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…nequalities involving Aharonov–Bohm magnetic potentials in dimensions 2 and 3

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… is devoted to a collection of results on nonlinear interpolation inequalities …with Schrödinger operators involving Aharonov–Bohm magnetic potentials, … consequences. As symmetry plays an important role for establishing optimal… we shall consider various cases corresponding to a circle, a two-dimensional … two-dimensional torus, and also the Euclidean spaces of dimensions 2 and … the results are new and we put the emphasis on the methods, as very lit… … on symmetry, rigidity and optimality in the presence of a magnetic field. …pectacular applications are new magnetic Hardy inequalities in dimensions 2 …

…Aharonov–Bohm magnetic potential; radial symmetry; cylindrical symmetry; …reaking; magnetic Hardy inequality; magnetic interpolation inequality; opti…nts; magnetic Schrödinger operator; magnetic Keller–Lieb–Thirring inequal…ic rings.

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tion

oblems involving magnetic fields play a peculiar role in the calculus of
s fair to say that there are no simple physical intuitions that may serve
d one has to extract information through exact computations, such as
the Landau Hamiltonian. A case in point are the *symmetry properties*
where in the absence of magnetic fields one has fairly robust methods
The isoperimetric inequality is one of the main sources of intuition;
ts are essentially different versions of the isoperimetric problem. This
ot the case in the presence of magnetic fields and there are very few
direction.

le is the work of Avron, Herbst and Simon [2] who proved that the
of the hydrogenic atom in a constant magnetic field has cylindri-
. The proof is quite involved. Another result that comes to mind is
who proved the equivalent of the Faber–Krahn inequality for the
perator with a constant magnetic field and with a Dirichlet boundary
a domain. The disk yields the smallest ground state energy among
qual area. Again, the proof is quite involved and some arguments are
linear setting. In this connection one should mention the recent result
Nys and van Schaftingen [8] who showed perturbatively that in some
iational problem involving a small constant magnetic field the mini-
the symmetry of the problem. Besides the constant magnetic field
class of physically relevant variational problems involve Aharonov–
ic fields and the purpose of this paper is to give an up-to-date account
dge.

onov–Bohm effect states that the wave function of a charged quan-
passing by a thin magnetic solenoid experiences a phase shift. This,
here is no apparent interaction with the solenoid except through the
the particle with the 'unphysical' vector potential. This prediction
ginally by Ehrenberg and Siday in 1949 (see [18]) and then again in
onov–Bohm (see [1]) and we will stay with the custom of calling it the
hm effect. It cannot be explained in terms of classical mechanics, but
ess experimentally verified (see [4]). It counts as one of the important
hanical effects.

uestion one may pose is the influence of the Aharonov–Bohm poten-
nergies of systems, say, of a particle in a potential interacting with
It is relatively straightforward to write the Hamiltonian for this situ-
may ask for the effect of the Aharonov–Bohm field on the *spectrum*
onian. One fruitful approach is to relate the ground state energy of
echanical particle in an external potential to the minimization of a
l variational problem. This also works in the presence of magnetic
particular, with the Aharonov–Bohm field. In this context, this leads

hich are the dual versions of the *Keller–Lieb–Thirring spectral esti-*
ous symmetry settings, we are interested in getting as much insight
out the best constants in the inequalities and also about the quali-
ies of their extremal functions. Indeed, in many cases studying the
perties of those extremal functions allows us to get very accurate,
s even sharp, estimates for the best constants in the inequalities (see
]).

clidean space $\mathbb{R}^d$, the *magnetic Laplacian* is defined via a *magnetic*
y

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi.$$

he case of dimensions $d=2$ and $d=3$. The *magnetic field* is $\mathbf{B} =$
adratic form associated with $-\Delta_{\mathbf{A}}$ is given by $\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\,\psi|^2$ and well
functions in the space

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \{\psi\in\mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\,\psi\in\mathrm{L}^2(\mathbb{R}^d)\}$$

gnetic gradient takes the form

$$\nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}.$$

–Bohm magnetic field can be considered as a singular measure sup-
set $x_1 = x_2 = 0$, where $(x_i)_{i=1}^d$ is a system of cartesian coordinates.
potential is defined as follows.

us consider *polar coordinates* (r,θ) such that

$$r = |x| = \sqrt{x_1^2+x_2^2} \quad\text{and}\quad r\,e^{i\theta} = x_1 + i\,x_2$$

ronov–Bohm magnetic potential

$$\mathbf{A} = \frac{a}{r^2}\,(x_2,-x_1) = \frac{a}{r}\,\mathbf{e}_\theta \tag{1.1}$$

real constant and $\{\mathbf{e}_r,\mathbf{e}_\theta\}$, with $\mathbf{e}_r = \frac{x}{r}$, denotes the orthogonal basis
ith our polar coordinates. The magnetic gradient and the magnetic
re explicitly given by

$$\frac{\partial}{\partial r},\frac{1}{r}\left(\frac{\partial}{\partial\theta} - i\,a\right)\Big), \quad -\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\,\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} - i\,a\right)^2.$$

us consider *cylindrical coordinates* (ρ,θ,z) where

$$\rho = \sqrt{x_1^2+x_2^2}, \quad \rho\,e^{i\theta} = x_1 + i\,x_2 \quad\text{and}\quad z = x_3$$

ronov–Bohm magnetic potential

tic gradient and the magnetic Laplacian are explicitly given by

$$\nabla_{\mathbf{A}} = \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\left(\frac{\partial}{\partial\theta} - i\,a\right), \frac{\partial}{\partial z}\right),$$

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\left(\frac{\partial}{\partial\theta} - i\,a\right)^2 - \frac{\partial^2}{\partial z^2}.$$

consider Aharonov–Bohm type magnetic potentials on compact mani- on the circle, the sphere and the torus. The expression of the magnetic be given case by case.

r is intended to provide a general overview of the mathematical results concerning various functional inequalities involving Aharonov–Bohm ls:

round state energy estimates.
nagnetic interpolation inequalities.
sults for optimal functions.
Keller–Lieb–Thirring inequalities.
Hardy inequalities.

ardy inequalities, all of the above inequalities will be considered on on the two-dimensional sphere $\mathbb{S}^2$, on the two-dimensional flat torus d on $\mathbb{R}^3$, with consequences on Hardy inequalities on the Euclidean d $\mathbb{R}^3$. It is crucial to consider precise geometric settings as we are optimal inequalities, which rely on non-trivial symmetry results. A ear interpolation inequality is

$$\|\nabla_{\mathbf{A}}\,u\|^2_{\mathrm{L}^2(\mathcal{X})} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{X})} \geq \mu_{\mathbf{A}}(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathcal{X})} \tag{1.3}$$

on u in the appropriate $\mathrm{H}^1_{\mathbf{A}}$ space, for any $\lambda > 0$, and for any $p > 2$, mpact manifold $\mathcal{X}$ in order to fix ideas. Assuming that $\mathrm{vol}(\mathcal{X}) = 1$, *e optimal inequality* is to the determine the largest value of $\lambda > 0$ for e $\mu_{\mathbf{A}}(\lambda) = \lambda + C$ for some constant C which is computed in terms of nds on $\mathcal{X}$. Equality is then realized by the constants. It is usually not ove that the equality is achieved in the inequality if $\mu_{\mathbf{A}}(\lambda)$ denotes onstant, for any $\lambda > 0$. If we consider the Euler–Lagrange equation, eformulated as the slightly more general *rigidity* question. For which we know that any solution is actually a constant? To obtain rigidity, to establish symmetry properties, which is usually the most difficult roof. In the non-compact case, optimal functions are not constants, dditional difficulty, but the problem can also be reduced to a symmetry

nother interpretation of the rigidity issue in terms of a *phase transition*. ct manifold case, the optimal function for (1.3) is always a constant if

and one can prove in many cases that there is a bifurcation from a
ase (solutions are constant) to a non-symmetric phase for a threshold
responding to the optimal inequality.

b–Thirring (KLT) inequalities are estimates of the ground state
$V]$ for the magnetic Schrödinger operator $-\Delta_{\mathbf{A}} - V$ in terms of
are obtained by duality from (1.3) with $q = p/(p-2)$. KLT inequalities
npletely equivalent to (1.3), including for optimality issues and related
ons, and essential for proving various *magnetic Hardy inequalities*,
of the highlights and the main motivations of this paper. However,
the fact that the accurate spectral information is carried by the KLT

ctually consider not only the *superquadratic* case $p > 2$, but also the
case $p < 2$ in which the role of the L^2 and L^p norms are exchanged.
ding nonlinear interpolation inequality is

$$\|\nabla_{\mathbf{A}} u\|^2_{\mathrm{L}^2(\mathcal{X})} + \mu \|u\|^2_{\mathrm{L}^p(\mathcal{X})} \ge \lambda_{\mathbf{A}}(\mu) \|u\|^2_{\mathrm{L}^2(\mathcal{X})} \tag{1.4}$$

$1, 2)$ and we can also draw a whole series of consequences (rigidity,
as in the superquadratic case. In particular, we are able to prove
nd interesting results on optimality and rigidity.

collects many results on functional inequalities with magnetic fields
ometric settings. Therefore it is difficult to pick particularly significant
over we believe that the interest of the paper lies as much in the
ds as in the results because very little is known on optimal inequalities
f Aharonov–Bohm magnetic fields and on the symmetry properties
ponding optimal functions. The most visible outcome of our work
nequalities, which are important tools in functional analysis. The
magnetic field is a key feature, for instance in dimension $d = 2$. Let us
the attention of the reader to some results that are prominent in this

1 deals with nonlinear magnetic interpolation inequalities, optimal
nd rigidity results on $\mathbb{S}^1$ in the subquadratic case. This is a new
ualities which complements the theory on *magnetic rings* in the
atic case studied in [14]. It was natural to study it in view of the *carré*
echnique by Bakry and Emery in [3], but as far as we know, it is an
result in the presence of a magnetic potential when $p < 2$.

2 is the counterpart of Theorem 3.1 in the case of the torus $\mathbb{T}^2 \approx$
s remarkable that we achieve an optimality result here as symmetry
roducts of manifolds are known to be difficult.

quadratic interpolation inequalities for proving KLT and then

l to consider also what happens on $\mathbb{S}^2$: see Proposition 2.2 and Corol- results in the superquadratic case.
magnetic Hardy inequalities of Theorem 4.2 are a new and striking of the nonlinear Hardy–Sobolev interpolation inequalities of [7] on an space $\mathbb{R}^2$.
.1 and 5.2 are two examples of application of the nonlinear magnetic on inequalities to magnetic Hardy inequalities on $\mathbb{R}^3$, which signifi- rove upon the results in [19, 22].

clude this introduction by some mathematical observations and some ferences. The overall question is to determine the functional spaces pted to magnetic Schrödinger operators in the spirit of [13]. Magnetic inequalities (without optimal constants) are usually not an issue as leduced from the non-magnetic interpolation inequalities by the *dia- quality*: see for instance [27]. However we are interested in retaining bout the magnetic field and characterizing optimality cases, which is e difficult target. As a convention, we shall speak of *Hardy–Sobolev* hen a term $\int |x|^{-2} |u|^2 \, dx$ is subtracted from the kinetic energy and of *hn–Nirenberg* inequalities when various pure power weights are taken On $\mathbb{R}^d$, $\Delta_{\mathbf{A}}$ has the same scaling properties as the non-magnetic Lapla- n Aharonov–Bohm magnetic potential and its spectrum is explicit. ties are spectral estimates but it has to be emphasized that they differ ssical estimates as they inherit nonlinear properties of the interpo- ities. Finally, it is a remarkable fact that, in presence of a magnetic Hardy inequality can be established in the two-dimensional case (see 5] for related papers).
r is organized as follows. Section 2 is devoted to some preliminary e circle $\mathbb{S}^1$ and on the two-dimensional sphere $\mathbb{S}^2$. New interpolation re established on the sphere $\mathbb{S}^2$, with an optimality result. Section 3 the study of a class of subquadratic magnetic interpolation inequal- d on the flat torus $\mathbb{T}^2$. We are able to identify a sharp condition of d deduce several Hardy inequalities in dimensions $d = 2$ and $d = 3$. start by recalling earlier, non-optimal but numerically almost sharp, erpolation inequalities on $\mathbb{R}^2$ in the presence of an Aharonov–Bohm d in order to establish some Hardy inequalities on $\mathbb{R}^2$. Section 5 is ardy inequalities on $\mathbb{R}^3$ with singularities which are either spherically y symmetric.

Set-Up and Preliminary Results

s devoted to various results on the sphere $\mathbb{S}^d$ without magnetic field only for completeness but also as introductory material for Secs. 2.3

agnetic interpolation inequalities on $\mathbb{S}^d$

$\mathbb{S}^d$, we consider the uniform probability measure $d\sigma$, which is the ced by the Lebesgue measure in $\mathbb{R}^{d+1}$, duly normalized and denote by orresponding L^q norm. Here we state known results for later use.

lation inequalities without weights

tion inequalities

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \tag{2.1}$$

$\in [1,2)\cup(2,+\infty)$ if $d=1$ and $d=2$, and for any $p\in[1,2)\cup(2,2^*]$ if $2^* := 2\,d/(d-2)$ is the Sobolev critical exponent. See [5, 6] for $p>2$ 1 or $d\ge 2$ and $p\le(2\,d^2+1)/(d-1)^2$.

we know from [12] that there exists a concave monotone increasing $\mu_{0,p}(\lambda)$ on $(0,+\infty)$ such that $\mu_{0,p}(\lambda)$ is the optimal constant in the

$$u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu_{0,p}(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}, \quad \forall\,u\in\mathrm{H}^1(\mathbb{S}^d) \tag{2.2}$$

$\lambda) = \lambda$ if and only if $\lambda \le d/(p-2)$. In this range, equality is achieved u is a constant function: this is a *symmetry* range. On the opposite, 2), the optimal function is not constant and we shall say that there is *aking.*

$1 \le p < 2$ is similar: there exists a concave monotone increasing $\lambda_{0,p}(\mu)$ on $(0,+\infty)$ such that $\lambda_{0,p}(\mu)$ is the optimal constant in the

$$u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \lambda_{0,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\,u\in\mathrm{H}^1(\mathbb{S}^d) \tag{2.3}$$

$\mu) = \mu$ if and only if $\mu\le d/(2-p)$. In this symmetry range, constants l functions, while there is symmetry breaking if $\mu > d/(2-p)$: optimal non-constant.

metry range, positive constants are actually the only *positive* solutions agrange equation

$$-\varepsilon\,\Delta u + \lambda\,u = u^{p-1}$$

1 is the sign of $(p-2)$, while there are multiple solutions in the aking range. The limit case $p=2$ can be obtained by taking the limit

hted Poincaré inequality for the ultra-spherical operator

ical coordinates $(z,\omega) \in [-1,1] \times \mathbb{S}^{d-1}$, we can rewrite the Laplace–
rator on $\mathbb{S}^d$ as

$$\Delta = \mathcal{L}_d + \frac{1}{1-z^2}\,\Delta_\omega \quad \text{with } \mathcal{L}_d\, u := \left(1 - z^2\right) u'' - d\, z\, u'$$

notes the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ and $\mathcal{L}_d$ is the *ultra-*
rator. In other words, $\mathcal{L}_d$ is the Laplace–Beltrami operator on $\mathbb{S}^d$
functions which depend only on z. The operator $\mathcal{L}_d$ has a basis of
s $G_{\ell,d}$, the Gegenbauer polynomials, associated with the eigenvalues
or any $\ell \in \mathbb{N}$ (see [28]). Here d is not necessarily an integer.
nsider the eigenvalue problem

$$-\mathcal{L}_2\, f + \frac{4\,\mathsf{a}^2}{1-z^2}\, f = \lambda\, f\,. \tag{2.4}$$

the unknown function according to $f(z) = \left(1-z^2\right)^{\mathsf{a}}\, g(z)$, we obtain

$$-\mathcal{L}_{2\,(2\,\mathsf{a}+1)}\, g + 2\,\mathsf{a}\,(1+2\,\mathsf{a})\, g = \lambda\, g$$

ines the eigenvalues $\lambda = \lambda_{\ell,\mathsf{a}}$ given by

$$2\,(2\,\mathsf{a}+1) - 1) + 2\,\mathsf{a}\,(1+2\,\mathsf{a}) = (\ell + 2\,\mathsf{a})\,(\ell + 2\,\mathsf{a} + 1), \quad \ell \in \mathbb{N}. \tag{2.5}$$

te by $g_{\ell,\mathsf{a}}(z) = G_{\ell,2\,(2\,\mathsf{a}+1)}(z)$ the associated eigenfunctions and define
$- z^2)^{\mathsf{a}} g_{\ell,\mathsf{a}}(z)$. By considering the lowest positive eigenvalue, we obtain
oincaré inequality.

For any $\mathsf{a} \in \mathbb{R}$ *and any function* $f \in \mathrm{H}^1_0[-1,1]$, *we have*

$$- z^2)|f'(z)|^2 + \frac{4\,\mathsf{a}^2}{1-z^2}\,|f(z)|^2\Big)\, dz \ge \lambda_{1,\mathsf{a}} \int_{-1}^{1} |f(z) - \bar{f}(z)|^2\, dz,$$

$$\bar{f}(z) = (1-z^2)^{\mathsf{a}}\ \frac{\displaystyle\int_{-1}^{1} f(z)(1-z^2)^{\mathsf{a}}\, dz}{\displaystyle\int_{-1}^{1} (1-z^2)^{2\,\mathsf{a}}\, dz}.$$

chieved by a function f *if and only if* f *is proportional to* $f_{1,\mathsf{a}}(z) =$

consistency that, if $f(z) = \left(1-z^2\right)^{\mathsf{a}}\, g(z)$, then

$$\left(\left(1-z^2\right)|f'(z)|^2 + \frac{4\,\mathsf{a}^2}{1-z^2}\,|f(z)|^2\right) dz$$

nt-hand side is the Dirichlet form associated with the operator

$$-\mathcal{L}_{2\,(2\,\mathsf{a}+1)} + 2\,\mathsf{a}\,(1+2\,\mathsf{a}).$$

ic rings: *Superquadratic inequalities on* $\mathbb{S}^1$

, we review a series of results which have been obtained in [14] in the c case $p>2$, in preparation for an extension to the subquadratic case will be studied in Sec. 3.

ic interpolation inequalities and consequences

er the superquadratic case $p>2$ in dimension $d=1$. We recall that θ where $\theta\in[0,2\pi)\approx\mathbb{S}^1$. As in [14] we consider the space $\mathrm{H}^1(\mathbb{S}^1)$ of ic functions $u\in C^{0,1/2}(\mathbb{S}^1)$, such that $u'\in\mathrm{L}^2(\mathbb{S}^1)$. Inequality (2.2) en as

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \lambda\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\,,\quad \forall\,u\in\mathrm{H}^1(\mathbb{S}^1) \tag{2.6}$$

$, 1/(p-2)]$. We also have the inequality

$$u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} \ge \frac{1}{4}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\,,\quad \forall\,u\in\mathrm{H}^1(\mathbb{S}^1), \tag{2.7}$$

21], with the convention that $\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}=0$ if u^{-2} is not integrable ial case, if u changes sign. Notice that inequality (2.7) is formally the $d-2)$ and $\lambda=d/(p-2)$ of (2.2) when $d=1$ (see [14, Appendix A]). was shown that the inequality (for complex-valued functions)

$$\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\lambda)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\,,\quad \forall\,\psi\in\mathrm{H}^1(\mathbb{S}^1,\mathbb{C}) \tag{2.8}$$

after eliminating the phase, to the inequality

$$a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\,,\quad \forall\,u\in\mathrm{H}^1(\mathbb{S}^1).$$

ce is relatively easy to prove if ψ does not vanish, but some care herwise: see [14] for details. Here we denote by $\mu_{a,p}(\lambda)$ the optimal .8). Using (2.8) and then (2.6), we obtain that

$$+\,a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$$-\,4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + 4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\,\|u^{-1}\|^{2}_{\mathrm{L}^2(\mathbb{S}^1)}\right)$$

$$-\,4\,a^2)\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{a^2+\lambda}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right) \ge (a^2+\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

dition $(a^2+\lambda)/(1-4\,a^2)\le 1/(p-2)$, which provides an estimate of

$2) + \lambda\,(p-2) \le 1$, *then* $\mu_{a,p}(\lambda) = a^2 + \lambda$ *and equality in* (2.8) *is only by the constants.*

$2) + \lambda\,(p-2) > 1$, *then* $\mu_{a,p}(\lambda) < a^2 + \lambda$ *and equality in* (2.8) *is not by the constants.*

tion $a \in [0,1/2]$ is not a restriction. First, replacing ψ by $e^{iks}\,\psi(s)$ for ows that $\mu_{a+k,p}(\mu) = \mu_{a,p}(\mu)$ so that we can assume that $a \in [0,1]$. idering $\chi(s) = e^{-is}\,\overline{\psi}(s)$, we find that

$$|\psi' + i\,a\,\psi|^2 = |\chi' + i\,(1-a)\,\chi|^2,$$

$= \mu_{1-a,p}(\mu)$.

tic Hardy inequalities on $\mathbb{S}^1$ *and* $\mathbb{R}^2$

can draw an easy consequence of Proposition 2.1 on a Keller–Lieb– inequality. By Hölder's inequality applied with $q = p/(p-2)$, we

$$_{(\mathbb{S}^1)} - \mu^{-1} \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\theta \ge \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu^{-1}\,\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}.$$

vith $\lambda = 0$ and μ such that $\mu^{-1}\,\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)} = \mu_{a,p}(0)$, we know that d side is non-negative. See [14] for more details. Altogether we obtain *magnetic Hardy inequality on* $\mathbb{S}^1$: *for any* $a \in \mathbb{R}$, *any* $p > 2$ *and , if* ϕ *is a non-trivial potential in* $\mathrm{L}^q(\mathbb{S}^1)$, *then*

$$' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}} \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\sigma, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^1). \tag{2.9}$$

ial case of the more general interpolation inequality

$$\begin{aligned} i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\theta &\ge \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \\ &\ge -\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \end{aligned} \tag{2.10}$$

$_{\mathrm{L}^q(\mathbb{S}^1)}$, where we denote by $\lambda_{a,p}(\mu)$ the inverse function of $\lambda \mapsto \mu_{a,p}(\lambda)$, Proposition 2.1. See [13] for details.

ard non-magnetic Hardy inequality on $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} |\nabla\psi|^2\,dx \ge \frac{1}{4}\,(d-2)^2 \int_{\mathbb{R}^d} \frac{|\psi|^2}{|x|^2}\,dx, \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d),$$

$d = 2$, but this degeneracy is lifted in the presence of a Aharonov– tic field. According to [26], we have ask whether an improvement can be obtained if the singularity $|x|^{-2}$ a weight which has an angular dependence. Using polar coordinates interpolation inequalities of [12], the inequality

$$\nabla\psi|^2\,dx \ge \frac{(d-2)^2}{4\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}}\int_{\mathbb{R}^d}\frac{\varphi(\theta)}{|x|^2}\,|\psi|^2\,dx,\quad \forall\,\psi\in \mathrm{H}^1(\mathbb{R}^d)$$

[24], under the condition that $q \ge 1+\frac{1}{2}\,(d-2)^2/(d-1)$, again with easure on $\mathbb{S}^{d-1}$. Magnetic and non-radial improvements have been 14]. Let us give a statement in preparation for similar extensions to mension $d=3$.

1 ([14]). *Let* $\mathbf{A}$ *as in* (1.1), $a\in[0,1/2]$, $p>2$, $q=p/(p-2)$ *and is a non-negative function in* $\mathrm{L}^q(\mathbb{S}^1)$. *With the above notations, the*

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \tau\int_{\mathbb{R}^2}\frac{\varphi(\theta)}{|x|^2}\,|\psi|^2\,dx,\quad \forall\,\psi\in \mathrm{H}^1(\mathbb{R}^d)$$

onstant $\tau>0$ *which is the unique solution of the equation*

$$\lambda_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)})=0.$$

$= a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ *if* $a^2\le 1/(p+2)$.

ic interpolation inequalities on $\mathbb{S}^2$

of Secs. 2.1 and 2.2, we state some new results concerning the two-phere with main results in Proposition 2.2 and Corollary 2.2.

etic ground state estimate

er the magnetic Laplacian on $\mathbb{S}^2$ and the associated Dirichlet form where $d\sigma$ is the uniform probability measure on $\mathbb{S}^2$. Using cylindrical $,z)\in[0,2\pi)\times[-1,1]$, we can write that $d\sigma=\frac{1}{4\pi}\,dz\,d\theta$ and assume etic gradient takes the form

$$\nabla_{\mathbf{A}}\,u=\begin{pmatrix}\sqrt{1-z^2}\,\dfrac{\partial u}{\partial z}\\[2ex] \dfrac{1}{\sqrt{1-z^2}}\left(\dfrac{\partial u}{\partial\theta}-i\,a\,u\right)\end{pmatrix}\tag{2.11}$$

s a magnetic flux, so that

$$|\nabla_{\mathbf{A}}\,u|^2=\left(1-z^2\right)\left|\frac{\partial u}{\partial z}\right|^2+\frac{1}{1-z^2}\left|\frac{\partial u}{\partial\theta}-i\,a\,u\right|^2.$$

Assume that $a\in\mathbb{R}$. *With the notation* (2.11), *we have* constant

$$\Lambda_a = \min_{k\in\mathbb{Z}} |k-a|(|k-a|+1). \tag{2.12}$$

t $\Lambda_a \le \Lambda_{1/2} = 3/4$.

an write u using a Fourier decomposition

$$u(z,\theta) = \sum_{\ell\in\mathbb{N}} \sum_{k\in\mathbb{Z}} u_{k,\ell}(z)\, e^{i\,k\,\theta}$$

hat

$$\mathbf{A}\, u|^2 = \sum_{\ell\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \left((1-z^2)|u_{k,\ell}'(z)|^2 + \frac{(k-a)^2}{1-z^2} |u_{k,\ell}(z)|^2 \right)$$

$$_{k,\ell}(z) = f_{\ell,|k-a|/2}(z) \int_{\mathbb{S}^2} u(z,\theta)\, \frac{2\, f_{\ell,|k-a|/2}(z)\, e^{-i\,k\,\theta}}{\displaystyle\int_{-1}^{1} \left(1-z^2\right)^{|k-a|}\, dz}\, d\sigma$$

is an eigenfunction of (2.4) with eigenvalue $\lambda = \lambda_{\ell,\mathsf{a}}$ such that $2\,\mathsf{a} =$ (2.5), we conclude that the spectrum of $-\Delta_{\mathbf{A}}$ is given by

$$(\ell+|k-a|)(\ell+|k-a|+1), \quad k\in\mathbb{Z}, \quad \ell\in\mathbb{N}. \qquad \square$$

uadratic interpolation inequalities and consequences

2.2. *Let $a\in\mathbb{R}$ and $p > 2$. With the notation* (2.11)*, there exists a tone increasing function $\lambda \mapsto \mu_{a,p}(\lambda)$ on $(-\Lambda_a,+\infty)$ such that $\mu_{a,p}(\lambda)$ constant in the inequality*

$$u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \lambda\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \ge \mu_{a,p}(\lambda)\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^2)}, \quad \forall\, u\in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^2).$$

$\mu_{a,p}(\lambda) \ge 2\,(\lambda+\Lambda_a)/\big(2+(p-2)\,\Lambda_a\big)$ *and* $\lim_{\lambda\to-\Lambda_a} \mu_{a,p}(\lambda) = 0$, *by* (2.12).

proof is adapted from [13, Proposition 3.1]. For an arbitrary $t\in(0,1)$, that

$$\begin{aligned} {}_{\mathbb{S}^2)} + \lambda\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} &\ge t\left(\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} - \Lambda_a\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \right) \\ &\quad + (1-t)\left(\|\nabla|u|\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \frac{\lambda+t\,\Lambda_a}{1-t}\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \right) \\ &\ge (1-t)\, \mu_{0,p}\left(\frac{\lambda+t\,\Lambda_a}{1-t} \right) \|u\|^2_{\mathrm{L}^p(\mathbb{S}^2)}, \end{aligned}$$

ence of Lemma 2.2 and of the *diamagnetic inequality* (see, e.g., [27, 2), the estimate is obtained by choosing t such that

$$\frac{\lambda + t\,\Lambda_a}{1-t} = \frac{2}{p-2}$$

that $\mu_{0,p}\,(2/(p-2)) = 2/(p-2)$. The limit as $\lambda \to -\Lambda_a$ is obtained ground state of $-\Delta_{\mathbf{A}}$ on $\mathrm{H}^1(\mathbb{S}^2)$ as test function. □

same method as for the proof of (2.9), we can deduce a Hardy-type

2. *Let $a \in \mathbb{R}$, $p > 2$ and $q = p/(p-2)$. With the notation* (2.11), *if ϕ al potential in $\mathrm{L}^q(\mathbb{S}^2)$, then*

$$\nabla_{\mathbf{A}}\,u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \ge \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^2)}} \int_{\mathbb{S}^2} \phi\,|u|^2\,d\sigma, \quad \forall\,u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^2).$$

ratic Magnetic Interpolation Inequalities

s devoted to new and optimal interpolation inequalities involving L^p *subquadratic* range $p \in (1,2)$ on the circle and on the torus (Secs. 3.1 Keller–Lieb–Thirring inequalities yield new magnetic Hardy inequal-d $\mathbb{R}^3$ (Sec. 3.3).

ic rings: Subquadratic interpolation inequalities on $\mathbb{S}^1$

e results of Sec. 2.2.1 to the subquadratic range $1 < p < 2$ using the 4].

ent of the inequality

ase of (2.1) corresponding to $d = 1$, we have the non-magnetic inter-uality

$$-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}, \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^1) \tag{3.1}$$

2). Our first result is the magnetic counterpart of this inequality.

Let $a \in \mathbb{R}$ and $p \in [1,2)$. Then there exists a concave monotone ction $\mu \mapsto \lambda_{a,p}(\mu)$ on $\mathbb{R}^+$ such that

$$\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}, \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{S}^1,\mathbb{C}). \tag{3.2}$$

enote by $\lambda_{a,p}(\mu)$ the optimal constant in (3.1).

xistence of $\lambda_{a,p}(\mu)$ is a consequence of (3.1) and of the *diamagnetic* $\rho = |\psi|$ and ϕ be such that $\psi = \rho(\theta)\,\exp\big(i\,\phi(\theta)\big)$. Since

$$|\psi' - i\,a\,\psi|^2 = |\rho'|^2 + |\phi' - a|^2\,\rho^2 \ge |\rho'|^2,$$

$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \|\,|\psi|'\,\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$. The concavity of $\mu \mapsto \lambda_{a,p}(\mu)$ is a *ce of an optimal function*

For all $a \in [0,1/2]$, $p \in [1,2)$ and $\mu \geq -a^2$, equality in (3.2) is t least one function in $\mathrm{H}^1(\mathbb{S}^1)$.

onsider a minimizing sequence $\{\psi_n\}$ for

$$\mathrm{nf}\left\{\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\;:\;\psi \in \mathrm{H}^1(\mathbb{S}^1),\,\|\psi\|_{\mathrm{L}^2(\mathbb{S}^1)} = 1\right\}.$$

gnetic inequality we know that the sequence $(\psi_n)_{n\in\mathbb{N}}$ is bounded in ne compact Sobolev embeddings, this sequence is relatively compact in $\mathrm{L}^2(\mathbb{S}^1)$. The map $\psi \mapsto \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is lower semi-continuous nma, which proves the claim. □

vanishing property

Asssume that $a \in (0,1/2)$, $p \in [1,2)$ and $\mu \geq -a^2$. If $\psi \in \mathrm{H}^1(\mathbb{S}^1)$ is nction for (3.2) with $\|\psi\|_{\mathrm{L}^p(\mathbb{S}^1)} = 1$, then $\psi(s) \neq 0$ for any $s \in \mathbb{S}^1$.

roof goes as in [14]. Let us decompose $v(s) = \psi(s)\,e^{ias}$ as a real and an t, respectively v_1 and v_2, which both solve the same Euler–Lagrange

$$-v_j'' - \mu(v_1^2 + v_2^2)^{\frac{p}{2}-1}\,v_j = \lambda_{a,p}(\mu)\,v_j, \quad j = 1,2.$$

$\in C^{0,1/2}(\mathbb{S}^1)$ and the nonlinear term is continuous, hence v is smooth. n $w = (v_1\,v_2' - v_1'\,v_2)$ is constant. If both v_1 and v_2 vanish at the same vanishes identically, which means that v_1 and v_2 are proportional. $/2)$, ψ is not 2π-periodic, a contradiction. □

ction to a scalar minimization problem

ec. 2.1.1 if $a = 0$ and assume in the proofs that $a > 0$. The main steps on are similar to the case $p > 2$ of [14]. We repeat the key points for Let us define

$$\mathcal{Q}_{a,p,\mu}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}.$$

$u \in \mathrm{H}^1(\mathbb{S}^1)$ is such that $u(s_0) = 0$ for some $s_0 \in (-\pi,\pi]$, then

$$|u(s)|^2 = \left(\int_{s_0}^{s} u'\,ds\right)^2 \leq \sqrt{2\pi}\,\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\,\sqrt{|s - s_0|}$$

integrable. In this case, as mentioned earlier, we adopt the convention

For any $a \in [0,1/2]$, $p \in [1,2)$, $\mu > -a^2$,

$$\lambda_{a,p}(\mu) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\mu}[u].$$

onsider functions on $\mathbb{S}^1$ as 2π-periodic functions on $\mathbb{R}$. If $\psi \in \mathrm{H}^1(\mathbb{S}^1)$,
$(s)\, e^{ias}$ satisfies the condition

$$v(s+2\pi) = e^{2i\pi a}\, v(s), \quad \forall\, s \in \mathbb{R} \tag{3.4}$$

$$\lambda_{a,p}(\mu) = \min \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

imization is taken on the set of the functions $v \in C^{0,1/2}(\mathbb{R})$ such that
) and (3.4) holds.
$u\, e^{i\phi}$ written in polar form, the boundary condition becomes

$$u(\pi) = u(-\pi), \quad \phi(\pi) = 2\pi\,(a+k) + \phi(-\pi) \tag{3.5}$$

$\mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ so that

$$\lambda_{a,p}(\mu) = \min \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

nimization is taken on the set of the functions $(u,\phi) \in C(\mathbb{R})^2$ such
$\mathrm{L}^2(\mathbb{S}^1)$ and (3.5) holds.
ultiplication of u by a constant so that $\|u\|_{\mathrm{L}^p(\mathbb{S}^1)} = 1$, the Euler–
ations are

$$u'' + |\phi'|^2\, u + \mu\, |u|^{p-2}\, u = \lambda_{a,p}(\mu)\, u \quad \text{and} \quad (\phi'\, u^2)' = 0.$$

, by integrating the second equation and using Lemma 3.3, we find a
ch that $\phi' = L/u^2$. Taking (3.5) into account, we deduce from

$$L \int_{-\pi}^{\pi} \frac{ds}{u^2} = \int_{-\pi}^{\pi} \phi'\, ds = 2\pi\,(a+k)$$

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = L^2 \int_{-\pi}^{\pi} \frac{ds}{u^2} = \frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}.$$

es that

$$\lambda_{a,p}(\mu) = \min_{u,\,k} \mathcal{Q}_{a+k,p,\mu}[u]$$

nimization is taken on all $k \in \mathbb{Z}$ and on all functions $u \in \mathrm{H}^1(\mathbb{S}^1)$.
e restriction $a \in (0,1/2)$, the minimum is achieved by $k = 0$.
$= 1/2$ is a limit case that can be handled as in [14, Theorem III.7].
e result holds also true, with the minimizer being in $\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}$, and

ity result

, as in [14], the study of (3.2) is reduced to the study of the inequality

$$+\, a^2\, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \lambda_{a,p}(\mu)\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\,, \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^1) \tag{3.6}$$

ow a real valued function. Necessary adaptations to the trivial case the limit case $a = 1/2$ are straightforward and left to the reader. The s the analogue of Proposition 2.1 in the subcritical range.

1. *Let $p \in (1,2)$, $a \in (0,1/2)$, and $\mu > 0$.*

$p) + 4\,a^2 \le 1$, then $\lambda_{a,p}(\mu) = a^2 + \mu$ and equality in (3.6) is achieved ne constants.
$) + 4\,a^2 > 1$, then $\lambda_{a,p}(\mu) < a^2 + \mu$ and equality in (3.6) is not achieved nstants.

se (i) we can write

$$\begin{aligned} &\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \\ &\quad = (1 - 4\,a^2) \left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{\mu}{1 - 4\,a^2}\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \right) \\ &\qquad + 4\,a^2 \left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\, \|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \right), \end{aligned}$$

(3.1) that

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{\mu}{1 - 4\,a^2}\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \frac{\mu}{1 - 4\,a^2}\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$) \le 1/(2-p)$ and conclude using (2.7).
), let us consider the test function $u_\varepsilon := 1 + \varepsilon\, w_1$, where w_1 is the corresponding to the first non-zero eigenvalue of $-\,d^2/ds^2$ on $\mathrm{H}^1(\mathbb{S}^1)$, boundary conditions, namely, $w_1(s) = \cos s$ and $\lambda_1 = 1$. A Taylor ows that

$$\mathcal{Q}_{a,p,\mu}[u_\varepsilon] = \big(1 + a^2 -\, \mu\,(2-p)\big)\, \varepsilon^2 + o(\varepsilon^2),$$

the result. Notice that the Taylor expansion is also valid if $a = 0$, so the optimal constant in (3.1), and also that a similar Taylor expansion of (2.7), which formally corresponds to $p = -\,2$. □

ov–Bohm magnetic interpolation inequalities on $\mathbb{T}^2$

a toy model for Aharonov–Bohm magnetic fields on the flat torus

sider the flat torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \approx [-\pi,\pi) \times [-\pi,\pi) \ni (x,y)$ with
dary conditions in x and y. We denote by $d\sigma$ the uniform probability
$dx\,dy/(4\pi^2)$ and consider the magnetic gradient

$$\nabla_{\mathbf{A}}\,\psi := (\psi_x, \psi_y - i\,a\,\psi) \tag{3.7}$$

etic kinetic energy

$$\|_{\mathrm{L}^2(\mathbb{T}^2)}^2 = \iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma = \iint_{\mathbb{T}^2} \big(|\psi_x|^2 + |\psi_y - \,i\,a\,\psi|^2\big)\,d\sigma.$$

etic ground state estimate

Assume that $a \in [0,1/2]$. *With the notation* (3.7), *we have*

$$\iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma \ge a^2 \iint_{\mathbb{T}^2} |\psi|^2\,d\sigma, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2).$$

ake a Fourier decomposition on the basis $(e^{i\,\ell\,x}\,e^{i\,k\,y})_{k,\ell\in\mathbb{Z}}$. We find
t modes are given by

$$k = 0, \quad \ell = 0 : \lambda_{00} = a^2,$$

$$k = 1, \quad \ell = 0 : \lambda_{10} = (1-a)^2 \ge a^2 \quad \text{since } a \in [0,1/2],$$

$$k = 0, \quad \ell = 1 : \lambda_{01} = 1 + a^2.$$

is the lowest mode. □

kry–Emery method applied to the 2*-dimensional torus*

he flow given by

$$\frac{\partial u}{\partial t} = \Delta u + (p-1)\,\frac{|\nabla u|^2}{u}$$

hat

$$\frac{d}{dt}\,\|u(t,\cdot)\|_{\mathrm{L}^p(\mathbb{T}^2)}^2 = 0$$

nd, and

$$\Big(\|\nabla u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{T}^2)}^2 - \lambda\,\|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{T}^2)}^2\Big)$$

$$\|\Delta u\|_{\mathrm{L}^2(\mathbb{T}^2)}^2 + (p-1)\iint_{\mathbb{T}^2} \Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma - \lambda\,(2-p)\,\|\nabla u\|_{\mathrm{L}^2(\mathbb{T}^2)}^2$$

and. Integrations by parts show that

$$\|\Delta u\|_{\mathrm{L}^2(\mathbb{T}^2)}^2 = \|\mathrm{Hess}\,u\|_{\mathrm{L}^2(\mathbb{T}^2)}^2$$

$$(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\Big)$$

$$)\left(\|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right) + (p-1)\left\|\mathrm{Hess}\,u - \frac{\nabla u\otimes\nabla u}{u}\right\|^2_{\mathrm{L}^2(\mathbb{T}^2)}.$$

n the Poincaré inequality that

$$\|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \ge \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)},$$

constant 1, so we can conclude in the case $1 \le p < 2$ that $_{2)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$ is monotone non-increasing if $0 \le \lambda \le 1$. As a we have the following result.

3.1. *For any $p \in [1,2)$, we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}, \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2).$$

orization result without magnetic potential

r than Proposition 3.1 follows from a tensorization argument that can 1, 17].

3.2. *For any $p \in [1,2)$, we have*

$$-p)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}, \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2). \tag{3.8}$$

factor $(2-p)$ is the optimal constant.

king on $\mathbb{T}^2$ a function depending only on $x \in \mathbb{S}^1$, it is clear that the 3.8) cannot be improved. The proof of (3.8) can be done with the method applied to $\mathbb{S}^1$ and goes as follows.

sider the flow given by

$$\frac{\partial u}{\partial t} = u'' + (p-1)\,\frac{|u'|^2}{u}$$

hat $\frac{d}{dt}\,\|u(t,\cdot)\|^2_{\mathrm{L}^p(\mathbb{S}^1)} = 0$ on the one hand, and

$$\frac{d}{dt}(\|u'(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)})$$

$$= \|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + (p-1)\int_{\mathbb{S}^1} u''\,\frac{|u'|^2}{u}\,d\sigma - \lambda\,(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$$= \|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{3}\,(p-1)\int_{\mathbb{S}^1} \frac{|u'|^4}{u^2}\,d\sigma - \lambda\,(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

hand. Hence

1, because of the Poincaré inequality $\|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$. Up to of λ, this computation also holds if $p > 2$ or if $p = -2$, as noticed is straightforward to extend it to the limit case $p = 2$ corresponding mic Sobolev inequality.

to [11, Proposition 3.1] or [17, Theorem 2.1] and up to a straightfor- on to the periodic setting, the optimal constant for the inequality on is the same as for the inequality on $\mathbb{S}^1$, provided $1 \le p < 2$. $\square$

quence of Proposition 3.2, we have the inequality

$$\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \Lambda_{0,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}, \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2), \tag{3.9}$$

$_{,p}(\mu)$ is a concave monotone increasing function on $(0,+\infty)$ such that r any $\mu \in \big(0, 1/(2-p)\big)$.

etic interpolation inequality in the flat torus

nsider the generalization of (3.9) to the case $a \neq 0$.

Assume that $p \in [1,2)$ *and* $a \in [0,1/2]$*. With the notation* (3.7)*, concave monotone increasing function* $\mu \mapsto \Lambda_{a,p}(\mu)$ *on* $(0,+\infty)$ *such* $\Lambda_{a,p}(\mu) = a^2$ *where* $\Lambda_{a,p}(\mu)$ *is the optimal constant in the inequality*

$$u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \Lambda_{a,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}, \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2). \tag{3.10}$$

have that

$$\Lambda_{a,p}(\mu) \ge \mu + (1 - \mu\,(2-p))\,a^2 \quad \textit{for any } \mu \le \frac{1}{2-p}.$$

n arbitrary $t \in (0,1)$, we can write that

$$\begin{aligned} & u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \\ & \ge t\left(\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - a^2\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right) \\ & \quad + (1-t)\left(\|\nabla|u|\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mu}{1-t}\,\|u\|^p_{\mathrm{L}^2(\mathbb{T}^2)}\right) + t\,a^2\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \\ & \ge \left[(1-t)\,\Lambda_{0,p}\left(\frac{\mu}{1-t}\right) + t\,a^2\right]\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \end{aligned}$$

netry result in the subquadratic regime

tion of the results on magnetic rings of Theorem 3.1, we can prove a ult for the optimal functions in (3.10) in the case $p<2$. Let $\Lambda_{a,p}(\mu)$ l constant in (3.10).

2. *Assume that $a\in[0,1/2]$ and $p\in[1,2)$. Then*

$$\Lambda_{a,p}(\mu)=\lambda_{a,p}(\mu)\quad \textit{if } \mu\le\frac{1}{p-2}$$

nal function for (3.2) *is then constant with respect to x. Moreover, $+\,\mu$ if and only if $\mu\,(2-p)+4\,a^2\le1$ and equality in* (3.10) *is then by the constants.*

s use the notation $⨍ f\,dx:=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\,dx$ in order to denote a normal-on with respect to the single variable x, where y is considered as a r almost every $x\in\mathbb S^1$ we can apply (3.2) to the function $\psi(x,\cdot)$ and

$$\mathbb T^2)+\mu\,\|\psi\|^2_{\mathrm L^p(\mathbb T^2)}$$

$$\psi\|^2_{\mathrm L^2(\mathbb T^2)}+\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm L^2(\mathbb T^2)}+\mu\,\|\psi\|^2_{\mathrm L^p(\mathbb T^2)}-\mu⨍\left(⨍|\psi|^p\,dy\right)^{\frac2p}\,dx.$$

$u:=|\psi|$, $v(x):=\left(⨍|u(x,y)|^p\,dy\right)^{1/p}$ and observe that

$$u^{p-1}\,u_x\,dy\le v^{1-p}\left(⨍u^p\,dy\right)^{\frac{p-1}{p}}\left(⨍|u_x|^2\,dy\right)^{\frac12}\left(⨍1\,dy\right)^{\frac12-\frac{p-1}{p}}$$

nequality, under the condition $p\le2$, that is,

$$|v_x|^2\le⨍|u_x|^2\,dy\le⨍|\partial_x\psi|^2\,dy.$$

that if $\mu\le1/(2-p)$,

$$v_x|^2\,d\sigma+\mu\left(\int_{\mathbb S^1}|v|^p\,d\sigma\right)^{2/p}-\mu\int_{\mathbb S^1}|v|^2\,d\sigma+\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm L^2(\mathbb T^2)}$$

$$\ge\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm L^2(\mathbb T^2)}$$

he equality is achieved by functions v which are constant with respect orem 3.1 applies. □

tic Hardy inequalities in dimensions **2** *and* **3**

n, we draw some consequences of our results on magnetic rings of

Lieb–Thirring inequalities on the circle

duality we obtain a spectral estimate.

3.3. *Assume that $a \in [0,1/2]$ and $p \in [1,2)$. If ϕ is a non-negative that $\phi^{-1} \in \mathrm{L}^q(\mathbb{S}^1)$, then the lowest eigenvalue λ_1 of $-(\partial_y - i\,a)^2 + \phi$ m below according to*

$$\lambda_1 \geq \lambda_{a,p}\left(\|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)}^{-1}\right)$$

s achieved by a constant potential ϕ if $\|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)}^{-1}\,(2-p) + 4\,a^2 \leq 1$.

Hölder's inequality with exponents $2/(2-p)$ and $2/p$, we get that

$$_{\mathbb{S}^1)} = \left(\int_{\mathbb{S}^1} \phi^{-\frac{p}{2}}\,\left(\phi\,|\psi|^2\right)^{\frac{p}{2}}\,d\sigma\right)^{2/p} \leq \|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)} \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\sigma$$

$-\,p)$, and with $\mu = \|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)}^{-1}$,

$$- i\,a\,\psi|^2\,d\sigma + \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\sigma \geq \int_{\mathbb{S}^1} |\psi' - i\,a\,\psi|^2\,d\sigma + \mu\,\|\psi\|_{\mathrm{L}^p(\mathbb{S}^1)}^2$$

$$\geq \lambda_{a,p}(\mu) \int_{\mathbb{S}^1} |\psi|^2\,d\sigma. \qquad (3.11)$$

t, then there is equality in Hölder's inequality. □

al estimate (3.11) is of a different nature than (2.10) because the gy and the magnetic kinetic energy have the same sign. By considering case $\mu\,(2-p) + 4\,a^2 = 1$, we obtain an interesting estimate.

1. *Let $a \in [0,1/2]$, $p \in (1,2)$ and $q = p/(2-p)$. If ϕ is a non-negative that $\phi^{-1} \in \mathrm{L}^q(\mathbb{S}^1)$, then*

$$\int_{\mathbb{S}^1} |\psi' - i\,a\,\psi|^2\,d\sigma + \frac{1-4\,a^2}{2-p}\,\|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)} \int_{\mathbb{S}^1} \phi\,|\psi|^2\,d\sigma$$
$$\geq \left(\frac{1-4\,a^2}{2-p} + a^2\right)\|\psi\|_{\mathrm{L}^2(\mathbb{S}^1)}^2, \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{S}^1).$$

ic Hardy-type inequalities in dimensions two and three

by $\theta \in [-\pi,\pi)$ the angular coordinate associated with $x \in \mathbb{R}^2$. can deduce a Hardy-type inequality for Aharonov–Bohm magnetic limension $d = 2$.

2. *Let $\mathbf{A}$ as in (1.1), $a \in [0,1/2]$, $p \in (1,2)$ and $q = p/(2-p)$. If ϕ tive potential such that $\phi^{-1} \in \mathrm{L}^q(\mathbb{S}^1)$ with $\|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)} = 1$, then for alued function $\psi \in \mathrm{H}^1(\mathbb{R}^2)$ we have* nsider cylindrical coordinates $(\rho,\theta,z)\in\mathbb{R}^+\times[0,2\pi)\times\mathbb{R}$ such that . In this system of coordinates the magnetic kinetic energy is

$$|\nabla_{\mathbf{A}}\,\psi|^2\,dx=\int_{\mathbb{R}^3}\left(\left|\frac{\partial\psi}{\partial\rho}\right|^2+\frac{1}{\rho^2}\left|\frac{\partial\psi}{\partial\theta}-i a\,\psi\right|^2+\left|\frac{\partial\psi}{\partial z}\right|^2\right)dx$$

$d\rho\,d\theta\,dz$. The following result was proved in [22, Sec. 2.2].

For any $\psi\in\mathrm{H}^1(\mathbb{R}^3)$, *we have*

$$\left(\left|\frac{\partial\psi}{\partial\rho}\right|^2+\left|\frac{\partial\psi}{\partial z}\right|^2\right)d\mu\geq\frac{1}{4}\iiint_{\mathbb{R}^+\times[0,2\pi)\times\mathbb{R}}\frac{|\psi|^2}{\rho^2+z^2}\,d\mu,\quad\forall\,\psi\in\mathrm{H}^1(\mathbb{R}^3).$$

ive an elementary proof. Assume that ψ is smooth and has compact inequality follows from the expansion of the square

$$\left(\left|\frac{\partial\psi}{\partial\rho}+\frac{\rho\,\psi}{2\,(\rho^2+z^2)}\right|^2+\left|\frac{\partial\psi}{\partial z}+\frac{z\,\psi}{2\,(\rho^2+z^2)}\right|^2\right)d\mu\geq 0,\quad\forall\,\psi\in\mathrm{H}^1(\mathbb{R}^3)$$

gration by parts of the cross terms. □

7 is an improved version of the standard Hardy inequality in the left-hand side of the inequality does not involve the angular part of ergy. A consequence of Corollary 3.1 and Lemma 3.7 is a Hardy-like imension $d=3$. For the angular part we argue as in Corollary 3.2. proof are left to the reader.

3. *Let* $\mathbf{A}$ *as in* (1.2), $a\in[0,1/2]$, $p\in(1,2)$ *and* $q=p/(2-p)$. *If* ϕ *such that* $\phi^{-1}\in\mathrm{L}^q(\mathbb{S}^1)$ *with* $\|\phi^{-1}\|_{\mathrm{L}^q(\mathbb{S}^1)}=1$, *then for any complex* *on* $\psi\in\mathrm{H}^1(\mathbb{R}^3)$ *we have*

$$\int_{\mathbb{R}^3}|\nabla_{\mathbf{A}}\,\psi|^2\,dx+\frac{1-4\,a^2}{2-p}\int_{\mathbb{R}^3}\frac{\phi(\theta)}{\rho^2}\,|\psi(x)|^2\,dx$$
$$\geq\frac{1}{4}\int_{\mathbb{R}^3}\frac{|\psi|^2}{|x|^2}\,dx+\left(\frac{1-4\,a^2}{2-p}+a^2\right)\int_{\mathbb{R}^3}\frac{|\psi|^2}{\rho^2}\,dx.$$

case is $\phi\equiv 1$, for which we obtain that

$$\nabla_{\mathbf{A}}\,\psi|^2\,dx\geq\frac{1}{4}\int_{\mathbb{R}^3}\frac{|\psi|^2}{|x|^2}\,dx+a^2\int_{\mathbb{R}^3}\frac{|\psi|^2}{|\rho|^2}\,dx\quad\forall\,\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)\,.$$

v–Bohm Magnetic Interpolation Inequalities in $\mathbb{R}^2$

rpolation inequalities on $\mathbb{R}^2$ are considered without weights in Sec. 4.1. hen introduced as in [7] in order to prove the new magnetic Caffarelli–

ic interpolation inequalities without weights

er on $\mathbb{R}^2$ the Aharonov–Bohm magnetic potential $\mathbf{A}$ given by (1.1).
magnetic inequality

$$|\nabla_{\mathbf{A}}\,\psi|^2 \ge |\nabla|\psi||^2 \quad \text{a.e. in } \mathbb{R}^2$$

$\in (2,\infty)$ and $\lambda > 0$, the Gagliardo–Nirenberg inequality

$$\mathbb{R}^2) + \lambda\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^2)} \ge \mathsf{C}_p\,\lambda^{\frac{p}{2}}\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)}, \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2) \quad (4.1)$$

constant C_p, we deduce that

$$\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \lambda\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \ge \mu_{a,p}(\lambda)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^2)}, \quad \forall\,\psi \in \mathrm{H}^1_a(\mathbb{R}^2). \quad (4.2)$$

3] for details. Here $\mu_{a,p}(\lambda)$ is the optimal constant in (4.2) for any
λ and, as a function of λ, $\mu_{a,p}(\lambda)$ is monotone increasing and con-
hat right-hand sides in (4.1) and (4.2) involve norms with respect to
easure. It turns out that $\mu_{a,p}(\lambda)$ is equal to the best constant of the
problem.

4.1. *Let $a \in \mathbb{R}$ and $p \in (2,\infty)$. The optimal constant in* (4.2) *is*

$$\mu_{a,p}(\lambda) = \mathsf{C}_p\,\lambda^{\frac{p}{2}}, \quad \forall\,\lambda > 0$$

s not achieved on $\mathrm{H}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$ *if $a \in \mathbb{R}\setminus\mathbb{Z}$.*

nstruction we know that $\mu_{a,p}(\lambda) \ge \mathsf{C}_p\,\lambda^{p/2}$. By taking an optimal
(4.1) and considering $\psi_n(x) = \psi(x + n\,\mathsf{e})$ with $n \in \mathbb{N}$ and $\mathsf{e} \in \mathbb{S}^1$, we
is equality.
ve by contradiction that equality is not achieved. If $\psi \in \mathrm{H}^1(\mathbb{R}^2) \cap$
mal, let $\phi = e^{i\,a\,\theta}\psi$. Since

$$\|\nabla_{\mathbf{A}}\,\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\partial_r\psi|^2 + \frac{|\partial_\theta\phi|^2}{|x|^2}\,dx$$

in (4.1) is achieved by functions with a constant phase only, this
$\phi = 0$ a.e., a contradiction with the periodicity of ψ with respect to
$\notin \mathbb{Z}$. □

n 4.1 means that the Aharonov–Bohm magnetic potential plays no
ighted interpolation inequalities. This is why it is natural to introduce
ns with adapted scaling properties.

ic Caffarelli–Kohn–Nirenberg inequalities in $\mathbb{R}^2$

–Kohn–Nirenberg inequality

blished in [9] and, earlier, in [25]. The exponent $\mathsf{b} = \mathsf{a} + 2/p$ is deter- scaling invariance and as p varies in $(2,\infty)$, the parameters a and b $\mathsf{a} < \mathsf{b} \le \mathsf{a} + 1$ and $\mathsf{a} < 0$. The case $\mathsf{a} > 0$ can be considered in an unctional space after a Kelvin-type transformation: see [10, 16], but onsider this case here. As noticed for instance in [16], by considering $x)$, Ineq. (4.3) is equivalent to the *Hardy-Sobolev inequality*

$$u|^2\,dx + \mathsf{a}^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx \ge \mathsf{C}_{\mathsf{a}} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2}\,dx\right)^{2/p}, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^2). \tag{4.4}$$

unctions for (4.3) are radially symmetric if and only if

$$\mathsf{b} \ge \mathsf{b}_{\mathrm{FS}}(\mathsf{a}) := \mathsf{a} - \frac{\mathsf{a}}{\sqrt{1+\mathsf{a}^2}}$$

[15, 23]. We refer to [7] for more details and for the proof of the netic Hardy–Sobolev inequality.

1 ([7]). *Let $a \in [0,1/2]$, $\mathbf{A}$ as in (1.1) and $p > 2$. For any $\lambda > -a^2$, otimal function $\lambda \mapsto \mu(\lambda)$ which is monotone increasing and concave*

$$dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx \ge \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}, \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2). \tag{4.5}$$

, the optimal function in (4.5) is

$$= \left(|x|^\alpha + |x|^{-\alpha}\right)^{-\frac{2}{p-2}}, \quad \forall\, x \in \mathbb{R}^2, \quad \textit{with } \alpha = \frac{p-2}{2}\sqrt{\lambda + a^2},$$

g and a multiplication by a constant, if

$$\lambda \le \lambda_\star := 4\,\frac{1 - 4\,a^2}{p^2 - 4} - a^2.$$

$a \in [0,1/2]$ and $\lambda > \lambda_\bullet$ with

$$\frac{8\left(\sqrt{p^4 - a^2\,(p-2)^2\,(p+2)\,(3\,p-2)} + 2\right) - 4\,p\,(p+4)}{(p-2)^3\,(p+2)} - a^2,$$

etry breaking, i.e. the optimal functions are not radially symmetric.

t computation shows that $\lambda_\star < \lambda_\bullet$ for any $a \in (0,1/2)$, and so there is we do not know whether the optimal functions in (4.5) are symmetric theless, as shown in [7], the values of $\lambda_\star$ and $\lambda_\bullet$ are numerically very other. If $\lambda \le \lambda_\star$, the expression of $\mu(\lambda)$ is explicit and given by

$$) = \frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,(\lambda + a^2)^{1+\frac{2}{p}} \left(\frac{2\,\sqrt{\pi}\,\Gamma\left(\frac{p}{p-2}\right)}{(P-2)\,\Gamma\left(\frac{p}{p-2} + \frac{1}{2}\right)}\right)^{1-\frac{2}{p}}.$$

y the equivalence of (4.3) and (4.4), we prove that the magnetic v inequality (4.5) is equivalent to an interpolation inequality of n–Nirenberg type in the presence of the Aharonov-Bohm magnetic

1 (Magnetic Caffarelli–Kohn–Nirenberg inequality). *Let* $p \in$ $\mathbf{A}$ *as in* (1.1) *for some* $a \in [0,1/2]$ *and* $\mathsf{a} \le 0$. *With* μ *as in Theo-* *ny* $\gamma < \mathsf{a}^2 + a^2$, *we have that*

$$\ldots\frac{{}_{\mathbf{A}}\,\phi|^2}{|^{2\mathsf{a}}}\,dx \ge \gamma \int_{\mathbb{R}^2} \frac{|\phi|^2}{|x|^{2\mathsf{a}+2}}\,dx + \mu(\mathsf{a}^2-\gamma)\left(\int_{\mathbb{R}^2} \frac{|\phi|^p}{|x|^{\mathsf{a}\,p+2}}\,dx\right)^{2/p}, \quad \forall\,\phi \in \mathcal{D}(\mathbb{R}^2;\mathbb{C})$$

$\lambda = \mathsf{a}^2 - \gamma$ *is the optimal constant.*

of symmetry and symmetry breaking in Theorem 4.1 have their exact Corollary 4.1. Details are left to the reader.

s consider the function $\phi(x) = |x|^{\mathsf{a}}\,\psi(x)$ and observe that

$$\int_{\mathbb{R}^2} \frac{|\nabla_{\mathbf{A}}\,\phi|^2}{|x|^{2\mathsf{a}}}\,dx = \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \mathsf{a}^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx$$

by applying (4.5) to ψ with $\lambda = \mathsf{a}^2 - \gamma$. □

etic Hardy inequality in $\mathbb{R}^2$

equence of Theorem 4.1 is the following *magnetic Keller–Lieb–* *ality*, which can be found in [7, Theorem 1]. Let $q = p/(p-2)$. The energy λ_1 of the magnetic Schrödinger operator $-\Delta_{\mathbf{A}} - \phi$ on $\mathbb{R}^2$ is

$$\Delta_{\mathbf{A}} - \phi) \ge -\lambda(\mu) \quad \text{where } \mu = \left(\int_{\mathbb{R}^2} |\phi|^q\,|x|^{2\,(q-1)}\,dx\right)^{1/q} \tag{4.6}$$

$\mu)$ is a convex monotone increasing function on $\mathbb{R}^+$ such that $= -a^2$, defined as the inverse of $\lambda \mapsto \mu(\lambda)$ of Theorem 4.1. Again al in (4.6) and the cases of symmetry and symmetry breaking are in e with the ones of Theorem 4.1.

ely, let us consider a function ϕ on $\mathbb{R}^2$. We can estimate an associated ödinger energy from below by

$$\ldots{}_{\mathbf{A}}\,\psi|^2 - \tau\,\frac{\phi}{|x|^2}\,|\psi|^2\Big)\,dx \ge \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx$$

nequality, with $q = p/(p-2)$, for an arbitrary parameter $\tau > 0$. For e choice of τ, we obtain the following result.

2 (A Magnetic Hardy Inequality). *Let $q \in (1,2)$ and* $\mathbf{A}$ *as ome $a \in [0,1/2]$. Then for any function $\phi \in \mathrm{L}^q\left(\mathbb{R}^2\,|x|^{-2}\,dx\right)$, we have*

$$\psi|^2\,dx \geq \mu(0)\left(\int_{\mathbb{R}^2} \frac{|\phi|^q}{|x|^2}\,dx\right)^{-\frac{1}{q}} \int_{\mathbb{R}^2} \frac{\phi}{|x|^2}\,|\psi|^2\,dx \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2),$$

the best constant in (4.5)*. Finally, when $a^2 \leq 4/(12+p^2)$, we know* (0) *explicitly*:

$$\mu(0) = \frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,a^{2+\frac{4}{p}}\left(\frac{2\,\sqrt{\pi}\;\Gamma\left(\frac{p}{p-2}\right)}{(p-2)\,\Gamma\left(\frac{p}{p-2}+\frac{1}{2}\right)}\right)^{1-\frac{2}{p}}.$$

v–Bohm Magnetic Hardy Inequalities in $\mathbb{R}^3$

n we address the issue of improved magnetic Hardy inequalities with –Bohm magnetic potential in dimension $d=3$ as defined by (1.2). nprove upon [19, Sec. V.B], including the case of a constant magnetic

proved Hardy inequality with radial symmetry

.B], it is proved that *for all $a > 0$, there is a constant $\mathcal{C}(a)$ such that* $\in [0,1/2]$ *and*

$$_3\,|\nabla_{\mathbf{A}}\,\psi|^2\,dx \geq \left(\frac{1}{4}+\mathcal{C}(a)\right)\int_{\mathbb{R}^3}\frac{|\psi|^2}{|x|^2}\,dx, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3). \tag{5.1}$$

r an angular dependence, we have the following result.

1. *Let* $\mathbf{A}$ *as in* (1.2)*, $a \in [0,1/2]$ and $q \in (1,+\infty)$. Then, for all*

$$\psi|^2\,dx \geq \int_{\mathbb{R}^3}\left(\frac{1}{4}+\frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^2)}}\,\phi(\omega)\right)\frac{|\psi|^2}{|x|^2}\,dx, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3).$$

$x|$ *and $\mu_{a,p}$ is defined as in Proposition* 2.2*.*

e $a \in [0,1/2]$, according to Proposition 2.2, we find in the limit case hat $\mu_{a,2}(0) \geq \Lambda_a = a\,(a+1)$ and improve the estimate (5.1) to 1) if $\phi \equiv 1$.

s use spherical coordinates $(r,\omega) \in [0,+\infty)\times\mathbb{S}^2$. The result follows nsion of the square and an integration by parts in part of the Dirichlet integral, and from Corollary 2.2 for the angular

$\square$

roved Hardy inequality with cylindrical symmetry

l Hardy inequality (without angular kinetic energy) of Lemma 3.7 be combined into the following improved Hardy inequality in presence potential.

2. *Let* $\mathbf{A}$ *as in* (1.2), $a \in [0, 1/2]$, $p > 2$, $q = p/(p-2)$ *and* $\phi \in \mathrm{L}^q(\mathbb{S}^1)$. $\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)$, *we have*

$$dx \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|^2}\,dx + \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}} \iiint_{\mathbb{R}^+\times[0,2\pi)\times\mathbb{R}} \frac{\phi(\theta)}{\rho^2}\,|\psi(\rho,\theta,z)|^2\,d\mu.$$

t the inequality is a strict improvement upon the Hardy inequality gnetic potential combined with the diamagnetic inequality. A sim- n is particularly illuminating is $\phi \equiv 1$ with $a^2 \leq 1/(p+2)$ so that ccording to Proposition 2.1, in which case we obtain that

$$\nabla_{\mathbf{A}}\,\psi|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|^2}\,dx + a^2\int_{\mathbb{R}^3} \frac{|\psi|^2}{|\rho|^2}\,dx, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3).$$

ments

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ov and D. Bohm, Significance of electromagnetic potentials in the quantum ys. Rev. **115** (1959) 485–491.

n, I. W. Herbst and B. Simon, Schrödinger operators with magnetic fields: s in homogeneous magnetic field, *Comm. Math. Phys.* **79** (1981) 529–572.

and M. Émery, Inégalités de Sobolev pour un semi-groupe symétrique, *d. Sci. Paris Sér. I Math.* **301** (1985) 411–413.

an and A. Tonomura, The Aharonov–Bohm effects: Variations on a subtle *ys. Today* **62** (2009) 38–43.

er, Sharp Sobolev inequalities on the sphere and the Moser–Trudinger *Ann. of Math.* **138**(2) (1993) 213–242.

aut-Véron and L. Véron, Nonlinear elliptic equations on compact Rie-

ure, J. Dolbeault, M. J. Esteban, A. Laptev and M. Loss, Symmetry results
mensional inequalities for Aharonov–Bohm magnetic fields, *Comm. Math.*
5 (2020) 2071–2087.
ure, M. Nys and J. Van Schaftingen, Properties of ground states of nonlinear
ger equations under a weak constant magnetic field, *J. Math. Pures Appl.*
2019) 123–168.
elli, R. Kohn and L. Nirenberg, First order interpolation inequalities with
Compositio Math. **53** (1984) 259–275.
a and Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: Sharp
, existence (and nonexistence), and symmetry of extremal functions, *Comm.*
l. Math. **54** (2001) 229–258.
, Entropies, convexity, and functional inequalities: On Φ-entropies and Φ-
nequalities, *J. Math. Kyoto Univ.* **44** (2004) 325–363.
ult, M. J. Esteban and A. Laptev, Spectral estimates on the sphere, *Anal.*
014) 435–460.
ult, M. J. Esteban, A. Laptev and M. Loss, Interpolation inequalities and
stimates for magnetic operators, *Ann. Henri Poincaré* **19** (2018) 1439–1463.
ult, M. J. Esteban, A. Laptev and M. Loss, Magnetic rings, *J. Math. Phys.*
051504.
ult, M. J. Esteban and M. Loss, Rigidity versus symmetry breaking via
flows on cylinders and Euclidean spaces, *Invent. Math.* **206** (2016) 397–440.
ault, M. J. Esteban, M. Loss and G. Tarantello, On the symmetry of
for the Caffarelli–Kohn–Nirenberg inequalities, *Adv. Nonlinear Stud.* **9**
3–726.
ult and X. Li, Φ-Entropies: Convexity, coercivity and hypocoercivity for
lanck and kinetic Fokker–Planck equations, *Math. Models Methods Appl.*
018) 2637–2666.
berg and R. Siday, The refractive index in electron optics and the principles
ics, *Proc. Phys. Soc. B* **62** (1949) 8–21.
n and F. Portmann, A magnetic contribution to the Hardy inequality, *J.*
ys. **55** (2014) 022101.
Rayleigh-type isoperimetric inequality with a homogeneous magnetic field,
Partial Differential Equations **4** (1996) 283–292.
E. M. Harrell and M. Loss, Optimal eigenvalues for some Laplacians and
ger operators depending on curvature, in *Mathematical Results in Quantum*
s (*Prague*, *1998*), Oper. Theory Adv. Appl., Vol. 108 (Birkhäuser, Basel,
. 47–58.
, D. Krejčiřík, A. Laptev and L. Vega, On the improvement of the Hardy
due to singular magnetic fields, *Comm. Partial Differential Equations* **45**
02–1212.
nd M. Schneider, Perturbation results of critical elliptic equations of Caffa-
n–Nirenberg type, *J. Differential Equations* **191** (2003) 121–142.
ann-Ostenhof and A. Laptev, Hardy inequalities with homogeneous weights,
Anal. **268** (2015) 3278–3289.
, Some integral inequalities and their applications in the theory of differ-
unctions of several variables, *Mat. Sb.* (*N.S.*) **54**(96) (1961) 331–380.
and T. Weidl, Hardy inequalities for magnetic Dirichlet forms, in *Math-*
Results in Quantum Mechanics (*Prague*, *1998*), Oper. Theory Adv. Appl.,

and M. Loss, *Analysis*, 2nd edn., Graduate Studies in Mathematics, Vol. 14
Mathematical Society, Providence, RI, 2001).
Orthogonal Polynomials, 4th edn. American Mathematical Society, Collo-
blications, Vol. XXIII (American Mathematical Society, Providence, R.I.,