF LEMMA 2.8. By (24),

$$\begin{aligned} &E\{|v_1|^4 \mid v_N = v\} \\ &\quad = \int_{\mathcal{S}_{N-1}}\left(\eta^4(v)|\vec{y}|^4 + \frac{\eta^2(v)}{(N-1)^2}|\vec{y}\cdot v|^2 + \frac{|v|^4}{(N-1)^4} + 2\frac{\eta^2(v)}{(N-1)^2}|\vec{y}|^2\right)\mathrm{d}\sigma_{N-1}, \end{aligned} \tag{132}$$

where

$$\eta^2(v) = \frac{N - |v|^2 - |v|^2/(N-1)}{N-1}.$$

Define $M_N := \int_{\mathcal{S}_{N-2}}|\vec{y}|^4\,\mathrm{d}\sigma_{N-2}$ which is bounded uniformly in N:

$$\lim_{N\to\infty}\int_{\mathcal{S}_{N-1}}|\vec{y}|^4\,\mathrm{d}\sigma_{N-1} = (2\pi/3)^{-3/2}\int_{\mathbb{R}^3}|y|^4 e^{-3|y|^2/2}.$$

Then the right-hand side of (132) becomes

$$M_N\eta^4(v) + \frac{1}{3(N-1)^2}\eta^2(v)|v|^2 + \frac{|v|^4}{(N-1)^4} + 2\frac{\eta^2(v)}{(N-1)^2}. \tag{133}$$

Note that for some constant C independent of N,

$$\frac{1}{3(N-1)^2}\eta^2(v)|v|^2 + \frac{|v|^4}{(N-1)^4} + 2\frac{\eta^2(v)}{(N-1)^2} \le \frac{C}{N}. \tag{134}$$

Next,

$$\eta^4(v) = \frac{N^2 + |v|^4 - 2N|v|^2}{(N-1)^2} + \frac{|v|^4}{(N-1)^4} + 2\frac{(N-|v|^2)|v|^2}{(N-1)^3}.$$

Again, for some constant C independent of N,

$$\frac{|v|^4}{(N-1)^4} + 2\frac{(N-|v|^2)|v|^2}{(N-1)^3} \le \frac{C}{N}.$$

□

PROOF OF LEMMA 2.9. By a simple adaptation of (24),

$$\begin{aligned}
&E\{|v_1|^4 \mid (v_{N-1}, v_N) = (v, w)\} \\
&\quad = \int_{\mathcal{S}_{N-2}} \biggl(\beta^4(v,w)|\vec{y}|^4 + \frac{\beta^2(v,w)}{(N-2)^2}|\vec{y}\cdot(v+w)|^2 \\
&\qquad + \frac{|v+w|^4}{(N-2)^4} + 2\frac{\beta^2(v,w)}{(N-2)^2}|\vec{y}|^2\biggr)\,\mathrm{d}\sigma_{N-2},
\end{aligned} \tag{135}$$

where

$$\beta^2(v,w) = \frac{N - |v|^2 - |w|^2 - |v+w|^2/(N-2)}{N-2},$$

which is nonnegative on the allowed values for (v, w). Note that $\beta^2(v,w) \le N/(N-2)$. Define $M_N := \int_{\mathcal{S}_{N-2}} |\vec{y}|^4 \,\mathrm{d}\sigma_{N-2}$ which is bounded uniformly in N:

$$\lim_{N\to\infty} \int_{\mathcal{S}_{N-2}} |\vec{y}|^4 \,\mathrm{d}\sigma_{N-2} = (2\pi/3)^{-3/2} \int_{\mathbb{R}^3} |y|^4 e^{-3|y|^2/2}.$$

Then the right-hand side of (135) becomes

$$M_N \beta^4(v,w) + \frac{1}{3(N-2)^2}\beta^2(v,w)|v+w|^2 + \frac{|v+w|^4}{(N-2)^4} + 2\frac{\beta^2(v,w)}{(N-2)^2}. \tag{136}$$

Note that for some constant C independent of N,

$$\frac{1}{3(N-2)^2}\beta^2(v,w)|v+w|^2 + \frac{|v+w|^4}{(N-2)^4} + 2\frac{\beta^2(v,w)}{(N-2)^2} \le \frac{C}{N}. \tag{137}$$

Next,

$$\begin{aligned}
\beta^4(v,w) &= \frac{N^2 + |v|^4 + |w|^4 + 2N|v|^2 + 2N|w|^2 + 2|v|^2|w|^2}{(N-2)^2} \\
&\quad + \frac{|v+w|^4}{(N-2)^4} + 2\frac{(N-|v|^2-|w|^2)|v+w|^2}{(N-2)^3}.
\end{aligned} \tag{138}$$

Again, for some constant C independent of N,

$$\frac{|v+w|^4}{(N-2)^4} + 2\frac{(N-|v|^2-|w|^2)|v+w|^2}{(N-2)^3} \le \frac{C}{N}.$$

□

APPENDIX B: QUANTITATIVE ESTIMATES ON $\widehat{\Delta}_{N,2}$

B.1. An explicit bound for $N \geq 4$. By (82), $P^{(\alpha)}$ defined in (65) satisfies

$$0 \leq P^{(\alpha)} \leq \left(\frac{N}{N-1}\right)^{\alpha/2} P^{(0)} \tag{139}$$

for all $\alpha \in [0, 2]$. As we have seen, the second largest eigenvalue of $P^{(0)}$, denoted $\mu_N^{(0)}$, is given by

$$\mu_N^{(0)} = \frac{3N-1}{3(N-1)^2}. \tag{140}$$

It follows from (139) and (140) that for all f orthogonal to the constants,

$$\langle f, P^{(\alpha)} f\rangle \leq \left(\frac{N}{N-1}\right)^{\alpha/2} \frac{3N-1}{3(N-1)^2} \|f\|_2^2, \tag{141}$$

for all $\alpha \in [0, 2]$. For $\alpha = 2$, we have $\langle f, P^{(2)} f\rangle \leq \frac{N(3N-1)}{3(N-1)^3} \|f\|_2^2$. Note that

$$\frac{N(3N-1)}{3(N-1)^3} = \frac{1}{N-1} + \frac{5}{3}\frac{1}{(N-1)^2} + \frac{2}{3}\frac{1}{(N-1)^3},$$

which evidently decreases monotonically as N increases. Next, since $W^{(2)}(\vec{v}) = 1 - \frac{1}{(N-1)^2}$, we have that for all f orthogonal to the constants

$$\begin{aligned} -\langle f, \widehat{L}_{N,2} f\rangle &= \langle f, (W^{(2)} - P^{(2)}) f\rangle \\ &\geq \left(1 - \frac{1}{N-1} - \frac{8}{3}\frac{1}{(N-1)^2} - \frac{2}{3}\frac{1}{(N-1)^3}\right)\|f\|_2^2. \end{aligned} \tag{142}$$

For $N = 3$, this yields only the useless bound $-\langle f, \widehat{L}_{3,2} f\rangle \geq -\frac{1}{4}\|f\|_2^2$. But already for $N = 4$, it yields

$$-\langle f, \widehat{L}_{4,2} f\rangle \geq \frac{28}{81}\|f\|_2^2.$$

Since the right-hand side of (142) increases as N increases, this, together with the comparison from Lemma 2.18, proves the following.

THEOREM B.1. *For all $N \geq 4$,*

$$\widehat{\Delta}_{N,2} \geq 1 - \frac{1}{N-1} - \frac{8}{3}\frac{1}{(N-1)^2} - \frac{2}{3}\frac{1}{(N-1)^3} > 0,$$

and for all $\alpha \in (0, 2)$,

$$\widehat{\Delta}_{N,\alpha} \geq \left(\frac{N-1}{N}\right)^{1-\alpha/2}\left(1 - \frac{1}{N-1} - \frac{8}{3}\frac{1}{(N-1)^2} - \frac{2}{3}\frac{1}{(N-1)^3}\right) > 0.$$

At this point, the only estimate we lack for a fully quantitative result is a quantitative estimate on $\widehat{\Delta}_{3,2}$.

B.2. An explicit bound for $N = 3$. By what has been explained earlier, $\widehat{\Delta}_{3,2} = \frac{3}{4} - \nu_3$ where

$$\nu_3 = \sup\{\langle f, P^{(2)} f\rangle_{L^2(\sigma_3)} : \|f\|_2 = 1, \langle f, 1\rangle_{L^2(\sigma_3)} = 0\}, \tag{143}$$

and by Lemma 2.17, $\widehat{\Delta}_{3,2} > 0$, or, what is the same $\nu_3 < \frac{3}{4}$.

If $\nu_3 \le \frac{1}{2}$, then evidently $\widehat{\Delta}_{3,2} \ge \frac{1}{4}$. Therefore, we need only consider the possibility that $\nu_3 > \frac{1}{2}$, and as we have seen, in this case ν_3 is an eigenvalue of $P^{(2)}$, and necessarily $\nu_3 < \frac{3}{4}$.

In seeking the second largest eigenvalue of $P^{(2)}$, we need only consider functions f of the form

$$f(\vec{v}) = \sum_{j=1}^{N} \varphi(v_j) \tag{144}$$

or

$$f(\vec{v}) = \varphi(v_1) - \varphi(v_2), \tag{145}$$

where in the second case we have taken advantage of the the symmetry of $P^{(2)}$ to assume without loss of generality that f is antisymmetric under interchange of v_1 and v_2.

LEMMA B.2. *For $N = 3$, the largest eigenvalue of $P^{(2)}$ on the orthogonal complement of the symmetric sector is no greater than* 0.735. *Thus, either $\widehat{\Delta}_{3,2} \ge 0.015$, or else the gap eigenfunction is symmetric.*

PROOF OF LEMMA B.2. For later use, we begin the proof for $N \ge 3$, and specialize to $N = 3$ later. Let f be given by (145), where we may assume that φ is orthogonal to the constants. Then

$$\frac{1}{N} w_{N,2}(v_1)(1-K)\varphi(v_1) - \frac{1}{N} w_{N,2}(v_2)(1-K)\varphi(v_2) = \lambda\big(\varphi(v_1) - \varphi(v_2)\big).$$

Multiplying by $\varphi(v_1)$ and integrating,

$$\frac{1}{N}\int_{\mathcal{S}_N} w_{N,2}(v_1)\big|(1-K)\varphi(v_1)\big|^2 = \lambda\langle\varphi, (1-K)\varphi\rangle. \tag{146}$$

By (80),

$$w_{N,2}(v) = \frac{N}{N-1} - \frac{N}{(N-1)^2}|v_k|^2. \tag{147}$$

Using (147) in (146) yields

$$\frac{1}{N-1}\langle\varphi, (1-K)^2\varphi\rangle - \frac{1}{(N-1)^2}\langle(1-K)\varphi, |v|^2(1-K)\varphi\rangle = \lambda\langle\varphi, (1-K)\varphi\rangle. \tag{148}$$

Now write $\sqrt{1-K}\varphi = \psi + \zeta$ where ψ is orthogonal to the constants, the three components of v and $|v|^2$. Then ζ is an eigenvector of K with eigenvalue $-1/(N-1)$, and hence

$$\frac{1}{N-1}\langle\varphi, (1-K)^2\varphi\rangle \ge \frac{1}{N-1}\langle\psi, (1-K)\psi\rangle, \tag{149}$$

and

$$\begin{aligned}\big\langle(1-K)\varphi, |v|^2(1-K)\varphi\big\rangle &= \big\langle\sqrt{1-K}\psi, |v|^2\sqrt{1-K}\psi\big\rangle + \frac{N-2}{(N-1)^2}\big\||v|\zeta\big\|_2^2 \\ &\quad - 2\big\||v|\sqrt{1-K}\psi\big\|_2 \frac{\sqrt{N-1}}{N-2}\big\||v|\zeta\big\|_2^2\end{aligned} \tag{150}$$

$$\geq \left(1 - \frac{1}{t}\right)\langle\sqrt{1-K}\psi, |v|^2\sqrt{1-K}\psi\rangle$$
$$+ (1-t)\frac{N-2}{(N-1)^2}\||v|\zeta\|_2^2,$$

for all $t > 0$, where we have used the arithmetic-geometric mean inequality.

At this point, we specialize to $N = 3$, and carry out some explicit computations that could be done for all $N \geq 3$, but are then more cumbersome.

Write $\zeta = \sum_{j=1}^4 a_j\eta_j(v)$ where, as before, $\eta(j)v) = \mathbf{e}\cdot v$ for $j = 1, 2, 3$, and where $\eta_4(v) = |v|^2 - 1$. One readily computes that

$$\||v|\zeta\|_2^2 = \sum_{j=1}^4 |a_j|^2\||v|\eta_j\|_2^2 \tag{151}$$

and that $\int_{\mathcal{S}_3}|v_1|^4\,\mathrm{d}\sigma_3 = \frac{5}{4}$ and $\int_{\mathcal{S}_3}|v_1|^6\,\mathrm{d}\sigma_3 = \frac{7}{4}$. From here, it follows that

$$\||v|\eta_j\|_2^2 = \frac{5}{4} \quad \text{for } j = 1, 2, 3 \quad \text{and} \quad \||v|\eta_4\|_2^2 = 1.$$

Using this in (151) finally yields $\||v|\zeta\|_2^2 \geq \|\zeta\|_2^2$, and evidently

$$\langle\sqrt{1-K}\psi, |v|^2\sqrt{1-K}\psi\rangle \leq 3\|\sqrt{1-K}\psi\|_2^2.$$

Therefore, for $0 < t < 1$, we have from (150) that

$$\langle(1-K)\varphi, |v|^2(1-K)\varphi\rangle \geq 3\left(1 - \frac{1}{\lambda}\right)\|\sqrt{1-K}\psi\|_2^2 + (1-\lambda)\|\zeta\|_2^2.$$

Using this estimate in (148) yields

$$\frac{1}{2}\|\sqrt{1-K}\psi\|_2^2 + \frac{3}{4}\|\zeta\|_2^2 - 3\left(1 - \frac{1}{t}\right)\|\sqrt{1-K}\psi\|_2^2 - (1-t)\|\zeta\|_2^2 \geq \lambda(\|\psi\|_2^2 + \|\zeta\|_2^2).$$

The second most negative eigenvalue of K for $N = 3$ is $-\frac{3}{8}$; see [7], Section 8, where this eigenvalue is denoted $\kappa_{1,2}$. It follows that $\|\sqrt{1-K}\psi\|_2^2 \leq \frac{11}{8}\|\psi\|_2^2$. Therefore,

$$\left(\frac{1}{16} - 3 + \frac{3}{t}\right)\|\psi\|_2^2 + \left(\frac{3}{4} - 1 + t\right)\|\zeta\|_2^2 \geq \lambda(\|\psi\|_2^2 + \|\zeta\|_2^2).$$

Choosing $t = 0.985$, we have that $0.735(\|\psi\|_2^2 + \|\zeta\|_2^2) \geq \lambda(\|\psi\|_2^2 + \|\zeta\|_2^2)$. $\square$

The remaining task is to bound the second largest eigenvalue of $P^{(2)}$ in the symmetric sector. We begin considering general $N \geq 3$ and shall specialize to $N = 3$ later.

Let f be given as in (144). Then $P^{(2)}f = \lambda f$ becomes

$$\frac{1}{N}\sum_{k=1}^N w_{N,2}(v_k)\big(\varphi(v_k) + (N-1)K\varphi(v_k)\big) = \lambda\sum_{j=1}^N \varphi(v_j). \tag{152}$$

By Theorem 2.5,

$$\frac{1}{N}w_{N,2}(v)\big(\varphi(v) + (N-1)K\varphi(v)\big) = \lambda\varphi(v). \tag{153}$$

Therefore, multiplying both sides of (153) by $\varphi(v)$ and integrating, we obtain

$$\frac{1}{N}\int_{\mathcal{S}_N}\varphi(v_1)w_{N,2}(v_1)\big(\varphi(v_1) + (N-1)K\varphi(v_1)\big)\,\mathrm{d}\sigma_N = \lambda\|\varphi\|_2^2. \tag{154}$$

By (80), (154) becomes

$$\langle \varphi, K\varphi\rangle - \frac{1}{N-1}\int_{\mathcal{S}_N} \varphi(v_1)|v_1|^2 K\varphi(v_1)\,\mathrm{d}\sigma_N = \left(\lambda - \frac{1}{N-1}\right)\|\varphi\|_2^2 + \frac{1}{(N-1)^2}\int_{\mathcal{S}_N} |v_1|^2\varphi^2(v_1)\,\mathrm{d}\sigma_N. \tag{155}$$

Define an operator M by $M\phi(v) = |v|^2(1 + (N-1)K)\phi(v)$.

Then (155) becomes

$$\left(\lambda - \frac{1}{N-1}\right) = \frac{\langle \varphi, K\varphi\rangle - (N-1)^{-2}\langle \varphi, M\varphi\rangle}{\|\varphi\|_2^2}.$$

Thus $\lambda - (N-1)^{-1}$ can be computed by computing the supremum of the right-hand side as φ ranges over functions that are orthogonal to 1, the three components of v and $|v|^2$.

Note that M commutes with rotations so the different angular momentum sectors are mutually orthogonal, and can be considered separately. In each sector, by the usual recursion relations for orthogonal polynomials, the matrix representing M in the eigenbasis of K is tridiagonal and explicitly computable, and the bounds proved in [7], Section 8, can be used to limit the number of angular momentum sectors that need to be considered. Hence one could obtain explicit bounds this way.

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