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EXPERIMENTAL FORCED RESPONSE ANALYSIS OF TWO-DEGREE-OF-FREEDOM PIECEWISE-LINEAR SYSTEMS WITH A GAP

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ABSTRACT

In this paper, an experimental forced response analysis for a two degree of freedom piecewise-linear oscillator is discussed. First, a mathematical model of the piecewise linear oscillator is presented. Second, the experimental setup developed for the forced response study is presented. The experimental setup is capable of investigating a two degree of freedom piecewise linear oscillator model. The piecewise linearity is achieved by attaching mechanical stops between two masses that move along common shafts. Forced response tests have been conducted, and the results are presented. Discussion of characteristics of the oscillators are provided based on frequency response, spectrogram, time histories, phase portraits, and Poincaré sections. Period doubling bifurcation has been observed when the excitation frequency changes from a frequency with multiple contacts between the masses to a frequency with single contact between the masses occurs.

INTRODUCTION

Engineered structures involving mechanical contacts between the components, or internal cracks show piecewise-linear (PWL) nonlinearity in their oscillatory responses. Hence, prediction of their dynamics requires a proper modeling of such nonlinearity. Followed by a pioneering work by Shaw and Holmes [1] about impact oscillators, there have been many attempts to date to investigate the behavior of PWL systems for their fundamental nature and practical importance [2–6]. One of the difficulties in the prediction of PWL system dynamics is that the system characteristics is not smooth, because there are sudden changes in the stiffness or the masses of the system depending on the states of the system. Therefore, typical numerical methods, such as the time integration method, require long computational time. This hinders analysts from conducting comprehensive parametric studies or optimizations of the systems involving such piecewise linearity.

The authors proposed methods to approximate the nonlinear resonant frequencies of PWL systems, by using the concept of *bilinear frequency* [1], where the system states are assumed to have only two-states: open state and sliding state [7]. This concept has been extended to form a reduction bases of the PWL system [8]. Jung *et al.* then proposed a method called *bilinear amplitude approximation* to approximate the amplitude of forced response of PWL systems, where two linear systems are stitched together to form the response of the PWL system [9]. Tien and D'Souza then generalized the concept of bilinear amplitude and frequency approximations for bilinear systems with gaps or prestress [10], and for bilinear systems involving complex behaviors [11]. To date, however, experimental investigations on such

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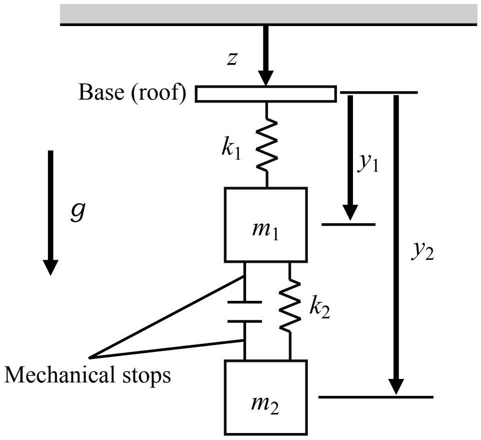


FIGURE 1. MASS SPRING MODEL WITH MECHANICAL STOPS

systems have not fully been conducted. Hence, those methods need to be experimentally validated. Therefore, the aim of this research is to develop experimental methods that enable the investigation of the nonlinear dynamic behavior of PWL systems, and to gain understanding of such systems.

This paper is organized as follows. In Section 1, the mathematical model of the PWL system of interest is described. In Section 2, the experimental setup developed for the analysis is presented. In Section 3, results of the forced response analysis are shown. Conclusions of the work are then drawn in Section 4.

1 MATHEMATICAL MODEL

This section provides the mathematical model of the PWL system to be investigated. Figure 1 shows the schematic diagram of the system of interest. Let y_1 and y_2 denote displacements of the two masses that are denoted as m_1 and m_2 measured from the base (roof) whose displacement is denoted as z. m_1 is connected to the base (roof) by a spring with k_1 being its spring constant. m_1 is connected also to m_2 by another spring with k_2 being its spring constant. Between the masses, there are mechanical stops that hinder inter-penetration of the masses. This system undergoes repetitive switching between two linear systems, which we call the *open* and *closed* states. The open state refers to the case where the masses are not in contact with each other, while the closed state refers to the case where the masses are in contact with each other.

For the open state, the equations of motion of the masses are written as follows.

$$m_1\ddot{y}_1 = -k_1(y_1-\ell_1)+k_2(y_2-y_1-\ell_2)+m_1(g-\ddot{z}), \quad (1)$$
$$m_2\ddot{y}_2 = -k_2(y_2-y_1-\ell_2)+m_2(g-\ddot{z}), \quad (2)$$

where ℓ_1 and ℓ_2 denote the unstretched lengths of springs with stiffnesses given by k_1 and k_2, respectively, and $y_2 \geqslant y_1$. The equilibrium states are obtained by solving Eqs. (1) and (2) with respect to y_1 and y_2 by setting any time derivative terms to be zero, i.e.,

$$\tilde{y}_1 = \ell_1 + (m_1+m_2)g/k_1, \quad (3)$$
$$\tilde{y}_2 = 2\ell_1+\ell_2+(m_1+m_2)g/k_1+m_2g/k_2. \quad (4)$$

Introducing relative displacements x_1 and x_2 measured from each equilibrium, i.e., $y_1 = \tilde{y}_1 + x_1$, $y_2 = \tilde{y}_2 + x_2$, and substituting these back into Eqs. (1) and (2), we obtain the equations of motion of the oscillators,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\begin{bmatrix} m_1\ddot{z} \\ m_2\ddot{z} \end{bmatrix} \quad (5)$$

where

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}. \quad (6)$$

When the masses are in contact with each other, the system is in the closed state. For the closed state, the masses move together and the system indeed becomes a single system. The corresponding equation of motion is written as

$$(m_1+m_2)\ddot{x}_1 + k_1x_1 = -(m_1+m_2)\ddot{z}, \quad (7)$$

where $x_1 = x_2$. As can be seen in Eqs. (5) and (7), the system equations change at the point when $x_1 = x_2$, which makes this system piecewise linear. In this paper, we focus on the experimental realization of this mathematical model, as described in the following section.

2 EXPERIMENTAL SETUP

Figure 2 shows the experimental setup developed for the investigation of the forced response of the two DOF PWL system. It contains mechanical components that achieve the two DOF PWL oscillator, and the measurement system. The entire system is directly mounted on the head of an electrodynamic shaker (San-Esu, SSV-750, Japan), which is capable of exciting the entire system in the vertical direction. A mass is hung from the roof of the housing by four linear springs. The mass is denoted as m_1, and the total spring constant of the four springs is denoted as k_1. The movement of m_1 is restricted to move only in the vertical direction by attaching four linear bushings on through holes of m_1

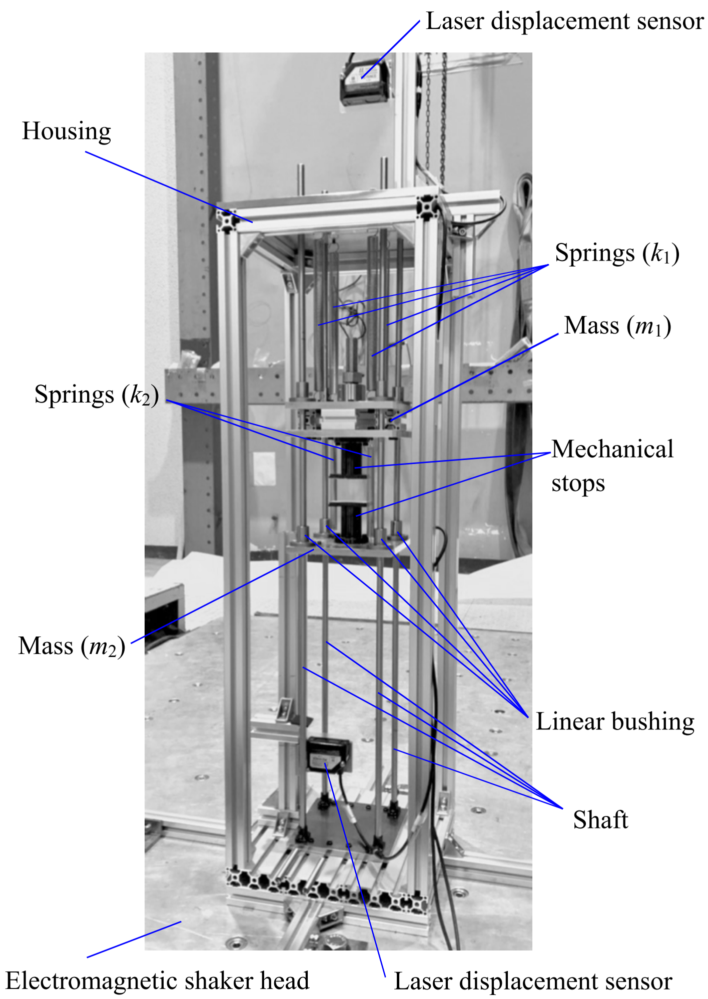


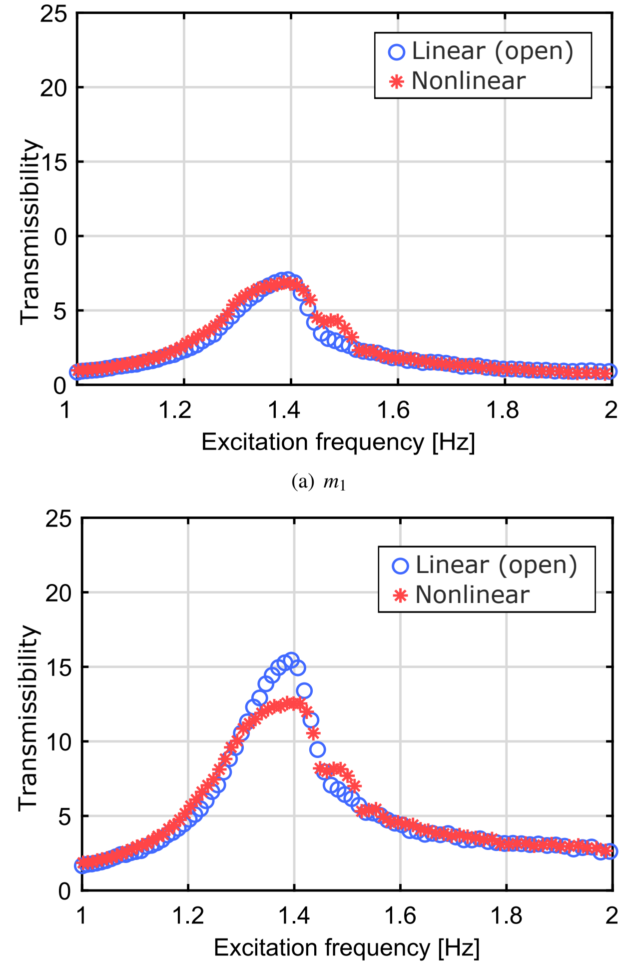
FIGURE 2. DEVELOPED TWO DOF OSCILLATORS SUBJECT TO VERTICAL BASE EXCITATION

and connecting them to linear shafts. The shafts stand vertically and rigidly fixed to the bottom of the setup and the roof. Also, another mass, which is denoted as m_2, is hung from m_1 by two linear springs, the total spring constant of which is denoted as k_2. The movement of m_2 is also restricted to motion in only the vertical direction by attaching linear bushings on through holes of m_2 and connecting them to the shafts. There are mechanical stops attached to the masses, which hinder the movement of the masses when they make contact with each other. Viscoelastic sheets with low stiffness are attached on both surfaces of the mechanical stops to minimize the rebound effect during impacts.

The key parameters of the oscillators are shown in Table 1. By substituting these values into Eq. (6), we obtain the first natural frequency of the open system to be expected as $f_o = 1.37$Hz. Furthermore, the natural frequency for the closed system can be

TABLE 1. PARAMETER VALUES OF THE OSCILLATORS

m_1 [kg]	m_2 [kg]	k_1 [N/m]	k_2 [N/m]
2.90	1.27	400	186



(a) m_1

(b) m_2

FIGURE 3. FORCED RESPONSE OF THE OSCILLATORS

obtained from Eq. (7) to be expected as $f_c = 1.56$Hz.

Displacements of the masses are measured by using laser displacement sensors (Keyence, IL-300, Japan), and recorded into a data-logger (KYOWA, EDX-200A, Japan).

3 RESULTS

First, results of forced response tests are discussed. A forced response test has been conducted for the frequency range from 1Hz to 2Hz, which contains the first natural frequency of the open system, and that of the closed system. Figure 3 shows the transmissibility versus the excitation frequency of the base.

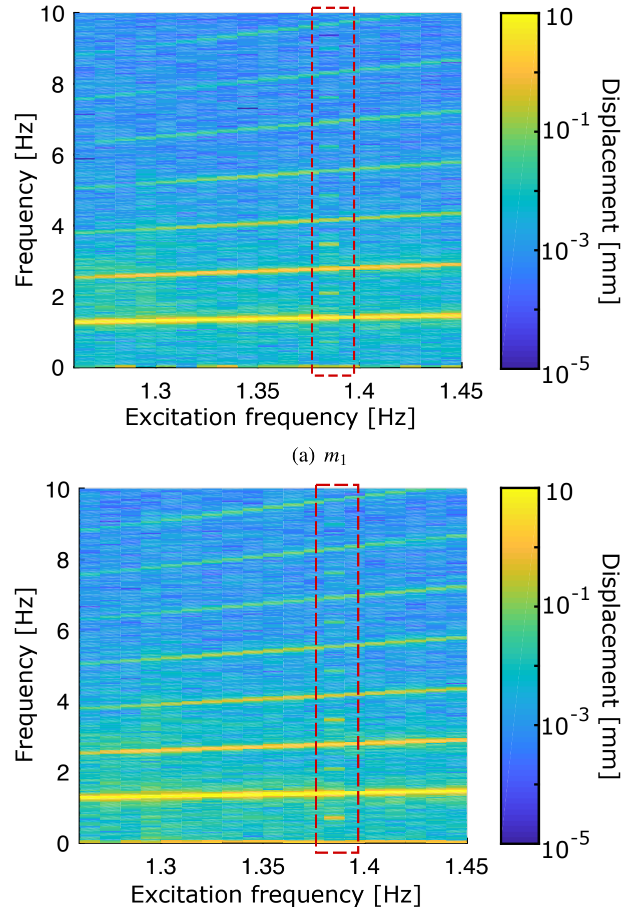
Transmissibility here is defined as the ratio of each mass's displacement amplitude to that of the base displacement. As can be seen from Fig. 3, both masses have primary resonance at 1.37Hz for the linear case. This corresponds to the expected natural frequency of the open system, or f_o. On the other hand, for the nonlinear case where intermittent contacts between masses occur, the first resonant frequency for the masses is slightly larger than that of the linear case. This appears to correspond to the *bilinear frequency* approximation [1] of the system, which is an approximated resonant frequency of bilinear systems that can be computed from the natural frequencies of the open and closed system, as follows:

$$f_{bi} = \frac{2 f_o f_c}{f_c + f_o} = 1.45\text{Hz}. \tag{8}$$

The magnitude of the transmissibility at the primary resonance is smaller than that for the linear case. This makes sense because the movement of the masses is hindered by the mechanical stops. Interestingly, following a slight jump in the response, a secondary resonance at around 1.5Hz is observed for both masses. This appears to correspond to the natural frequency of the closed system, or f_c=1.56Hz.

To investigate the frequency content in the responses for the nonlinear cases, a fast Fourier transform (FFT) was conducted on the displacement time histories for each excitation frequency, and shown in Fig. 4. As expected, a strong peak is observed at the frequency that corresponds to the excitation frequency. Furthermore, superharmonic components, or frequency components that are integer multiples of the excitation frequency, are also observed. This is mostly due to the distortion in the displacement time histories Moreover, as indicated in Fig. 4, subharmonic components, or frequency components that are fractional multiples of the excitation frequency, are also observed at around 1.38Hz. This indicates the occurrence of a *period doubling bifurcation*. The period doubling bifurcation has been observed numerically for PWL systems by Zuo and Curnier [12] and Jiang *et al.* [2], for instance.

Time histories of the masses near the bifurcation point are shown in Fig. 5. First, when the excitation frequency is 1.26Hz, the oscillators collide with each other only once during a vibration cycle, as shown in Fig. 5(a). After increasing the excitation frequency to 1.29Hz, the oscillators start to collide with each other twice during a vibration cycle, as shown in Fig. 5(b). Then, when the excitation frequency is 1.37Hz, as can be seen in Fig. 5(c), contact between the masses occur twice during a period of excitation. Also, we can see that the waveforms repeat themselves exactly after a period of excitation. Next, when the excitation frequency is 1.38Hz, on the other hand, waveforms of the displacements are slightly distorted, and the waveforms do not repeat themselves after a period of excitation, as can be seen



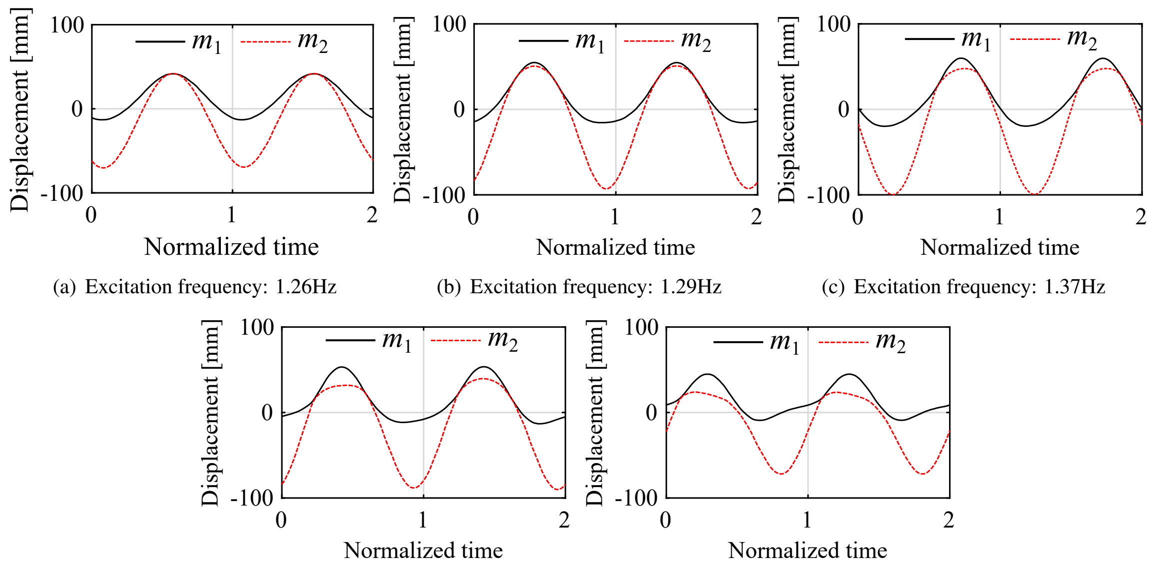
(a) m_1

(b) m_2

FIGURE 4. FFT OF THE FORCED RESPONSE

in Fig. 5(d). Again, the masses come in contact twice during a period of excitation for this excitation frequency. Finally, when the excitation frequency is 1.39Hz, the state of contact turns back to the state where the masses collide only once during a period of excitation, as shown in Fig. 5(e). Furthermore, the waveforms appears to repeat themselves after a period of excitation. It means that the period of oscillation turns back to that of the excitation. In summary, when the excitation frequency is smaller than the resonant frequency, the masses collide with each other once during a vibration cycle. As the excitation frequency increases, the masses start to collide twice during a vibration cycle. Then, when the excitation frequency reaches the resonance a period doubling bifurcation occurs, while the masses still collide with each other twice during a cycle. When the excitation frequency is increased slightly above the resonance, the masses start to collide only once during a cycle.

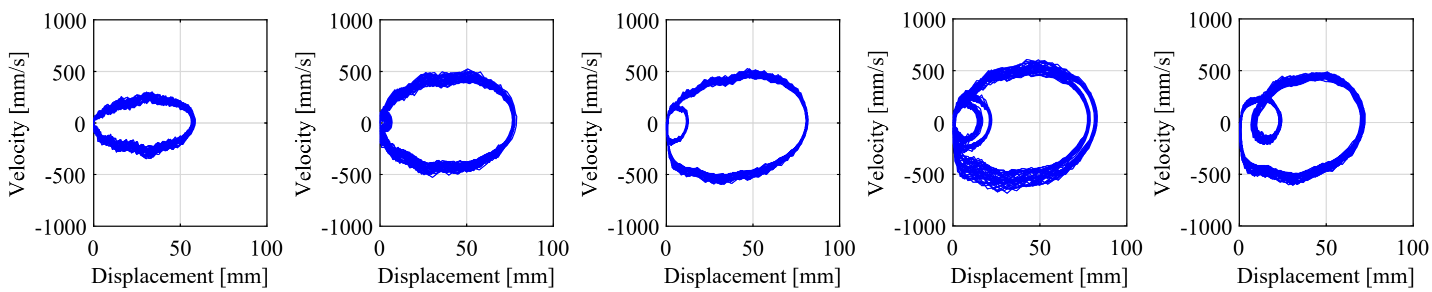
To see the oscillators' behaviors near the bifurcation point in detail, phase portraits of the *gap function* have been plotted, and



(a) Excitation frequency: 1.26Hz (b) Excitation frequency: 1.29Hz (c) Excitation frequency: 1.37Hz

(d) Excitation frequency: 1.38Hz (e) Excitation frequency: 1.39Hz

FIGURE 5. TIME HISTORIES OF THE MASSES NEAR THE BIFURCATION POINT



(a) 1.26Hz (b) 1.29Hz (c) 1.37Hz (d) 1.38Hz (e) 1.39Hz

FIGURE 6. PHASE PORTRAITS OF THE GAP FUNCTION NEAR THE BIFURCATION POINT

shown in Fig. 6. The gap function is defined as

$$g(t) = y_2(t) - y_1(t). \tag{9}$$

As can be seen in Fig. 6(a), when the excitation frequency is 1.26Hz, the trajectory of the gap function forms a closed limit cycle in the phase plane. Clearly, there is only one collision during a vibration cycle. On the other hand, as shown in Fig. 6(b), when the excitation frequency is 1.29Hz, the rebound effect results in a small loop near the origin of the phase plane. When the excitation frequency is 1.37Hz, the small loop near the origin grows, as shown in Fig. 6(c). When the excitation frequency is 1.38Hz, some parts of the limit cycle are split into multiple paths, as shown in Fig. 6(d). This also indicates the occurrence of a period doubling bifurcation. Finally, as shown in Fig. 6(e), when the excitation frequency is 1.39Hz, the split paths of the limit cycle observed in Fig. 6(b) disappear. This means that the doubled period of the oscillation turns back to a single period.

To better understand the bifurcation characteristics discussed above, a Poincaré section [13], or stroboscopic map has been taken from the phase portrait of the gap function. Namely, the snapshots of the trajectory of the gap function in the phase space are taken at time instants that are integer multiples of the excitation period. The Poincaré section of the gap when the excitation frequency is 1.29Hz is shown in Fig. 7(a). As seen in the figure, there is only a single group of points in the plot. This means that the periodicity of the motion is still kept for this excitation frequency. As can be seen in Figs. 7(b) and (c), the periodicity is still kept at 1.29Hz and 1.37Hz. On the other hand, as shown in Fig. 7(d), when the excitation frequency is 1.38Hz, there are two groups of points in the plots. This again indicates the occurrence of a period doubling bifurcation. When the excitation frequency is 1.39Hz, there is again only a single group of points in the plots, as seen in Fig. 7(e).

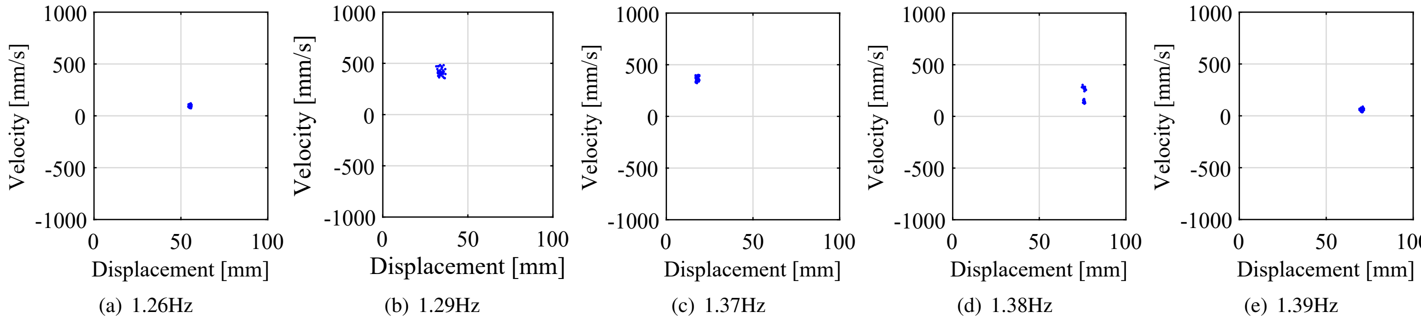


FIGURE 7. POINCARÉ SECTION OF THE GAP FUNCTION NEAR THE BIFURCATION POINT

4 CONCLUSION

In this paper, an experimental investigation of the forced response of a piecewise linear system has been presented. Forced response tests were conducted to show that the experimental setup was capable of simulating the dynamics of a piecewise linear oscillator. Furthermore, it was found that a period doubling bifurcation occurs between frequencies where oscillators are in contact twice during a vibration cycle, and where oscillators are in contact only once during a vibration cycle. Further investigations of this system using numerical simulation are planned as a part of future work.

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